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EXISTENCE OF PERFECT EQUILIBRIA: A DIRECT PROOF*

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Abstract. We formulate and prove a modification of Eilenberg-Montgomery fixed-point theorem, which is a generalization of Kakutani's theorem. It enables us to provide a *direct proof* of the existence of perfect equilibria in finite normal form games and extensive games with perfect recall.

We construct a correspondence whose fixed points are precisely the perfect equilibria of a given finite game. Existence of a fixed point is secured by the modified version of Eilenberg-Montgomery theorem.

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1. Introduction

Perfect equilibria were introduced in 1975 by Selten [**?**] as a refinement of subgame perfect and Nash equilibria. Perfect equilibria play important role in contemporary game theory due to its stability with respect to slight imperfections of rationality, or "trembling-hand perfection".

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Perfect equilibria exist for every normal form game, and for every extensive form game with perfect recall there exists a perfect equilibrium in behavior strategies. However, the proof of existence, given by Selten [**?**], is indirect and relies on the existence of a Nash equilibrium in every perturbed game. This makes the perfect equilibria not as "tangible" and complicates their treatment.

We are the first to provide a direct proof of the existence of perfect equilibria in normal form games and in extensive form games with perfect recall. We construct a correspondence whose fixed points are precisely the perfect equilibria of a given game. This correspondence coincides with the best response correspondence on the interior of the strategy space, however the constructed correspondence possesses much more general boundary behavior. In order to prove existence of a fixed point, we formulate and prove a version of Eilenberg-Montgomery fixed point theorem. Its proof is inspired by and borrows the idea of Selten's proof of existence of perfect equilibria [**?**].

The paper is organized as follows. Chapter 2 provides definitions of the game-theoretic concepts invoked in this paper. In Chapter 3 we introduce the necessary tools of algebraic topology, formulate and prove a version of Eilenberg-Montgomery fixed point theorem. In Chapter 4 we construct a correspondence whose fixed points are the perfect equilibria of a given game, and apply the version of Eilenberg-Montgomery theorem introduced in the previous chapter to proof the existence of a fixed point.

2. Definitions and Methodology

Definition 2.1. *A **normal form** Γ of a finite n-player game is a tuple $(S_1, \cdots, S_n, \hat{h})$, where S_i is a finite set of pure strategies of player i, $S = \prod_{i=1}^n S_i$, and $\hat{h} : S \twoheadrightarrow \mathbb{R}^n$ is the payoff function that assigns to every $s \in S$ the vector of payoffs $\hat{h}(s) = (\hat{h}_1(s), \cdots, \hat{h}_n(s))$.*

Definition 2.2. *A **mixed strategy** a_i for player i is a probability distribution over S_i. The set of all such probability distributions is denoted by $A_i \equiv \Delta_{c_i}$, where c_i is the cardinality of S_i. The set of mixed strategies for the game Γ is $A = \prod_{i=1}^n A_i$.*

Mixed strategy $a_i \in A_i$ for player i is completely mixed if a_i is in the interior of A_i, i.e., $a_i \in A_i^\circ$. Mixed strategy $a \in A$ is completely mixed if for every player i, a_i is completely mixed.

We can now define an expected payoff function h, which is an extension of the payoff function $\hat{h}$ to all of A.

Definition 2.3. *An* ***expected payoff function*** *is a function* $h : A \twoheadrightarrow \mathbb{R}^n$ *such that*

$$h(a) = \prod_{s \in S} p_1(s) p_2(s) \cdots p_n(s) \hat{h}(s),$$

where $p_i(s)$ *is the probability that* a *assigns to the* i^{th} *component of* s*, i.e., the probability with which player* i *chooses* s_i*.*

Definition 2.4. *A* ***best response correspondence*** *of player* i *is the correspondence* $\beta_i : A_{-i} = \prod_{j \neq i} A_j \twoheadrightarrow A_i$ *defined for each* $a_{-i} \in A_{-i}$ *as*

$$\beta_i(a_{-i}) = \left\{ \tilde{a}_i \in A_i \, : \, h_i(\tilde{a}_i, a_{-i}) \geq h_i(a_i, a_{-i}) \; \forall a_i \in A_i \right\}.$$

Definition 2.5. *A* ***best response correspondence*** *for an* n*-person normal form game is the correspondence* $\beta : A \twoheadrightarrow A$ *defined for each* $a \in A$ *as an* n*-tuple* $(\beta_1(a_{-1}), \beta_2(a_{-2}), \cdots, \beta_n(a_{-n}))$*, where for each* i*,* $\beta_i(a_{-i})$ *is player* i*'s best response correspondence defined above.*

We will define perfect equilibrium in terms of substitute sequences (as appeared in Selten (1975) as an alternative characterization of perfection).

Definition 2.6. *A* ***substitute sequence*** *for a strategy profile* $\bar{a} \in A$ *is a sequence of completely mixed strategy profiles approaching* $\bar{a}$*, i.e, a sequence* $\left\{a^k\right\} \subseteq A^\circ$ *such that* $a^k \to \bar{a}$ *as* $k \to \infty$*.*

Definition 2.7. *A mixed strategy* $a^* \in A$ *is called a* ***perfect equilibrium point*** *of a normal form game* Γ *if* a^* *is a best response to at least one substitute sequence for* a^**.*

3. A New Version of the Eilenberg-Montgomery Fixed Point Theorem

We first provide some definitions invoked in Eilenberg-Montgomery fixed-point theorem and the formulation of the theorem itself, and then formulate and prove a slight modification of the theorem.

Definition 3.1. *Given a topological space X, a nonempty-valued correspondence $\varphi : X \twoheadrightarrow X$ is **closed** if it has a closed graph.*

Definition 3.2. *Let X be a topological space, then a nonempty-valued correspondence $\varphi : X \twoheadrightarrow X$ is **closed** if it has a closed graph.*

Definition 3.3. *Let X be topological space and A a subspace of X. Then a continuous map $r : X \to A$ is a **retraction** if the restriction of r to A is the identity map on A. Equivalently, if we denote the inclusion map on A by $\iota : A \hookrightarrow X$, a retraction is a continuous map $r : X \to A$ such that $r \circ \iota$ is homotopic to the identity map on A.*

Definition 3.4. *A subspace A of a topological space X is called a **retract** if such a retraction exists.*

Definition 3.5. *Let X be topological space and A a subspace of X. Then A is called a **neighborhood retract** of X if there exists an open set $U \subseteq X$ such that $A \subseteq U$ and A is a retract of U.*

The notion of a neighborhood retract is a weakening of a retract: let A be a subspace of a topological space X such that A is a retract of X. Then, take X to be the neighborhood of A in Definition to deduce that A is a neighborhood retract of X.

Definition 3.6. *A topological space A is an **absolute neighborhood retract (or ANR)** if for every embedding of A as a closed subset of a normal space X the image of A is a neighborhood retract of X.*

Definition 3.7. *A topological space A is a **deformation retract** of X if there exists a continuous function $r : X \to A$ which is homotopic to the identity map of A and fixes A. Equivalently, there exists a continuous map $H : X \times I \to X$ (where $I = [0, 1]$) such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$. The homotopy H is called a **deformation retraction** of X onto A.*

Some authors do not require a deformation retract to fix A, and call the space described above a strong deformation retract. An important property of a deformation retract as stated in Definition is that it is homotopy invariant and preserves homology groups, which is crucial in proving the new version of Eilenberg-Montgomery theorem.

Definition 3.8. *A nonempty compact metric space X is said to be* ***acyclic*** *provided:*

(1) *the homology groups $H_n(x)$ vanish for all $n > 0$, $x \in X$, and*
(2) *the reduced 0^{th} homology group $\hat{H}_0(x)$ vanishes for $x \in X$.*

Theorem 3.9. ***(Eilenberg-Montgomery fixed point theorem)*** *Let X be an acyclic absolute neighborhood retract and $\varphi : X \twoheadrightarrow X$ be a closed correspondence such that for every $x \in X$ the set $\varphi(x)$ is acyclic. Then φ has a fixed point.*

The proof can be found in [**?**]. Notice also that according to Definition 3.7 if X is acyclic, then it is understood that X is a nonempty compact metric space.

We are now ready to formulate and prove a version of Eilenberg-Montgomery fixed point theorem. It allows much more general boundary behavior of the correspondence (as long as the closedness of the graph is preserved), however it requires existence of a nicely behaved deformation retract. Notice, however, that the latter assumption is always fulfilled in certain cases (for instance, when X is a convex set; see Example 3.11), which produces a stronger version of Eilenberg-Montgomery theorem for those cases.

Theorem 3.10. *Let X be an acyclic absolute neighborhood retract, C be a dense subset of X, and $A \subset C$ be a deformation retract of X such that $H(x,t) \in C$ for every $x \in X$ and $t > 0$. Let $\varphi : X \twoheadrightarrow X$ be a closed correspondence such that for every $x \in C$ the set $\varphi(x)$ is acyclic. Then φ has a fixed point.*

Proof. Fix $t \in I$, $t > 0$, and consider $H(X,t)$. Clearly, $H(X,t)$ itself is a deformation retract of X, hence it is ANR and acyclic (since deformation retraction preserves homology groups). Observe that $H(X,t) \subseteq C$, and apply Theorem **??** to $H(X,t)$ in place of X, and φ restricted to $H(X,t)$. Denote the corresponding fixed point of φ on $H(X,t)$ by a^t.

Consider the net $\{a^t\}_{t\in(0,1]}$ in C. Since $\bar{C} = X$ is compact, the net $\{a^t\}$ has a limit point in X (see Theorem 2.31 in Aliprantis and Border (2006)). So passing to a subsequence without loss of generality, we get $a^t \to a^*$ as $t \to \infty$. Denoting by G_φ the graph of the correspondence φ, we see that $(a^t, a^t) \in G_\varphi$ since a^t is a fixed point of φ over the set $H(X,t)$. Therefore by the closedness of G_φ we have $(a^*, a^*) \in G_\varphi$, so that a^* is a fixed point of φ over X, which completes the proof.

■

Example 3.11. *(Existence of a nicely behaved deformation retract)* Let X be a convex subset of some topological vector space. If X is a singleton, then the only dense subset of it is X itself. Assume X is not a singleton, then convexity implies $X^\circ \neq \emptyset$. Let $C = X^\circ$, fix $x^* \in X^\circ$ and $\alpha \in (0,1)$. Clearly C is dense in X.

Consider a straight-line homotopy $H : X \times I \twoheadrightarrow X$ defined for each $x \in X$ as $H(x,t) = ta^* + (1-t)x$. Clearly, H is continuous, $H(x,0) = x$ and $H(x,1) = a^*$ for all $x \in X$. Notice also that $H(a^*, t) = a^*$ for all $t \in A$. Therefore the homotopy H is a deformation retraction, and the singleton a^* is a deformation retract of X. Notice also that $H(X,t) \subseteq C$ for all $0 < t \leq 1$. Therefore X, $C = X^\circ$, $A = a^*$ and H satisfy the hypotheses of Theorem 3.10.

4. A Direct Proof of the Existence of Perfect Equilibria

Recall that given an n-player normal or extensive form game Γ with a strategy space A_i for each player i (as usual let $A = \prod_{i=1}^n A_i$), the graph of the best response correspondence

$$G_\beta = \{(a,b) \in A \times A : \, a \in A, b \in \beta(a)\},$$

where $\beta : A \twoheadrightarrow A$ is the best response correspondence as stated in Definition 2.5.

Now consider the graph of the best response correspondence restricted to $A^\circ \times A$, i.e.,

$$G = \{(a,b) \in A^\circ \times A : \, a \in A^\circ, b \in \beta(a)\}.$$

Define G_θ to be the closure of the set G in $A \times A$, and identify with G_θ a correspondence from A to A, call it θ, having G_θ as its graph. Clearly, such correspondence is well-defined as long as G_θ as a subset of $A \times A$ is well defined, but it is so because a closure of the set G in $A \times A$ is well-defined.

From the above it follows immediately that θ is a closed correspondence (i.e., has a closed graph). Indeed, $G_\theta = \bar{G}$ in $A \times A$ implies that G_θ is closed in $A \times A$.

Lemma 4.1. *The correspondence θ is nonempty-valued and closed.*

Proof. It remains only to show that θ is nonempty-valued. So, fix any $a \in A$. If $a \in A^\circ$, then $\theta(a) \neq \emptyset$ since in the interior of A, θ coincides with the best response correspondence, which is nonempty-valued.

Let $a \in \partial A$. Take a sequence $\{a_k\}$ of completely mixed strategies (i.e., $a_k \in A^\circ$ for each $k \in \mathbb{N}$) such that $\lim_{k\to\infty} a_k = a$. Since each a_k is in the interior of A, then $\theta(a_k) \neq \emptyset$. For each $k \in \mathbb{N}$ pick some $b_k \in \theta(a_k)$. Thus we obtained a sequence of points $\{b_k\}$ in a compact set A, hence this sequence has a convergent subsequence, without loss of generality $b_{k_m} \to b$ as $m \to \infty$ for some $b \in A$.

Notice also that $(a_k, b_k) \in G_\theta$ for each k and by relabelling $(a_k, b_k) \to (a, b)$ as $k \to \infty$. However, since G_θ is closed, then $(a, b) \in G_\theta$, i.e., $b \in \theta(a)$. This establishes that θ is nonempty-valued.

■

Notice that the Closed Graph Theorem (see Theorem 17.11, p. 561 of Aliprantis and Border(2006)) implies that θ is upper hemicontinuous and closed-valued.

The following theorem shows that for any normal form game Γ, the set of its perfect equilibria coincides with the set of fixed points of the constructed correspondence θ.

Theorem 4.2. *A strategy profile $a^* \in A$ is a fixed point of θ if and only if a^* is a perfect equilibrium point of Γ.*

Proof. ($\Rightarrow$) Let the strategy profile $a^* \in A$ be such that $a^* \in \theta(a^*)$, that is, $(a^*, a^*) \in G_\theta$. Since G_θ is the closure of G in $A \times A$, there exists a sequence $(x^k, y^k) \subseteq G$ such that $(x^k, y^k) \to (a^*, a^*)$ as $k \to \infty$.

Observe that for each k, $\beta(x^k)$ is a Cartesian product of some faces of the unit simplices A_i, hence there exists $m \in \mathbb{N}$ such that

$$a^* \in \beta(x^k)$$

for all $k \geq m$, and $x^k \to a^*$.

This shows that a^* is a perfect equilibrium point of Γ.

($\Leftarrow$) Assume $a^* \in A$ is a perfect equilibrium point of Γ, then there exists a substitute sequence for a^*, say $\{a^k\}$ such that $a^* \in \beta(a^k)$ for all $k \in \mathbb{N}$.

Notice that $(a^k, a^*) \in G$ for all k and $(a^k, a^*) \to (a^*, a^*)$ as $k \to \infty$. Therefore $(a^*, a^*) \in \bar{G} = G_\theta$, q.e.d.

■

The difficulty with proving that the correspondence θ has a fixed point arises from the fact that θ need not be convex-valued or even acyclic-valued. However, it is nicely behaved in the interior of the strategy space.

Notice that the correspondence θ satisfies the hypotheses of Theorem **??**. Indeed, A is a nonempty, convex and compact metric space, hence it is acyclic ANR, and a deformation retract satisfying the hypothesis of Theorem **??** exists (see Example 3.11). Since the correspondences β and θ coincide on C, then $\theta(x)$ is acyclic for every $x \in C$. Therefore by Theorem **??** correspondence θ has a fixed point over A. This establishes existence of a perfect equilibrium for a finite normal form game.

Our result extends to prove existence of a perfect equilibrium in behavior strategies in extensive games with perfect recall. To show this, we employ a fundamental result of Selten [**?**]. It establishes a bijection (one-to-one and onto map) between the set of perfect equilibria (in behavior strategies) of an extensive game with perfect recall and the set of perfect equilibria of the corresponding agent normal form. Then existence of perfect equilibria for an extensive game with perfect recall is implied by perfect equilibrium existence for the corresponding agent normal form game, established earlier.

References

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis: A Hitchhikers Guide*, 3rd Edition, Springer–Verlag, New York & London, 2006.

[2] C. D. Aliprantis and S. K. Chakrabarti, *Games and Decision Making*, Oxford University Press, New York & London, 2000.

[3] C. D. Aliprantis and I. Topolyan, Continuity and Equilibrium Stability, *Purdue Working Paper.*

[4] S. Eilenberg and D. Montgomery, Fixed-point theorems for multivalued transformations, *American Journal of Mathematics* **68** (1946), 214–222.

[5] D. Fudenberg and J. Tirole, *Game Theory*, MIT Press, Cambridge, MA, 1991.

[6] E. Klein and A.C. Thompson, *Theory of Correspondences*, John Wiley & Sons, New York, 1984.

[7] R. Myerson, *Game Theory: Analysis of Conflict*, Harvard University Press, Cambridge, MA, 1997.

[8] J. Nash, Equilibrium goints in N-person games, *Proceedings of the National Academy of Sciences* **36** (1950), 48–49.

[9] J. Nash, Non-cooperative games, *Annals of Mathematics* **54(2)** (1951), 286–295.

[10] M. J. Osborne, *An Introduction to Game Theory*, Oxford University Press, Oxford & New York, 2004.

[11] R. Selten, Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragetragheit, *Zeitschrift fur die gesamte Staatswissenschaft* **121** (1965), 301–324.

[12] R. Selten, Re-examination of the perfectness concept for the equilibrium points in extensive games, *International Journal of Game Theory* **4** (1975), 24–55.