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Portfolio Symmetry
and Momentum



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Abstract

This paper presents a theoretical framework to model the evolution of a portfolio whose weights vary over time. Such a portfolio is called a dynamic portfolio. In a first step, considering a given investment policy, we define the set of the investable portfolios. Then, considering portfolio vicinity in terms of turnover, we represent the investment policy as a graph. It permits us to model the evolution of a dynamic portfolio as a stochastic process in the set of the investable portfolios. Our first model for the evolution of a dynamic portfolio is a random walk on the graph corresponding to the investment policy chosen. Next, using graph theory and quantum probability, we compute the probabilities for a dynamic portfolio to be in the different regions of the graph. The resulting distribution is called spectral distribution. It depends on the geometrical properties of the graph and thus in those of the investment policy. The framework is next applied to an investment policy similar to the Jeegadeesh and Titman's momentum strategy. We define the optimal dynamic portfolio as the sequence of portfolios, from the set of the investable portfolios, which gives the best returns over a respective sequence of time periods. Under the assumption that the optimal dynamic portfolio follows a random walk, we can compute its spectral distribution. We found then that the strategy symmetry is a source of momentum.

Keywords

Graph Theory, Momentum, Dynamic Portfolio, Quantum Probability, Spectral Analysis

JEL Codes

C14, C44

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1 Introduction

In 1993, Jeegadeesh and Titman [JT1993] first presented the momentum strategy. Considering a given market, this strategy is buying the stocks that performed well in the past and selling those that performed poorly in the past. They showed that this strategy generates significant positive returns. This persistent arbitrage opportunity has raised a lot of interest as it goes against the market efficiency.

The Jeegadeesh and Titman's momentum strategy [JT1993] is a zero-dollar long/short equally-weighted strategy being long the best stock decile of a given market and being short the worse stock decile. The strategy being zero-dollar, the value of the sold stocks is the value of the bought stocks. The strategy being equally-weighted, the value invested in each stock is the same, in absolute value. In order to determine the best and worse stock deciles, they rank the stocks according to their past performance. Typically, a stock past performance is its return over the last six months.

Jeegadeesh and Titman used their momentum strategy [JT1993] on a market made of all the available stocks from the CRSP¹ database. With an arousing interest, this strategy has been used in many other markets. It is shown to generate significant positive returns in most of the international stock markets [R1998] [CAW2000], in commodities markets [KSS2007] and in currency markets [OW2003]. For an extensive review of the research about momentum, please refer to [KSS2007]. For now, the main explanation of momentum comes from behavioral finance. According to Barberis, Shleifer and Vishny [BSV1998] and Daniel, Hirshleifer and Subrahmanyam [DHS1998], momentum is due to the investors under-reaction to news.

The innovation of this paper is to present a framework that tackles the study of the market dynamics directly by the means of portfolios, and not by the study of the individual stocks of the market. Considering a given investment policy, it allows us to model the evolution of a dynamic portfolio as a path on a graph generated by the investment strategy. Moreover, this framework takes into account the symmetries of the given strategy. These properties allows us to compute the probabilities for the occurrence of given changes in a portfolio. Considering a momentum strategy, we find that the symmetry of this strategy is a source of momentum effect.

This paper is divided in 4 sections. In the first one, we propose the general framework of the study. In the second section, we provide the tools required in order to apply the framework. These tools are mainly based on graph theory and quantum probability. In the third section, we use the framework for a momentum strategy. We verify that the strategy symmetry induces some of the momentum effect. Section four concludes.

2 The Framework

In this section, we introduce a general framework which permits us to associate a graph to a given investment strategy.

2.1 Definition of a Strategy

Here, we call a strategy an investment policy that defines the set of all the investable portfolios. This investment policy sets the constraints on the portfolio weights.

For instance, the Jeegadeesh and Titman's momentum strategy [JT1993] can be defined as following:

- 20% of the stocks have non-zero weights (C1)
- the long and the short positions have equal weights, ω_0 , in absolute value (C2)

¹Center for Reseach in Security Prices

- the weights imply a 2:1 leverage² (C3)
- the number of long and short positions are the same (C4)

or, in a more formal way:

Let consider a market of N assets and denote $\gamma(i)$ the given weight associated to asset i in a portfolio respecting the Jeegadeesh and Titman's momentum strategy [JT1993], then we have:

- $\exists J \subset \{1, \dots, N\}$ such that $\gamma(j) = 0, \forall j \in J$ and $\frac{|J|}{N} = 0.8$ (C1)
- $|\gamma(i)| = \omega_0, \forall i \in \{1, \dots, N\} \setminus J, \omega_0 > 0$. (C2)
- $\sum_{i=1}^N |\gamma(i)| = 2$ (C3)
- $\sum_{i=1}^N \gamma(i) = 0$ (C2+C4)

2.2 The Set of Portfolios generated by a given Two-Weight Strategy

In this subsection, we are interested in the set of portfolios generated by a given strategy. We focus on two-weight strategies that invest in all the assets of the considered market. These two weights are noted ω_0 and ω_1 . In the following, we note Γ the set of portfolios generated by a specified strategy. Given a market of N assets, we note γ a portfolio of Γ represented by the vector of its weights: $\gamma = (\gamma(1), \dots, \gamma(N))'$. A' stands for the transpose of A . $\gamma(i)$ corresponds to the weight of the portfolio γ in the i -th asset, $\forall i = 1; \dots; N$.

Let F_2^N be the binary vector space of dimension N . As we are interested in two-weight strategies, there exists an isomorphism between Γ and a subset of F_2^N . We consider that a vector ϕ of F_2^N represents a portfolio γ with $\gamma(i) = \omega_0$ if $\phi(i) = 0$ and $\gamma(i) = \omega_1$ if $\phi(i) = 1, i \in \{1, \dots, N\}$.

Now, as in Jeegadeesh and Titman [JT1993], we consider a momentum strategy being a zero-dollar long/short equally-weighted strategy. However, instead of having positions in two deciles of the market stocks (implying three weights), the strategy we consider has positions in each of the market stocks. This strategy differs from the Jeegadeesh and Titman's momentum strategy [JT1993] by the constraint (C1) which becomes:

- 100% of the stocks have non-zero weights (C1)

We call this strategy the full market momentum strategy with leverage 2:1.

Definition 1 : Having a market of N assets, N even, we define Γ the set of portfolios generated by the full-market momentum strategy with leverage 2:1 by:

$$\Gamma = \left\{ \gamma \in \left\{ -\frac{2}{N}, \frac{2}{N} \right\}^N \mid \sum_{i=1}^N \gamma(i) = 0 \right\}.$$

The associated subset of F_2^N which is isomorph to Γ is noted Φ , and we have:

$$\Phi = \left\{ \phi \in F_2^N \mid \sum_{i=1}^N \phi(i) = \frac{N}{2} \right\}.$$

²The notation 2:1 means that the amount of capital backing the portfolio represents 50% of the portfolio value. It is the minimum required under the U.S. regulation (Regulation T).

2.3 Strategy representation

In order to get the distribution of the distance between a dynamic portfolio and its original portfolio - given a strategy and the portfolio dynamics - we are going to use the graph theory which permits to obtain an elegant representation of the strategy. First of all, we specify the distance between two portfolios, then the notion of graph associated to a strategy, finally we illustrate our approach with an example.

Let consider $\gamma_1, \gamma_2 \in \Gamma$. The turn-over between the two portfolios, noted $TO(\gamma_1, \gamma_2)$, defines a distance between γ_1 and γ_2 and we can express it as the 1-norm distance:

$$TO(\gamma_1, \gamma_2) = \sum_{i=1}^N |\gamma_1(i) - \gamma_2(i)|.$$

We will say that two portfolios are at a distance 1, one of each other, if they differ by the minimal difference between two portfolios of Γ . The choice of the distance is not important as we are only interested in the definition of neighbor portfolios that we bind with a edge whose weight is 1.

Now, we denote E the set of edges such that each edge binds two vertices whose turnover is minimal. We can define the undirected graph $G = (\Gamma, E)$ which characterizes the investment strategy. Here below, we provide examples for two different investment strategies.

1. In the case of a two-weight strategy with no constraint, the weight allocated to each stock of the market is either ω_0 or ω_1 and the graph generated by this strategy is the hypercube graph, also known as the Hamming graph. Let consider a market of 3 assets: A, B and C. The portfolios generated by this strategy are provided in Table 1.

	Asset A	Asset B	Asset C
Ptf 1	ω_0	ω_0	ω_0
Ptf 2	ω_1	ω_0	ω_0
Ptf 3	ω_0	ω_1	ω_0
Ptf 4	ω_0	ω_0	ω_1
Ptf 5	ω_1	ω_1	ω_0
Ptf 6	ω_1	ω_0	ω_1
Ptf 7	ω_0	ω_1	ω_1
Ptf 8	ω_1	ω_1	ω_1

Table 1: Portfolios of a 2-weight strategy with no constraint in a market of 3 assets

The graph associated with this strategy is the hypercube presented in Figure 1.

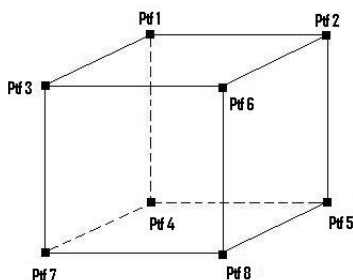


Figure 1: Graph associated to the 2-weight strategy with no constraint

2. In the case of the full market momentum strategy, in a market of 4 assets (A, B, C and D), the generated portfolios are given in Table 2. We represent the Full Market Momentum Strategy by the graph given in Figure 2.

	Asset A	Asset B	Asset C	Asset D
Ptf 1	ω_0	ω_0	ω_1	ω_1
Ptf 2	ω_1	ω_0	ω_1	ω_0
Ptf 3	ω_0	ω_1	ω_1	ω_0
Ptf 4	ω_1	ω_0	ω_0	ω_1
Ptf 5	ω_0	ω_1	ω_1	ω_0
Ptf 6	ω_1	ω_1	ω_0	ω_0

Table 2: Portfolios of the Full Market Momentum strategy in a market of 4 assets

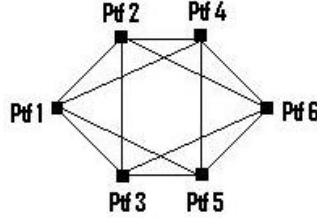


Figure 2: Graph associated to the Full Market Momentum Strategy

2.4 Dynamic Portfolio

In the following, we also use the notion of dynamic portfolio. We define a dynamic portfolio as a "portfolio" whose weights change over time. It is represented as a sequence of portfolios.

For instance, in a market of 4 assets, we can represent the dynamic portfolio over the periods $t = \{1, \dots, 5\}$: $\gamma = (\gamma_{t=1}, \gamma_{t=2}, \gamma_{t=3}, \gamma_{t=4}, \gamma_{t=5})$, as follows:

$$\gamma_{t=1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \gamma_{t=2} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \gamma_{t=3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}; \gamma_{t=4} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}; \gamma_{t=5} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

Here, we call $\gamma_{t=1}$ the origin portfolio of γ .

3 The Spectral Distribution

Given a graph associated to a strategy, we now introduce a way to compute the distribution function of the distance traveled by a dynamic portfolio which follows a random walk on this graph. For the convenience, this distribution will be called spectral distribution in the following. First, we propose to model the evolution of a dynamic portfolio by a random walk. Next, we provide its spectral distribution.

3.1 Portfolio dynamics: a random walk

Let note γ a dynamic portfolio. Our first assumption is to suppose that the considered dynamic portfolio γ follows a walk on the graph $G = (\Gamma, E)$, as defined by \mathbf{H}_0 .

H₀ : Let consider two consecutive periods T_1 and T_2 , with same length T , such that $T_1 = [t_1, t_2]$ and $T_2 = [t_2, t_3]$. Let γ_{T_1} and γ_{T_2} be the respective portfolios of the dynamic portfolio γ for these two periods. We say that γ follows a walk on the graph $G = (\Gamma, E)$ if it exists a finite sequence $\tau_1 = t_1 < \tau_2 < \dots < \tau_p = t_2$ such that $d(\gamma_{[\tau_i, \tau_i+T]}, \gamma_{[\tau_{i+1}, \tau_{i+1}+T]}) = 1$, and $\gamma_{[\tau_i, \tau_i+T]} \in \Gamma, \forall i \in \{1, \dots, p\}$.

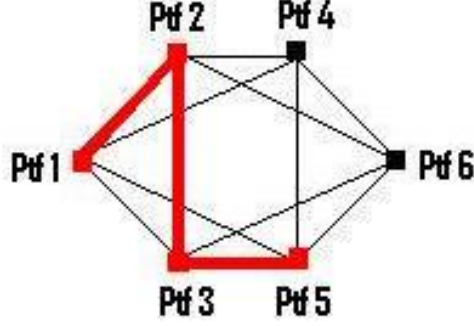


Figure 3: Illustration of a walk in a graph

In addition, we suppose that this walk is random with an equal probability to go in any directions, then we get the following assumption:

H₁ : The dynamic portfolio γ follows a random walk with an equal probability to go in any directions.

The assumption **H₁** provides us with a model for the dynamics of γ . In Figure 3, we exhibit a random walk for the portfolio γ .

3.2 The Spectral Distribution of a Dynamic Portfolio

We present now an expression for the spectral distribution of a dynamic portfolio in the general framework defined before. Next, we apply it to the particular case of the Full Market Momentum Strategy.

3.2.1 General expression of the spectral distribution

Let consider a graph $G = (\Gamma, E)$ generated by a given strategy, and a dynamic portfolio γ_r , with a specified original portfolio γ_o , verifying the hypothesis **H₁**.

The first thing to note is that if the graph G is distance-regular, then it can be characterized by the eigenvalues of its adjacency matrix, and their multiplicities. It means that the properties of an investment strategy represented by such a graph can also be resumed by these eigenvalues. Moreover, from Hora and Obata [HO2007], we have the following result:

Theorem 1 *If G is a finite distance-regular graph with the intersection table*

$$\begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_d \\ a_0 & a_1 & a_2 & \dots & a_d \\ b_0 & b_1 & b_2 & \dots & b_d \end{pmatrix}$$

then the spectral distribution of G is a probability measure such that the probability for dynamic portfolio γ_r on G to be at a distance n from the original portfolio γ_o after r random moves is :

$$P(d(\gamma_r, \gamma_o)) = \left| \frac{1}{\|P_n\|} \int_{-\infty}^{+\infty} x^r P_n(x) \mu(dx) \right|^2$$

i.e.

$$P(d(\gamma_r, \gamma_o)) = \left| \frac{1}{\|P_n\| |\Gamma|} \sum_{i=0}^d m_i \lambda_i^r P_n(\lambda_i) \right|^2$$

where λ_i and m_i are the eigenvalues of the adjacency matrix of the graph G and $\{P_n(x)\}_{n=0}^d$ are the Gram-Schmidt polynomials associated with μ , with

$$\mu = \frac{1}{|\Gamma|} \sum_{i=0}^s m_i \delta_{\lambda_i},$$

and δ_λ stands for the delta-measure.

3.2.2 The spectral distribution of a dynamic portfolio in the case of a full-market momentum strategy

We compute now the spectral distribution provided by the previous theorem for a full-market momentum strategy. Let consider $G = (\Gamma, E)$ the graph generated by the full-market momentum strategy on a market of N assets. In order to apply theorem 1, we need first to verify that G is a distance-regular graph and we need also to compute the eigenvalues, with their multiplicities, of the adjacency matrix of G . This last computation uses the 2-antipodal property of G . These properties are obtained in the following lemma whose proof is postponed in the Section 6, as also the computation of the eigenvalues of the adjacent matrix of G .

Lemma 1 *Given a graph generated by the 2-Weight full market momentum strategy, then we have the following properties:*

- (i) *It is a distance-regular graph,*
- (ii) *It is a 2-antipodal graph.*

Now, as soon as we know that G is distance-regular, we are able to compute the eigenvalues, with their multiplicities, of the adjacency matrix of G . So, we can use theorem 1 with the full market momentum strategy. In the following, we apply results of Theorem 1 on a market of 10 assets ($N = 10$) and also the results obtained in the previous application (Section 3). Assuming that a dynamic portfolio follows a random walk on $G = (\Gamma, E)$, we provide on figure 4 the probabilities to find it at a distance n from its origin after $r = 1, 3, 5, 10$ moves.

We observe that, after one move, the dynamic portfolio can only be at a distance 1 from its origin. We also note that the spectral distribution tends to be symmetric for an increasing number of moves. This last observation illustrates the ergodicity of a random walk in this graph.

4 The Full Market Momentum strategy and the Momentum effect

First, we introduce the optimal dynamic portfolio. Next, we propose to link the momentum effect with the evolution of the optimal dynamic portfolio. Finally, we verify that it corresponds to the momentum effect we observe in the market of the 10 datastream world indices.

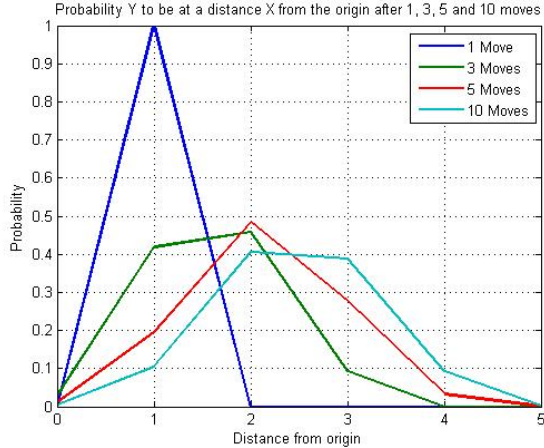


Figure 4: Probabilities for a dynamic portfolio to be at a distance n from its origin after 1, 3, 5, 10 moves

4.1 The Optimal Dynamic Portfolio

Given the returns of the N assets of the market over a given period, we define the optimal portfolio as the portfolio, among those in Γ , which provides the highest return. The optimal portfolio over a period $[t_1, t_2]$ is noted $\gamma_{[t_1, t_2]}^o$. The optimal portfolio is easily constructed giving the weight $\frac{2}{N}$ to the $\frac{N}{2}$ best performers and the weight $-\frac{2}{N}$ to the $\frac{N}{2}$ worse performers. The sequence $\gamma^o = \left(\gamma_{[t_1, t_2]}^o, \gamma_{[t_2, t_3]}^o, \dots, \gamma_{[t_{p-1}, t_p]}^o \right)$ of the optimal portfolios is called the optimal dynamic portfolio.

4.2 The Momentum Effect

The momentum effect is defined as the persistence of the returns of a given portfolio. Jeegadeesh and Titman [JT1993] show empirically that the optimal dynamic portfolio of their momentum strategy presents some momentum effect. So, the optimal portfolio which was previously the best portfolio should provide a positive return the next period. Their strategy defines a symmetrical set of portfolios. It means that the new optimal portfolio should be close rather than far from the previous optimal portfolio. In other words, the spectral distribution of the optimal dynamic portfolio should be skewed toward 0.

Moreover, here, as a first approximation, we consider the momentum effect in terms of distance instead of returns. As this distance represents turn-over, it corresponds to the number of assets returns spreads from the optimal portfolio, these spreads being between the half best performers and the half worse performers. Assuming that the spreads are all the same, the approximation seems valid. Indeed, with a market of N assets, a portfolio at a distance smaller than $\frac{N}{4}$ from the optimal portfolio has a positive return. Without the assumption of equal spreads, the approximation is still valid in average because the existence of a large spread implies the existence of a smaller spread.

4.3 Empirical verification with a 10-asset market

Let consider the market made by the 10 Datastream world indices over 407 months (from 01/1975 to 11/2008). In this market, applying the full market momentum strategy, i.e. buying the half best performers and selling the half worse performers of the last month and holding them the next month, leads to a monthly return of 0.45% with a t-stat of 3.88. So, this market exhibits momentum.

We are interested in the evolution the optimal dynamic portfolio of the full market momentum strategy, month after month. We first verify that it follows a random walk as defined in \mathbf{H}_0 . So, we count the number of moves affecting the optimal dynamic portfolio over a 22-day rolling window. We report the result in table 3.

Number of moves	0	1	2	3+
Percentage of occurrence	56%	41%	3%	N/A

Table 3: Percentage of occurrence of a certain number of moves in the optimal dynamic portfolio over a 22-day rolling window

As \mathbf{H}_0 is violated only 3% of the time, and only by 2-moves, we suppose that with a higher frequency rolling window these 2-moves would be splitted in two. Then, we can consider that assumption \mathbf{H}_0 is holding. In addition, we suppose that this walk is random (\mathbf{H}_1).

So, as the market considered is a 10-asset market, the probabilities to find the optimal dynamic portfolio at a distance n from its origin after $r = 1, 3, 5, 10$ moves can be computed from theorem 1 and have been already provided in figure 4.

Now, we report, in figure 5, the number of single moves that this optimal dynamic portfolio walks each month. We observe that 18.5% of the number of moves are lower or equal to 6, and 25.8% of the moves lower or equal to 7. It corresponds to the respective probabilities 61.8% and 57.1% to be at a distance from origin below the average, being 2.5. So, according to our modeling, we can observe momentum for around 20% of the observations. Indeed, 20% of the time, the number of moves in the optimal dynamic portfolio is small enough to imply momentum effect. For the rest of the time, the number of moves leads to a symmetric distribution. Thus, the final portfolio has equal probabilities to provide a positive or negative return. As a consequence, overall, we would expect this strategy to be profitable.

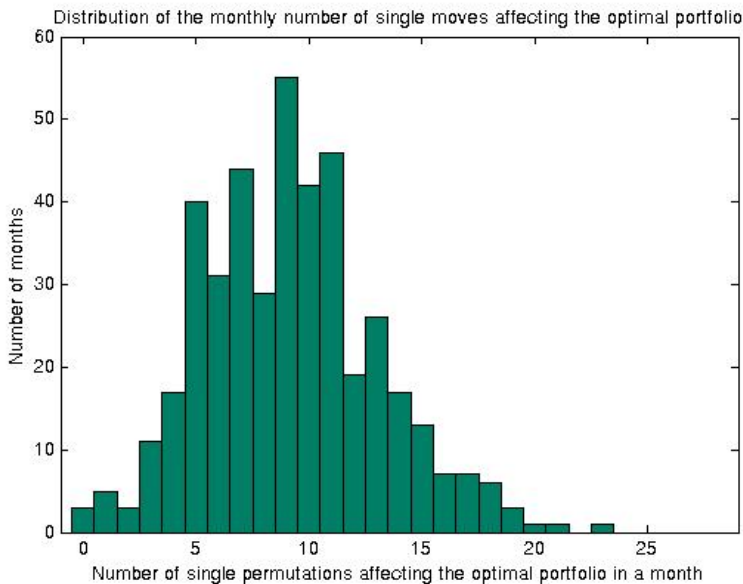


Figure 5: Number of single moves of the optimal portfolios over a month

from this example, we see that the symmetry of the full market momentum strategy induces some of the momentum effect observed in the evolution of the optimal dynamic portfolio. Indeed, a random walk in

the graph generated by such a strategy produces a spectral distribution which is skewed enough to be at the origin of the momentum effect. Finally, we have verified that the optimal dynamic portfolio evolves according to a walk which, under the random assumption, would lead to the momentum effect effectively observed.

5 Conclusion

In this paper, we propose a novel framework to model the walk of dynamic portfolios on graphs corresponding to determined strategies. The graph vertices consist in all the portfolios generated by a given strategy and the graph edges bind any two portfolios of this set that require the minimal turn-over to pass from one to the other. Under the assumption that a dynamic portfolio evolves passing from a portfolio of the graph to an adjacent one, we can use recent developments in graph theory and quantum probability in order to get the probabilities for the dynamic portfolio to be at a certain distance from a previous location. These probabilities depend on the graph eigenvalues which represents the symmetries of the graph, i.e. the symmetries of the investment strategy.

As an illustration, we have seen that the momentum strategy symmetry induces some of the momentum effect. Indeed a random walk in the graph generated by such a strategy produces probabilities (for the dynamic portfolio to be at a certain distance from a previous location) skewed enough to be at the origin of the momentum effect. Moreover, some of the momentum effect observed in the Jeegadeesh and Titman's momentum strategy [JT1993] can also be explained by the strategy symmetries. Indeed, considering as the entire market the one made of the assets selected in Jeegadeesh and Titman's strategy [JT1993], the conclusions obtained for the full-market momentum strategy stand.

The momentum effect being a short-term property of the markets, it would also be interesting, as future work, to model trends as long run walk, considering random walks as the noise surrounding this long run walk. Eventually, one might even think about modeling a business cycle as a cycle in terms of walk in a graph.

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6 APPENDIX

6.1 Proof of Lemma 1

(i) Let note G the graph generated by the 2-Weight Full Market Momentum Strategy in a market of N assets. We suppose N even. Let i, j, k be non-negative integers. We know that a graph $G = (V, E)$ is *distance-regular* if for any choice of $x, y \in V$ with $d(x, y) = k$ the number of vertices $z \in V$ such that $d(x, z) = i$ and $d(y, z) = j$ is independent of the choice of x, y . Thus, taking $x, y \in V$ with $d(x, y) = k$ the number $p_{ij}^k = |\{z \in V; d(x, z) = i, d(y, z) = j\}|$ is defined only depending on i, j, k .

Let consider $\gamma, \gamma_1, \gamma_2 \in \Gamma$, three portfolios such that $d(\gamma, \gamma_1) = i$, $d(\gamma, \gamma_2) = j$ and $d(\gamma_1, \gamma_2) = k$. We note ϕ, ϕ_1 and ϕ_2 the respective representations of γ, γ_1 and γ_2 in F_2^N . Let note $\Gamma_i(\gamma_1)$ and $\Gamma_j(\gamma_2)$ the sets of the portfolios which are, respectively, at a distance i from γ_1 and at a distance j from γ_2 .

$$\Gamma_i(\gamma_1) = \{\gamma \in \Gamma | d(\gamma_1, \gamma) = i\}$$

$$\Gamma_j(\gamma_2) = \{\gamma \in \Gamma | d(\gamma_2, \gamma) = j\}.$$

We divide $\Gamma_i(\gamma_1) \cap \Gamma_j(\gamma_2)$ in 4 parts:

$$P_1 = \{i \in \{1, \dots, N\} | \phi_1(i) = 0 \text{ and } \phi_2(i) = 1\}$$

$$P_2 = \{i \in \{1, \dots, N\} | \phi_1(i) = 1 \text{ and } \phi_2(i) = 0\}$$

$$P_3 = \{i \in \{1, \dots, N\} | \phi_1(i) = 0 \text{ and } \phi_2(i) = 0\}$$

$$P_4 = \{i \in \{1, \dots, N\} | \phi_1(i) = 1 \text{ and } \phi_2(i) = 1\}.$$

As $d(\gamma_1, \gamma_2) = k$, we have:

$$|P_1| = k, |P_2| = k, |P_3| = \frac{N}{2} - k \text{ and } |P_4| = \frac{N}{2} - k.$$

Now, we study the allocation of, let say, the long positions of γ among these 4 parts. The strategy is two-weighted so, once the long positions are allocated, the short positions also are.

Let note:

$$A = \{i \in \{1, \dots, N\} | \phi(i) = 1 \text{ and } \phi_1(i) = 0 \text{ and } \phi_2(i) = 1\}$$

$$B = \{i \in \{1, \dots, N\} | \phi(i) = 1 \text{ and } \phi_1(i) = 1 \text{ and } \phi_2(i) = 0\}$$

$$C = \{i \in \{1, \dots, N\} | \phi(i) = 1 \text{ and } \phi_1(i) = 0 \text{ and } \phi_2(i) = 0\}$$

$$D = \{i \in \{1, \dots, N\} | \phi(i) = 1 \text{ and } \phi_1(i) = 1 \text{ and } \phi_2(i) = 1\}.$$

With these notations, we have:

$$|\Gamma_i(\gamma_1) \cap \Gamma_j(\gamma_2)| = \sum_{|A|,|B|,|C|,|D|} \binom{k}{|A|} \binom{k}{|B|} \binom{\frac{N}{2}-k}{|C|} \binom{\frac{N}{2}-k}{|D|}. \quad (1)$$

The strategy is a full-market momentum strategy, so ϕ has $\frac{N}{2}$ long positions and $\frac{N}{2}$ short positions, so:

$$|A| + |B| + |C| + |D| = \frac{N}{2}. \quad (2)$$

Moreover $d(\gamma, \gamma_1) = i$, so we have:

$$|A| + |C| = i. \quad (3)$$

As $d(\gamma, \gamma_2) = j$, we also have:

$$|B| + |C| = j. \quad (4)$$

From (3) and (4), we get:

$$|B| = |A| + j - i \quad (5)$$

$$|C| = i - |A|. \quad (6)$$

From (2), (5) and (6), we get:

$$|D| = \frac{N}{2} - j - |A|. \quad (7)$$

Noting $|A|$ and inserting (5), (6) and (7) in (1), we finally get:

$$\begin{aligned} |\Gamma_i(\gamma_1) \cap \Gamma_j(\gamma_2)| &= \sum_n \binom{k}{n} \binom{k}{n+j-i} \binom{\frac{N}{2}-k}{i-n} \binom{\frac{N}{2}-k}{\frac{N}{2}-j-n} \\ &= \sum_{n=0}^{k-j+i} \binom{k}{n} \binom{k}{k+i-j-n} \binom{\frac{N}{2}-k}{i-n} \binom{\frac{N}{2}-k}{n+j-k} \end{aligned} \quad (8).$$

From (8), we have that $|\Gamma_i(\gamma_1) \cap \Gamma_j(\gamma_2)|$ only depends on i, j and k , thus G is a distance-regular graph.

(ii) A graph G is k -antipodal when each of its vertex is diametrically opposed to $k-1$ others which are themselves diametrically opposed.

Considering the graph defined by a Full-Market Momentum strategy, the distance considered (half the Hamming distance) consists in permutations between long and short positions. Then, the maximum number of permutations that can be applied to a given portfolio is by permuting all its long position with all its short positions, as there are $\frac{N}{2}$ long positions and $\frac{N}{2}$ short positions. So the maximum distance between any two portfolios is $\frac{N}{2}$. It corresponds to the diameter of the graph. Moreover, only one portfolio corresponds to the one got by permuting all the long positions with all the short positions of a given portfolio. Thus, the graph is 2-antipodal.

6.2 Computation of the spectrum of the graph generated by the 2-Weight Full Market Momentum Strategy

Let consider the graph G defined by a Full-Market Momentum strategy. Its diameter is noted d . From Lemma 1 (i), we know that it is a distance-regular graph. Let note its intersection table as following

$$\begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_d \\ a_0 & a_1 & a_2 & \dots & a_d \\ b_0 & b_1 & b_2 & \dots & b_d \end{pmatrix}.$$

We know that the eigenvalues of the adjacency matrix of the distance-regular graph G are the eigenvalues of the following matrix T :

$$T = \begin{pmatrix} a_0 & b_1 & & & & \\ c_1 & a_2 & b_2 & & & \\ & c_2 & a_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & & a_{d-1} & b_{d-1} \\ & & & & c_{d-1} & a_d \end{pmatrix}.$$

The matrix T is a $(d+1) \times (d+1)$ matrix. From Lemma 1 (ii), we have $d = \frac{N}{2}$. So, for markets with a reasonable number of assets, we are able to compute the graph eigenvalues.

From Lemma 1 (ii), we also know that G is 2-antipodal. And from Dalfó, Fiol and Garriga 2008 [DFG2008], we know that a distance-regular graph G of diameter d with spectrum

$$Spec(G) = \begin{pmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_d \\ m_0 & m_1 & \dots & m_d \end{pmatrix}$$

is r -antipodal ($r \geq 2$) if and only if its eigenvalues satisfy:

$$\begin{cases} m_k = \frac{\pi_0}{\pi_k} & (\text{k even}) \\ m_k = (r-1)\frac{\pi_0}{\pi_k} & (\text{k odd}) \end{cases}$$

where $\pi_k = \prod_{i=0, i \neq k}^d |\lambda_k - \lambda_i|$. So we can compute the multiplicities of the eigenvalues as following:

$$m_k = \frac{\prod_{i=1}^d |\lambda_0 - \lambda_i|}{\prod_{i=0, i \neq k}^d |\lambda_k - \lambda_i|}, \forall k \in \{1, \dots, d\}$$