Feedback Nash Equilibria for Non-Linear Differential Games in Pollution Control*

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Abstract

Dynamic problems of pollution and resource management with stock externalities often require a differential games framework of analysis. In addition they are represented realistically by non-linear transition equations. However, feedback Nash equilibrium (FBNE)

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solutions, which are the desired ones in this case, are difficult to obtain in problems with non-linear-quadratic structure. We develop a method to obtain numerically non-linear FBNE for a class of such problems, with a specific example for shallow lake pollution control. We compare FBNE solutions, by considering the entire equilibrium trajectories, with optimal management and open-loop solutions and we show that the value of the best FBNE is in general worse than the open-loop and optimal management solutions.

Keywords: Differential games, Pollution control, Non-linear feedback Nash equilibrium solution, Abel equation, Open-loop, Optimal management.

JEL Classification: Q25, C73, C61.

1 Introduction

Many pollution control models have a similar format. Emissions as a by-product of production or consumption accumulate into a stock of pollutants, which is damaging in some way. Production of energy with fossil fuels, for example, releases CO_2 and this contributes to the stock of greenhouse gases, which may cause damage through climate change. To take another example, agricultural activities release phosphorus into lakes, and the resulting stock of phosphorus causes a loss of ecological services, provided by these lakes. Resource extraction problems also have a similar format, if the stock of the resource has some direct utility besides the option to extract in the future. For example, a forest may have an amenity or a biodiversity value besides the option to extract wood. Another important characteristic of these problems is that the damage usually is a public bad. This implies that the optimal control models, that can be set-up to handle the trade-offs above, have stock externalities, which turns the framework of analysis into a differential game. The techniques developed in this paper apply to this general class of problems but in order to keep the presentation transparent, the paper focuses on pollution control problems and, more specifically, on the eutrophication of lakes.

Differential games have been extensively studied during the recent decades to analyze economic problems in areas such as industrial organization, resource and environmental economics or macroeconomic policy. The solution concept that is most often used is the open-loop Nash equilibrium (OLNE), where controls only depend on time (and the initial state of the system). As it is well known, the OLNE is weakly time-consistent but not strongly time-consistent (Başar, 1989): it does not possess the Markov perfect property and is not robust against unexpected changes in the state of the system. Therefore, the feedback Nash equilibrium (FBNE) is a more satisfactory solution concept. It is derived in a dynamic programming framework, so that controls depend on time and state and the solution is Markov perfect by construction. However, solutions are usually very difficult to derive. It is straightforward to find linear feedback equilibria for problems with linear dynamic systems and quadratic objectives, but two types of possible non-linearities complicate matters considerably. The first one is the possibility of *strategic* non-linearities. Tsutsui and Mino (1990) have shown that non-linear equilibria may exist for a linear-quadratic formulation of the dynamic duopoly with sticky prices and their result has later been applied to other problems, such as the problem of international pollution control (Dockner and Long, 1993). The second one is the possibility of *system* non-linearities.

Recent advances in environmental and resource economics emphasize the need for a realistic representation of the ecological system. Realistic modeling of natural systems in most cases indicates that the use of linear dynamics for these natural processes in unified economic-ecological models might not be a good approximation. Non-linearities in the transition equations mainly relate to the existence of non-linear feedbacks, which are physical processes that further impact an initial change of the system under investigation. Feedbacks could be positive if the impact is such that the initial perturbation is enhanced, or negative if the initial perturbation is reduced. For example, in the study of climate change a positive feedback occurs when a higher temperature, due to increased accumulation of greenhouse gases, causes evaporation from the oceans which enhances the greenhouse effects. To take another example, in the analysis of eutrophication of lakes, positive feedbacks are related to the release of phosphorus that has been slowly accumulated in sediments and submerged vegetation. Ignoring these non-linearities might obscure very important characteristics that we observe in reality such as bifurcations and irreversibilities or hysteresis. As a consequence. the design of policies that do not take the impact of non-linearities into account might lead to erroneous results and non-desirable states of the ecosystem. Note that the term *feedback* both refers to physical processes that yield non-linearities in the dynamics of the system and to strategies of economic agents that depend on the state of the system.

This paper shows how to derive the set of (non-linear) feedback Nash equilibria for differential games with non-linear feedbacks in the state transition. It will be clear that the solution cannot be derived analytically but that a numerical approach is needed. This approach is valid for a general class of symmetric non-linear differential games, but the paper will focus on the well-known model for the eutrophication of (shallow) lakes. The basic idea is that the dynamic programming equations can be rewritten as an ordinary differential equation in the feedback control of the economic agents. This proves to be an Abel differential equation and the paper presents an algorithm to solve it. The absence of a boundary condition implies that multiple feedback Nash equilibria exist. The boundary condition can be linked to the resulting steady state, so that the set of feedback Nash equilibria can be parametrized by the feasible steady states.

The results allow us to assess the efficiency of the different Nash equilibria. It is interesting to start with the steady states and to compare this with the literature on international pollution control, a linear-quadratic problem. This literature developed in three steps. First, van der Ploeg and de Zeeuw (1992) derived the linear feedback Nash equilibrium and showed that the steady state lies further away from the optimal management steady state than in the open-loop Nash equilibrium. Dockner and Long (1993) characterized the non-linear feedback Nash equilibria for this problem and showed that the optimal management steady state can be approximated by a feedback Nash equilibrium steady state, for small enough discount rates. Rubio and Casino (2002) modified this result by showing that it does not hold if the state trajectory starts below the optimal management steady state. The last observation returns in the non-linear differential game of this paper but the conclusion for initial states above the optimal management steady state must be modified as well. It can happen (for certain values of the parameters of the problem) that the steady states of the feedback Nash equilibria do not pass the open-loop Nash equilibrium steady state and can therefore not approximate the optimal management steady state. Dockner and Wagener (2006) derive necessary conditions for (non-linear) feedback Nash equilibria through an auxiliary system of differential equations, and apply this approach to a number of problems, among which the shallow lake problem we study in this paper.

Furthermore, our approach allows us to compare the values of the different solutions, and not only the steady states. Note that because we consider symmetric solutions to a symmetric problem, in equilibrium the economic agents have the same value of the objective, which can be denoted as the value of that equilibrium. In this way, equilibria can be ranked. Even if the steady states are close, the equilibrium trajectories and the resulting values can be very different. In our approach we derive the feedback control functions numerically and we can therefore also calculate each equilibrium trajectory and resulting value, as a function of the initial state. It will be shown in this paper that the value of the best feedback Nash equilibrium is generally worse than the value of the open-loop Nash equilibrium, and therefore *a fortiori* worse than the value under optimal management.

2 A Class of Non-Linear Differential Games in Pollution Control

Consider a situation where n economic agents take actions a_i, $i = 1, 2, ..., n$, at each point in time t, with which they affect the state of a natural system, that is shared by all the agents. The actions could, for example, be emissions of greenhouse gases due to industrial activities, or phosphorus loadings into a lake due to agricultural activities. The economic agents in these cases are countries, concerned about climate change, or communities, concerned about the eutrophication of a lake that they share. The action a_i generates benefits according to a strictly increasing and concave utility function $U(a_i)$, which is assumed to be the same for all agents. The evolution of pollutant in the natural system is described by the non-linear transition equation

$$\dot{x}(t) = \sum_{i=1}^{n} a_i(t) - bx(t) + f(x(t)), x(0) = x_0. \tag{1}$$

The state variable x could be interpreted, for example, as accumulated greenhouse gases or accumulated phosphorus in a lake. Besides the standard linear degradation term $-bx$, non-linear feedbacks occur that are represented by the function $f(x)$, which is an increasing non-linear function of the state variable x. In the application that follows, the function $f(x)$ is a convex-concave function with a switching point in between, where $f'(x)$ is maximal. The stock of pollutants x causes environmental damage (or equivalently, reduces the flow of useful services generated by the natural system) according to a strictly increasing and convex damage function $D(x)$, which is also assumed to be the same for all agents. It follows that the flow of net benefits accruing to each agent at each point in time is given by $U(a_i(t)) - D(x(t))$. Each agent is choosing a strategy a_i (at this point assumed to be only a function of time) in order to maximize the present value of net benefits over an infinite time horizon, or

$$\max_{a_i(.)} \int_0^\infty e^{-\rho t}[U(a_i(t)) - D(x(t))]dt, i = 1, 2, ..., n, \tag{2}$$

subject to (1), where $\rho > 0$ is a discount rate, common for all agents.

The game aspect is standard: all actions add to the public bad, so that each agent generates a negative externality for the other agents. Optimal management requires to choose the set of strategies $\{a_1, a_2, ..., a_n\}$ in order to maximize the sum of individual net benefits, or

$$\max_{\{a_1(.),...a_n(.)\}} \int_0^\infty e^{-\rho t}[\sum_{i=1}^n U(a_i(t)) - nD(x(t))]dt, \tag{3}$$

subject to (1). The current-value Hamiltonian H for this problem is given by

$$H = \sum_{i=1}^n U(a_i) - nD(x) + \lambda[a - bx + f(x)], a = \sum_{i=1}^n a_i, \tag{4}$$

and Pontryagin's maximum principle yields the necessary conditions

$$U'(a_i) + \lambda = 0, i = 1, 2, ..., n, \tag{5}$$

$$\dot{x}(t) = a(t) - bx(t) + f(x(t)), x(0) = x_0, \tag{6}$$

$$\dot{\lambda}(t) = [\rho + b - f'(x(t))]\, \lambda(t) + nD'(x(t)), \tag{7}$$

Solving (5) for a_i, and substituting the result in equation (6) yields the Modified Hamiltonian Dynamic System (MHDS), in the state-costate space (x, λ), for the optimal control problem associated with optimal management. It is convenient, for demonstrating the results later, to rewrite this into the MHDS in the state-control space (x, a). Differentiating equation (5) with respect to time, and substituting the result and (5) in equation (7) leads to

$$U''(a_i(t))\dot{a}_i(t) = [\rho + b - f'(x(t))]\, U'(a_i(t)) - nD'(x(t)), i = 1, 2, ..., n. \tag{8}$$

In order to demonstrate the results, it is convenient to specify the functions U, D and f. The equations simplify considerably if it is assumed that the utility function U has a constant elasticity of marginal utility equal to 1, so that $U(a_i) = \ln a_i$. Furthermore, it is assumed that the damage function has a simple quadratic form: $D(x) = cx^2$. The parameter c can be interpreted as the relative importance the agent attaches to the environmental damage in relation to the utility of the action that causes the damage.

Equation (8) reduces to

$$\dot{a}_i(t) = -\left[\rho + b - f'(x(t))\right] a_i(t) + n a_i^2(t) 2cx(t), i = 1, 2, ..., n. \tag{9}$$

Because of symmetry, multiplication by n yields

$$\dot{a}(t) = -\left[\rho + b - f'(x(t))\right] a(t) + a^2(t) 2cx(t). \tag{10}$$

Equations (6) and (10) form the MHDS in the state-control space (x, a), where a denotes the total action of all the agents together. As it is shown in Brock and Starrett (2003), under the assumptions made on the $U(a_i)$, $D(x)$ and $f(x)$ functions, this MHDS has an odd number of steady states. The first and the last steady states are locally stable. The locally stable steady states have the saddle-point property, with a one-dimensional globally stable manifold, and the locally unstable steady states, with possibly complex eigenvalues, lie between two locally stable steady states.

It is straightforward to derive the open-loop Nash equilibrium (OLNE) of this game by applying Pontryagin's maximum principle to the individual optimal control problems (2). This leads to the following simple modifications in the necessary conditions

$$U'(a_i) + \lambda_i = 0, i = 1, 2, ..., n, \tag{11}$$

$$\dot{x}(t) = a(t) - bx(t) + f(x(t)), x(0) = x_0, \tag{12}$$

$$\dot{\lambda}_i(t) = \left[\rho + b - f'(x(t))\right] \lambda_i(t) + 2cx(t), \tag{13}$$

where λ_i denotes the costate of the optimal control problem for agent i. The same manipulations as above finally yield the MHDS in the state-control space (x, a), consisting of equation(6) and

$$\dot{a}(t) = -\left[\rho + b - f'(x(t))\right] a(t) + \frac{1}{n} a^2(t) 2cx(t). \tag{14}$$

The function $f(x)$ represents the internal feedbacks in the natural system. Many natural systems have been carefully investigated and therefore $f(x)$ will be chosen according to the specification that has a good fit to the observations of one of these natural systems, namely the shallow lake.

2.1 The Shallow Lake

Shallow lakes have been intensively studied over the last two decades and it has been shown that the essential dynamics of the eutrophication process can be modelled by the differential equation

$$\dot{P}(t) = L(t) - sP(t) + r\frac{P^2(t)}{P^2(t) + m^2}, P(0) = P_0, \tag{15}$$

where P is the amount of phosphorus sequestered in algae, L is the input of phosphorus (the "loading"), s is the rate of loss consisting of sedimentation, outflow and sequestration in other biomass, r is the maximum rate of internal loading and m is the anoxic level (see for an extensive treatment of the lake model Carpenter and Cottingham, 1997, or Scheffer, 1997). Less is known about deep lakes but from what is known it can be expected that the same type of model is adequate. By substituting $x = \frac{P}{m}$, $a = \frac{L}{r}$, $b = \frac{sm}{r}$ and by changing the time scale to $\frac{rt}{m}$, equation (15) can be rewritten as

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t)+1}, x(0) = x_0. \tag{16}$$

The parameter b can vary considerably across lakes. It indicates whether the dynamics of the lake has bifurcations with hysteresis effects or irreversibilities. If a is constant, three situations can occur. For high values of b, equation (16) has one stable steady state for each value of a. For medium values of b, however, some values of a yield two stable steady states, so that a hysteresis effect occurs. For low values of b, the high steady states are irreversible (for details see Mäler, Xepapadeas and de Zeeuw (2003)). By comparing equation (16) with equation (1), it can be seen that the shallow lake model is an example of a natural system with internal feedbacks, where the non-linear convex-concave function $f(x)$ is specified as $f(x) = \frac{x^2}{x^2+1}$. If the lake is shared by a number of communities, say n, that develop agricultural activities around the lake, the loading of phosphorus a is the total loading of these communities: $a = \sum_{i=1}^{n} a_i$.

Mäler, Xepapadeas and de Zeeuw (2003) present and compare the optimal management solution and the open-loop Nash equilibrium for the shallow lake. With the specification for the function $f(x)$, the general analysis above immediately provides the MHDS for these two outcomes, consisting of equation (16) and

$$\dot{a}(t) = -(\rho + b - \frac{2x(t)}{(x^2(t)+1)^2})a(t) + 2ca^2(t)x(t) \tag{17}$$

for optimal management and

$$\dot{a}(t) = -(\rho + b - \frac{2x(t)}{(x^2(t)+1)^2})a(t) + \frac{1}{n}2ca^2(t)x(t) \tag{18}$$

for the open-loop Nash equilibrium. The parameter b is fixed at $b = 0.6$ (the hysteresis case) and the discount rate ρ is fixed at $\rho = 0.03$. If the parameter c is high enough (e.g. $c = 1$), the MHDS for optimal management has one saddle-point stable steady state in the so-called oligotrophic region (low levels of pollution). However, the open-loop Nash equilibrium with two communities has in this case two saddle-point stable steady states (with an unstable steady state in between): one in the oligotrophic region and one in the eutrophic region (high levels of pollution). Wagener (2003) shows that a solution trajectory of the shallow lake system described by the MHDS which starts at a point $(x_0, a_0) > 0$, either ends up on one of the two saddle points, or produces a control which goes to infinity in finite time, or does not satisfy the transversality condition at infinity $\lim_{t\to\infty} \lambda(t) e^{-\rho t} = 0$. Note that the two-player problem is equivalent to the optimal management problem with the parameter c divided by two. A Skiba point exists which means that for low initial levels of pollution it is best to follow Nash equilibrium strategies towards the oligotrophic steady state, but for high initial levels of pollution it is best to follow Nash equilibrium strategies towards the eutrophic steady state. The outcomes are depicted in Figures 1a (phase diagram) and 1b (stable manifolds) for optimal management, and in Figures 2a (phase diagram) and 2b (stable manifolds) for the open-loop Nash equilibrium.

Insert Figures 1 and 2 here.

These outcomes are important benchmarks for the feedback Nash equilibria that are derived in the next section. A full analysis of Skiba points in the shallow lake model can be found in Wagener (2003). A stochastic shallow lake is analysed in Dechert and O'Donnell (2006) and an empirical analysis for a Dutch lake is presented in Hein (2006).

3 Feedback Nash Equilibria

The feedback Nash equilibria (FBNE) for the class of non-linear differential games, specified in Section 2, result from solving the dynamic programming or Hamilton-Jacobi-Bellman equations in the value functions V_i. The functions and parameters in the problem formulation (2) do not directly depend on time, so that the problem is stationary. Therefore the equilibrium strategies can be represented in a time-stationary feedback form $a_i = h_i(x)$, $i = 1, 2, ..., n$, and the value functions V_i only depend on the state x. Furthermore, the problem is symmetric and only symmetric equilibria are considered, so that the index i can be dropped for the functions V and h. Finally, it is assumed that the functions h and V are differentiable.[1] The dynamic programming or Hamilton-Jacobi-Bellman equation for each agent i becomes

[1] Differentiability of the value function may only hold for a certain range of parameters. Furthermore, since the value function for the shallow lake problem is continuous but not differentiable, for certain parameters, at the so-called Skiba points (Wagener 2003), discontinuities of the feedback rule are expected. We focus our analysis on a parameter range where the value function is differentiable, since the derivation of equilibrium feedback strategies with jumps is beyond the scope of this paper and constitutes an area for further research. It can be said that in the present paper we determine a class of continuous feedback rules which hold for a certain parameter range, but piecewise continuous feedback equilibrium strategies may exist for more general parameter ranges.

$$\rho V(x) = \max_{a_i} \{U(a_i) - D(x) + V'(x)[a_i + (n-1)h(x) - bx + f(x)]\}. \quad (19)$$

The optimality condition is

$$U'(a_i) + V'(x) = 0. \quad (20)$$

In equilibrium $a_i = h(x)$, so that

$$V'(x) = -U'(h(x)). \quad (21)$$

The dynamic programming or Hamilton-Jacobi-Bellman equation becomes

$$\rho V(x) = U(h(x)) - D(x) - U'(h(x))[nh(x) - bx + f(x)]. \quad (22)$$

By differentiating (22) with respect to x, using the optimality condition (21) again and rearranging terms, a non-linear ordinary differential equation in $h(x)$ is obtained:

$$\begin{aligned} &[(nh(x) - bx + f(x))U''(h(x)) + (n-1)U'(h(x))]h'(x) \quad (23) \\ &= (\rho + b - f'(x))U'(h(x)) - D'(x). \end{aligned}$$

This equation is called the Euler equation (see Miranda and Fackler, 2002, p. 325-326). The absence of a boundary condition to this equation implies that multiple feedback Nash equilibria may exist. Only feedback Nash equilibria for which the level of pollutants x converges to a steady state are considered. This implies that for such a steady state x_f the equation

$$h(x_f) = \frac{bx_f - f(x_f)}{n}. \tag{24}$$

can be used as a boundary condition for the differential equation (23). Equation (24) follows from equation (16) with $a = nh(x)$ and the assumption that x_f is a steady state of the lake.

The contours of the algorithm, that will be presented shortly, will be clear by now. Starting from a steady state x_f, the differential equation (23) has a boundary condition (24) and can be solved, which yields a candidate feedback Nash equilibrium corresponding to that steady state.

Before specifying the functions and parameters in (23) and (24) according to the shallow lake problem, it is interesting to return for a moment to the linear-quadratic differential game of international pollution control (van der Ploeg and de Zeeuw, 1992), that was mentioned in the introduction. In that problem the utility and damage functions are both assumed to be quadratic ($U(a_i) = \beta a_i - \frac{1}{2}a_i^2, D(x) = \frac{1}{2}\gamma x^2$), the internal feedbacks are assumed not to exist ($f(x) = 0$), and the number of countries $n = 2$. The differential equation (23) becomes

$$[3h(x) - bx - \beta]h'(x) = (\rho + b)h(x) + \gamma x - (\rho + b)\beta, \tag{25}$$

with hyperbola as solutions, and the steady states are characterized by a line, $h(x) = \frac{b}{2}x$, in the (x, h) space. This situation is depicted in Fig. 1 in

Dockner and Long (1993), as well as in Rubio and Casino (2002).[2]

3.1 The Shallow Lake Continued

The specifications for the shallow lake problem ($U(a_i) = \ln a_i,$ and $D(x) = cx^2, f(x) = \frac{x^2}{x^2+1}$) turn the non-linear ordinary differential equation (23) into

$$[-h(x) + bx - \frac{x^2}{x^2+1}]h'(x) = (\rho + b - 2cxh(x) - \frac{2x}{(x^2+1)^2})h(x) \tag{26}$$

and the boundary condition (24) into

$$h(x_f) = \frac{1}{n}(bx_f - \frac{x_f^2}{x_f^2+1}). \tag{27}$$

Note that equation (26) does not explicitly depend on n.

Equation (26) is an Abel differential equation of the second kind (Murphy, 1960), which cannot be solved analytically. In this paper the ode solver *ode15s* of Matlab is used to find a numerical solution of (26), with boundary condition (27) (see The MathWorks, 2002, Polyanin and Zaitsev, 1995). In order to be able to make a comparison with the benchmark cases of optimal management and the open-loop Nash equilibrium in Section 2.1, the para-

[2] As suggested by one reviewer, one way of making the connection between the linear-quadratic version of this differential game with the non-linear version discussed here, would be to multiply the non-linear term, $f(x)$, by a small parameter, say $\varepsilon > 0$, and obtain the Nash equilibria as a function of ε (assuming that the payoff functions are quadratic, possibly perturbed by small non-quadratic terms, also multiplied by ε). One can obtain the Nash equilibria in this case (both open-loop and feedback) iteratively, with the *zero-th-order* solution (corresponding to $\varepsilon = 0$) being the solution to the linear-quadratic game. Corrections to this *zero-th-order* solution will then also involve simpler problems, which are more readily solvable.

meters are fixed at the same values as in Section 2.1: $b = 0.6$, $\rho = 0.03$, $c = 1$ and $n = 2$. The parameter x_f denotes the steady state of the feedback Nash equilibrium. According to the boundary condition (27), it must lie on the curve $h(x) = \frac{1}{n}(bx - \frac{x^2}{x^2+1})$ in the (x, h) plane, but it is otherwise free so that a multiplicity of Nash equilibria may result. Note that h represents the individual loading, so that this curve denotes the steady states of the lake model (16) because total loading $a = nh$. Note that when the curve for the steady states of the lake model (16) in the (x, a) plane, $a = bx - \frac{x^2}{x^2+1}$, is superimposed on the (x, h) plane, this curve represents the points at which singularities can be expected, because at these points the coefficient of $h'(x)$ in the differential equation (26) is equal to 0.

As it was mentioned in the introduction, comparing steady states is one thing but ultimately the resulting values of the different outcomes have to be compared, taking account of what happens on the trajectories towards the steady state. These values are a function of the initial state x_0. The values for the feedback Nash equilibria are given by

$$V_f(x_0, x_f) = \int_0^\infty e^{-\rho t}[\ln h(x(t)) - cx^2(t)]dt, \tag{28}$$

where $h(x)$ is the solution of the differential equation (26), with boundary condition (27), and $x(t)$ is the solution of the differential equation

$$\dot{x}(t) = nh(x(t)) - bx(t) + \frac{x^2(t)}{x^2(t) + 1}, x(0) = x_0. \tag{29}$$

It will be clear that in general not every steady state x_f can be reached from any initial state x_0. It will also be clear that in general not every steady state x_f will be stable. If, however, a number of stable steady states can be reached from some initial state, it is assumed that the agents will be able to coordinate on the best feedback Nash equilibrium, if it exists. This means that in that case the value will be only a function of the initial state:

$$V_f(x_0) = \max_{x_f} V_f(x_0, x_f) \tag{30}$$

where x_f must be reachable from x_0 and stable, and where it is assumed that the maximum exists.

The numerical algorithm to characterize the best feedback Nash equilibrium for the shallow lake problem consists of the following steps:[3]

Step 1. For each candidate x_f the non-linear ordinary differential equation (26) with boundary condition (27) is solved, with the ode solver *ode15s* of Matlab, in the intervals $[p, x_f]$ and $[x_f, q]$, where p and q are chosen appropriately.

Step 2. The numerical solution for $h(x)$ is then used to solve the transition equation (29) in the interval $[0, T]$, where T is chosen appropriately.

Step 3. Then the value (28) is computed, using a Matlab *quad* function.

Step 4. Finally, the set of values is maximized over the set of admissible x_f, according to (30).

The results will also be presented in steps. At the end a comparison can be made with optimal management and the open-loop Nash equilibrium.

[3] The software code is available through the JEDC supplement archives. Any questions regarding the code may be directed to M. Plexousakis

3.1.1 Candidate feedback control strategies

Figures 3 to 5 plot the solutions of the non-linear ordinary differential equation (26), with boundary condition (27), for three different regions for the candidate steady states x_f. Recall that these candidate steady states have to lie on the curve $h(x) \equiv g_s(x)/n = \frac{1}{n}(bx - \frac{x^2}{x^2+1})$, and that the curve $h(x) \equiv g_s(x) = bx - \frac{x^2}{x^2+1}$ denotes the possible singularities of the differential equation.[4]

Region 1. For $x_f < 0.17$, the profiles of the solutions are given by Figure 3. The solution extends backward to the origin and extends forward until it meets the curve of singularities at an infinite slope. The candidate steady states in this region are not stable. The origin, where the level of pollution is zero, is not feasible because of the assumptions on the utility function U.

Insert Figure 3 here.

Region 2. For $0.17 < x_f < 0.72$, the profiles of the solutions are given by Figure 4. The series of profiles corresponds to two series of candidate steady states: a series of increasing unstable steady states , starting at $x_f = 0.17$, and a series of decreasing stable steady states, starting at $x_f = 0.72$. At the end, at $x_f = 0.38$, where the profile of the solution is tangent to the curve $h(x) \equiv g_s(x)/n = \frac{1}{n}(bx - \frac{x^2}{x^2+1})$, a steady state results that can be reached from the right but not from the left.

Insert Figure 4 here.

[4] It should be noted that the derived profiles $h(x)$ do not blow up in finite time, or do not approach zero as x increases. This suggests that the $h(x)$ profiles derived by the algorithm presented in this paper have a behavior similar to the one expected by control trajectories of the optimal management or the open-loop problems which satisfy the conditions discussed in Wagener (2003, see Appendix A).

Region 3. For $x_f > 0.72$, the profiles of the solutions are given by Figure 5. The solution extends backward until it meets the curve of singularities at an infinite slope. The candidate steady states in this region are stable.

Insert Figure 5 here.

3.1.2 Best feedback Nash equilibrium

In Steps 2 to 4 of the numerical algorithm, the initial state x_0 has to be determined. In Region 1, only initial states are feasible that are smaller than the x-value of the intersection point of the profile, corresponding to the steady state $x_f = 0.17$, with the curve of singularities. In Region 2, only initial states are feasible that are larger than the value of the unstable steady state of the first profile ($x_0 = 0.17$). In Region 3, only initial states are feasible that are larger than the x-value of the intersection point of the profile, corresponding to the steady state $x_f = 0.72$, with the curve of singularities. By combining these observations, it follows that only initial states are feasible that are larger than $x_0 = 0.17$.

This can be better explained with the help of figure 6, which is an enlarged version of figure 4 around $x_f = 0.38$.

Insert Figure 6 here.

For $0.17 < x_0 < 0.38$, profiles can be chosen that start at the unstable steady state $x_0 = x_f$ on the curve of the steady states $h(x) \equiv g_s(x)/n = \frac{1}{n}(bx - \frac{x^2}{x^2+1})$, which is the dashed line labeled SS, leading to a stable steady state x_f on the curve SS between 0.38 and 0.72. Furthermore, profiles can be chosen that start above that point (if possible) resulting in a higher stable steady state. The maximization in Step 4 shows that it is best to choose the profile that starts at the unstable steady state. In figure 6 this means that, with $x_0 = 0.24$, profiles could start at all intersections of the vertical line through $x_0 = 0.24$. Intersection points of that vertical line with profiles below A are not feasible because the dynamics is unstable. Furthermore, by maximization it is best to choose point A on SS. This point is unstable and leads to a stable steady state at B as indicated by the arrows. For $x_0 > 0.38$ (point E in figure 6), profiles can be chosen that lead to a stable steady state above the initial state, and profiles can be chosen that lead to a stable steady state between 0.38 and the initial state. The maximization in Step 4 shows that it is best to choose the profile that leads (from the right) to the stable steady state 0.38, which is the profile tangent to the curve of steady states, SS, from all the possible starting points, at point E.

Figure 7 depicts the relation between the initial state and the resulting steady state of the best feedback Nash equilibrium.

Insert Figure 7 here.

It is interesting to note that the selection of the best feedback Nash equilibrium leads to a form of time-inconsistency for $0.17 < x_0 < 0.38$. Rubio and Casino (2002) obtained a similar result for the international pollution control game. Starting at the initial state x_0, it is best to choose the profile that starts on the curve of steady states $h(x) = \frac{1}{n}(bx - \frac{x^2}{x^2+1})$ at the point $h(x_0)$ lying on the intersection of the vertical line through x_0 and the curve of steady states, like point A in figure 6. However, when the strategies are reconsidered after some time has elapsed and the state has reached a point between x_0 and 0.38, say point $x_0^{'}$, it is best not to follow that same profile anymore but rather to switch down to the profile that starts on the curve

$h(x) = \frac{1}{n}(bx - \frac{x^2}{x^2+1})$ at the point $h\left(x_0'\right)$. Moreover, if the state moves beyond 0.38, it is best to switch down to the profile that leads the system back to the stable steady state 0.38. A way to resolve this time-inconsistency is reoptimizing the feedback Nash equilibrium over time. The resulting trajectory approximately follows the curve $h(x) = \frac{1}{n}(bx - \frac{x^2}{x^2+1})$ and then converges to the steady state 0.38. This implies that the steady state 0.38 is an important point. It is the steady state of the best feedback Nash equilibrium for initial states $x_0 > 0.38$, but also for initial states $0.17 < x_0 < 0.38$ in case the reoptimization is applied.

3.1.3 Comparison with optimal management and the open-loop Nash equilibrium

Figures 1 and 2 depict the outcomes for optimal management and the open-loop Nash equilibrium. For the parameter values $b = 0.6$, $\rho = 0.03$, $c = 1$, $n = 2$, the saddle-point stable steady state for optimal management is equal to 0.353, and the saddle-point stable steady states for the open-loop Nash equilibrium are equal to 0.393 and 1.58. Another interesting point is the *flip* point: this is the local maximum of the curve for the steady states of the lake model (16). It is called the *flip* point because when total loading a is gradually increased from a low level, at that point the steady state of the lake will flip to a substantially higher level of pollution. It is equal to 0.408 for these parameter values.

When these values are compared with the previous results of this section, the general conclusion for the international pollution control game (Dockner and Long, 1993) is confirmed. The steady state for the best feedback Nash equilibrium (0.38) lies closer to the steady state for optimal management (0.353) than the steady states for the open-loop Nash equilibrium (0.393 and 1.58). Moreover, if the discount rate ρ approaches 0, then the steady state of the best feedback Nash equilibrium approaches the steady state of optimal management: namely for $\rho = 0.02$, the values are 0.365 and 0.344, respectively, and for $\rho = 0.01$, the values are 0.344 and 0.336.

However, this picture does not hold for all parameter values. If, for example, the number of communities sharing the lake is increased to $n = 3$, the steady state of the best feedback Nash equilibrium becomes 0.417, whereas the best steady state of the open-loop Nash equilibrium becomes 0.412 (the optimal management steady state remains 0.353, because this is independent of the number of communities). The reason is that the steady state of the open-loop Nash equilibrium has moved beyond the flip point. It is easy to see why this happens by looking more closely at equations and graphs above. First look at the profiles of the feedback Nash equilibrium strategies in Figure 4. These profiles have a local maximum for a value of x between 0.17 and 0.72. At such a local maximum $h'(x) = 0$, so that according to (26) individual loading there is given by

$$h(x) = \frac{\rho + b - \frac{2x}{(x^2+1)^2}}{2cx}. \tag{31}$$

This corresponds to the following relation between total loading $a = nh$ and the state x:

$$a = n\frac{\rho + b - \frac{2x}{(x^2+1)^2}}{2cx}. \tag{32}$$

Note that equation (32) represents the steady states of the differential equation for total loading (18) in the MHDS of the open-loop Nash equilibrium. The argument now goes as follows. In the (x, h) plane, the best steady state of the open-loop Nash equilibrium is represented by the first intersection of the curves (31) and $h(x) = \frac{1}{n}(bx - \frac{x^2}{x^2+1})$. According to equation (31), this point is also the local maximum of one of the profiles of the possible feedback Nash equilibrium strategies. Because the steady state of the open-loop Nash equilibrium lies to the right of the flip point, the curve $h(x) = \frac{1}{n}(bx - \frac{x^2}{x^2+1})$ is decreasing at this point. It follows that this profile cannot be the profile of the best feedback Nash equilibrium, because the last one must be tangent to this curve. Moreover, it follows that the steady state of the best feedback Nash equilibrium is the tangency point of a lower profile and is therefore larger than the best steady state of the open-loop Nash equilibrium.

It is still true that lowering the discount rate ρ moves the steady state of the best feedback Nash equilibrium towards the steady state of optimal management. At first, the effect of increasing the number of communities n is neutralized, so that the best steady state of the open-loop Nash equilibrium moves below the flip point again, and then the story in the beginning of this section applies. However, for any fixed discount rate ρ, the story works vice versa: increasing the number of communities n moves the steady state of the best feedback Nash equilibrium beyond the open-loop one.

Comparisons between the different outcomes have focused on steady states up to now but it is, of course, more important to look at the values $V(x_0)$. In Step 4 of the numerical algorithm above, the value for the best feedback Nash equilibrium $V_f(x_0)$ is explicitly calculated. The same can be done for the values for optimal management $V_o(x_0)$ and for the open-loop Nash equilibrium $V_n(x_0)$. In Figure 8 the results are presented for a number of values of the initial state x_0 (and for the original set of parameter values).

Insert Figure 8 here

The best feedback Nash equilibrium is generally performing worse than the open-loop Nash equilibrium, and therefore *a fortiori* worse than optimal management, although the differences are small for initial states between 0.19 and 0.46. Only in the neighborhood of the steady states, the performance is about the same. This shows that focusing on steady states does not give the right picture. It cannot generally be concluded that, in this type of models, Nash equilibria have been found that support optimal management.

4 Conclusion

It is often not realistic to use linear models for natural systems in pollution control problems. This implies that it is necessary to be able to handle non-linear differential games in analyzing pollution control problems with negative externalities or public bads. This paper provides a numerical solution procedure to derive feedback Nash equilibria for the shallow lake problem, a typical example of a non-linear differential game in environmental and resource economics.

The literature on linear-quadratic differential games has shown that non-linear feedback Nash equilibria exist with steady states that are close to the optimal management steady state. However, in this paper it is shown that for non-linear differential games the steady state of the best feedback

Nash equilibrium is not necessarily close to the optimal management steady state. Moreover, this paper shows that even if these steady states are close, the value of the corresponding feedback Nash equilibrium is generally much worse than the value of optimal management.

The good news is that we can handle an important class of non-linear differential games, but the bad news is that generally the best feedback Nash equilibrium does not support optimal management. More research is needed to get better insight into this complicated type of problems.

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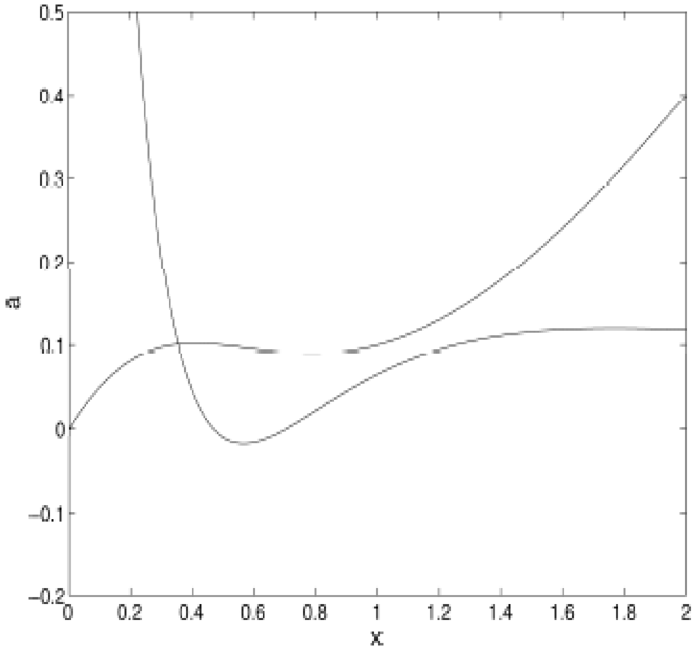

Figure 1a: Steady state for the optimal management

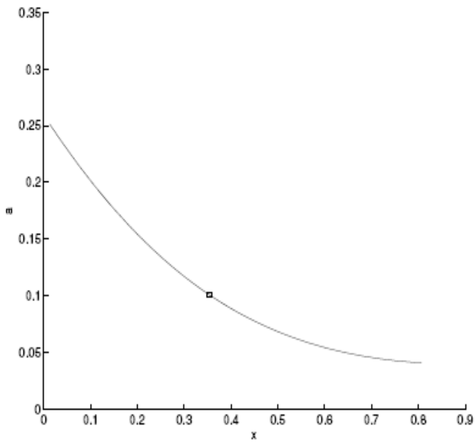

Figure 1b: Stable manifold for the optimal management problem

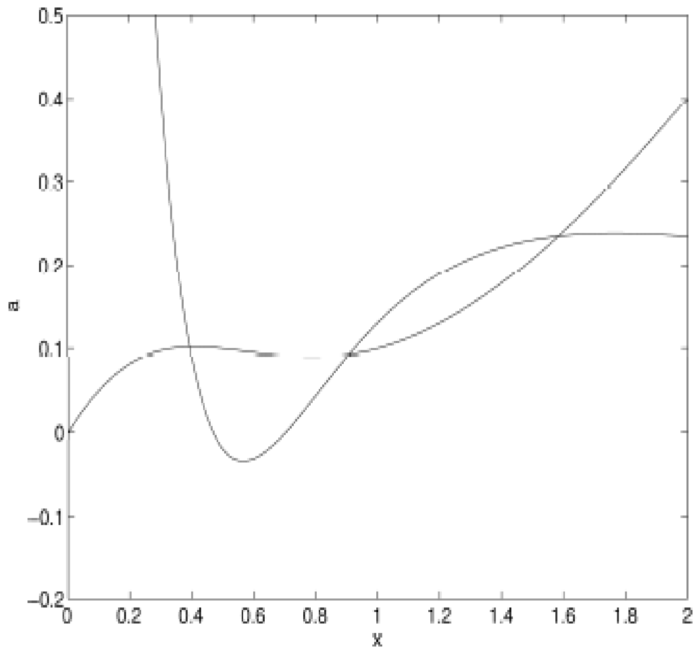

Figure 2a: Steady states for the Open Loop Nash Equilibrium

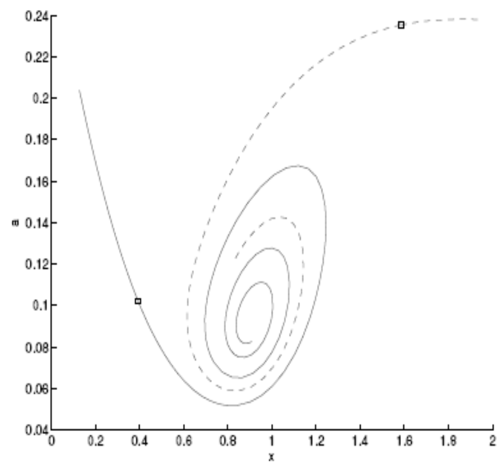

Figure 2b: Stable manifolds for the open loop Nash equilibrium

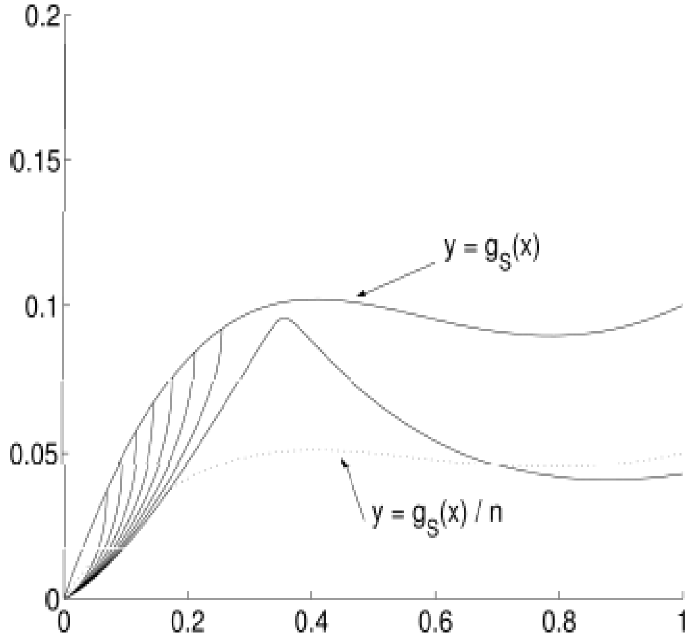

Figure 3: Profiles of $h\,(x)$ for $x_p < 0.17$

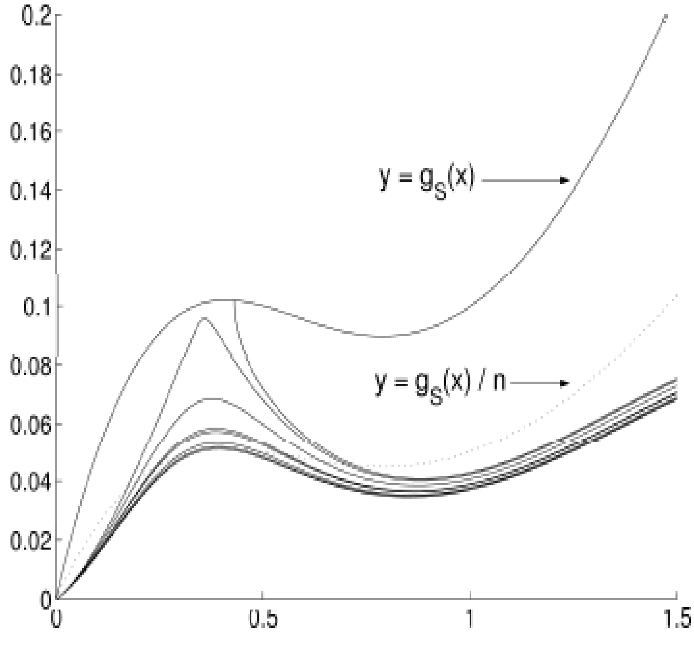

Figure 4: Profiles of $h\,(x)$ for $0.17 < x_p < 0.72$

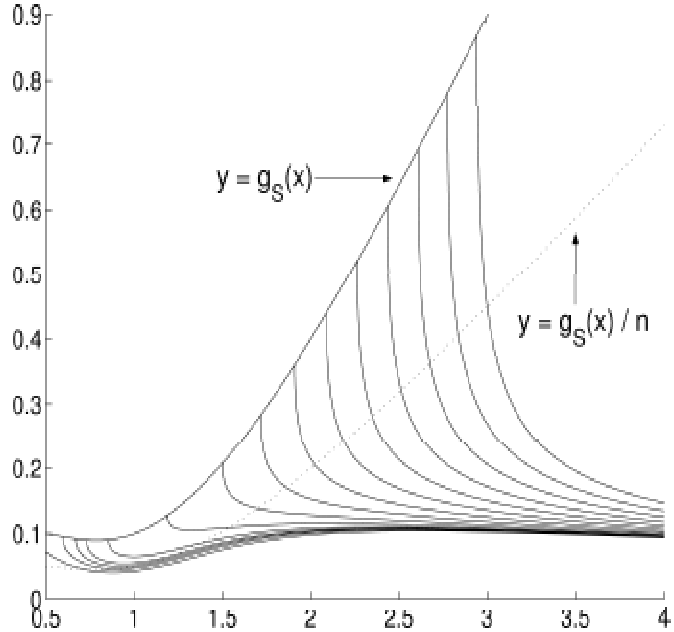

Figure 5: Profiles of $h\,(x)$ for $x_p > 0.72$

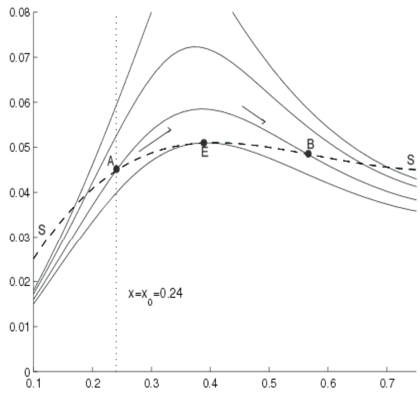

Figure 6: Best feedback Nash Equilibrium

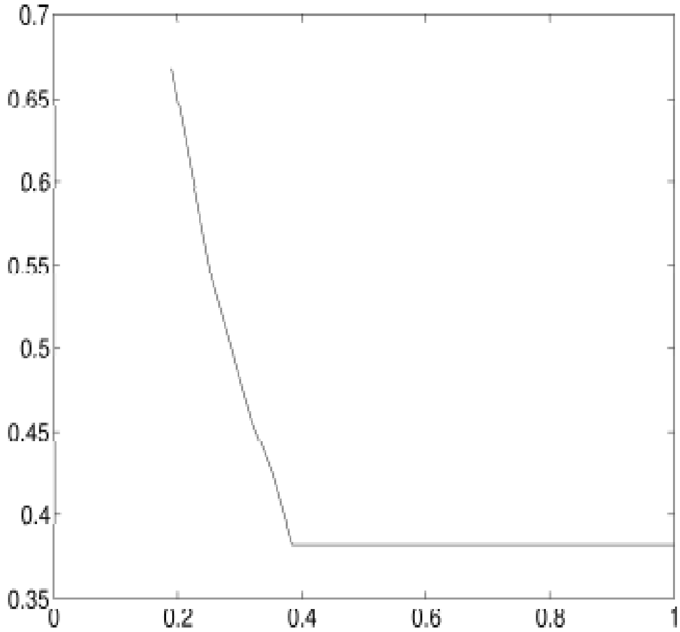

Figure 7: x_F vs x_0

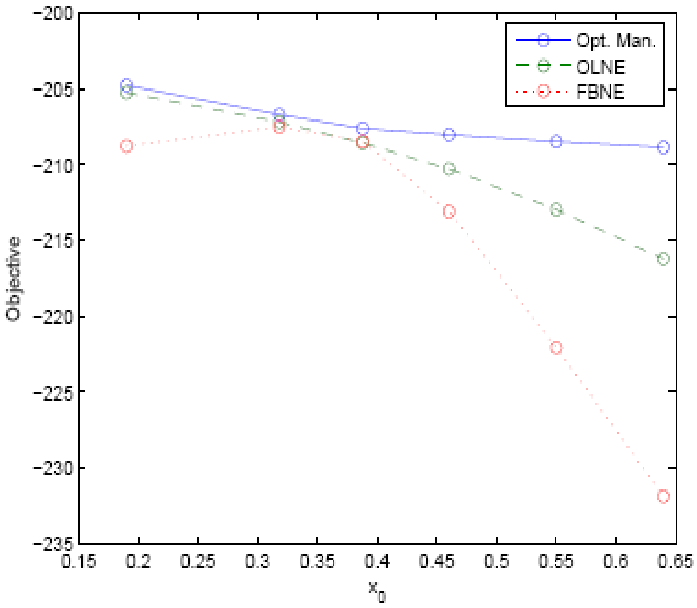

Figure 8: Comparison of the Objectives