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SECTORS MAY USE MULTIPLE TECHNOLOGIES SIMULTANEOUSLY: THE RECTANGULAR CHOICE-OF-TECHNOLOGY MODEL WITH BINDING FACTOR CONSTRAINTS (REVISED)

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Sectors May Use Multiple Technologies Simultaneously: The Rectangular Choice-of-Technology Model with Binding Factor Constraints

Abstract. We develop the rectangular choice-of-technology model with factor constraints, or RCOT, a linear programming input-output model for analysis of the economy of a single region. It allows for one or more sectors to operate more than one technology simultaneously, with the relatively lowest-cost one supplemented by others if it encounters a binding factor constraint. The RCOT model solves for sector outputs, goods prices that are set by the highest-cost technologies in use, and scarcity rents that correspond to binding factor constraints experienced by the lower-cost technologies. The model is motivated by the fact that mineral deposits of different qualities may be exploited simultaneously, as may primary and recycled sources for the same materials or irrigated and rainfed techniques for producing the same crop. RCOT generalizes Carter's square choice-of-technology model, in particular adding the factor constraints that allow several alternatives to operate simultaneously. The Appendix gives a numerical example.

Keywords: input-output model, linear programming model, choice of technology, prices and rents, binding factor constraints

JEL codes: C67, O33, Q32Sectors May Use Multiple Technologies Simultaneously: The Rectangular Choice-of-Technology Model with Binding Factor Constraints

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1. Introduction

Every good may be produced in different establishments using somewhat different technologies. However, for some goods the range of variation in simultaneously utilized input structures is especially wide, with significant economic and environmental implications. This is particularly true for resource extraction sectors exploiting deposits of heterogeneous qualities and limited supply. If a low-cost mine does not produce enough to satisfy demand, it may be necessary to supplement this output using a different technology at other mines with less accessible, lower grade deposits. An economy may rely on both low-input, rain-fed agriculture while also making use of higher-input, irrigated methods for producing the same crop on other land. Metal-processing industries may exploit recycled materials as available, supplemented by extraction of primary ores to fill their remaining needs. These choices among alternatives will reflect cost differences, and in the case of scarce factors several options may be simultaneously operational. It is especially important to capture wide variations in cost structures for producing a particular good, rather than relying on an average only, in the context of a model of the world economy based on comparative advantage, since in this framework it is the relatively highest-cost producer who sets the world price. Relying on an average technology in a producing country would set the world price artificially low.

One of the features of the basic input-output model is that each sector is assumed to produce a single characteristic output and to do so using a single average input structure, or technology. It is well understood that both of these are simplifying assumptions. Any model substantially simplifies the complexities of a real economy, and the widespread use of square input-output tables and matrices, not only by economists but increasingly also by industrial ecologists, suggests that the resulting representation is deemed adequate for most purposes. This is true because the variations among co-existing technologies within a sector are generally much smaller than those among sectors producing different outputs. But it can be important to represent the options among alternative input structures in selected cases, especially for the extractive sectors, which need to match technologies to the heterogeneous qualities of factor endowments (land, water, in-ground and offshore resources) and shift to different technologies, often with step increases in costs, as higher quality endowments are exhausted.

The possibility of choosing among alternate technologies, according to some criterion, leads us from the basic input-output (IO) model to a linear programming (LP) formulation of the input-output model. This paper develops the quantity and price LP/IO models for what we call the rectangular choice-of-technology model with factor constraints, or RCOT model, and illustrates its use for analysis of a single region's economy and then for simultaneous, interdependent choices in multiple regions. The term rectangular refers to the additional columns, but not rows, that are inserted into the initially square **A** matrix to represent the additional technologies. The number of rows remains unchanged because the purchasers of a product are assumed to be indifferent to the technology used in its production. The choice criterion is represented by an objective function and since the modeling framework is intended for evaluating alternative scenarios for sustainable development, we systematically choose to minimize costs for given consumption demand rather than to maximize consumption. The resulting model is compared to previous models, referred to in this paper as square models, that allow a choice of technology by means of multiple, square **A** matrices to account for a given number (the same number for all sectors in the case of square models) of additional technologies.

For the questions we set out to address, described below, we show that the rectangular model of a single region's options is superior to the square model because: (1) it obtains a well-defined, unique solution by contrast with the indeterminateness inherent in the square model, and (2) it eliminates the need to maintain and manipulate redundant information that characterizes the square model. Only in the special case when all sectors have the same number of distinct technological options are the two methods mathematically equivalent. In the general case when each sector may have any number of options, and many if not most have only one, a single rectangular **A** matrix with contiguous representation of all the options available to each sector has clear advantages: it is not only more compact but also conceptually and practically more compelling than the "trick" of using multiple square matrices with a lot of redundancy.

In Section 2 we review the small literature on square choice-of-technology models, where all sectors have exactly two options, and indicate the differences in both objectives and assumptions relative to the RCOT model. Section 3 presents the RCOT model, first without factor constraints as is the case in applications of the square model, and it then introduces factor constraints and describes their implications. Both the quantity model and the price model are given, and their properties are derived. Section 4 generalizes the square choice-of–technology model, first for the case where all sectors have an arbitrarily large but identical number of alternative technologies and then for the more general case where different sectors have different numbers of choices. The section concludes by comparing the rectangular model and square model. Section 5 extends the rectangular model from a single region to multiple, interdependent regions, situating them in the context of an input-output model of the world economy, the World Trade Model, or WTM (Duchin 2005). Section 6 describes the properties of factor rents and goods prices in the RCOT price model, and the final section summarizes and discusses data sources and next steps. The Appendix formulates the basic input-output model as an LP/IO and contains numerical examples to illustrate the various models.

2. Applications of the Square Choice-of-Technology Model with Two Choices

In her important book-length study of technological change in the U.S. economy, Carter introduced into the basic, one-region input-output model the choice between two alternative technologies for producing a sector's characteristic product (Carter 1970). With the objective of explaining the technological changes that had taken place between 1947 and 1958, Carter provided to each sector in 1958 the choice between the input structure in the 1958 input-output table and that in the table for 1947. She formulated the problem as a linear program (LP) in terms of two coefficient matrices, $\mathbf{A}_{1947}$ and $\mathbf{A}_{1958}$, representing the old and new options, respectively. The program assured that 1958 final demand was satisfied at the lowest cost in terms of value-added, the latter being calculated as the sum of labor costs (labor inputs times a wage rate) and interest charges:

Min $Z = \mathbf{v}'_{1947}\mathbf{x}_{1947} + \mathbf{v}'_{1958}\mathbf{x}_{1958}$

s.t. $(\mathbf{I} - \mathbf{A}_{1947})\mathbf{x}_{1947} + (\mathbf{I} - \mathbf{A}_{1958})\mathbf{x}_{1958} \geq \mathbf{y}_{1958}$.

The optimal solution relied upon most sectors' selecting the 1958 technology. However, for several sectors including iron and petroleum mining, the 1947 technologies would have been more efficient choices in 1958 but were probably no longer plausible options. The superiority of the 1958 technologies turned out to be robust under a range of wage rates and different assumptions about the interest rates in the two years.

Leontief (1986) used the same model with two alternatives for each sector to analyze the choice between an old technology, represented by a column in the U.S. input-output matrix for 1977, and a new technology, in a projected matrix for 2000. He examined how the choices would change under different assumptions about relative prices for capital and labor, employing an explicit matrix of capital coefficients and an index of real wages, and allowing for trade-offs between the two factor prices. He found that health-care and education sectors would choose the old options in all cases and explained that outcome by government subsidies to the newer option to enable quality improvements. At the other extreme, the sectors producing computers and semiconductors would adopt the new technologies under the full range of combinations of factor prices that were examined (ranging from zero to 40% as the rate of return on capital).

Duchin and Lange (1995) employed the same model with technological choices for 1963 and 1977, based on input-output tables for those two years, and then 1977 and 2000, using a projected matrix for 2000. They sought to distinguish whether opting for the newer technology in 1977 (or 2000) was advantageous for a sector if only that sector adopted the new technology, or if the benefit of that choice was dependent on other sectors' also adopting the newer technological options. They found that typically the newer technology was cost-saving in both cases, but that some sectors would have made clearly suboptimal choices in selecting the newer technology if other sectors – in particular the computer-related sectors -- did not also do so.

In each of these three studies, all sectors have exactly two distinct technologies to choose between, the options correspond to the average technologies available in two different years, and the alternatives are represented as the columns of two square matrices of dimension *n* x *n*. The solution selects one and only one of the two technologies to be employed in each sector. Since it is known in advance that the newer technology was in fact put in place in the case of an *ex post* analysis, the square model makes it possible to discover those instances where this decision does not appear to have been cost saving for individual sectors. Carter's LP formulation has the important property of allowing producers a choice of technologies, to which consumers of the homogeneous output are indifferent.

More recently, Julia and Duchin (2007) developed a square formulation that is more general in several ways. They allow for any number of alternative technologies for a given sector in a single region, where alternatives represent current options and not all sectors are required to have the same number of options. Individual regions make these choices *simultaneously* within the context of a model of the world economy that permits inter-regional trade. This framework for the first time offers a nesting of intra-regional choices, within a framework of inter-regional choices, among technological options. To study the effects of climate change on global agriculture, Julia and Duchin allow each region up to six options for crop production and six for livestock production, with the choices depending ultimately upon competition for different qualities of land. The representation for a single region takes the form of six region-specific **A** matrices of order n x n. (It also requires six region-specific matrices of factor requirements per unit of output. The representation of factor requirements and the imposition of factor constraints are discussed below.) For those sectors that have only a single technological option, the same column of coefficients is repeated in all matrices. Applying the World Trade Model (Duchin 2005) to twelve regions, the cost structures for all the alternative technologies in all regions are considered simultaneously to determine the lowest-cost international distribution of production and corresponding world prices. The WTM also imposes factor constraints, with implications described below.

All the models described in this section utilize multiple square **A** matrices in order to describe the alternative technologies. In the next section we replace these square matrices by a single rectangular form of the **A** matrix. None of the one-region studies described above considered the implications of constraints on factor availability, and in the absence of factor constraints each sector operates using a single, lowest-cost technology. Factor constraints, however, may require a sector to use more than one of its available technologies, starting with the cheapest technology until it runs into a binding constraint and then choosing the next best option. For each constraint that is actually binding, some sector will utilize one additional technology. All sectors may compete for some factors, such as capital and labor, while other factors may be sector-specific or even technology-specific: for example, the oil extraction sector will apply one technology (on average) to extract crude oil from land-based wells, limited by the amount of crude and the production capacity, and if more oil is needed, different technology for extracting the distinct reserves in deep-sea offshore oil fields. No sector other than oil extraction will be directly constrained by these sector-specific capacity constraints. Other constraints may affect more than one but not all sectors. This may be the case if there is competition among all extraction sectors for, say, mining engineers who are in short supply. We introduce an explicit representation of factor requirements in the models that follow.

3. Rectangular Choice-of Technology Model in a Single Region

3.1. The Basic Input-Output Model with a Single Technology for Each Sector

An economy with n sectors, each using a single given technology, and k factors of production is described by

$$(\mathbf{I} - \mathbf{A})\,\mathbf{x} = \mathbf{y} \text{ and}$$
$$\boldsymbol{\varphi} = \mathbf{Fx},$$

where the output vector **x** and final demand vector **y** are n x 1, the coefficient matrix **A** is n x n, the factor use vector **φ** is k x 1, and the factor requirement matrix **F** is k x n. With **y** and **A** square, **x** and **φ** can be uniquely determined as

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} \quad \text{and}$$
$$\boldsymbol{\varphi} = \mathbf{F}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}.$$

This model makes use of **F**, the matrix of factor requirements per unit of output, where the factors, including resources, will generally be measured in physical units. To maintain the representation of factors in physical units throughout the analysis, we introduce a vector of factor prices, $\boldsymbol{\pi}$; $\mathbf{F}'\boldsymbol{\pi}$ will replace the vector of value-added, **v**, in Carter's objective function as the measure of factor costs.

The equations above determining output and factor use comprise the input-output quantity, or primal, model. The price, or dual, model,

$$\mathbf{p} = (\mathbf{I} - \mathbf{A}')^{-1}\mathbf{F}'\boldsymbol{\pi},$$

determines the unit prices of products.

Section A of the Appendix shows how this input-output model can be formulated as a linear program, describing it in terms of the LP nomenclature that will be used later in this paper, and provides a numerical example.

3.2. Rectangular Model with Up to q Alternative Technologies for Each Sector

Say that sector i has the choice among t_i alternative technologies. Then

$$t = \sum_{i=1}^{n} t_i\,, \;\; t_i \geq 1, \quad t \geq n$$

is the total number of technologies available to the economy as a whole. We can rewrite the basic model to accommodate these alternative technologies if we augment **x** by inserting additional components, and augment **I, A** and **F** by inserting additional columns, corresponding to the alternate technologies. The resulting model can be expressed formally as

$$(\mathbf{I^*} - \mathbf{A^*})\mathbf{x^*} = \mathbf{y}$$
$$\boldsymbol{\varphi} = \mathbf{F}^*\mathbf{x}^*$$

where $\mathbf{x^*}$ is t x 1, $\mathbf{I^*}$ and $\mathbf{A^*}$ are n x t, $\mathbf{y}$ remains n x 1, $\mathbf{F^*}$ is k x t, and $\boldsymbol{\varphi}$ remains k x 1. (In general, k can be expected to increase from the basic model if some technology-specific factors are included for the new options. For simplicity, we retain k instead of using a more complex notation.) Thus the ith row of I* will contain as many 1's as there are alternative technologies for sector i. The rectangular structure of $\mathbf{A^*}$ means that in general there is no unique solution for $\mathbf{x^*}$ unless we specify a criterion for selecting among the feasible solutions. We choose to minimize total factor use, where individual factors are weighted by factor prices, $\boldsymbol{\pi}$, as this objective is consistent with our sustainable development framework. The resulting LP/IO formulation, model (1), is

$$\begin{aligned} &\min Z = \boldsymbol{\pi}' \mathbf{F^*x^*} \\ &\text{s.t. } (\mathbf{I^*} - \mathbf{A^*})\mathbf{x^*} \geq \mathbf{y}, \end{aligned} \tag{1}$$

where the inequality imposes the final demand constraints.

In the theorems that follow we will assume that output of every sector is strictly positive. If a sector were to have no output that would constitute a degenerate case; it would be dealt with by removal of the sector.

Theorem 1: The RCOT model without factor constraints (model (1)) has the following properties:

(a) Each of the n sectors will use one and only one of its available technologies.
(b) Each sector will produce exactly that amount of output required to satisfy final demand.
(c) The technology it uses will be its lowest-cost option, given the choices for the other sectors.

Proof:
(a) This problem has $t \geq n$ decision variables ($\mathbf{x^*}$) and n functional constraints (see Appendix for definitions). LP theory specifies that the number of variables with non-zero values (*i.e.*, basic variables) cannot exceed the number of functional constraints (Hillier and Liebermann, 2010, p. 96). Since each of the n sectors has non-zero output by assumption, it must be using at least one of its technologies. But the number of technologies in use cannot exceed n, so no sector can use multiple technologies.
(b) We write the final demand constraints in model (1) as equality constraints, where $\mathbf{s}$ is an n x 1 vector of slack variables:

$$(\mathbf{I^*} - \mathbf{A^*})\mathbf{x^*} = \mathbf{y} + \mathbf{s}.$$

This LP has a total of $t + n$ variables, t decision variables and n slack variables**.** Since only n variables can be non-zero and, from part a, n of the decision variables are non-zero, all n elements of $\mathbf{s}$ must be zero, meaning that no surplus output is produced.
(c) The cost of a sector's inputs is the cost of factors required directly or indirectly. Since the objective function minimizes total factor cost, each sector will use that one of its

technologies that assures the lowest total cost for the economy as a whole, given the choices of the other sectors. End of proof.

Note that the choice of technology for a given sector is interdependent with those for other sectors. If, say, two sectors have technological options, the lowest-cost choice for each sector is dependent on that for the other. For this reason, one cannot speak of a sector's lowest-cost technology except in the context of the economy as a whole. In the remainder of this paper, this is what is meant by a sector's lowest-cost technology. The economy-wide context also applies when a sector's highest-cost technology is discussed.

As each sector uses one technology, the corresponding columns comprise a square matrix, and prices can be determined from the input-output dual. Alternatively, these prices can be determined from the LP/IO dual without first reducing A* to a square matrix. (In fact, these prices can be determined directly from model (1) but it is illustrative to show the dual model explicitly.) For every LP problem (the primal problem), there is a well-defined dual LP problem (Hillier and Lieberman, 2010). Model (2), the dual to model (1), is

$$\begin{aligned} &\max W = \mathbf{p}'\mathbf{y} \\ &\text{s.t. } (\mathbf{I^*} - \mathbf{A^*})'\mathbf{p} \leq \mathbf{F^*}'\boldsymbol{\pi}, \end{aligned} \tag{2}$$

which determines **p**, the *n x 1* vector of product prices. One of the numerous relationships between a primal and its dual is that min Z = max W. A numerical example of both the quantity and price models is provided in Section B of the Appendix.

3.3. Rectangular Model with Up to q Alternative Technologies for Each Sector and with Factor Constraints

If in addition to production constraints we impose constraints on factor availability, then model (3) becomes

$$\begin{aligned} &\min Z = \boldsymbol{\pi}'\mathbf{F^*x^*} \\ &\text{s.t. } (\mathbf{I^*} - \mathbf{A^*})\mathbf{x^*} \geq \mathbf{y} \\ &\qquad \mathbf{F^*x^*} \leq \mathbf{f}, \end{aligned} \tag{3}$$

where **x*** is t x 1, **I*** and **A*** are n x t, **y** remains n x 1, **F*** is k x t, and **f** is a k x 1 vector of factor endowments. Again, we can rewrite the n + k functional constraints as the equality constraints

$$\begin{aligned} &(\mathbf{I^*} - \mathbf{A^*})\mathbf{x^*} = \mathbf{y} + \mathbf{s_1} \\ &\qquad \mathbf{F^*x^*} = \mathbf{f} - \mathbf{s_2} \end{aligned}$$

where $\mathbf{s_1} \geq 0$ is an n x 1 vector of slack output variables and $\mathbf{s_2} \geq 0$ is a k x 1 vector of slack factor use variables.

As in the previous case, there is a well-defined dual price LP/IO model corresponding to the primal of model (3):

$$\begin{aligned} &\max W = \mathbf{p}'\mathbf{y} - \mathbf{r}'\mathbf{f} \\ &\text{s. t.} \quad (\mathbf{I^*} - \mathbf{A^*})'\mathbf{p} - \mathbf{F^*}'\mathbf{r} \leq \mathbf{F^*}'\boldsymbol{\pi}, \end{aligned} \tag{4}$$

where **p** is the *n x 1* vector of sectoral product prices and **r** is the *k x 1* vector of factor scarcity rents. (Model (4), unlike model (2), can generally not be written as a standard input-output price model even after solving model (3) because in this case it is not possible to reduce **A*** to a square matrix if any sector has more than one technology simultaneously in operation.) Note that the vector of factor prices, $\pi + r$, has two components, where π is exogenous and r is endogenous. The economic interpretation of this expression is discussed in Section 6.

Theorem 2: The RCOT primal, or quantity, model with factor constraints (model (3)) has the following properties:

a) If no factor constraints are binding, a sector will use one and only one technology, and it will be its lowest-cost technology.
b) If there are any binding factor constraints, then one or more sectors may use multiple technologies. These will be those combinations of technologies that minimize factor costs for the economy as a whole.

Proof:

a) This LP has *t* decision variables (x*) and n + k slack variables. Since there are $n + k$ functional constraints, no more than $n + k$ variables can be basic (i.e., non-zero). If no factor constraints are binding, k slack variables are non-zero, meaning that only up to n decision variables may be non-zero, returning to the case of Theorem 1.
b) Each binding factor constraint results in one slack variable being zero (*i.e*., non-basic). Thus, if *m* factor constraints are binding, only $k – m$ slack variables may be non-zero, allowing the number of non-zero elements of x* to increase to $n + m$. Minimal cost follows from the definition of the objective function. End of proof.

If a factor used by more than one sector becomes scarce (*i.e.*, the factor constraint is binding), and if there is still a feasible solution, one of the sectors using the scarce factor may need to operate two technologies simultaneously. If it is a sector-specific factor that is scarce, that sector may operate two technologies at once with both using the scarce factor. However, if it is a technology-specific factor that is scarce, a second technology for that sector, not dependent on the scarce factor, will need to be put into operation.

Theorem 3: The RCOT dual, or price, model with factor constraints (model (4))

simultaneously determines prices of products and scarcity rents on fully utilized factors.

a) The price for each product is that determined by the highest-cost technology actually in production for the sector producing that product.
b) For any lower-cost technologies simultaneously in use, their scarce factors earn rents that jointly account for the difference between their costs and the prices established by the highest-cost technology in use for that product.

Proof:

a) By the Complementary Slackness theorem of LP (Hillier and Lieberman, 2010), a dual price slack is non-basic if it corresponds to a decision variable in the primal that is basic. In other words for any technology that is in use, the price slack is zero and the unit price is equal to the sum of costs. This must be true in particular for the most costly technology, which (except for a degenerate case) depends on no factors that are fully utilized. With no non-zero scarcity rents in its price equation, its costs set the price for that sector.
b) Any lower-cost technology also in operation for a given sector must use at least one scarce factor, and therefore includes at least one non-zero scarcity rent in its price equation. Since by a) the price slack is zero, the endogenous rent on that factor (or those factors) is determined such that the total unit costs, including the scarcity rent(s) on factors, are equal to the sectoral price set by the highest-cost technology. End of proof.

The numerical example in Section C of the Appendix illustrates the case of an economy with three sectors, two of which have more than one technological option, where one sector has more than one technology simultaneously in use. The example quantifies the output using each technology, the price for each sector's product, and the rent on the scarce factor. It also shows how endogenous factor rents and goods prices change in response to a change in final demand, in this case a reduction in final demand that switches the status of a factor from scarce to abundant.

4. Square Choice-of-Technology Model in a Single Region

The square model can be used to offer *exactly* q choices, or *up to* q choices, to each sector. Following Carter, these models are described in the absence of factor constraints. The Appendix provides a numerical example applying the square model using the same data as for the rectangular model of Section 3.

4.1. Square Model with Exactly q Distinct Technologies for all Sectors

First, consider the case for a single region, where every one of its n sectors has exactly q distinct production technologies, numbered from 1 to q. If the options correspond to different years, the order from 1 to q can be chronological; otherwise it will be arbitrary. Thus the total number of available technological alternatives is

$$d = n*q, \quad d \geq n.$$

We form q n x n matrices $\mathbf{A^{(1)}}$ ...... $\mathbf{A^{(q)}}$, where the n columns of $\mathbf{A^{(j)}}$ consist of the j^{th} alternative for each of the n sectors. In similar manner we form the q k x n matrices $\mathbf{F^{(1)}}$ … $\mathbf{F^{(q)}}$, and the q n x 1 vectors $\mathbf{x^{(1)}}$ ... $\mathbf{x^{(q)}}$, where $x_i^{(j)}$, the i^{th} element of $\mathbf{x^{(j)}}$, is the output of sector i using its j^{th} technological alternative. Then model (5) is

$$\min \ Z = \boldsymbol{\pi}'\tilde{\mathbf{F}}\tilde{\mathbf{x}} = \boldsymbol{\pi}'\left[\mathbf{F^{(1)}} \quad \mathbf{F^{(2)}} \quad .. \quad .. \quad \mathbf{F^{(q)}}\right]\begin{bmatrix} \mathbf{x^{(1)}} \\ \mathbf{x^{(2)}} \\ : \\ : \\ \mathbf{x^{(q)}} \end{bmatrix} \tag{5}$$

$$\text{s.t.} \qquad [\mathbf{I} - \mathbf{A^{(1)}} \quad \mathbf{I} - \mathbf{A^{(2)}} \ ...... \ \mathbf{I} - \mathbf{A^{(q)}}]\begin{bmatrix} \mathbf{x^{(1)}} \\ \mathbf{x^{(2)}} \\ : \\ : \\ \mathbf{x^{(q)}} \end{bmatrix} \geq \mathbf{y}\,.$$

Note that when all n sectors have exactly q technologies then the rectangular model **I* - A*** has the exact same dimensions, n x nq, as $[\mathbf{I} - \mathbf{A^{(1)}} \ \ \mathbf{I} - \mathbf{A^{(2)}} \ ….. \ \mathbf{I} - \mathbf{A^{(q)}}]$. Similarly, **F*** has the same dimensions as $\tilde{\mathbf{F}}$ and **x*** has the same dimensions as $\tilde{\mathbf{x}}$. Reordering the columns in **I* - A*** and **F***, and the corresponding rows in **x***, leads to models (1) and (5) being identical. Thus in this case there is no advantage for the RCOT model.

4.2. Square Model with Up to q Alternative Technologies for Each Sector

If sectors are offered the technological choices that were available at two or more specific times, and assuming that each sector has only one technology available to it at each time, then each industry would have the same number of technological alternatives. If, by contrast, we seek to consider the desirability of actual contemporaneous alternatives, different sectors will typically consider different numbers of alternatives, and for some a single average technology will be adequate. For an economy with n sectors, where the i^{th} sector has t_i technological alternatives, let

$$q = \max_i \{t_i\} \text{ and, as before, } t = \sum_{i=1}^{n} t_i\,.$$

Then, since the model is formulated in terms of square coefficient matrices, $d = n*q \geq t$ remains the number of technologies in use, *i.e.*, of decision variables.

Say that sector *i* has fewer than *q* technological options. In order that all the $\mathbf{A}^{(j)}$, *j* = 1 ... *q*, remain square (*n* x *n*) matrices, any one of its t_i distinct technology options may be *replicated* $q - t_i$ times. The model will remain mathematically in the form of model (5). However, the *n* x *d* matrix

$$\left[\mathbf{I}-\mathbf{A}^{(1)} \quad \mathbf{I}-\mathbf{A}^{(2)} \quad \ldots \quad \ldots \quad \mathbf{I}-\mathbf{A}^{(q)}\right]$$

will have the column representing technology option *i repeated* $q - t_i + 1$ times. $\mathbf{A}^{(q)}$ contains new column options for only those sectors with *q* distinct options; for a sector with fewer than *q* distinct options it contains the same column as appeared in $\mathbf{A}^{(q-1)}$, which may in turn be the same as in one or more preceding matrices.

As in the rectangular case, in the absence of factor constraints, there can at most be *n* non-zero sector outputs. This means that again each of the *n* sectors may produce output using one technology only. The objective function assures that it will be the one with the lowest factor costs. However, if this low-cost technology is one that is replicated, the system described by model (5) will be indeterminate: the solution is not unique, and the output for a sector with a technological option that is replicated may be attributed to the output variable for any one of these repetitions. Operationally this indeterminacy does not pose a major problem, requiring us only to be aware of which decision variables refer to repetitions of the same technology. However, this requirement complicates interpretation of the results and reflects an inefficient representation. The ratio *t/d* is a rough measure of the relative efficiency of the square model relative to the rectangular one, equaling 1.0 when all sectors have the same number of distinct options. However, if an economy has 100 sectors, five of which are mining sectors with four technological options each while each remaining sector has one average technology, *t/d* = (95*1 + 5*4)/(100*5) = 115/500 = 0.23. In this case, which is the general case for analysis of scenarios about the future, the rectangular model is substantially more efficient.

5. Simultaneous Choices in Multiple Regions

We return to the earlier discussion of Julia and Duchin (2007), who introduced the choice of up to *q* technologies for each sector in each of m individual regions in square-matrix format, for the simultaneous choice among intra-regional as well as inter-regional options subject to factor constraints: it is shown below as model (6). (The WTM price model is not shown; like the price dual of the RCOT model, model (4), it also solves simultaneously for product prices and scarcity rents on factors (see Duchin 2005).)

$$\begin{aligned} & Min\ Z = \sum_{i=1}^{m}\sum_{j=1}^{q} \boldsymbol{\pi}_{\mathbf{i}}^{(\mathbf{j})\prime}\mathbf{F}_{\mathbf{i}}^{(\mathbf{j})}\mathbf{x}_{\mathbf{i}}^{(\mathbf{j})} \\ & s.t. \sum_{i=1}^{m}\sum_{j=1}^{q}(\mathbf{I}-\mathbf{A}_{\mathbf{i}}^{(\mathbf{j})})\mathbf{x}_{\mathbf{i}}^{(\mathbf{j})} \geq \sum_{i=1}^{m}\mathbf{y}_{\mathbf{i}} \\ & \quad \mathbf{F}_{\mathbf{i}}^{(\mathbf{j})}\mathbf{x}_{\mathbf{i}}^{(\mathbf{j})} \leq \mathbf{f}_{\mathbf{i}}^{(\mathbf{j})} \quad j = 1,.....,q,\ i = 1,.....,m \end{aligned} \tag{6}$$

Because technological options were introduced only for agricultural sectors, and these options have sector-specific factor constraints, each agricultural sector in a region may utilize several technologies simultaneously. That is, some crops can be produced using each land quality, starting with the relatively lowest-cost choice and proceeding to the next lowest-cost as the more productive land is fully utilized. Note that, since only agricultural sectors have choices among alternative technologies, this is an inefficient and indeterminate representation.

Now we are able to rewrite that model in rectangular format as model (7), which is an input-output model of the world economy with choice of technology in individual regions using the RCOT formulation. The factors of production may be sector-specific or even technology-specific in some cases, while in other cases many or all sectors, and technologies, will compete for limited endowments of labor, capital, fresh water, land, and other resources. The model is completely general in that the numbers of choices may differ both by region and by sector. The crucial feature of this formulation is that the lowest-cost technological choices for a given region are interdependent with those for all other regions and in general are different from those that would be selected for each region in a one-region model.

$$\begin{aligned} & Min\ Z = \sum_{i=1}^{m} \boldsymbol{\pi}_{\mathbf{i}}^{*\prime}\mathbf{F}_{\mathbf{i}}^{*}\mathbf{x}_{\mathbf{i}}^{*} \\ & s.t.\ \sum_{i=1}^{m}(\mathbf{I}^{*}-\mathbf{A}_{\mathbf{i}}^{*})\mathbf{x}_{\mathbf{i}}^{*} \geq \sum_{i=1}^{m}\mathbf{y}_{\mathbf{i}} \\ & \quad \mathbf{F}_{\mathbf{i}}^{*}\mathbf{x}_{\mathbf{i}}^{*} \leq \mathbf{f}_{\mathbf{i}} \quad i = 1,.....,m. \end{aligned} \tag{7}$$

6. Factor Rents and Goods Prices

The RCOT model is an LP/IO model that selects the lowest-cost choices of technologies for an economy described by technology input structures, factor endowments, a non-scarcity part of factor prices (discussed below), and final demand. It determines output quantities by technology, goods prices, and factor rents. It is distinctive among input-output models in that production is constrained by factor availability, and the level and mix of final demand, the exogenous portion of factor prices, and factor endowments impact both quantities and prices. We call it an input-output model because it reflects an approach to economic modeling that distinguishes it from neoclassical models that also make use of input-output databases. All input-output models are consistently meso-level, and technological options and consumption patterns are given exogenously as scenario-specific assumptions rather than assuming they can be explained endogenously by a formal expression. (For a discussion of the distinctive features of input-output models, see Duchin 2010.)

One of the novel features of the RCOT model is that each factor has a two-part price, $\pi_i + r_i$ for the i^{th} factor, where the former is exogenous while the latter is the endogenous scarcity rent. In the basic input-output price model, the value of factor requirements per unit of output is entirely exogenous (while in neoclassical economic models, factor prices are entirely endogenous), so such a "hybrid price" calls for explanation. Note first that the endogenous scarcity rent will be equal to zero if a factor is not fully employed, in which case π_i is the effective price of that factor. Stipulating an exogenous portion allows for non-zero factor prices in the common case when factors are not fully exploited, such as the wage rate in the absence of "full employment." For analysis of sustainable development, the models minimize factor use (rather than maximizing growth or consumption or "utility"). As a consequence, scenarios may well leave endowments, such as resource stocks or potentially arable land, less than fully utilized. It is obviously not the case that the utilized portions are given away free of cost.

By contrast, a factor that is fully utilized in the RCOT formulation will earn an endogenous, non-zero scarcity rent, r_i, which reflects by how much the availability of an additional unit of this factor would reduce factor costs for the economy as a whole. To relate the exogenous factor price to the choice of technologies, we point out that r_i also indicates by how much π_i could be increased before the technology choices are affected. To see this, assume a given problem with solutions $\mathbf{x^*}$, $\mathbf{p}$, and $\mathbf{r}$. Now re-run the problem repeatedly with π_i increasing. The identities of the basic variables will not change – although the value of r_i will fall to compensate the rise in π_i -- until the exogenous value for π_i exceeds the original value of $\pi_i + r_i$, at which point new technology choices will be required and the endogenous rent of factor i will change: in particular, now that it is so costly it may not be fully utilized so r_i may fall to zero. The exogenous part of the factor price affects the scarcity rent but it does not determine its magnitude, which depends also on the technological options, endowments, other factor prices, and demand.

The exogenous factor price, based essentially on quality expectations, can be thought of as negotiated at the beginning of the production period between the owner of the factor and the sector that wishes to exploit it, whether a wage rate for workers or a royalty paid by a corporation to the owner of a resource, say an oil-extraction company to the Saudi government. If that price is such that the asset can be exploited but not fully, that price will constitute the entire price. If it turns out that the asset can be exploited and is fully utilized, higher-cost producers, who do not rely on the scarce factor in question, will set the prices of products relying on the scarce asset, and a bonus or windfall we call the scarcity rent is earned. Such a windfall could be provided for in the pre-exploitation agreement, but its value cannot be anticipated beforehand because it depends on economy-wide conditions. (The windfall may be shared between the factor owner and the asset exploiter, again subject to negotiation.) These two-part prices for factors are also properties of the World Trade Model family of multiregional models.

The role of final demand in the RCOT model is also distinctive, relative to its representation in the basic input-output model (or to neoclassical models). Various "non-substitution theorems" describe the rather restrictive assumptions under which the prices of goods in a competitive economy can be shown to be independent of the level and mix of final demand. While these theorems are often interpreted as *justification* for the "fixed coefficients" of the basic input-output model, in fact the objective of these theorems is to portray input-output models as a *special* case of general equilibrium models of a competitive economy. In our view, input-output models are by contrast *alternatives to* general equilibrium models of a competitive economy. In any case, final demand in the RCOT model *does* impact on goods prices and *does not* have fixed coefficients, so these theorems are not relevant to it. Of course the RCOT model also does not satisfy the other restrictive assumptions of these theorems, in that it accommodates multiple factors of production and, unlike neoclassical models, does not by assumption require that they be fully utilized. Like other input-output models, the RCOT model aims to relax conceptual restrictions that may conceal the feasibility of potentially desirable outcomes, for example, reductions rather than increases in consumption.

7. Conclusions and Next Steps

This paper develops the rectangular choice-of-technology model with factor constraints, or RCOT, for analysis of the economy of a single region (or several interdependent regions) that is represented by a rectangular, rather than a square, input-output matrix. RCOT is an LP/IO model that allows for the choice among technological alternatives for one or more sectors. In the case where all sectors have the same number of distinct alternatives, RCOT is equivalent to Carter's square choice-of-technology model. In the empirically important case where only a subset of sectors has explicit alternatives, RCOT provides the same result as the square model but using an approach that is more, and often substantially more, parsimonious. The most distinctive contribution of RCOT, however, emerges with the introduction of explicit factor constraints. When one or more factor constraints are binding, at least one of the sectors that are dependent on a scarce factor may operate more than one technology simultaneously. Consistent with this quantity result, the RCOT model determines each sector's price, which is set by the highest-cost technology in use in that sector, and scarcity rents, which are earned by the factors whose scarcity constrains expanded production by the lower-cost technologies. The Appendix provides numerical examples illustrating all these features.

The key characteristic of the rectangular choice-of-technology representation for a single region, whether for use in a one-region model or for capturing the interdependence among regions within a model of the world economy, is the replacement of the familiar square matrices $\mathbf{I}$ and $\mathbf{A}$, once for each of q options, by a single instance of the rectangular, non-square matrices $\mathbf{I^*}$ and $\mathbf{A^*}$. In the limit, where only one sector has q

alternatives and the other n-1 sectors have only one each, still q n x n **A** matrices (and q k x n **F** matrices) would be required, with n-1 of their n columns identical. The use of **I***, **A***, and **F***, as defined in this paper, removes all redundancies and provides a determinate solution while also rendering the logic of the solution algorithm transparent.

The choice-of-technology model's reliance on rectangular, non-square input-output matrices offers another, less evident advantage. In the past virtually all extensions of the basic input-output model – from the so-called dynamic inverse that makes investment endogenous, to the first input-output model of the world economy with endogenous imports and exports, to the Social Accounting Matrix that provides closure for households and government -- have relied exclusively on square, invertible coefficient matrices. The desirable features of the rectangular choice-of-technology model may demonstrate the potential fruitfulness of other departures from the legacy of square, invertible matrices only. One consequence of the move to the LP/IO model is the introduction of both choices and choice criteria, enlarging the scope of an analysis.

The RCOT model described in this paper was developed to facilitate scenario analysis applying the World Trade Model to a new, environmentally extended, input-output database for the world economy (Tukker et al. 2009) that will include a large set of resource inputs and pollutant outputs. The RCOT model offers the possibility to add columns specifying technological alternatives for sectors that are represented in the baseline database by a single average technology. Estimates of factor endowments also need to be added to the input-output database.

For scenarios about entirely new technologies not already in use in the base year, or for more detail than is available in existing input-output tables, additional columns of coefficients must come directly from technological information. However, another source of such information, which tends to be overlooked, is the product-by-industry use matrix; as ten Raa (1994) pointed out, the input-output modeler, who until now has been content with only square input-output tables, can benefit by examining the underlying rectangular supply and use matrices that are compiled by statistical offices. For example, the input structures for fossil fuel electric power generation and nuclear electric power generation are often distinguished in this source, and in forming the symmetric table they will be weighted into one column using the base year weights. For our purposes, the RCOT model can use this information to represent distinct technological alternatives, and the reliance on one relative to the other will be determined endogenously under alternative scenarios. In the future it would be useful if these tables contained much more information on alternative technologies for producing homogeneous products, especially for the extractive and agricultural sectors that systematically use different means to deal with inputs of varying qualities.

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Appendix: Numerical Example

A. No alternative technologies

The basic input-output model is comprised of the primal quantity equation

$$\mathbf{x} = \mathbf{Ax} + \mathbf{y}$$

and the dual price equation

$$\mathbf{p} = \mathbf{A}'\mathbf{p} + \mathbf{v},$$

where **v**, or value-added, is the product of a quantity and a price ($\mathbf{v} = \mathbf{F}'\boldsymbol{\pi}$ in our notation). The **A** matrix is square and it allows for no alternative technologies. To consider alternative technologies represented by non-square (rectangular) **A** matrices, we move from the input-output model to a linear programming (LP) formulation which we refer to as the LP/IO model. The LP/IO primal problem corresponding to the basic input-output primal model is

$$\text{minimize } Z = \mathbf{v}'\mathbf{x}$$

subject to:

$$(\mathbf{I} - \mathbf{A})\,\mathbf{x} \geq \mathbf{y},$$
$$\mathbf{x} \geq 0.$$

Z is known as the *objective function* and establishes the criterion for choice. We have selected an objective function that minimizes factor costs. In general the choice of objective function will affect the optimum value of **x**, but when **A** is a square matrix it has no effect, as there are no choices to be made. The elements of the output vector **x** are, in LP nomenclature, the *decision variables*. The final demand constraints, $(\mathbf{I} - \mathbf{A})\,\mathbf{x} \geq \mathbf{y}$, are called *functional constraints*. (A second type of LP constraints, the non-negativity constraints, $\mathbf{x} \geq 0$, requires that output cannot be negative.) Any **x** meeting all the constraints is referred to as a feasible solution.

The final demand constraints are formulated as inequalities above, but they can be written as equalities through the introduction of *slack variables*, **s,** leading to

$$(\mathbf{I} - \mathbf{A})\,\mathbf{x} = \mathbf{y} + \mathbf{s}.$$

The total number of variables in an LP model is the sum of the decision variables and the slack variables. LP also classifies the variables into *basic variables*, with non-zero values (except in the degenerate case), and *non-basic variables*, those equal to zero. LP theory states that the number of basic variables equals the number of slack variables (or, equivalently, the number of functional constraints). Thus for each slack variable that is zero, an additional decision variable may be non-zero. When the **A** matrix is square the number of decision variables equals the number of slack variables and therefore the number of basic variables. Assuming that (final plus intermediate) demand for all sectors' outputs is greater than zero, all the decision variables will be greater than zero and basic; thus all the slack variables will be non-basic and zero. This means that all the final demand constraints are binding and all the "greater than or equals" are in fact "equals." As a result the final demand constraints can be written as the equality

$$(\mathbf{I} - \mathbf{A})\,\mathbf{x} = \mathbf{y},$$

and the LP/IO model for square **A** matrices has the unique feasible solution,

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y},$$

independent of the choice of objective function and identical to the solution of the input-output model. The existence of a unique feasible solution does not hold in general for non-square **A** matrices.

All LP problems have an associated dual; in our example the LP/IO dual problem is

$$\text{maximize W} = \mathbf{p}'\mathbf{y}$$

subject to:

$$(\mathbf{I} - \mathbf{A})'\,\mathbf{p} \le \mathbf{v}.$$

Here the elements of **p** are the decision variables. With **A** square, the unique feasible solution is $\mathbf{p} = [(\mathbf{I} - \mathbf{A})']^{-1}\,\mathbf{v}.$ LP duality theory specifies that minimum Z = maximum W, and it follows that

$$\mathbf{v}'\,\mathbf{x} = \mathbf{p}'\mathbf{y}.$$

As a point of reference, we provide a numerical example. Consider an economy with 3 sectors and 2 factors, $n = 3$ and $k = 2$, where

$$\mathbf{A} = \begin{bmatrix} 0.35 & 0.15 & 0.26 \\ 0.25 & 0.22 & 0.22 \\ 0.20 & 0.26 & 0.31 \end{bmatrix},\ \mathbf{F} = \begin{bmatrix} 2.1 & 3.2 & 1.2 \\ 1.2 & 2.2 & 1.3 \end{bmatrix},\ \boldsymbol{\pi} = \begin{bmatrix} 1.0 \\ 0.9 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 20 \\ 25 \\ 22 \end{bmatrix}.$$

The basic primal model has the following solution:

$$\mathbf{x} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} = \begin{bmatrix} 85.693 \\ 84.496 \\ 88.562 \end{bmatrix},$$

$$\boldsymbol{\varphi} = \mathbf{Fx} = \begin{bmatrix} 556.62 \\ 403.85 \end{bmatrix},$$

and $Z = \mathbf{v'x} = \boldsymbol{\pi}'\,\mathbf{Fx}$ = 920.083. Finally, from the price dual,

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \mathbf{(I - A')^{-1}F'}\boldsymbol{\pi} = \begin{bmatrix} 14.255 \\ 13.783 \\ 13.201 \end{bmatrix}.$$

B. Rectangular model for alternative technologies with no factor constraints

We now introduce one alternative technology for sector 2 and two alternative technologies for sector 3. Therefore, $t_1 = 1$. $t_2 = 2$, and $t_3 = 3$, so $q = 3$ and $t = 1+2+3 = 6$. We solve using model (1):

$$\mathbf{I^*} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A^*} = \begin{bmatrix} 0.35 & 0.15 & 0.23 & 0.26 & 0.28 & 0.24 \\ 0.25 & 0.22 & 0.16 & 0.22 & 0.21 & 0.25 \\ 0.20 & 0.26 & 0.30 & 0.31 & 0.33 & 0.30 \end{bmatrix}$$

$$\mathbf{F^*} = \begin{bmatrix} 2.1 & 3.2 & 1.9 & 1.2 & 0.8 & 1.4 \\ 1.2 & 2.2 & 1.3 & 1.3 & 1.1 & 1.1 \end{bmatrix}$$

Note that the pattern of 1's in the i^{th} row of **I*** indicates which columns in **A*** and **F*** are associated with the i^{th} sector. This problem is written as

$$\min Z = \boldsymbol{\pi}'\,\mathbf{F^*x^*}$$
$$\text{s.t. } \mathbf{(I^* - A^*)x^*} \geq \mathbf{y}$$

leading to

$$\mathbf{x}^* = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_2^{(2)} \\ x_3^{(1)} \\ x_3^{(2)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 106.331 \\ 0 \\ 87.328 \\ 0 \\ 103.679 \\ 0 \end{bmatrix},$$

$$\boldsymbol{\varphi}^* = \mathbf{F}^*\mathbf{x}^* = \begin{bmatrix} 472.16 \\ 355.17 \end{bmatrix},$$

and Z = 791.814.

In the absence of factor constraints each sector uses only its lowest cost technology. The reduced value of Z compared to the case with n alternatives is the result of sector 2 switching to the second of its two technologies and sector 3 switching to the second of its three technologies. As a result the requirements for both factors are substantially reduced.

As noted in the text, the prices can be obtained from the input-output dual if we first reduce **A*** and **F*** to $\mathbf{A_e}$ and $\mathbf{F_e}$ respectively by eliminating columns 2, 4, and 6, corresponding to the unused technologies:

$$\mathbf{A_e} = \begin{bmatrix} 0.35 & 0.23 & 0.28 \\ 0.25 & 0.16 & 0.21 \\ 0.20 & 0.30 & 0.33 \end{bmatrix} \text{ and } \mathbf{F_e} = \begin{bmatrix} 2.1 & 1.9 & 0.8 \\ 1.2 & 1.3 & 1.1 \end{bmatrix},$$

and since $\mathbf{A_e}$ is square

$$\mathbf{p} = \begin{bmatrix} \mathrm{p}_1 \\ \mathrm{p}_2 \\ \mathrm{p}_3 \end{bmatrix} = (\mathbf{I} - \mathbf{A_e}')^{-1}\mathbf{F_e}'\boldsymbol{\pi} = \begin{bmatrix} 12.785 \\ 11.280 \\ 11.550 \end{bmatrix}.$$

The same result can be obtained from the LP/IO dual, model (2), without first reducing **A*** to a square matrix:

$$\max Z = \mathbf{p}'\mathbf{y}$$
$$\text{s.t. } (\mathbf{I}^* - \mathbf{A}^*)'\mathbf{p} \le \mathbf{F}^{*\prime}\boldsymbol{\pi}.$$

Solving for **p** leads to

$$\mathbf{p} = \begin{bmatrix} 12.785 \\ 11.280 \\ 11.550 \end{bmatrix}.$$

C. Rectangular model for alternative technologies with factor constraints

With the addition of factor constraints, we have model (3):

$$\min Z = \boldsymbol{\pi}' \mathbf{F^*x^*}$$

$$\text{s.t. } (\mathbf{I^*} - \mathbf{A^*})\mathbf{x^*} \geq \mathbf{y}$$
$$\mathbf{F^*x^*} \leq \mathbf{f}.$$

Assuming, for the two factor endowments,

$$\mathbf{f} = \begin{bmatrix} 540 \\ 342 \end{bmatrix}$$

leads to

$$\mathbf{x^*} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_2^{(2)} \\ x_3^{(1)} \\ x_3^{(2)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 99.788 \\ 0 \\ 87.536 \\ 0 \\ 26.644 \\ 71.953 \end{bmatrix},$$

$$\boldsymbol{\varphi^*} = \mathbf{F^*x^*} = \begin{bmatrix} 498.92 \\ 342.00 \end{bmatrix},$$

and $Z = 805.724$.

Note that while there are now five constraints, only four of the constraints are binding. The first of the two factor constraints retains a surplus of 540 – 498.92 = 42.08. As a result, there are only four non-zero outputs. Due to the binding constraint on factor 2, sector 3 is simultaneously using its third as well as its second technology, and the higher cost of this technology is reflected in the increased value of *Z* as compared to the model with no factor constraints.

The price dual (model (4)),

$$\max W = \mathbf{p'y} - \mathbf{f'r}$$

$$\text{s. t.} \qquad \mathbf{(I^* - A^*)'p - F^{*\prime}r} \le \mathbf{F^{*\prime}\pi},$$

can then be used to determine the price vector **p** and the factor scarcity rent vector **r**:

$$\mathbf{p} = \begin{bmatrix} 18.553 \\ 16.566 \\ 17.351 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 0 \\ 1.056 \end{bmatrix}.$$

We next reduce final demand for sector 3 from 22 to 18 so that

$$\mathbf{y} = \begin{bmatrix} 20 \\ 25 \\ 18 \end{bmatrix}.$$

Now factor 2 is no longer a binding constraint, and solving the quantity primal (model (3)) leads to

$$\mathbf{x}^* = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_2^{(2)} \\ x_3^{(1)} \\ x_3^{(2)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 100.896 \\ 0 \\ 883.369 \\ 0 \\ 94.313 \\ 0 \end{bmatrix}.$$

The absence of binding factor constraints means that sector 3 uses only its low cost technology. Solving the price dual (model (4)) leads to

$$\mathbf{p} = \begin{bmatrix} 12.785 \\ 11.280 \\ 11.550 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The rent for factor 2 falls to 0 and the prices return to the values of the example with no factor constraints.

D. Square model for alternative technologies with no factor constraints

This example illustrates the model discussed in Section 4.2. Again, t_1 = 1. t_2 = 2, and t_3 = 3, q = 3 and t = 6. We have

$$\mathbf{A^{(1)}} = \begin{bmatrix} 0.35 & 0.15 & 0.26 \\ 0.25 & 0.22 & 0.22 \\ 0.20 & 0.26 & 0.31 \end{bmatrix}, \ \mathbf{A^{(2)}} = \begin{bmatrix} 0.35 & 0.23 & 0.28 \\ 0.25 & 0.16 & 0.21 \\ 0.20 & 0.30 & 0.33 \end{bmatrix}, \text{ and } \mathbf{A^{(3)}} = \begin{bmatrix} 0.35 & 0.23 & 0.24 \\ 0.25 & 0.16 & 0.25 \\ 0.20 & 0.30 & 0.30 \end{bmatrix},$$

as well as

$$\mathbf{F^{(1)}} = \begin{bmatrix} 2.1 & 3.2 & 1.2 \\ 1.2 & 2.2 & 1.3 \end{bmatrix}, \ \mathbf{F^{(2)}} = \begin{bmatrix} 2.1 & 1.9 & 0.8 \\ 1.2 & 1.3 & 1.1 \end{bmatrix}, \text{ and } \mathbf{F^{(3)}} = \begin{bmatrix} 2.1 & 1.9 & 1.4 \\ 1.2 & 1.3 & 1.1 \end{bmatrix}.$$

Then

$$\min Z = \boldsymbol{\pi}'\tilde{\mathbf{F}}\tilde{\mathbf{x}} = \boldsymbol{\pi}' [\ \mathbf{F^{(1)}}\ \ \mathbf{F^{(2)}}\ \ \mathbf{F^{(3)}}]\,\tilde{\mathbf{x}}$$

$$\text{s.t. } [\mathbf{I} - \mathbf{A^{(1)}}\ \ \mathbf{I} - \mathbf{A^{(2)}}\ \ \mathbf{I} - \mathbf{A^{(3)}}]\,\tilde{\mathbf{x}} \geq \mathbf{y}$$

leads to

$$\tilde{\mathbf{x}} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \\ x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 103.679 \\ 106.331 \\ 87.328 \\ 0 \end{bmatrix},$$

$$\tilde{\boldsymbol{\varphi}} = \tilde{\mathbf{F}}\tilde{\mathbf{x}} = \begin{bmatrix} 472.16 \\ 355.17 \end{bmatrix},$$

and Z = 791.814.

Note that, since sector 1 has only a single technology, the assignment of sector 1's output to $x_1^{(3)}$ rather than to $x_1^{(1)}$ or $x_1^{(2)}$ is arbitrary, reflecting the indeterminate nature of the formulation. Similarly, the output of $x_2^{(3)}$ could have equally well been assigned to $x_2^{(2)}$. Finally, note that the results of the rectangular (model (3)) and square models (model (5)) are equivalent in the information contained in the results, but the rectangular representation is more compact and offers a determinate solution.