Nonstationary Oblivious Equilibrium

Gabriel Y. Weintraub
Columbia University

C. Lanier Benkard
Stanford University
and NBER

Przemyslaw Jeziorski
Stanford University

Benjamin Van Roy
Stanford University

November, 2008
PRELIMINARY AND INCOMPLETE

1 Introduction

In a recent paper, Weintraub, Benkard, and Van Roy (2008b) propose an approximation method for analyzing Ericson and Pakes (1995)-style dynamic models of imperfect competition. In that paper, we defined a new notion of equilibrium, *oblivious equilibrium* (henceforth, OE), in which each firm is assumed to make decisions based only on its own state and knowledge of the long-run average industry state, but where firms ignore current information about competitors' states. The great advantage of OE is that they are much easier to compute than are Markov perfect equilibria (henceforth, MPE). Moreover, we showed that an OE provides meaningful approximations of *long-run* Markov perfect dynamics of an industry with many firms if, alongside some technical requirements, the equilibrium distribution of firm states obeys certain "light-tail condition".

To facilitate using OE in practice, in Weintraub, Benkard, and Van Roy (2008a) we provide a computational algorithm for solving for OE, and approximation bounds that can be computed to provide researchers with a numerical measure of how close OE is to MPE in their particular application. We also provided computational evidence supporting the conclusion that OE often yields good approximations of MPE behavior for industries like those that empirical researchers would like to study.

While our computational results suggest that OE will be useful in many applications on its own, we believe that a major contribution of OE will be as a starting point with which to build even better approximations. As a matter of fact, in Weintraub, Benkard, and Van Roy (2008a) we extended our base model as well as algorithms for computing OE and error bounds to incorporate aggregate shocks common to all firms.

Such an extension is important, for example, when analyzing the dynamic effects of industry-wide business cycles.

In this paper we introduce another important extension to OE. OE offers a way to approximate *long-run* Markov perfect industry dynamics with many firms, and it could be used if one is interested in long-run economic indicators, such as long-run average investment. These quantities are independent of the initial state of the industry. In other cases, one may be interested in the short-run dynamic behavior of an industry starting from a given initial condition. For example, one may want to asses how an industry would evolve over a few years after a policy or environmental change. With this motivation, we introduce a nonstationary notion of OE in which every firm knows the industry state in the initial period but does not update this knowledge after that point. We call this new equilibrium concept, *nonstationary oblivious equilibrium* (henceforth, NOE). NOE is based on the same idea as oblivious equilibrium but it offers a way to approximate *short-run* transitional dynamics that may result, for example, from shocks or policy changes.

The model and assumptions in this paper are the same as in Section 3 of Weintraub, Benkard, and Van Roy (2008b). For the sake of completeness, we present the model in the Appendix.

In Section 2 we define nonstationary oblivious equilibrium (henceforth, NOE). Moreover, in our computational experiments we focus on NOE that become stationary as time progresses. In Section 2 we also show that under mild technical conditions such NOE exist. In the following sections we present algorithms and results for NOE in a similar spirit to those presented in Weintraub, Benkard, and Van Roy (2008b) and Weintraub, Benkard, and Van Roy (2008a) for OE.

In Section 3 we provide an algorithm to compute NOE that become stationary as time progresses. The algorithm is computationally efficient; it can compute NOE in few minutes even for industries with hundreds of firms.

In Section 4 we provide an efficient simulation-based algorithm to compute a bound on approximation error. Error here is measured in terms of the expected incremental value that an individual firm in the industry can capture by unilaterally deviating from the NOE strategy. The algorithm for bounding approximation error allow us to verify accuracy of NOE as an approximation for each problem instance.

In Section 5 we provide a computational study and, using the error bounds, show that NOE offers useful approximations for relevant models of industries with hundreds or even tens of firms. We also show that NOE can endogenously generate industry dynamics, like industry shake-outs, similar to those observed in data sets. Our results show that by using NOE it is possible to further expand the set of dynamic industries that can be analyzed computationally.

While the previous results provide support for using NOE in practice, in Section 6 we provide an asymptotic result that provides a theoretical justification for the approximation. We show that, if alongside some technical requirements, the equilibrium distribution of firm states obeys certain “light-tail” condition, then the approximation error vanishes as the market sizes grows. We note that important parts of the proof of this result require different techniques to the ones used in its analog theorem for OE in Weintraub, Benkard, and Van Roy (2008b).

Finally, in Section 7 we provide conclusions and some thoughts for future research.

2 Nonstationary Oblivious Equilibrium

Oblivious equilibrium, offers a way to approximate *long-run* Markov perfect industry dynamics. In this section we introduce a new equilibrium concept, nonstationary oblivious equilibrium, that can be used to approximate the *short-run* dynamic behavior of an industry starting from an initial state of interest.

Recall that an oblivious equilibrium was based on the idea that when there are a large number of firms (and no aggregate shocks), simultaneous changes in individual firm quality levels can average out such that in the long-run the industry state remains roughly constant over time. Based on a similar idea we introduce a method to approximate the short-run behavior of an industry that starts from a given state of interest. If there are a large number of firms (and no aggregate shocks), the industry state starting from a given initial state roughly follows a deterministic trajectory. In this setting, each firm can make near-optimal decisions based only on its own quality level and by knowing the deterministic trajectory followed by the industry state. With this motivation, we consider restricting firm strategies so that each firm's decisions depend only on the firm's quality level and the time period. We call such restricted strategies *nonstationary oblivious* since they involve decisions made without full knowledge of the circumstances — in particular, the state of the industry. Note that nonstationary oblivious strategies differ from oblivious strategies because they depend on the time period. To simplify notation we assume that the industry is at the initial state of interest at time period $t = 0$.

2.1 Nonstationary Oblivious Strategies and Entry Rate Functions

Let $\tilde{\mathcal{M}}_{ns} = \tilde{\mathcal{M}}^{\infty} \subset \mathcal{M}^{\infty}$ and $\tilde{\Lambda}_{ns} = \tilde{\Lambda}^{\infty} \subset \Lambda^{\infty}$ denote the set of nonstationary oblivious strategies and the set of nonstationary oblivious entry rate functions.[1] A nonstationary oblivious strategy is a sequence

[1]Recall that $\tilde{\mathcal{M}} \subset \mathcal{M}$ and $\tilde{\Lambda} \subset \Lambda$ denote the set of oblivious strategies and the set of oblivious entry rate functions; $\mathcal{M}$ and Λ are the set of Markov strategies and the set of entry rate functions (see Weintraub, Benkard, and Van Roy (2008b)). The set X^{∞} is the infinite cross product of X, i.e. $X^{\infty} = X \times X \times X....$

of oblivious strategies. Hence, if $\mu \in \tilde{\mathcal{M}}_{ns}$ is a nonstationary oblivious strategy, then $\mu = \{\mu_0, \mu_1, ...\}$, where for each time period $t \geq 0$, $\mu_t \in \tilde{\mathcal{M}}$ is an oblivious strategy. For example, if firm i uses strategy $\mu \in \tilde{\mathcal{M}}_{ns}$ then at time period t, firm i takes action $\mu_t(x_{it})$, where x_{it} is the state of firm i at time t (so the action depends both on the time period and the state). In a NOE firms will make decisions assuming that the industry state evolves deterministically. Moreover, firms will assume the industry state at time period t is the *expected* industry state after t time periods of evolution given the competitors' strategy and starting from the industry state of interest. Therefore, under this assumption, the time period determines the industry state.

A nonstationary oblivious entry rate function is a sequence of oblivious entry rate functions. Hence, if $\lambda \in \tilde{\Lambda}_{ns}$ is a nonstationary oblivious entry rate function, then $\lambda = \{\lambda_0, \lambda_1, ...\}$ where for every period $t \geq 0$, λ_t is real-valued.

2.2 Nonstationary Oblivious Equilibrium

Note that if all firms use a common strategy $\mu \in \tilde{\mathcal{M}}_{ns}$, the quality level of each evolves as an independent transient non-homogenous Markov chain. Let the transition sub-probabilities of this transient Markov chain for period t be denoted by $P_{\mu_t}(x, y)$. If there were an infinite number of firms, though each evolves stochastically, the percentage of firms that transition from any given quality level to another would be deterministic. Similarly, the percentage of firms that exit would be deterministic. Motivated by this fact, for $\mu \in \tilde{\mathcal{M}}_{ns}$, $\lambda \in \tilde{\Lambda}_{ns}$, and $s \in \overline{\mathcal{S}}$ we define the following sequence of industry states:

(2.1)
$$\tilde{s}_{t+1}(x) = \begin{cases} \sum_{y \in \mathbb{N}} P_{\mu_t}(y, x)\tilde{s}_t(y) + \lambda_t & \text{if } x = x^e \\ \sum_{y \in \mathbb{N}} P_{\mu_t}(y, x)\tilde{s}_t(y) & \text{otherwise,} \end{cases}$$

where $\tilde{s}_0 = s \in \overline{\mathcal{S}}$. Note that $\tilde{s}_t$ is the expected industry state at time t given strategy μ and it can be easily computed by matrix multiplication. For all $x \in \mathbb{N}$, we let $\tilde{s}_{(\mu,\lambda,s),t}(x) = \tilde{s}_t(x)$, where for all $t \geq 0$, $\tilde{s}_t(x)$ is given by equation (2.1).

For nonstationary oblivious strategies $\mu', \mu \in \tilde{\mathcal{M}}_{ns}$, a nonstationary oblivious entry rate function $\lambda \in \tilde{\Lambda}_{ns}$, and an initial industry state s, we define a *nonstationary oblivious value function* for period t

(2.2)
$$\tilde{V}_t(x|\mu', \mu, \lambda, s) \;=\; E_{\mu'}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(x_{ik}, \tilde{s}_{(\mu,\lambda,s),k}) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i}\middle| x_{it} = x\right].$$

This value function should be interpreted as the expected net present value of a firm that is at quality level x at time t and follows nonstationary oblivious strategy μ', under the assumption that, for all $t \geq 0$, its competitors' state will be given by $\tilde{s}_{(\mu,\lambda,s),t}$ at time t. Note that even though the firm's state trajectory only depends on the firm's own strategy μ', the nonstationary oblivious value function remains a function of the competitors' strategy μ and the entry rate λ through the expected industry state trajectory $\tilde{s}_{(\mu,\lambda,s),\cdot}$. We abuse notation by using $\tilde{V}_t(x|\mu, \lambda, s) \equiv \tilde{V}_t(x|\mu, \mu, \lambda, s)$ to refer to the nonstationary oblivious value function when firm i follows the same strategy μ as its competitors.

We now define a new solution concept. To avoid pathological behavior in which an entry rate grows unboundedly large and is followed by massive exit, we restrict all entry rates to be less than a predetermined upper bound $\lambda_{max} \geq \sup_{x,s} \pi(x, s)/(1 - \beta) + \overline{\phi}$. We introduce the following assumption that is kept throughout the paper unless otherwise explicitly noted:[2]

Assumption 2.1. $\tilde{\Lambda} = [0, \lambda_{max}]$.

An $s-$*nonstationary oblivious equilibrium* consists of a strategy $\mu \in \tilde{\mathcal{M}}_{ns}$ and an entry rate function $\lambda \in \tilde{\Lambda}_{ns}$ that satisfy the following conditions:

1. Firm strategies optimize a nonstationary oblivious value function:

$$\sup_{\mu' \in \tilde{\mathcal{M}}_{ns}} \tilde{V}_0(x|\mu', \mu, \lambda, s) = \tilde{V}_0(x|\mu, \lambda, s), \qquad \forall x \in \mathbb{N}. \tag{2.3}$$

2. At every period of time, the nonstationary oblivious expected value of entry is zero or boundary conditions are satisfied. For all $t \geq 0$,

$$\lambda_t \in (0, \lambda_{max}) \text{ implies } \beta\tilde{V}_{t+1}(x^e|\mu, \lambda, s) - \kappa = 0$$
$$\beta\tilde{V}_{t+1}(x^e|\mu, \lambda, s) - \kappa < 0 \text{ implies } \lambda_t = 0$$
$$\beta\tilde{V}_{t+1}(x^e|\mu, \lambda, s) - \kappa > 0 \text{ implies } \lambda_t = \lambda_{max}\ .$$

Note that the optimization of $\tilde{V}_0$ implies, by dynamic programming principles, that firms optimize $\tilde{V}_t$ for all $t \geq 0$.

In this paper, we focus on NOE that become stationary as time progresses. That is, we focus on NOE $(\mu, \lambda) \in \tilde{\mathcal{M}}_{ns} \times \tilde{\Lambda}_{ns}$ that converge to an OE $(\tilde{\mu}, \tilde{\lambda}) \in \tilde{\mathcal{M}} \times \tilde{\Lambda}$ as time progresses in the following sense:

[2] It is simple to show that an OE entry rate must be smaller than λ_{max}. Moreover, in our computational experiments we never observed NOE entry rates growing unboundedly large and they were always much smaller than λ_{max}. For this reason and to simplify the explanation, λ_{max} is omitted in the description of the algorithm in Section 3. We make use of λ_{max}, however, in the existence proof that follows.

for all $x \in \mathbb{N}$, $\lim_{t\to\infty} \mu_t(x) = \tilde{\mu}(x)$, and $\lim_{t\to\infty} \lambda_t = \tilde{\lambda}$. In the next subsection, we show the existence of such NOE.

2.3 Existence of NOE that Become Stationary

By Assumptions A.1 and A.2, investments and expected discounted profits are uniformly bounded over all states by $\hat{\iota}$ and $\sup_{x,s} \pi(x,s)/(1-\beta) + \overline{\phi}$, respectively. Therefore, with out loss of generality, we restrict the range of $\mu \in \tilde{\mathcal{M}}$ to $[0, \hat{\iota}] \times [0, \sup_{x,s} \pi(x,s)/(1-\beta) + \overline{\phi}]$.

We introduce the following additional assumption.

Assumption 2.2. *For all $i \in \mathbb{N}$ and $t \in \mathbb{N}$, $x_{it} \leq x_{max}$.*

Assumption 2.2 implies that all the analysis can be restricted to the finite space of quality levels $\{0, ..., x_{max}\}$.

We define the set of converging nonstationary strategies and entry rate functions:

$$\hat{\mathcal{M}}_{ns} = \{\mu \in \tilde{\mathcal{M}}_{ns} : \text{ for which there exists } \mu \in \tilde{\mathcal{M}}, \text{ such that, for all } x, \lim_{t\to\infty} \mu_t(x) = \mu(x)\} ,$$

$$\hat{\Lambda}_{ns} = \{\lambda \in \tilde{\Lambda}_{ns} : \text{ for which there exists } \lambda \in \tilde{\Lambda}, \text{ such that, } \lim_{t\to\infty} \lambda_t = \lambda\} .$$

We endowed this sets with the metric compatible with the product topology.

The following is the main result of this section. The proof is provided in the Appendix.

Theorem 2.1. *Suppose Assumptions A.1, A.2, A.3, 2.1, and 2.2 hold. Then, there exists a NOE $(\mu, \lambda) \in \hat{\mathcal{M}}_{ns} \times \hat{\Lambda}_{ns}$. Moreover, for all x, $\lim_{t\to\infty} \mu_t(x) = \tilde{\mu}(x)$ and $\lim_{t\to\infty} \lambda_t = \tilde{\lambda}$, where $(\tilde{\mu}, \tilde{\lambda}) \in \tilde{\mathcal{M}} \times \tilde{\Lambda}$ is an OE.*

3 Algorithm to Compute NOE

We propose an algorithm to compute NOE that become stationary and converge to OE. We impose this form of convergence in the algorithm and then solve backwards. In this way, the problem of finding a NOE is reduced to a finite horizon problem.

Suppose we are mostly interested in the behavior of the industry in the interval between time periods $t = 0$ and $t = \underline{T}$. Let $\tilde{V}, \tilde{\mu}, \tilde{\lambda}, \tilde{s}$ be a (stationary) OE value function, strategy, entry rate, and expected state, respectively. Let $\overline{T} := \min\{t | \beta^{t-\underline{T}} \tilde{V}(x^e + t\overline{w}) \leq \delta\}$, where $\delta > 0$ is a predetermined precision. We assume there is a finite time horizon of length $\overline{T}$, and that after $\overline{T}$ the NOE coincides with an OE. More specifically, for all $t > \overline{T}$, $\mu_t = \tilde{\mu}$, $\lambda_t = \tilde{\lambda}$, and $\tilde{s}_t = \tilde{s}$. In addition, for $t > \overline{T}$, firms garner profits according to the OE value function. This simplification should not have a significant impact on the behavior of the industry for the time periods of interest between $t = 0$ and $t = \underline{T}$. After this reduction computing a NOE is simple; it requires solving finite-horizon one-dimensional dynamic programming problems.

At each iteration of the algorithm, we (1) compute the strategies that maximize the nonstationary oblivious value functions (step 10) and (2) we compute new entry rates depending on the extent of the violation of the zero-profit conditions (step 16). Strategies and entry rates are updated "smoothly" (steps 20 and 21). The parameters N_1, N_2, γ_1, and γ_2 are set after some experimentation to speed up convergence.

If $\delta = 0, \epsilon_0 = 0$ and the termination condition of the outer loop is satisfied with $\epsilon_1 = \epsilon_2 = 0$, we have an $s-$nonstationary oblivious equilibrium. Small values of ϵ_0, ϵ_1, and ϵ_2 allow for small errors associated with limitations of numerical precision.

Algorithm 1 $s-$Nonstationary Oblivious Equilibrium Solver

1: $\lambda_t := \tilde{\lambda}$, for all t.
2: $\mu_t := \tilde{\mu}$, for all t.
3: Define $\tilde{V}_{\overline{T}+1}(x|\mu^*, \mu, \lambda, s) := \tilde{V}(x)$, for all x μ^*, μ, and λ.
4: $n := 1$.
5: **repeat**
6:     Compute $\tilde{s}_{(\mu,\lambda,s),t}$ for $t \in \{0, ..., \overline{T}\}$.
7:     $\Delta_0 := 0$; $\Delta_1 := 0$.
8:     $t := \overline{T}$.
9:     **repeat**
10:         Choose $\mu_t^* \in \tilde{\mathcal{M}}$ to maximize $\tilde{V}_t(x|\mu^*, \mu, \lambda, s)$ simultaneously for all x.
11:         $\psi_t = \beta\tilde{V}_{t+1}(x^e|\mu^*, \mu, \lambda, s)) - \kappa$
12:         $\Delta_0 = \max(\Delta_0, \psi_t)$.
13:         **if** $\lambda_t > \epsilon_0$ **then**
14:             $\Delta_1 = \max(\Delta_1, -\psi_t)$.
15:         **end if**
16:         $\lambda_t^* := \lambda_t(\beta\tilde{V}_{t+1}(x^e|\mu^*, \mu, \lambda, s))/\kappa$.
17:         Let $t := t - 1$.
18:     **until** $t = 0$.
19:     $\Delta_2 :=\parallel \mu - \mu^* \parallel_\infty$.
20:     $\mu := \mu + (\mu^* - \mu)/(n^{\gamma_1} + N_1)$.
21:     $\lambda := \lambda + (\lambda^* - \lambda)/(n^{\gamma_2} + N_2)$.
22:     $n := n + 1$.
23: **until** $\Delta_0 \leq \epsilon_1$ and $\Delta_1 \leq \epsilon_1$ and $\Delta_2 \leq \epsilon_2$.

We use an $s-$nonstationary oblivious equilibrium to approximate short-run behavior of an industry that starts from a given initial state s. In the next section, we provide error bounds that are useful to asses the accuracy of the approximation for any given applied problem.

4 Error Bounds

We derive error bounds in this section. To bound approximation error, we first define what is meant by approximation error in this context. Because an optimal strategy for a firm that unilaterally deviates from a NOE strategy depends on the time period (since its competitors are using nonstationary strategies), we introduce nonstationary Markov strategies. We define $\mathcal{M}_{ns} = \mathcal{M}^{\infty}$ and $\Lambda_{ns} = \Lambda^{\infty}$ as the set of nonstationary Markov strategies and entry rate functions, respectively. A nonstationary Markov strategy is a sequence of Markov strategies. Hence, if $\mu \in \mathcal{M}_{ns}$ is a nonstationary Markov strategy, then $\mu = \{\mu_0, \mu_1, ...\}$, where for all $t \geq 0$, $\mu_t \in \mathcal{M}$ is a Markov strategy. Similarly, a nonstationary entry rate function is a sequence of entry rate functions. Hence, if $\lambda \in \Lambda_{ns}$ is a nonstationary entry rate function, then $\lambda = \{\lambda_0, \lambda_1, ...\}$ where for all $t \geq 0$, $\lambda_t \in \Lambda$ is an entry rate function. For nonstationary Markov strategies $\mu', \mu \in \mathcal{M}_{ns}$ and nonstationary entry rate function $\lambda \in \Lambda_{ns}$, we define the nonstationary value function,

$$V_t(x, s|\mu', \mu, \lambda) = E_{\mu',\mu,\lambda}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(x_{ik}, s_{-i,k}) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i}\Big| x_{it} = x, s_{-i,t} = s\right],$$

where i is taken to be the index of a firm at quality level x at time t. In an abuse of notation, we will use the shorthand, $V_t(x, s|\mu, \lambda) \equiv V_t(x, s|\mu, \mu, \lambda)$. The nonstationary value function allows for dependence on nonstationary Markov strategies. We use this value function to evaluate the *actual* expected discounted profits garner by a firm that uses a nonstationary Markov strategy. Suppose we are interested in the short-run dynamic behavior of an industry that starts at state $s \in \overline{\mathcal{S}}$. Let $(\tilde{\mu}, \tilde{\lambda})$ be a NOE. We quantify approximation error at each state $x \in \mathbb{N}$ by

$$\sup_{\mu' \in \mathcal{M}_{ns}} V_0\left(x, s|\mu', \tilde{\mu}, \tilde{\lambda}\right) - V_0\left(x, s|\tilde{\mu}, \tilde{\lambda}\right).$$

Hence, approximation error is the amount by which a firm in state x with competitors in state s can improve its expected discounted profits by unilaterally deviating from the nonstationary oblivious strategy $\tilde{\mu}$ to an optimal nonstationary (non-oblivious) Markov strategy. We introduce our error bound. We denote $[x]^+ = \max(x, 0)$ and $\underline{x}(k) = [x - k\overline{w}]^+$.

Theorem 4.1. *Let Assumptions A.1, A.2, and A.3 hold. Let* $\tilde{\mu} \in \tilde{\mathcal{M}}_{ns}$ *and* $\tilde{\lambda} \in \tilde{\Lambda}_{ns}$ *be an* $s-$*nonstationary* *oblivious equilibrium. Then, for all $x \in \mathbb{N}$,*

$$(4.1) \qquad \sup_{\mu' \in \mathcal{M}_{ns}} V_0\left(x, s | \mu', \tilde{\mu}, \tilde{\lambda}\right) - V_0\left(x, s | \tilde{\mu}, \tilde{\lambda}\right) \leq$$

$$E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=0}^{\infty} \beta^k \left[\max_{x' \in \{\underline{x}(k), \ldots, x+k\overline{w}\}} \left(\pi(x', s_{-i,k}) - \pi(x', \tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k})\right)\right]^+ \Big| s_{-i,0} = s\right]$$

$$+ \; E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=0}^{\tau_i} \beta^k \left(\pi(x_{ik}, \tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k}) - \pi(x_{ik}, s_{-i,k})\right) \Big| x_{i0} = x, s_{-i,0} = s\right].$$

The proofs can be found in the Appendix. Given a NOE, the error bound can be computed using simulation. It requires simulating the industry evolution under NOE strategies starting from the initial industry state s.[3] It is worth mentioning that the result can be generalized a great deal. In particular, many of the prior assumptions can be dropped; for instance, most alternative entry processes will not change the result.

If the dynamics of the model include depreciation, that is, there is a positive probability the quality level of the firm goes down even if investment is arbitrarily large, tighter bounds can be derived. Let $\Delta_k(y, s) = \pi(y, s) - \pi(y, \tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k})$. Let $\hat{\mu}$ be a strategy such that the firm never exits the industry and invests an infinite amount at every state. We have the following result.

Theorem 4.2. *Let Assumptions A.1, A.2, and A.3 hold. Let $\tilde{\mu} \in \tilde{\mathcal{M}}_{ns}$ and $\tilde{\lambda} \in \tilde{\Lambda}_{ns}$ be an $s-$nonstationary oblivious equilibrium. Suppose that, for all $s \in \overline{\mathcal{S}}$ and for all $k \geq 0$, the function $\Delta_k(y, s)^+$ is nondecreasing in y. Then, for all $x \in \mathbb{N}$,*

$$(4.2) \qquad \sup_{\mu' \in \mathcal{M}_{ns}} V_0\left(x, s | \mu', \tilde{\mu}, \tilde{\lambda}\right) - V_0\left(x, s | \tilde{\mu}, \tilde{\lambda}\right)$$

$$\leq \sum_{k=0}^{\infty} \beta^k E_{\hat{\mu},\tilde{\mu},\tilde{\lambda}}\left[\left[\pi(x_{ik}, s_{-i,k}) - \pi(x_{ik}, \tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k})\right]^+ \Big| x_{i0} = x, s_{-i,0} = s\right]$$

$$+ \; E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=0}^{\tau_i} \beta^k \left(\pi(x_{ik}, \tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k}) - \pi(x_{ik}, s_{-i,k})\right) \Big| x_{i0} = x, s_{-i,0} = s\right].$$

Note that x_{it} is controlled by strategy $\hat{\mu}$, therefore, it is independent of everything else. If there is depreciation and $\Delta_k(y, s)^+$ is nondecreasing in y, bound (4.2) is generally tighter than bound (4.1). The latter takes a maximum over achievable states in the first sum. The former takes an expectation with respect to $\hat{\mu}$ and because of depreciation, larger achievable states have smaller weights, reducing the magnitude of the bound.

[3]Note that under our assumptions, for all $t > \overline{T}$, $\tilde{s}_t = \tilde{s}$, $\tilde{\mu}_t = \tilde{\mu}$, and $\tilde{\lambda}_t = \tilde{\lambda}$.

The expectation over x_{ik} can be written in closed form for the model in Section 6.1 of Weintraub, Benkard, and Van Roy (2008a) where firms can change their state by at most one quality level per time period facilitating its computation.

Corollary 4.1. *Let Assumptions A.1, A.2, and A.3 hold. Consider the industry model in Section 6.1 of Weintraub, Benkard, and Van Roy (2008a) where firms can change their state by at most one quality level per time period. Let $\tilde{\mu} \in \tilde{\mathcal{M}}_{ns}$ and $\tilde{\lambda} \in \tilde{\Lambda}_{ns}$ be an $s-$nonstationary oblivious equilibrium. Suppose that, for all $s \in \overline{\mathcal{S}}$ and for all $k \geq 0$, the function $\Delta_k(y, s)^+$ is nondecreasing in y. Then, for all $x \in \mathbb{N}$,*

$$
\begin{aligned}
(4.3)\quad & \sup_{\mu' \in \mathcal{M}_{ns}} V_0\left(x, s | \mu', \tilde{\mu}, \tilde{\lambda}\right) - V_0\left(x, s | \tilde{\mu}, \tilde{\lambda}\right) \\
& \leq \sum_{k=0}^{\infty} \beta^k \sum_{y \in \{x, \ldots, x+k\}} \binom{k}{y-x} (1-\delta)^{y-x} \delta^{k-(y-x)} E\left[\left[\pi(y, s_{-i,k}) - \pi(y, \tilde{s}_{(\tilde{\mu}, \tilde{\lambda}, s), k})\right]^+ \Big| s_{-i,0} = s\right] \\
& \qquad + \, E_{\tilde{\mu}, \tilde{\lambda}}\left[\sum_{k=0}^{\tau_i} \beta^k \left(\pi(x_{ik}, \tilde{s}_{(\tilde{\mu}, \tilde{\lambda}, s), k}) - \pi(x_{ik}, s_{-i,k})\right) \Big| x_{i0} = x, s_{-i,0} = s\right].
\end{aligned}
$$

5 Computational Experiments

In this section we conduct some computational experiments to evaluate how NOE performs in practice. In particular, we use NOE to analyze the short-run transitional industry dynamics from one long-run OE to another after a profit shock. More specifically, we compute OE for a given industry model. Then, we increase the market size by 25% and compute the new OE. We compute a NOE that converges to the new OE and for which the initial state is the original OE expected state. Figure 1 schematizes the industry evolution. We use the same model as in Section 6 of Weintraub, Benkard, and Van Roy (2008a). In Section 5.1 we study the behavior of the error bound for different parameter specifications. In Section 5.2 we show the different dynamics that NOE can generate.

5.1 Behavior of the Bound

Our first set of results investigate the behavior of the approximation error bound under several different model specifications. We use the same parameters as in Section 6.2 of Weintraub, Benkard, and Van Roy (2008a) with the "almost deterministic" entry process. We consider two different values of θ_1 and the investment cost d: $(\theta_1, d) = (0.2,\ 0.2)$ and $(\theta_1, d) = (0.7,\ 0.7)$. The former ("Low") is a situation where the level of vertical differentiation is low and it is inexpensive to invest to improve quality. The latter ("High") is the opposite.

For each set of parameters, we compute a NOE where the starting state is the OE expected state.[4] We then use the approximation error bound in Theorem 4.2 to compute an upper bound on the percentage error in the value function, $\frac{\sup_{\mu' \in \mathcal{M}_{ns}} V_0\left(x,s|\mu',\tilde{\mu},\tilde{\lambda}\right) - V_0\left(x,s|\tilde{\mu},\tilde{\lambda}\right)}{\tilde{V}_0(x|\mu,\lambda,s)}$, where $(\tilde{\mu}, \tilde{\lambda})$ are the NOE strategy and entry rate, respectively.[5] Percentage error is taken with respect to the nonstationary oblivious value function. We estimate the expectations using simulation.[6] We compute the previously mentioned percentage approximation error bound for different market sizes. As the market size increases, the expected number of firms in the original OE (and hence in the initial state) increases, and the approximation error bound decreases.

In Figure 2 (see Appendix) we present the percentage approximation error bound as a function of the number of firms at the initial state for the two levels of vertical differentiation. For the low vertical differentiation case it takes around 60 firms to bring the bound down to around 2%, and 200 firms to bring it to around 1%. For the high case it takes around 100 firms to bring the bound to 3% and 200 firms to bring it to 2%.

Most economic applications would involve from less than ten to several hundred firms. These results show that the approximation error bound may sometimes be small (<2%) in these cases, though this would depend on the model and parameter values for the industry under study.

5.2 Short-Run Transitional Industry Dynamics

In this section we study transitional dynamics that NOE can generate. We consider the same model and parameters as in the previous section with two variants: $m = 750,\ \theta_1 = 0.5,\ d = 0.5$, and $m = 1500,\ \theta_1 = 2,\ d = 1$.[7] In the same spirit as above, we refer to the former as a case of low level of vertical differentiation, and to the latter as a case of high level of vertical differentiation.

Figure 3 shows the evolution for the case of low level of vertical differentiation. First, note that because the market size increases the new OE holds more firms than the original OE. Potential entrants realize

[4] We round fractional numbers in the OE expected state to the closest integer.

[5] While we are not able to show that $\Delta_k(y,s)^+$ is nondecreasing in y, we check it computationally for all sampled states in the simulation.

[6] The bound is estimated with a relative precision of at most 10% and a confidence level of 98% (in cases where the bound is very small it is difficult to achieve better precision than this). Note that the percentage approximation error bound depends on the state x so for the purposes of this section we consider the percentage bound evaluated at the entry state. For the computations we took the maximum achievable state, x_{max}, to be a state such that the expected number of visits of a firm using the OE strategy was at most 10^{-5}. In computing the bounds, we assumed that the maximum achievable state under the best response (non-oblivious) strategy was also x_{max}.

[7] In the latter, we also included an additional noise term, ϵ_{it}, to the firm's evolution, which is independent of everything else. The noise term allows for random appreciation and depreciation in the following way: $\mathcal{P}[\epsilon_{it} = 1] = 0.25,\ \mathcal{P}[\epsilon_{it} = 0] = 0.5,\ \mathcal{P}[\epsilon_{it} = -1] = 0.25$. The noise term generates richer dynamics.

the existence of this profitable opportunity and the NOE entry rate is high at the beginning. Then, after approximately ten periods, it converges to the new OE entry rate. As a consequence, the NOE expected total number of firms increases very quickly and in five periods is very close to converge to the new OE expected total number of firms.

Figure 4 shows the evolution for the case of high level of vertical differentiation. Again, the new OE holds more firms than the original, firms realize this profitable opportunity and the NOE entry rate is high at the beginning and then converges to the new OE entry rate. However, in this case there is "too much" entry at the beginning. Because the level of vertical differentiation is high, becoming one of the largest firms in the industry entails huge profits and too many firms enter the industry hoping to become dominant. However, only some of them receive favorable idiosyncratic shocks and the rest slowly starts exiting. Hence, after a burst of entry the industry exhibits a shake-out until it converges to the new OE expected state.

In summary, NOE generates different dynamics depending on the model. In a case of low level of vertical differentiation, the number of firms in the NOE increases and exhibits a quick convergence to the new long-run OE. In a case of high level of vertical differentiation, the NOE exhibits too much entry at the beginning and the industry exhibits a slow shake-out until it converges to the new long-run OE.

6 Asymptotic Theorem

In this section we establish an asymptotic result that provides conditions under which approximation error converges to zero as the market size grows. The main condition is that the sequence of NOE generates firm size distributions that are "light-tailed" in a sense that we will make precise. The result provides a theoretical justification for using NOE to approximate short-run industry dynamics.

The model in Section A does not explicitly depend on market size. However, market size would typically enter the profit function, $\pi(x_{it}, s_{-i,t})$, through the underlying demand system; in particular, profit for a firm at a given state (x, s) would typically increase with market size. Therefore, in this section we consider a sequence of markets indexed by market sizes $m \in \Re_+$. All other model primitives are assumed to remain constant within this sequence except for the profit function, which depends on m. To convey this dependence, we denote profit functions by π_m.

We index functions and random variables associated with market size m with a superscript (m). For our asymptotic analysis, we consider a sequence of initial states $\tilde{s}_0^{(m)}$ that are indexed by the market size m, and for which $\sum_x \tilde{s}_0^{(m)}(x) = \Theta(m)$.[8] Hence, the number of firms at $\tilde{s}_0^{(m)}$ increases proportionally to the market

[8]In this notation $f(m) = \Theta(m)$ means that there exists $c_1, c_2 > 0$ and $\overline{m}$ such that, $c_1 m < f(m) < c_2 m$, for all $m > \overline{m}$.

size asymptotically. We do not impose further restrictions on $\tilde{s}_0^{(m)}$; for example, $\tilde{s}_0^{(m)}$ could have firms at higher quality levels in larger markets. We let $(\mu^{(m)}, \lambda^{(m)})$ denote a NOE for market size m. Let $V_t^{(m)}$ and $\tilde{V}_t^{(m)}$ represent the nonstationary value function and nonstationary oblivious value function, respectively. The random vector $s_t^{(m)}$ denotes the industry state at time t when every firm uses strategy $\mu^{(m)}$, the entry rate function is $\lambda^{(m)}$, and the initial state is $\tilde{s}_0^{(m)}$. Note that $s_0^{(m)} = \tilde{s}_0^{(m)}$.

It will be helpful to decompose $s_t^{(m)}$ according to $s_t^{(m)} = f_t^{(m)} n_t^{(m)}$, where $f_t^{(m)}$ is the random vector that represents the fraction of firms in each state, and $n_t^{(m)}$ is the total number of firms, respectively. Let $\tilde{n}_t^{(m)} \equiv E[n_t^{(m)}] = \sum_x \tilde{s}_t^{(m)}(x)$ denote the expected number of firms at time period t. Let $\tilde{f}_t^{(m)} = \tilde{s}_t^{(m)} / \tilde{n}_t^{(m)}$ denote the normalized expected industry state. With some abuse of notation, we define

$\pi_m(x_{it}, f_{-i,t}, n_{-i,t}) \equiv \pi_m(x_{it}, n_{-i,t} \cdot f_{-i,t})$.

6.1 Assumptions about the Sequence of Profit Functions

In addition to Assumption A.1, which applies to individual profit functions, we will make the following assumptions, which apply to sequences of profit functions. Let $\mathcal{S}_1 = \{f \in \mathcal{S} | \sum_{x \in \mathbb{N}} f(x) = 1\}$ and $\mathcal{S}_{1,z} = \{f \in \mathcal{S}_1 | \forall x > z, f(x) = 0\}$.

Assumption 6.1.

1. $\sup_{x \in \mathbb{N}, s \in \mathcal{S}} \pi_m(x, s) = O(m)$.[9]

2. $$\sup_{m \in \Re_+, x \in \mathbb{N}, f \in \mathcal{S}_1, n > 0} \left| \frac{d \ln \pi_m(x, f, n)}{d \ln n} \right| < \infty.$$

3. *For all $c > 0$ and $\overline{m} \in \mathbb{N}$, there exists a function $\overline{\pi} : \mathbb{N} \to \Re_+$, such that, for all sequences $n : \mathbb{N} \mapsto \mathbb{N}$ satisfying $n(m) \geq cm$ for all $m > \overline{m}$,*

$$\sup_{m \in \Re_+, f \in \mathcal{S}_1} \pi_m(x, f, n(m)) \leq \overline{\pi}(x), \forall x \in \mathbb{N}.$$

Moreover, for all $x \in \mathbb{N}$,

$$\sup_{\mu \in \mathcal{M}} E_\mu \left[\sum_{k=t}^{\infty} \beta^{k-t} \overline{\pi}(x_{ik}) \Big| x_{it} = x \right] < \infty.$$

4. *For all $z \in \mathbb{N}$, there exists $c > 0$ and $\overline{m} \in \mathbb{N}$, such that, if $n(m) < cm$ and $m > \overline{m}$, then,*

$$\pi_m(x^e, f, n(m)) > \kappa/\beta + 1, \ \forall f \in \mathcal{S}_{1,z}.$$

Assumption 6.1.1, which states that profits increase at most linearly with market size, should hold for virtually all relevant classes of profit functions. It is satisfied, for example, if the total disposable income of the

[9] In this notation, $f(m) = O(m)$ denotes $\limsup_m \frac{f(m)}{m} < \infty$.

consumer population grows linearly in market size.[10] Assumption 6.1.2 requires that profits are "smooth" with respect to the number of firms and, in particular, states that the relative rate of change of profit with respect to relative changes in the number of firms is uniformly bounded. Roughly speaking, Assumption 6.1.3 states that if number of firms grow at least linearly with the market size, maximum achievable expected discounted profits remain uniformly bounded over all market sizes. Assumption 6.1.4 states that if the number of firms is smaller than a fraction of the market size and the market size is large, then firm's profit at the entry state become large. The assumptions hold, for example, for a single-period profit function derived from a demand system given by a logit model where the spot market equilibrium is Nash in prices.

6.2 Asymptotic Nonstationary Markov Equilibrium Property

Our aim is to establish that, under certain conditions, NOE provides an accurate approximation as the market size grows. Motivated by our definition of approximation error, we define the following concept to formalize the sense in which this approximation becomes exact.

Definition 6.1. *A sequence $(\mu^{(m)}, \lambda^{(m)}) \in \mathcal{M}_{ns} \times \Lambda_{ns}$ possesses the asymptotic nonstationary Markov equilibrium (ANME) property if for all $x \in \mathbb{N}$,*

$$\lim_{m\to\infty} \sup_{\mu' \in \mathcal{M}_{ns}} V_0^{(m)}(x, s_0^{(m)}|\mu', \mu^{(m)}, \lambda^{(m)}) - V_0^{(m)}(x, s_0^{(m)}|\mu^{(m)}, \lambda^{(m)}) = 0 \, .$$

6.3 A Light-Tail Condition Implies ANME

Even when there are a large number of firms, if the market tends to be concentrated — for example, if the market is usually dominated by few firms — the ANME property is unlikely to hold. A strategy that does not keep track of the dominant firms will perform poorly. To ensure the ANME property, we need to impose a "light-tail" condition that rules out this kind of market concentration. In this section, we establish that under an appropriate light-tail condition the sequence of NOE possesses the ANME property.

Note that $\frac{d \ln \pi_m(y,f,n)}{df(x)}$ is the semi-elasticity of one period profits with respect to the fraction of firms in state x. We define the *maximal absolute semi-elasticity function*:

$$g(x) = \max_{m\in\Re_+, y\in\mathbb{N}, f\in\mathcal{S}_1, n>0} \left| \frac{d \ln \pi_m(y,f,n)}{df(x)} \right| .$$

For each x, $g(x)$ is the maximum rate of relative change of any firm's single-period profit that could result

[10]For example, if each consumer has income that is less than some upper bound $\overline{Y}$ then total disposable income of the consumer population (an upper bound to firm profits) is always less than $m \cdot \overline{Y}$.

from a small change in the fraction of firms at quality level x. Since larger competitors tend to have greater influence on firm profits, $g(x)$ typically increases with x, and can be unbounded.

Finally, we introduce our light-tail condition. For each m and t, let $\tilde{x}_t^{(m)} \sim \tilde{f}_t^{(m)}$, that is, $\tilde{x}_t^{(m)}$ is a random variable with probability mass function $\tilde{f}_t^{(m)}$. Recall that $\tilde{f}_t^{(m)}$ is a vector that represents the normalized expected industry state at time t for market size m.

Assumption 6.2. *For all quality levels x, $g(x) < \infty$. For all $\epsilon > 0$, there exists a quality level z such that*

$$E\left[g(\tilde{x}_t^{(m)})\mathbf{1}_{\{\tilde{x}_t^{(m)} > z\}}\right] \leq \epsilon,$$

for all market sizes m, and all time periods $t > 0$.

Put simply, the light tail condition requires that the number of firms in states where a small change in the fraction of firms has a large impact on the profits of other firms, must be relatively small in the sequence of expected states. In practice this typically means that very large firms (and hence high concentration) rarely occur when the industry starts from the chosen initial state.[11]

In Assumption 2.1, for each market size, we established an upper bound for entry rates, λ_{max}. This upper bound can vary with the market size. In the next assumption, we establish that the upper bound on entry rates $\lambda_{max}^{(m)}$ grows at the same rate as the market size asymptotically. The assumption simplifies our asymptotic analysis.

Assumption 6.3. *For each market size m, there is an upper bound on entry rates given by $\lambda_{max}^{(m)}$. Moreover, $\lambda_{max}^{(m)} = \Theta(m)$.*

In the next result, we establish that, for all time periods, the expected number of firms grows at least linearly in the market size asymptotically.

Proposition 6.1. *Let Assumptions A.1, A.2, A.3, 6.1.4, 6.2, and 6.3 hold. Then, there exists $c > 0$ and $\overline{m}$, such that, for all $t > 0$ and for all $m > \overline{m}$, $\tilde{n}_t^{(m)}/m \geq c$.*

All proofs of this section can be found in the Appendix. The result implies that if the light-tail condition is satisfied, then the expected number of firms under NOE strategies in each time period grows to infinity as the market size grows.

[11]Note that the quality level z in the light-tail assumption above is the same for all market sizes m and time periods t. The results in this section allow for any sequence of NOE. We conjecture that if we constraint the results to sequences of NOE that become stationary as time progresses (Section 2.3), then we could weaken the light-tail assumption and allow z to vary with t. Also note that the light tail condition is not assumed at $t = 0$; we do not need that condition to prove our results because at the initial period there is no uncertaity about the industry state.

The next result, establishes a form of convergence for the normalized industry states. First, we define $\|f\|_{1,g} = \sum_x |f(x)|g(x)$.

Proposition 6.2. *Let Assumptions A.1, A.2, A.3, 6.1, 6.2, and 6.3 hold. Then, for all $t > 0$, $n_t^{(m)}/\tilde{n}_t^{(m)} \rightarrow_p 1$ and $\|f_t^{(m)} - \tilde{f}_t^{(m)}\|_{1,g} \rightarrow_p 0$, as m grows.*[12]

The light-tail condition is key to prove the second part of the result, namely, convergence of the normalized industry states in the $\|\cdot\|_{1,g}$ weighted-norm. We also note that this part of the result requires a very different proof technique to its analog for OE in Weintraub, Benkard, and Van Roy (2008b). This form of convergence allows us to ensure the ANME property, which leads to the main result of this section.

Theorem 6.1. *Under Assumptions A.1, A.2, A.3, 6.1, 6.2, and 6.3 the sequence $\{\mu^{(m)}, \lambda^{(m)}\}$ of NOE possesses the ANME property.*

7 Closing Remarks

In this paper we introduced nonstationary oblivious equilibrium as a way to approximate short-run transitional dynamics that may result, for example, from shocks or policy changes. We provided an efficient algorithm to compute NOE and an efficient simulation-based algorithm to compute a bound on approximation error. The algorithm for bounding approximation error allow us to verify accuracy of NOE as an approximation for each problem instance. Using these methods, we provided a computational study and showed that NOE offers useful approximations for relevant models of industries with hundreds or even tens of firms. We also showed that NOE can endogenously generate interesting industry dynamics, like industry shake-outs. Our results show that by using NOE it is possible to further expand the set of dynamic industries that can be analyzed computationally.

While the previous results provide support for using NOE in practice, we also provided an asymptotic result that provides a theoretical justification for the approximation. We showed that, if alongside some technical requirements, the equilibrium distribution of firm states obeys certain "light-tail" condition, then the approximation error vanishes as the market sizes grows.

The previous result suggests that OE and NOE will not provide accurate approximations for industries with few dominant firms that have a significant market share. In Weintraub, Benkard, and Van Roy (2007) we develop an extended notion of nonstationary oblivious equilibrium that allows for there to be a set of

[12]We use $\rightarrow_p$ to denote convergence in probability.

"dominant firms", whose firm states are always monitored by every other firm. Our hope is that the dominant firm nonstationary OE will provide better approximations for more concentrated industries. Testing these approximations in empirical applications is the matter of current and future research.

A A Dynamic Model of Imperfect Competition

In this section we formulate a model of an industry in which firms compete in a single-good market. The model is general enough to encompass numerous applied problems in economics. Indeed, a blossoming recent literature on EP-type models has applied similar models to advertising, auctions, collusion, consumer learning, environmental policy, international trade policy, learning-by-doing, limit order markets, mergers, network externalities, and other applied problems.

Our model is close in spirit to that of Ericson and Pakes (1995), but with some differences. Most notably, we modify the entry and exit processes in Ericson and Pakes (1995) so as to make them more realistic when there are a large number of firms. Additionally, the asymptotic theorem in this paper does not hold when there are aggregate industry shocks, so our model includes only idiosyncratic shocks.[13]

A.1 Model and Notation

The industry evolves over discrete time periods and an infinite horizon. We index time periods with non-negative integers $t \in \mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, \ldots\}$). All random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a filtration $\{\mathcal{F}_t : t \geq 0\}$. We adopt a convention of indexing by t variables that are $\mathcal{F}_t$-measurable.

Each firm that enters the industry is assigned a unique positive integer-valued index. The set of indices of incumbent firms at time t is denoted by S_t. At each time $t \in \mathbb{N}$, we denote the number of incumbent firms as n_t.

Firm heterogeneity is reflected through firm states. To fix an interpretation, we will refer to a firm's state as its quality level. However, firm states might more generally reflect productivity, capacity, the size of its consumer network, or any other aspect of the firm that affects its profits. At time t, the quality level of firm $i \in S_t$ is denoted by $x_{it} \in \mathbb{N}$.

We define the *industry state* s_t to be a vector over quality levels that specifies, for each quality level $x \in \mathbb{N}$, the number of incumbent firms at quality level x in period t. We define the state space $\overline{\mathcal{S}} =$

[13] In Weintraub, Benkard, and Van Roy (2008a) we extend the model and analysis to include aggregate shocks.

$\left\{ s \in \mathbb{N}^\infty \middle| \sum_{x=0}^\infty s(x) < \infty \right\}$. Though in principle there are a countable number of industry states, we will also consider an extended state space $\mathcal{S} = \left\{ s \in \Re_+^\infty \middle| \sum_{x=0}^\infty s(x) < \infty \right\}$. This will allow us, for example, to consider derivatives of functions with respect to the industry state. For each $i \in S_t$, we define $s_{-i,t} \in \mathcal{S}$ to be the state of the *competitors* of firm i; that is, $s_{-i,t}(x) = s_t(x) - 1$ if $x_{it} = x$, and $s_{-i,t}(x) = s_t(x)$, otherwise. Similarly, $n_{-i,t}$ denotes the number of competitors of firm i.

In each period, each incumbent firm earns profits on a spot market. A firm's single period expected profit $\pi(x_{it}, s_{-i,t})$ depends on its quality level x_{it} and its competitors' state $s_{-i,t}$.

The model also allows for entry and exit. In each period, each incumbent firm $i \in S_t$ observes a positive real-valued sell-off value ϕ_{it} that is private information to the firm. If the sell-off value exceeds the value of continuing in the industry then the firm may choose to exit, in which case it earns the sell-off value and then ceases operations permanently.

If the firm instead decides to remain in the industry, then it can invest to improve its quality level. If a firm invests $\iota_{it} \in \Re_+$, then the firm's state at time $t+1$ is given by,

$$x_{i,t+1} = \max\left(0, x_{it} + w(\iota_{it}, \zeta_{i,t+1})\right),$$

where the function w captures the impact of investment on quality and $\zeta_{i,t+1}$ reflects uncertainty in the outcome of investment. Uncertainty may arise, for example, due to the risk associated with a research and development endeavor or a marketing campaign. Note that this specification is very general as w may take on either positive or negative values (e.g., allowing for positive depreciation). We denote the unit cost of investment by d.

In each period new firms can enter the industry by paying a setup cost κ. Entrants do not earn profits in the period that they enter. They appear in the following period at state $x^e \in \mathbb{N}$ and can earn profits thereafter.[14]

Each firm aims to maximize expected net present value. The interest rate is assumed to be positive and constant over time, resulting in a constant discount factor of $\beta \in (0,1)$ per time period.

In each period, events occur in the following order:

1. Each incumbent firms observes its sell-off value and then makes exit and investment decisions.
2. The number of entering firms is determined and each entrant pays an entry cost of κ.
3. Incumbent firms compete in the spot market and receive profits.
4. Exiting firms exit and receive their sell-off values.

[14]Note that it would not change any of our results to assume that the entry state was a random variable.

5. Investment outcomes are determined, new entrants enter, and the industry takes on a new state s_{t+1}.

A.2 Model Primitives

Our model above allows for a wide variety of applied problems. To study any particular problem it is necessary to further specify the primitives of the model, including the profit function π, the distribution of the sell-off value ϕ_{it}, the investment impact function w, the distribution of the investment uncertainty ζ_{it}, the unit investment cost d, the entry cost κ, and the discount factor β.

Note that in most applications the profit function would not be specified directly, but would instead result from a deeper set of primitives that specify a demand function, a cost function, and a static equilibrium concept. An important parameter of the demand function (and hence the profit function) that we will focus on below, is the size of the relevant market, which we will denote as m. Later on in the paper we subscript the profit function with the market size parameter, π_m, to explicitly recognize the dependence of profits on market size. For expositional clarity, the subscript is omitted in the assumptions listed below, implying that the market size is being held fixed.

A.3 Assumptions

We make several assumptions about the model primitives, beginning with the profit function. An industry state $s \in \mathcal{S}$ is said to *dominate* $s' \in \mathcal{S}$ if for all $x \in \mathbb{N}$, $\sum_{z \geq x} s(z) \geq \sum_{z \geq x} s'(z)$. We will denote this relation by $s \succeq s'$. Intuitively, competition associated with s is no weaker than competition associated with s'.

Assumption A.1.

1. *For all $s \in \mathcal{S}$, $\pi(x, s)$ is increasing in x.*
2. *For all $x \in \mathbb{N}$ and $s, s' \in \mathcal{S}$, if $s \succeq s'$ then $\pi(x, s) \leq \pi(x, s')$.*
3. *For all $x \in \mathbb{N}$ and $s \in \mathcal{S}$, $\pi(x, s) > 0$, and $\sup_{x,s} \pi(x, s) < \infty$.*
4. *For all $x \in \mathbb{N}$, the function $\ln \pi(x, :) : \mathcal{S} \to \Re_+$ is continuously Fréchet differentiable. Hence, for all $x \in \mathbb{N}$, $y \in \mathbb{N}$, and $s \in \mathcal{S}$, $\ln \pi(x, s)$ is continuously differentiable with respect to $s(y)$. Further, for any $x \in \mathbb{N}$, $s \in \mathcal{S}$, and $h \in \mathcal{S}$ such that $s + \gamma h \in \mathcal{S}$ for $\gamma > 0$ sufficiently small, if*

$$\sum_{y \in \mathbb{N}} h(y) \left| \frac{\partial \ln \pi(x, s)}{\partial s(y)} \right| < \infty,$$

then

$$\frac{d \ln \pi(x, s + \gamma h)}{d\gamma}\Big|_{\gamma=0} = \sum_{y \in \mathbb{N}} h(y) \frac{\partial \ln \pi(x, s)}{\partial s(y)}.$$

The assumptions are fairly weak. Assumption A.1.1 ensures that increases in quality lead to increases in profit. Assumption A.1.2 states that strengthened competition cannot result in increased profit. Assumption A.1.3 ensures that profits are positive and bounded. Assumption A.1.4 is technical and requires that log-profits are Fréchet differentiable. Note that it requires partial differentiability of the profit function with respect to each $s(y)$. Profit functions that are "smooth", such as ones arising from random utility demand models like the logit model, will satisfy this assumption.

We also make assumptions about investment and the distributions of the private shocks:

Assumption A.2.

1. *The random variables $\{\phi_{it}|t \geq 0, i \geq 1\}$ are i.i.d. and have finite expectations and well-defined density functions with support $\Re_+$.*
2. *The random variables $\{\zeta_{it}|t \geq 0, i \geq 1\}$ are i.i.d. and independent of $\{\phi_{it}|t \geq 0, i \geq 1\}$.*
3. *For all ζ, $w(\iota, \zeta)$ is nondecreasing in ι.*
4. *For all $\iota > 0$, $\mathcal{P}[w(\iota, \zeta_{i,t+1}) > 0] > 0$.*
5. *There exists a positive constant $\overline{w} \in \mathbb{N}$ such that $|w(\iota, \zeta)| \leq \overline{w}$, for all (ι, ζ). There exists a positive constant $\overline{\iota}$ such that $\iota_{it} < \overline{\iota}$, $\forall i, \forall t$.*
6. *For all $k \in \{-\overline{w}, \ldots, \overline{w}\}$, $\mathcal{P}[w(\iota, \zeta_{i,t+1}) = k]$ is continuous in ι.*
7. *The transitions generated by $w(\iota, \zeta)$ are unique investment choice admissible .*

Again the assumptions are natural and fairly weak. Assumptions 3.2.1 and 3.2.2 imply that investment and exit outcomes are idiosyncratic conditional on the state. Assumptions 3.2.3 and 3.2.4 imply that investment is productive. Note that positive depreciation is neither required nor ruled out. Assumption 3.2.5 places a finite bound on how much progress can be made or lost in a single period through investment. Assumption 3.2.6 ensures that the impact of investment on transition probabilities is continuous. Assumption 3.2.7 is an assumption introduced by Doraszelski and Satterthwaite (2007) that ensures a unique solution to the firms' investment decision problem. It is used to guarantee existence of an equilibrium in pure strategies, and is satisfied by many of the commonly used specifications in the literature.

We assume that there are a large number of potential entrants who play a symmetric mixed entry strategy. In that case the number of actual entrants is well approximated by the Poisson distribution (see the appendix for a derivation of this result). This leads to the following assumptions:

Assumption A.3.

1. *The number of firms entering during period t is a Poisson random variable that is conditionally independent of $\{\phi_{it}, \zeta_{it}|t \geq 0, i \geq 1\}$, conditioned on s_t.*
2. *$\kappa > \beta \cdot \overline{\phi}$, where $\overline{\phi}$ is the expected net present value of entering the market, investing zero and earning zero profits each period, and then exiting at an optimal stopping time.*

We denote the expected number of firms entering at industry state s_t, by $\lambda(s_t)$. This state-dependent entry rate will be endogenously determined, and our solution concept will require that it satisfies a zero expected discounted profits condition. Modeling the number of entrants as a Poisson random variable has the advantage that it leads to more elegant asymptotic results. Assumption 3.3.2 ensures that the sell-off value by itself is not sufficient reason to enter the industry.

B Proof Theorem 2.1

B.1 Preliminaries

In an abuse of notation, we consider a restricted state space $\mathcal{S} = \left\{ s \in \Re_+^\infty \middle| \sum_{x=0}^{x_{max}} s(x) < N \right\}$.[15] We endowed the set $\mathcal{S}$ with a metric for which single-period profit functions are continuous for all $s \in \mathcal{S}$.[16]

We define a set $\mathcal{S}^\infty = \{(s_0, s_1, s_2, ...) : s_t \in \mathcal{S};$ for which there exists $s \in \mathcal{S}, \lim_{t\to\infty} s_t = s\}$, endowed with the metric compatible with the product topology.[17] The elements of $\mathcal{S}^\infty$ are denoted by $S = \{s_0, s_1, ...\}$.

We define a new set of strategies $\mathcal{M}_\mathcal{S} : \mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^\infty \to [0, \hat{\iota}] \times [0, \sup_{x,s} \pi(x, s)/(1-\beta) + \overline{\phi}]$. Similarly, in an abuse of notation, for all $\mu \in \mathcal{M}_\mathcal{S}$, $x \in \mathbb{N} \cap [0, x_{max}]$ and $S \in \mathcal{S}^\infty$, we define a new value function,

$$\overline{V}(x, S|\mu) = E_\mu \left[\sum_{k=0}^{\tau_i} \beta^k \left(\pi(x_{ik}, s_k) - d\iota_{ik} \right) + \beta^{\tau_i - t} \phi_{i,\tau_i} \middle| x_{i0} = x \right].$$

Let $\overline{V}(x, S) = \sup_{\mu \in \mathcal{M}_\mathcal{S}} \overline{V}(x, S|\mu)$, $\forall x \in \mathbb{N} \cap [0, x_{max}]$, $\forall S \in \mathcal{S}^\infty$. Note that the state space of this dynamic programming problem is uncountable. However, because single-period profits, investments, and expected sell-off value are bounded, the supremum can always be attained simultaneously for all x and S by a common strategy μ (Bertsekas (2001)).

We define a translation operator $G : \mathcal{S}^\infty \to \mathcal{S}^\infty$, such that, $G(S) = \{s_1, s_2, ...\}$.

We define the following Bellman operator:

$$TV(x, S) = \pi(x, s_0) + E \left[\max \left\{ \phi_{it}, \sup_{\iota \in [0,\hat{\iota}]} \left(-d\iota + \beta E \left[V(x_{i,t+1}, G(S)) | x_{it} = x, \iota_{it} = \iota \right] \right) \right\} \right], \tag{B.1}$$

for all $x \in \mathbb{N} \cap [0, x_{max}]$ and $s \in \mathcal{S}^\infty$.

[15] Under our assumptions, this restriction is done without loss of generality for N large enough.

[16] In general, in our case this is the metric defined by the $\| \cdot \|_{1,g}$ norm. Since now we are restricting the state space to be finite dimensional, single period profits are continuous in any norm.

[17] Note that $\lim_{t\to\infty} s_t = s$ is defined with the metric for which single-period profit functions are continuous for all $s \in \mathcal{S}$.

We define $\bar{\iota} : \mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^\infty \to \Re_+$ as the greedy policy with respect to $\overline{V}$. That is,

$$\bar{\iota}(x, S) = \underset{\iota \in [0, \hat{\iota}]}{\operatorname{argmax}} \left(-d\iota + \beta E \left[\overline{V}(x_{i,t+1}, G(S)) | x_{it} = x, \iota_{it} = \iota \right] \right) . \tag{B.2}$$

By assumption A.2, $\bar{\iota}(x, S)$ exists and is unique for all $x \in \mathbb{N} \cap [0, x_{max}]$ and $s \in \mathcal{S}^\infty$. We also define the exit strategy $\overline{\rho} : \mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^\infty \to \Re_+$ as

$$\overline{\rho}(x, S) = \max_{\iota \in [0, \hat{\iota}]} \left(-d\iota + \beta E \left[\overline{V}(x_{i,t+1}, G(S)) | x_{it} = x, \iota_{it} = \iota \right] \right) . \tag{B.3}$$

Finally, we denote $\overline{\mu}(x, S) = (\bar{\iota}(x, S), \overline{\rho}(x, S))$.

For all $\mu \in \hat{\mathcal{M}}_{ns}$ and $\lambda \in \hat{\Lambda}_{ns}$ we define the following operator:

$$H_1(\mu, \lambda) = \left(\{s_t\}_{t=0}^\infty, \lambda \right), \text{ where}$$

$$s_{t+1} = s_t P_{\mu_t} + \mathbf{1}_{x^e} \lambda_t, \tag{B.4}$$

s_0 is the initial state in the NOE, and $\mathbf{1}_x(y) = 1$, if $y = x$ and $\mathbf{1}_x(y) = 0$, otherwise.

The first component of the operator H_1 maps a sequence of strategies and entry rates into a sequence of expected states. The second component applies the identity to the sequence of entry rates.

For all $S \in \mathcal{S}^\infty$ and $\lambda \in \hat{\Lambda}_{ns}$, we define the following operator

$$H_2(S, \lambda) = \left\{ \overline{\mu}(\cdot, G^t(S)), \max\{0, \min\{\lambda_t + \beta \overline{V}(x^e, G^{t+1}(S)) - \kappa, \lambda_{max}\}\} \right\}_{t=0}^\infty .$$

The operator H_2 maps a sequence of states S and entry rates λ into a sequence of optimal strategies and updated entry rates.

B.2 Outline of Proof

We prove Theorem 2.1 at the end of the section. In Section B.3, we prove useful lemmas. We provide an outline here. For $\mu \in \hat{\mathcal{M}}_{ns}$ and $\lambda \in \hat{\Lambda}_{ns}$, define the operator $H(\mu, \lambda) = H_2 \circ H_1(\mu, \lambda)$. Note that a fixed point of H is a NOE. The proof uses Brouwer-Schauder-Tychonoff's theorem (Aliprantis and Border (2006)) to show that a fixed point of H exists in $\hat{\mathcal{M}}_{ns} \times \hat{\Lambda}_{ns}$. The main steps of the proof are the following:

1. Prove that H_1 is a continuous operator (Lemma B.1).

2. Prove that H_2 is a continuous operator (Lemma B.4).

3. Prove that $H_2 \circ H_1$ maps elements from $\hat{\mathcal{M}}_{ns} \times \hat{\Lambda}_{ns}$ into itself (Lemmas B.1 and B.4 together).

4. Show that if NOE strategies and entry rates converge as time progresses, they converge to OE strategies and entry rate.

B.3 Lemmas

The assumptions in Theorem 2.1 hold throughout this section.

Lemma B.1. *The operator H_1 maps elements from $\hat{\mathcal{M}}_{ns} \times \hat{\Lambda}_{ns}$ into $\mathcal{S}^{\infty} \times \hat{\Lambda}_{ns}$ and is continuos.*

Proof. First, we prove that the first component of H_1 maps elements from $\hat{\mathcal{M}}_{ns} \times \hat{\Lambda}$ into $\mathcal{S}^{\infty}$, that is, $\{s_t\}_{t=0}^{\infty}$ in equation (B.4) is a converging sequence.

Note that because profits are bounded and ϕ_{it} has support in $\Re_+$, there is a probability uniformly bounded away from zero over all strategies $\mu \in \hat{\mathcal{M}}_{ns}$, states $x \in \mathbb{N} \cap [0, x_{max}]$, and time periods $t \in \mathbb{N}$, of exiting the industry at each time. Therefore, for all $\mu \in \hat{\mathcal{M}}_{ns}$, $\sup_{t \geq 0} \sigma\left(P_{\mu_t}\right) < 1$, where $\sigma(P)$ is the spectral radius of the matrix P.

Now,

$$s_t = s_0 \prod_{i=0}^{t-1} P_{\mu_i} + \sum_{i=0}^{t-1} \mathbf{1}_{x^e} \lambda_i \prod_{j=i+1}^{t-1} P_{\mu_j} \ . \tag{B.5}$$

Let

$$\tilde{s}_t = \tilde{\lambda} \mathbf{1}_{x^e} \sum_{i=0}^{t-1} P_{\tilde{\mu}}^{t-i-1} \ .$$

Note that because $\sigma\left(P_{\tilde{\mu}}\right) < 1$,

$$\lim_{t \to \infty} \tilde{s}_t = \tilde{\lambda} \mathbf{1}_{x^e} (I - P_{\tilde{\mu}})^{-1} = \tilde{s} \ .$$

We have that,

$$s_t - \tilde{s}_t = s_0 \prod_{i=0}^{t-1} P_{\mu_i} + \sum_{i=0}^{t-1} \lambda_i \mathbf{1}_{x^e} \prod_{j=i+1}^{t-1} P_{\mu_j} - \tilde{\lambda} \mathbf{1}_{x^e} \sum_{i=0}^{t-1} P_{\tilde{\mu}}^{t-i-1} \ .$$

Let $\epsilon > 0$. Because $\sup_{t \geq 0} \sigma\left(P_{\mu_t}\right) < 1$:

1. There exist s such that for all $t > s+2$,

$$\left|\left|\lambda_{max}\mathbf{1}_{x^e}\sum_{i=0}^{t-s-2}\prod_{j=i+1}^{t-1}P_{\mu_j}\right|\right| < \frac{\epsilon}{3}\,,$$

   and

$$\left|\left|\tilde{\lambda}\mathbf{1}_{x^e}\sum_{i=0}^{t-s-2}P_{\tilde{\mu}}^{t-i-1}\right|\right| < \frac{\epsilon}{3}\,.$$

2. There exists $T > s+2$, such that, for all $t \geq T$,

$$\left|\left|s_0\prod_{i=0}^{t-1}P_{\mu_i}\right|\right| < \frac{\epsilon}{3}\,.$$

Therefore, for $t > T$ we have that,

$$\begin{aligned}||s_t - \tilde{s}_t|| &< \epsilon + \left|\left|\sum_{i=t-s-1}^{t-1}\lambda_i\mathbf{1}_{x^e}\prod_{j=i+1}^{t-1}P_{\mu_j} - \tilde{\lambda}\mathbf{1}_{x^e}\sum_{i=t-s-1}^{t-1}P_{\tilde{\mu}}^{t-i-1}\right|\right| \\ &< \epsilon + \sum_{i=t-s-1}^{t-1}\left|\left|\lambda_i\mathbf{1}_{x^e}\prod_{j=i+1}^{t-1}P_{\mu_j} - \tilde{\lambda}\mathbf{1}_{x^e}P_{\tilde{\mu}}^{t-i-1}\right|\right|\,.\end{aligned}$$

Note that the sum has s terms, for all t. Additionally, $\lim_{t\to\infty}\lambda_i = \tilde{\lambda}$, and $\lim_{t\to\infty}P_{\mu_t} = P_{\tilde{\mu}}$, because P_μ is continuous in μ. Therefore,

$$\lim_{t\to\infty}\|s_t - \tilde{s}_t\| < \epsilon\,.$$

Take $\epsilon \to 0$ to conclude that $\lim_{t\to\infty}\|s_t - \tilde{s}_t\| = 0$, and hence, $\lim_{t\to\infty}s_t = \tilde{s}$.

Finally, by equation (B.5), s_t is continuous because P_μ is continuos in μ. Hence, H_1 is continuos. □

Now, we show that the operator H_2 is continuos. First, we show some preliminary lemmas.

Lemma B.2. *The transition operator G is continuous on $\mathcal{S}^\infty$.*

Proof. Consider the sequence $\{S^k \in \mathcal{S}^\infty : k \geq 0\}$, such that $\lim_{k\to\infty}S^k = S \in \mathcal{S}^\infty$. Therefore, $\lim_{k\to\infty}s_t^k = s_t$, for all $t \geq 0$. In particular, the latter holds for $t \geq 1$. Therefore, $\lim_{k\to\infty}G(S^k) = G(S)$. The result follows. □

Lemma B.3. *The value function $\overline{V}$ is the unique solution of Bellman's equation $V = TV$ within the class of bounded functions. Moreover, $\overline{V}$ and $\overline{\mu}$ are continuous in the metric compatible with the product topology in $\mathbb{N}\cap[0, x_{max}]\times\mathcal{S}^\infty$.*

Proof. Because single-period profits, investments, and expected sell-off value are bounded, $\overline{V}$ is the unique solution of Bellman's equation $V = TV$ within the class of bounded functions (Bertsekas (2001)). We use the contraction mapping theorem to prove that, additionally, $\overline{V}$ is continuous in the metric compatible with the product topology in $\mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^{\infty}$. In particular, we show that T has a fixed point within the class of *bounded and continuous* functions. Because the fixed point is $\overline{V}$, $\overline{V}$ is continous.

Let $C_b(\mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^{\infty}, \Re)$ be the space of continuous and bounded real-valued functions with domain $\mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^{\infty}$. Recall that $C_b(\mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^{\infty}, \Re)$ is a complete metric space with the metric defined by the supremum norm (Marsden and Hoffman (1993)). Also, recall that T is a contraction in the supremum norm because $\beta < 1$ (Bertsekas (2001)). Now, we show that T maps elements from $C_b(\mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^{\infty}, \Re)$ into $C_b(\mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^{\infty}, \Re)$.

Take a function $V \in C_b(\mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^{\infty}, \Re)$. By definition (B.1)

$$TV(x, S) = \pi(x, s_0) + E\left[\max\left\{\phi_{it}, \sup_{\iota \in [0, \hat{\iota}]} \left(-d\iota + \beta E\left[V(x_{i,t+1}, G(S)) | x_{it} = x, \iota_{it} = \iota\right]\right)\right\}\right] .$$

By Lemma B.2, the operator G is continuos. The profit function π and the value function V are continuos. By assumption A.2, $(-d\iota + \beta E\left[V(x_{i,t+1}, G(S)) | x_{it} = x, \iota_{it} = \iota\right])$ is continuos in ι and V, and the random variable ϕ_{it} is absolutely continuos. Moreover ι is optimized over a compact space. By Berge's Maximum Theorem, $TV(x, S)$ is continuos. Moreover, TV is bounded because profits, investments, and expected sell-off value are uniformly bounded over all states. Therefore, T maps elements from $C_b(\mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^{\infty}, \Re)$ into $C_b(\mathbb{N} \cap [0, x_{max}] \times \mathcal{S}^{\infty}, \Re)$.

Using the contraction mapping theorem (Marsden and Hoffman (1993)), we conclude that T has a fixed point among the class of continuous and bounded functions, therefore, $\overline{V}$ is continous. Using Berge's Maximum Theorem again, we conclude that $\bar{\iota}$ and $\overline{\rho}$ are also continuous. □

Lemma B.4. *The operator H_2 maps elements from $\mathcal{S}^{\infty} \times \hat{\Lambda}_{ns}$ into $\hat{\mathcal{M}}_{ns} \times \hat{\Lambda}_{ns}$, and is continuous.*

Proof. By Lemmas B.2 and B.3, for each $t \geq 0$, $\left\{\overline{\mu}(\cdot, G^t(S)), \max\{0, \min\{\lambda_t + \beta\overline{V}(x^e, G^{t+1}(S)) - \kappa, \lambda_{max}\}\}\right\}$ is continous in (S, λ). Hence, H_2 is continuous.

Now, $\lim_{t\to\infty} G^t(S) = \{s, s, s, ...\}$, for some $s \in \mathcal{S}$, because $S \in \mathcal{S}^{\infty}$. Hence, using the continuity of $\overline{\mu}$ (Lemma B.3), $\lim_{t\to\infty} \overline{\mu}(\cdot, G^t(S)) = \mu$, for some $\mu \in \tilde{\mathcal{M}}$. By a similar argument, the second component of H_2 also converges. Hence, H_2 maps elements from $\mathcal{S}^{\infty} \times \hat{\Lambda}_{ns}$ into $\hat{\mathcal{M}}_{ns} \times \hat{\Lambda}_{ns}$. □

B.4 Proof Theorem 2.1

Proof. Because the range of $\mu \in \tilde{\mathcal{M}}$ and $\lambda \in \tilde{\mathcal{M}}$ are bounded, by Tychonoff Theorem (Royden (1988)), $\hat{\mathcal{M}}_{ns} \times \hat{\Lambda}_{ns}$ is a compact set with the product topology. It is also a convex set and a subset of a locally convex Haussdorff space.

By Lemmas B.1 and B.4, H is a continuos operator that maps elements from $\hat{\mathcal{M}}_{ns} \times \hat{\Lambda}_{ns}$ into itself. Hence, by Brouwer-Schauder-Tychonoff' theorem, there exists a fixed point (μ, λ) of H in the set $\hat{\mathcal{M}}_{ns} \times \hat{\Lambda}_{ns}$. The fixed point is a NOE, such that, for all x, $\lim_{t\to\infty} \mu_t(x) = \tilde{\mu}(x)$ and $\lim_{t\to\infty} \lambda_t = \tilde{\lambda}$.

To finish the proof we show that $(\tilde{\mu}, \tilde{\lambda}) \in \mathcal{M} \times \Lambda$ is an OE. Using the argument in Lemma B.1, it is straightforward to show that the sequence S in the first component of $H_1(\mu, \lambda)$ converges to $\tilde{s} = \mathbf{1}_{x^e}\tilde{\lambda}\,(I - P_{\tilde{\mu}})^{-1}$, the long-run expected state under oblivious strategy and entry rate $(\tilde{\mu}, \tilde{\lambda})$. By Lemma B.4, $\mu_t(\cdot) = \overline{\mu}(\cdot, G^t(S))$. Taking $\lim_{t\to\infty}$, using the fact that $\lim_{t\to\infty} G^t(S) = \{\tilde{s}, \tilde{s}, ...\}$, and that $\overline{\mu}$ is continuous (Lemma B.3), we conclude that $\tilde{\mu}(\cdot)$ is an OE strategy. Because $\overline{V}$ is continuous (Lemma B.3), the associated OE value function is $\overline{V}(x, \{\tilde{s}, \tilde{s}, ...\})$. Because λ is a fixed point of H and taking $\lim_{t\to\infty}$, if $\tilde{\lambda} = 0$, then $\overline{V}(x^e, \{\tilde{s}, \tilde{s}, ...\}) \leq 0$. Similarly, if $\tilde{\lambda} > 0$, then $\beta\overline{V}(x^e, \{\tilde{s}, \tilde{s}, ...\}) = \kappa$. Therefore, $\tilde{\lambda}$ is an OE entry rate. The result follows. □

C Proofs Section 4

Proof of Theorem 4.1. First, let us write,

$$
\begin{aligned}
V_0\left(x, s|\mu', \tilde{\mu}, \tilde{\lambda}\right) - V_0\left(x, s|\tilde{\mu}, \tilde{\lambda}\right) &= V_0\left(x, s|\mu', \tilde{\mu}, \tilde{\lambda}\right) - \tilde{V}_0\left(x|\tilde{\mu}, \tilde{\lambda}, s\right) \\
&+ \tilde{V}_0\left(x|\tilde{\mu}, \tilde{\lambda}, s\right) - V_0\left(x, s|\tilde{\mu}, \tilde{\lambda}\right) \ .
\end{aligned} \tag{C.1}
$$

Because $\tilde{\mu}$ and $\tilde{\lambda}$ attain an $s-$nonstationary oblivious equilibrium, for all x,

$$
\tilde{V}_0\left(x|\tilde{\mu}, \tilde{\lambda}, s\right) = \sup_{\mu' \in \tilde{\mathcal{M}}_{ns}} \tilde{V}_0\left(x|\mu', \tilde{\mu}, \tilde{\lambda}, s\right) = \sup_{\mu' \in \mathcal{M}_{ns}} \tilde{V}_0\left(x|\mu', \tilde{\mu}, \tilde{\lambda}, s\right) \ ,
$$

where the last equation follows because there will always be an optimal nonstationary oblivious strategy when optimizing a nonstationary oblivious value function even if we consider more general strategies. Let

$\mu^* \in \mathcal{M}_{ns}$ be such that $\sup_{\mu' \in \mathcal{M}_{ns}} V_0\left(x, s|\mu', \tilde{\mu}, \tilde{\lambda}\right) = V_0\left(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}\right)$, for all $x \in \mathbb{N}$. We have,

$$\text{(C.2)} \quad V_0\left(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}\right) - \tilde{V}_0\left(x|\tilde{\mu}, \tilde{\lambda}, s\right) \leq \\ E_{\mu^*, \tilde{\mu}, \tilde{\lambda}}\left[\sum_{k=0}^{\tau_i} \beta^k \left(\pi(x_{ik}, s_{-i,k}) - \pi(x_{ik}, \tilde{s}_{(\tilde{\mu}, \tilde{\lambda}, s), k})\right) \Big| x_{i0} = x, s_{-i,0} = s\right].$$

Competitors of firm i are using nonstationary oblivious strategies, therefore, their evolution is not affected by firm i's evolution. Hence,

$$\text{(C.3)} \quad V_0\left(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}\right) - \tilde{V}_0\left(x|\tilde{\mu}, \tilde{\lambda}, s\right) \leq \\ E_{\tilde{\mu}, \tilde{\lambda}}\left[\sum_{k=0}^{\infty} \beta^k \left[\max_{x' \in \{\underline{x}(k), \ldots, x + k\overline{w}\}} \left(\pi(x', s_{-i,k}) - \pi(x', \tilde{s}_{(\tilde{\mu}, \tilde{\lambda}, s), k})\right)\right]^+ \Big| s_{-i,0} = s\right].$$

On the other hand,

$$\text{(C.4)} \quad \tilde{V}_0\left(x|\tilde{\mu}, \tilde{\lambda}, s\right) - V_0\left(x, s|\tilde{\mu}, \tilde{\lambda}\right) = \\ E_{\tilde{\mu}, \tilde{\lambda}}\left[\sum_{k=0}^{\tau_i} \beta^k \left(\pi(x_{ik}, \tilde{s}_{(\tilde{\mu}, \tilde{\lambda}, s), k}) - \pi(x_{ik}, s_{-i,k})\right) \Big| x_{i0} = x, s_{-i,0} = s\right].$$

The result follows by expressions (C.1), (C.3), and (C.4). □

Proof of Theorem 4.2. By equation (C.2):

$$V_0\left(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}\right) - \tilde{V}_0\left(x|\tilde{\mu}, \tilde{\lambda}, s\right) \leq \\ E_{\mu^*, \tilde{\mu}, \tilde{\lambda}}\left[\sum_{k=0}^{\tau_i} \beta^k \left(\pi(x_{ik}, s_{-i,k}) - \pi(x_{ik}, \tilde{s}_{(\tilde{\mu}, \tilde{\lambda}, s), k})\right) \Big| x_{i0} = x, s_{-i,0} = s\right].$$

Competitors of firm i are using nonstationary oblivious strategies, therefore, their evolution is not affected by firm i's evolution. Using this fact and a similar argument to Theorem 5.2 in Weintraub, Benkard, and Van Roy (2008a) we obtain:

$$\text{(C.5)} \quad V_0\left(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}\right) - \tilde{V}_0\left(x|\tilde{\mu}, \tilde{\lambda}, s\right) \\ \leq \sum_{k=0}^{\infty} \beta^k E_{\hat{\mu}, \tilde{\mu}, \tilde{\lambda}}\left[\left[\pi(x_{ik}, s_{-i,k}) - \pi(x_{ik}, \tilde{s}_{(\tilde{\mu}, \tilde{\lambda}, s), k})\right]^+ \Big| x_{i0} = x, s_{-i,0} = s\right]$$

The result of the proof is analogous to Theorem 4.1. □

Proof of Corollary 4.1. The proof is analogous to Theorem 4.2, but uses a similar argument to Corollary B.1 in Weintraub, Benkard, and Van Roy (2008a) to obtain:

$$
\begin{aligned}
\text{(C.6)} \quad & V_0\left(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}\right) - \tilde{V}_0\left(x|\tilde{\mu}, \tilde{\lambda}, s\right) \\
& \leq E_{\hat{\mu},\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=0}^{\infty} \beta^k \left[\pi(x_{ik}, s_{-i,k}) - \pi(x_{ik}, \tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k})\right]^+ \middle| x_{i0} = x, s_{-i,0} = s\right] \\
= & \sum_{k=0}^{\infty} \beta^k \sum_{y \in \{x,\ldots,x+k\}} \binom{k}{y-x} (1-\delta)^{y-x} \delta^{k-(y-x)} E\left[\left[\pi(y, s_{-i,k}) - \pi(y, \tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k})\right]^+ \middle| s_{-i,0} = s\right].
\end{aligned}
$$

□

D Proofs Section 6

Let $\ell_{1,g} = \{f \in \Re_+^\infty | \, \|f\|_{1,g} < \infty\}$. With some abuse of notation, let $\mathcal{S}_{1,g} = \mathcal{S}_1 \cap \ell_{1,g}$.

D.1 Proof of Proposition 6.1

Proof of Proposition 6.1. It is simple to show that Assumption A.1.4 implies that, for all $f, f' \in \mathcal{S}_{1,g}$, and $m, n \in \Re_+$ (see Lemma A.6 in Weintraub, Benkard, and Van Roy (2008b))

$$|\ln \pi_m(x^e, f, n) - \ln \pi_m(x^e, f', n)| \leq \|f - f'\|_{1,g}.$$

By the previous equation and Assumption 6.2, it follows that, for all $\epsilon > 0$, there exists $z \in \mathbb{N}$, such that,

$$\sup_{m,t} \inf_{\hat{f} \in \mathcal{S}_{1,z}} |\ln \pi_m(x^e, \tilde{f}_t^{(m)}, \tilde{n}_t^{(m)}) - \ln \pi_m(x^e, \hat{f}, \tilde{n}_t^{(m)})| \leq \epsilon.$$

Therefore, for all $m \in \Re_+, t > 0$, there exists $\hat{f}_t^{(m)} \in \mathcal{S}_{1,z}$, such that,

$$\exp(-\epsilon)\pi_m(x^e, \hat{f}_t^{(m)}, \tilde{n}_t^{(m)}) \leq \pi_m(x^e, \tilde{f}_t^{(m)}, \tilde{n}_t^{(m)}). \quad \text{(D.1)}$$

Assumption 6.1.4 implies that there exists $c_1 > 0$ and $\overline{m}_1 \in \mathbb{N}$, such that, if $\tilde{n}_t^{(m)} < c_1 m$ and $m > \overline{m}_1$, then,

$$\pi_m(x^e, \hat{f}_t^{(m)}, \tilde{n}_t^{(m)}) > \frac{\kappa}{\beta} + 1. \tag{D.2}$$

Assumption 6.3 implies that there exists $c_2 > 0$ and $\overline{m}_2$ such that, $\lambda_{\max}^{(m)} > c_2 m$, for all $m > \overline{m}_2$. Let $\overline{m} = \max\{\overline{m}_1, \overline{m}_2\}$ and $c = \min\{c_1, c_2\}$. We show that the proposition holds for c and $\overline{m}$ given by these quantities.

Let us assume for contradiction that there exists $t > 0$ and $m > \overline{m}$, such that, $\tilde{n}_t^{(m)}/m < c$. We need to consider two cases. First, consider $\lambda_{t-1}^{(m)} = \lambda_{max}^{(m)}$. In this case, $\tilde{n}_t^{(m)} \geq \lambda_{max}^{(m)} > cm$ by Assumption 6.3 and we arrive to a contradiction.

In the second case, $\lambda_{t-1}^{(m)} < \lambda_{max}^{(m)}$. By the zero expected value condition in the definition of NOE, it must be that

$$\beta \tilde{V}_t^{(m)}(x^e | \mu^{(m)}, \lambda^{(m)}, s_0^{(m)}) - \kappa \leq 0.$$

By equations (D.1) and (D.2), for ϵ sufficiently small

$$\frac{\kappa}{\beta} < \pi_m(x^e, \tilde{f}_t^{(m)}, \tilde{n}_t^{(m)}) \leq \tilde{V}_t^{(m)}(x^e | \mu^{(m)}, \lambda^{(m)}, s_0^{(m)}),$$

contradicting the zero expected value condition. The result follows. □

D.2 Proof of Proposition 6.2

We start by proving some preliminary lemmas.

Lemma D.1. *Let X be a binomial random variable with parameters (p, n). Then,*

- *If $p = 0$, for all $q \in \Re, n \geq 1$,*

  $$\frac{E \exp(qX)}{\exp[q(\delta + 1)EX]} = 1.$$

  and

  $$\frac{E \exp(qX)}{\exp[q(\delta - 1)EX]} = 1.$$

- *For all $\delta > 0$, there exists $C_1 > 0$, such that, for all $p \in (0, 1], n \geq 1$,*

  $$\frac{E \exp(qX)}{\exp[q(\delta + 1)EX]} \leq \exp(-C_1 EX), \tag{D.3}$$

  where $q = \ln(1 + \delta)$.

For all $\delta > 0$, there exists $C_2 > 0$, such that, for all $p \in (0, 1], n \geq 1$,

(D.4)
$$\frac{E\exp(qX)}{\exp[q(1-\delta)EX]} \leq \exp(-C_2 EX),$$

where $q = \ln(1-\delta)$.

Proof. The case $p = 0$ is trivial. Suppose $0 < p \leq 1$.

We use the following inequalities that are easy to show:

(D.5)
$$(1+x)\log(1+x) - x > 0 \text{ and } (1-x)\log(1-x) + x > 0, \forall x \in (0,1).$$

Using the moment generating function for the binomial distribution, we have that for all $q \in \Re$,

$$\frac{E\exp(qX)}{\exp[q(\delta+1)EX]} = \frac{(1-p+pe^q)^n}{\exp[q(\delta+1)EX]} = \exp\left\{-EX\left[q(1+\delta) - \frac{1}{p}\log(1-p+pe^q)\right]\right\}$$

Note that $\frac{1}{p}\log(1+(e^q-1)p)$ is decreasing in p, for all q. Its derivative has the opposite sign to

$$(1+(e^q-1)p)\log(1+(e^q-1)p) - (e^q-1)p > 0.$$

So using that $\lim_{p\to 0}\frac{1}{p}\log(1-p+pe^q) = \exp(q) - 1$ we have that:

$$q(1+\delta) - \frac{1}{p}\log(1-p+pe^q) \geq q(1+\delta) - \exp(q) + 1, \forall p \in (0,1],\ \forall q \in \Re.$$

The right-hand side of the above inequality is maximizied at $q = \ln(1+\delta)$ and its maximum is $C_1 = (1+\delta)\ln(1+\delta) - \delta > 0$, which is positive by equation (D.5). Inequality (D.3) then follows.

To show inequality (D.4) follow a similar argument, noting that,

$$\frac{E\exp(qX)}{\exp[q(1-\delta)EX]} = \frac{(1-p+pe^q)^n}{\exp[q(1-\delta)EX]} = \exp\left\{-EX\left[q(1-\delta) - \frac{1}{p}\log(1-p+pe^q)\right]\right\},$$

and choose $q = \log(1-\delta)$ and $C_2 = (1-\delta)\log(1-\delta) + \delta > 0$. $\square$

Lemma D.2. *Let Assumptions A.1, A.2, A.3, 6.1.4, 6.2, and 6.3 hold. Then, for all $\delta > 0$, there exists $c > 0$ and $\overline{m} \in \mathbb{N}$, such that for all $t > 0$ and $m > \overline{m}$,*

$$\mathcal{P}\left[\left|\frac{n_t^{(m)}}{\tilde{n}_t^{(m)}} - 1\right| \geq \delta\right] \leq 2e^{-cm}.$$

Proof. Let B_{xt} be a binomial random variable that represents the number of firms that at the initial period were at quality level x and are still inside the industry at time period t. Let Y_{kt} be a random variable that represents the number of firms that entered the industry at time period $0 < k \leq t$ and are still inside the industry at time period t. Because entry at every time period $t \geq 1$ is represented by a Poisson random variable and firms' trajectories are independent (because they use NOE strategies), we have that Y_{kt} is Poisson. Let $Y_t = \sum_{k=1}^{t} Y_{kt}$. The random variable Y_t is Poisson because it is a sum of independent Poisson random variables. Hence, we can write

$$n_t = \sum_x B_{xt} + Y_t,$$

where each random variable B_{xt} is binomial and Y_t is Poisson. Moreover, these random variables are independent. Note that for each market size m, $\sum_x \tilde{s}_0^{(m)}(x) < \infty$, hence, the sum above has a finite number of terms B_{xt} for each market size m.

We have that

$$\mathcal{P}\left(\frac{n_t^{(m)}}{\tilde{n}_t^{(m)}} - 1 \geq \delta\right) \leq \frac{E\exp\left(\sum_x qB_{xt}^{(m)} + qY_t^{(m)}\right)}{\exp\left[q(\delta+1)\left(\sum_x EB_{xt}^{(m)} + EY_t^{(m)}\right)\right]} =$$
$$= \frac{E\exp(qY_t^{(m)})}{\exp[q(\delta+1)EY_t^{(m)}]} \prod_x \frac{E\exp(qB_{xt}^{(m)})}{\exp[q(\delta+1)EB_{xt}^{(m)}]}$$

Now, using Lemma D.1 and a similar analysis for the Poisson distribution based on the moment generating function, there exists $C_1 > 0$, such that, for all $t > 0$ and $m \in \Re$,[18]

$$\mathcal{P}\left(\frac{n_t^{(m)}}{\tilde{n}_t^{(m)}} - 1 \geq \delta\right) \leq \exp(-C_1 EY_t^{(m)}) \prod_x \exp(-C_1 EB_{xt}^{(m)}) =$$
$$= \exp\left(-C_1 E\left[Y_t^{(m)} + \sum_x B_{xt}^{(m)}\right]\right) = \exp(-C_1 \tilde{n}_t^{(m)}).$$

Now, by Proposition 6.1, there exists $C_2 > 0$ and $\overline{m}$, such that, for all $m > \overline{m}$ and $t > 0$, $\tilde{n}_t^{(m)} \geq C_2 m$. Therefore, there exists $c = C_1 C_2$ and $\overline{m}$, such that, for all $t > 0$ and $m > \overline{m}$,

$$\text{(D.6)} \qquad \mathcal{P}\left(\frac{n_t^{(m)}}{\tilde{n}_t^{(m)}} - 1 \geq \delta\right) \leq \exp(-cm).$$

[18]The same constant C_1 as in inequality (D.3) serves to bound the term corresponding to the Poisson random variable.

A similar bound can be derived for $\mathcal{P}\left(1 - \frac{n_t^{(m)}}{\tilde{n}_t^{(m)}} \geq \delta\right)$. The result follows by putting together both bounds. $\square$

Proof of Proposition 6.2. Convergence of $n_t^{(m)}/\tilde{n}_t^{(m)}$ follows from Lemma D.2. To complete the proof, we will establish convergence of $\|f_t^{(m)} - \tilde{f}_t^{(m)}\|_{1,g}$, for all $t > 0$. Note that for any $z \in \mathbb{N}$,

$$\begin{aligned}
\|f_t^{(m)} - \tilde{f}_t^{(m)}\|_{1,g} &\leq z \max_{x \leq z} g(x)|f_t^{(m)}(x) - \tilde{f}_t^{(m)}(x)| + \sum_{x>z} g(x) f_t^{(m)}(x) + \sum_{x>z} g(x)\tilde{f}_t^{(m)}(x) \\
&\equiv A_z^{(m)} + B_z^{(m)} + C_z^{(m)},
\end{aligned}$$

where we have omitted the dependence on t to simplify notation. We will show that for any z, $A_z^{(m)}$ converges in probability to zero, that for any $\delta > 0$, for sufficiently large z, $\lim_{m\to\infty} \mathcal{P}[C_z^{(m)} \geq \delta] = 0$, and that for any $\delta > 0$ and $\epsilon > 0$, for sufficiently large z, $\limsup_{m\to\infty} \mathcal{P}[B_z^{(m)} \geq \delta] \leq \epsilon/\delta$. The assertion that $\|f_t^{(m)} - \tilde{f}_t^{(m)}\|_{1,g} \to_p 0$ follows from these facts.

By Assumption 6.2, for any $\delta > 0$, for sufficiently large z, $\limsup_{m\to\infty} C_z^{(m)} < \delta$, and therefore, $\lim_{m\to\infty} \mathcal{P}[C_z^{(m)} \geq \delta] = 0$. By Tonelli's Theorem, $E[B_z^{(m)}] = C_z^{(m)}$. Invoking Markov's inequality, for any $\delta > 0$ and $\epsilon > 0$, for sufficiently large z, $\limsup_{m\to\infty} \mathcal{P}[B_z^{(m)} \geq \delta] \leq \epsilon/\delta$.

To finish the proof we show that for any z, $A_z^{(m)}$ converges in probability to zero. In particular, we show that, for all x, $|f_t^{(m)}(x) - \tilde{f}_t^{(m)}(x)| \to_p 0$ using mathematical induction. First, note that the result holds for $t = 0$ because $\tilde{f}_0^{(m)}(x) = f_0^{(m)}(x)$ (recall that at $t = 0$, $s_0^{(m)} = \tilde{s}_0^{(m)}$). To complete our inductive argument, we now show that $|f_k^{(m)}(x) - \tilde{f}_k^{(m)}(x)| \to_p 0,\ \forall x$, implies $|f_{k+1}^{(m)}(x) - \tilde{f}_{k+1}^{(m)}(x)| \to_p 0,\ \forall x$.

We define

$$D_k^{(m)}(x) = \frac{1}{n_{k+1}^{(m)}} \left(\hat{n}_k^{(m)} \sum_{y \in [x \pm \overline{w}]} \hat{f}_k^{(m)}(y) \hat{p}_k^{(m)}(y, x) + \mathbf{1}_{\{x = x^e\}} e_k^{(m)} \right),$$

where $\hat{s}_k^{(m)}(y)$ is the number of firms at time k in state y in market m after exit decisions have been realized; $\hat{n}_k^{(m)} = \sum_y \hat{s}_k^{(m)}(y)$ is the total number of firms after exit decisions have been realized (and before new entrants enter the industry); $\hat{f}_k^{(m)}(y) = \frac{\hat{s}_k^{(m)}(y)}{\hat{n}_k^{(m)}}$ is the fraction of firms at time k in state y after exit decisions have been realized; $e_k^{(m)}$ is the number of entrants at time $k+1$ in market m; $\hat{p}_k^{(m)}(y, x)$ is the probability of transitioning from state y to state x conditional on staying in the industry; and $\mathbf{1}_{\{\cdot\}}$ is the indicator function. Let $p_k^{(m)}(y)$ be the probability of staying in the industry for a firm in state y in market m in time period k, and $p_k^{(m)}(y, x)$ the unconditional probability of transitioning from state y to state x. Then, $\hat{p}_k^{(m)}(y, x) = \frac{p_k^{(m)}(y,x)}{p_k^{(m)}(y)}$. Note that this conditional probability is well defined for all m, k, x, y, because by assumption, $\pi_m(x,s) > 0,\ \forall m,x,s$, and the random variables ϕ_{it} have support in $\Re_+$.

By the triangle inequality $|f_{k+1}^{(m)}(x) - \tilde{f}_{k+1}^{(m)}(x)| \leq |f_{k+1}^{(m)}(x) - D_k^{(m)}(x)| + |D_k^{(m)}(x) - \tilde{f}_{k+1}^{(m)}(x)|$. Therefore it suffices to show that every term at the right hand side of the inequality goes to zero in probability as the market size grows.

First, we show that $|f_{k+1}^{(m)}(x) - D_k^{(m)}(x)| \to_p 0$. We can decompose $f_{k+1}^{(m)}(x)$ as

$$f_{k+1}^{(m)}(x) = \frac{1}{n_{k+1}^{(m)}} \left(\sum_{y\in[x\pm\overline{w}]} Z_y(x) + \mathbf{1}_{\{x=x^e\}} e_k^{(m)} \right),$$

where $Z_y(x)$ is a binomial random variable with parameters $\{\hat{n}_k^{(m)}\hat{f}_k^{(m)}(y), \hat{p}_k^{(m)}(y,x)\}$. For any $\epsilon > 0$, conditional on $\hat{f}_k^{(m)}, \hat{n}_k^{(m)}, n_{k+1}^{(m)} > 0$,

$$\begin{aligned}
\mathcal{P}(|f_{k+1}^{(m)}(x) - D_k^{(m)}(x)| > \epsilon) &= \mathcal{P}\left(\left| \sum_{y\in[x\pm\overline{w}]} Z_y(x) - \hat{n}_k^{(m)} \sum_{y\in[x\pm\overline{w}]} \hat{f}_k^{(m)}(y)\hat{p}_k^{(m)}(y,x) \right| > \epsilon n_{k+1}^{(m)} \right) \\
&\leq \frac{\hat{n}_k^{(m)}}{\epsilon^2 (n_{k+1}^{(m)})^2} \sum_{y\in[x\pm\overline{w}]} \hat{f}_k^{(m)}(y)\hat{p}_k^{(m)}(y,x)(1-\hat{p}_k^{(m)}(y,x)) \leq \frac{2\overline{w}+1}{\epsilon^2 n_{k+1}^{(m)}},
\end{aligned}$$

where the first inequality follows by Chebyshev's inequality. Integrating over $\hat{f}_k^{(m)}, \hat{n}_k^{(m)}, n_{k+1}^{(m)}$, it follows that unconditionally,

$$\mathcal{P}(|f_{k+1}^{(m)}(x) - D_k^{(m)}(x)| > \epsilon) \leq (2\overline{w}+1)/\epsilon^2 E\left[1/n_{k+1}^{(m)} \mathbf{1}_{\{n_{k+1}^{(m)}>0\}}\right] + \mathcal{P}\left(n_{k+1}^{(m)} = 0\right).$$

Because $n_{k+1}^{(m)}/\tilde{n}_{k+1}^{(m)} \to_p 1$ and $\tilde{n}_{k+1}^{(m)} \to \infty$ (Proposition 6.1) it must be that $n_{k+1}^{(m)} \to_p \infty$. Therefore, $1/n_{k+1}^{(m)} \to_p 0$, and by the bounded convergence theorem we have that $E\left[1/n_{k+1}^{(m)} \mathbf{1}_{n_{k+1}^{(m)}>0}\right] \to 0$. It follows that $|f_{k+1}^{(m)}(x) - D_k^{(m)}(x)| \to_p 0$.

Now, we show that $|D_k^{(m)}(x) - \tilde{f}_{k+1}^{(m)}(x)| \to_p 0$. By definition,

$$\tilde{f}_{k+1}^{(m)}(x) = \frac{1}{\tilde{n}_{k+1}^{(m)}} \left(\sum_{y\in[x\pm\overline{w}]} \tilde{n}_k^{(m)} \tilde{f}_k^{(m)}(y) p_k^{(m)}(y,x) + \mathbf{1}_{\{x=x^e\}} \lambda_k^{(m)} \right)$$

Suppose $x \neq x^e$. Then, by the triangle inequality,

$$|D_k^{(m)} - \tilde{f}_{k+1}^{(m)}(x)| \leq \sum_{y\in[x\pm\overline{w}]} \hat{p}_k^{(m)}(y,x) \left| \frac{\hat{s}_k^{(m)}(y)}{n_{k+1}^{(m)}} - \frac{\tilde{n}_k^{(m)} \tilde{f}_k^{(m)}(y) p_k^{(m)}(y)}{\tilde{n}_{k+1}^{(m)}} \right|.$$

It suffice to show that each term in the sum converges to zero in probability.

We can write:

$$\text{(D.7)}\left|\frac{\hat{s}_k^{(m)}(y)}{n_{k+1}^{(m)}} - \frac{\tilde{n}_k^{(m)}\tilde{f}_k^{(m)}(y)p_k^{(m)}(y)}{\tilde{n}_{k+1}^{(m)}}\right| = \left|\frac{n_k^{(m)}f_k^{(m)}(y)}{n_{k+1}^{(m)}s_k^{(m)}(y)}\sum_{i=1}^{s_k^{(m)}(y)} W_{iky}^{(m)} - \frac{\tilde{n}_k^{(m)}}{\tilde{n}_{k+1}^{(m)}}\tilde{f}_k^{(m)}(y)p_k^{(m)}(y)\right|,$$

where $W_{iky}^{(m)}$ are i.i.d. Bernoulli random variables with mean $p_k^{(m)}(y)$ that are equal to one if firm i at state y stays inside the industry at time period k in market m. We consider two cases:

(i) Suppose $\liminf_{m\to\infty} \tilde{f}_k^{(m)}(y) > 0$. Because $|f_k^{(m)}(y) - \tilde{f}_k^{(m)}(y)| \to_p 0$ by the inductive hypothesis and $n_k^{(m)} \to_p \infty$, one can check that in this case $s_k^{(m)}(y) \to_p \infty$. Then, it is simple to verify using Chebyshev's inequality that $1/s_k^{(m)}(y)\sum_{i=1}^{s_k^{(m)}(y)} W_{iky}^{(m)} - p_k^{(m)}(y) \to_p 0$. Because for all k, $n_k^{(m)}/\tilde{n}_k^{(m)} \to_p 1$, we have that $(n_k^{(m)}/\tilde{n}_k^{(m)})(\tilde{n}_{k+1}^{(m)}/n_{k+1}^{(m)}) \to_p 1$. Also, $|f_k^{(m)}(y) - \tilde{f}_k^{(m)}(y)| \to_p 0$ by the inductive hypothesis. Finally, Proposition 6.1 together with Assumption 6.3 imply that $\tilde{n}_k^{(m)}/\tilde{n}_{k+1}^{(m)}$ remains uniformly bounded from above over all market sizes. The previous convergence results together with the latter uniform bound imply that (D.7)$\to_p 0$.

(ii) Suppose $\liminf_{m\to\infty} \tilde{f}_k^{(m)}(y) = 0$. For a subsequence for which $\lim_{r\to\infty} \tilde{f}_k^{(m_r)}(y) > 0$, we can apply argument (i). Now, we consider a subsequence for which $\lim_{r\to\infty} \tilde{f}_k^{(m_r)}(y) = 0$. Considering that $\tilde{n}_k^{(m)}/\tilde{n}_{k+1}^{(m)}$ remains uniformly bounded from above over all market sizes, we have that $\lim_{r\to\infty} \tilde{n}_k^{(m_r)}\tilde{f}_k^{(m_r)}(y)p_k^{(m_r)}(y)/\tilde{n}_{k+1}^{(m_r)} = 0$. Now, $|f_k^{(m)}(y) - \tilde{f}_k^{(m)}(y)| \to_p 0$ implies $f_k^{(m_r)} \to_p 0$. Also, $1/s_k^{(m)}(y)\sum_{i=1}^{s_k^{(m)}(y)} W_{iky}^{(m)} \leq 1$. Moreover, $(n_k^{(m)}/\tilde{n}_k^{(m)})(\tilde{n}_{k+1}^{(m)}/n_{k+1}^{(m)}) \to_p 1$. These facts with the converging together lemma imply that $\frac{n_k^{(m_r)}f_k^{(m_r)}(y)}{n_{k+1}^{(m_r)}s_k^{(m_r)}(y)}\sum_{i=1}^{s_k^{(m_r)}(y)} W_{iky}^{(m_r)} \to_p 0$. We conclude that (D.7)$\to_p 0$.

If $x \neq x_e$ we need to additionally show that $\left|\frac{\lambda_k^{(m)}}{\tilde{n}_{k+1}^{(m)}} - \frac{e_k^{(m)}}{n_{k+1}^{(m)}}\right| \to_p 0$. By the fact that $\frac{n_{k+1}^{(m)}}{\tilde{n}_{k+1}^{(m)}} \to_p 1$ it suffices to show that $\left|\frac{\lambda_k^{(m)}}{\tilde{n}_{k+1}^{(m)}} - \frac{e_k^{(m)}}{\tilde{n}_{k+1}^{(m)}}\right| \to_p 0$. Using Chebyshev's inequality and the fact that $e_k^{(m)}$ is distributed Poisson we have that

$$\mathcal{P}(|e_k^{(m)} - \lambda_k^{(m)}| > \epsilon\tilde{n}_{k+1}^{(m)}) \leq \frac{\lambda_k^{(m)}}{(\epsilon\tilde{n}_{k+1}^{(m)})^2} \leq \frac{1}{\epsilon\tilde{n}_{k+1}^{(m)}} \to 0.$$

Hence, for all x, $|f_t^{(m)}(x) - \tilde{f}_t^{(m)}(x)| \to_p 0$, concluding the proof. □

D.3 Proof of Theorem 6.1

We start by proving some preliminary lemmas.

Lemma D.3. *Let Assumptions A.1, A.2, A.3, 6.1, 6.2 and 6.3 hold. Then, for all $x \in \mathbb{N}$ and $t > 0$,*

$$\sup_m \sup_{\mu \in \mathcal{M}} E_\mu \left[\sum_{k=t}^{\infty} \beta^{k-t} \sup_{f \in \mathcal{S}_1} \pi_m(x_{ik}, f, \tilde{n}_k^{(m)}) \,\Big|\, x_{it} = x \right] < \infty.$$

Proof. Take $c > 0$ and $\overline{m}$ given by Proposition 6.1. Using Assumption 6.1.3 and the fact that for all $m > \overline{m}$ and $t > 0$, $\tilde{n}_t^{(m)} \geq cm$, we have that

$$\sup_m \sup_{\mu \in \mathcal{M}} E_\mu \left[\sum_{k=t}^{\infty} \beta^{k-t} \sup_{f \in \mathcal{S}_1} \pi_m(x_{ik}, f, \tilde{n}_k^{(m)}) \,\Big|\, x_{it} = x \right] \leq \sup_{\mu \in \mathcal{M}} E_\mu \left[\sum_{k=t}^{\infty} \beta^{k-t} \overline{\pi}(x_{ik}) \,\Big|\, x_{it} = x \right] < \infty.$$

The result follows. □

The following technical lemma follows immediately from Assumption 6.1.3. We omit the proof.

Lemma D.4. *Let Assumptions A.1.3 and 6.1.2 hold. Then, for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $n, \hat{n} \in \Re_+$ satisfying $|n/\hat{n} - 1| < \delta$,*

$$\sup_{m \in \Re_+, x \in \mathbb{N}, f \in \mathcal{S}_1} \left| \frac{\pi_m(x, f, n) - \pi_m(x, f, \hat{n})}{\pi_m(x, f, \hat{n})} \right| \leq \epsilon.$$

Lemma D.5. *Let Assumptions A.1, A.2, A.3, 6.1, 6.2, and 6.3 hold. Then, for all sequences $\{\hat{\mu}^{(m)} \in \mathcal{M}_{ns}\}$ and $x \in \mathbb{N}$,*

$$\lim_{m \to \infty} E_{\hat{\mu}^{(m)}, \mu^{(m)}, \lambda^{(m)}} \left[\sum_{k=0}^{\tau_i} \beta^k |\pi_m(x_{ik}, s_{-i,k}^{(m)}) - \pi_m(x_{ik}, f_{-i,k}^{(m)}, \tilde{n}_k^{(m)})| \,\Big|\, x_{i0} = x, s_{-i,0}^{(m)} = \tilde{s}_0^{(m)} \right] = 0.$$

Proof. For the purpose of this proof, we will assume that all expectations are conditioned on $x_{i0} = x$ and $s_{-i,0}^{(m)} = \tilde{s}_0^{(m)}$. Let $\Delta_{it}^{(m)} = |\pi_m(x_{it}, s_{-i,t}^{(m)}) - \pi_m(x_{it}, f_{-i,t}^{(m)}, \tilde{n}_t^{(m)})|$. Fix $\epsilon > 0$ and let $\delta > 0$ satisfy the assertion of Lemma D.4. Let $Z_t^{(m)}$ denote the event $\left| n_t^{(m)} / \tilde{n}_t^{(m)} - 1 \right| \geq \delta$. Applying Tonelli's Theorem and noting that $s_{-i,0}^{(m)} = \tilde{s}_0^{(m)}$ we obtain,

$$\begin{aligned}
E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\sum_{k=0}^{\tau_i}\beta^k\Delta_{ik}^{(m)}\right] &\leq \sum_{k=1}^{\infty}\beta^k E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\Delta_{ik}^{(m)}\right] \\
&= \sum_{k=1}^{\infty}\beta^k\left(E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\Delta_{ik}^{(m)}\mathbf{1}_{\neg Z_k^{(m)}}\right] + E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\Delta_{ik}^{(m)}\mathbf{1}_{Z_k^{(m)}}\right]\right) \\
&\leq \sum_{k=1}^{\infty}\beta^k\left(\epsilon E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\pi_m(x_{ik}, f_{-i,k}^{(m)}, \tilde{n}_k^{(m)})\right] + O(m)\mathcal{P}[Z_k^{(m)}]\right) \\
&\leq \epsilon E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\sum_{k=1}^{\infty}\beta^k\sup_{f\in\mathcal{S}_1}\pi_m(x_{ik}, f, \tilde{n}_k^{(m)})\right] + \sum_{k=1}^{\infty}\beta^k O(m)\mathcal{P}[Z_k^{(m)}],
\end{aligned}$$

where the second to last inequality follows from Assumption 6.1.1 and Lemma D.4. By Lemma D.2, there exist constants $c, \overline{m}$ such that $\mathcal{P}[Z_k^{(m)}] \leq 2e^{-cm}$, for all $k > 0$ and $m > \overline{m}$. Hence, the second sum above converges to zero. Moreover, ϵ is arbitrary and the expected sum in the first term is uniformly bounded over all market sizes (by Lemma D.3). The result follows. □

The following technical lemma is proved in Weintraub, Benkard, and Van Roy (2008b).

Lemma D.6. *Let Assumptions A.1.3 and A.1.4 hold. Then, for all $\epsilon > 0$ there exists $\delta > 0$ such that for $f, \hat{f} \in \mathcal{S}_{1,g}$ satisfying $\|f - \hat{f}\|_{1,g} < \delta$,*

$$\sup_{m\in\Re_+, x\in\mathbb{N}, n\in\Re_+}\left|\frac{\pi_m(x, f, n) - \pi_m(x, \hat{f}, n)}{\pi_m(x, \hat{f}, n)}\right| \leq \epsilon.$$

Lemma D.7. *Let Assumptions A.1, A.2, A.3, 6.1, 6.2, and 6.3 hold. Then, for all sequences $\{\hat{\mu}^{(m)} \in \mathcal{M}_{ns}\}$ and $x \in \mathbb{N}$,*

$$\lim_{m\to\infty} E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\sum_{k=0}^{\tau_i}\beta^k|\pi_m(x_{ik}, f_{-i,k}^{(m)}, \tilde{n}_k^{(m)}) - \pi_m(x_{ik}, \tilde{s}_k^{(m)})| \;\middle|\; x_{i0} = x, s_{-i,0}^{(m)} = \tilde{s}_0^{(m)}\right] = 0.$$

Proof. For the purpose of this proof, we will assume that all expectations are conditioned on $x_{i0} = x$ and $s_{-i,0}^{(m)} = \tilde{s}_0^{(m)}$. Let $\Delta_{it}^{(m)} = |\pi_m(x_{it}, f_{-i,t}^{(m)}, \tilde{n}_t^{(m)}) - \pi_m(x_{it}, \tilde{s}_t^{(m)})|$. Fix $\epsilon > 0$ and let δ satisfy the assertion of Lemma D.6. Let $Z_t^{(m)}$ denote the event $\|f_t^{(m)} - \tilde{f}_t^{(m)}\|_{1,g} \geq \delta$. Applying Tonelli's Theorem and noting that $s^{(m)}_{-i,0} = \tilde{s}^{(m)}_0$ we obtain,

$$\begin{aligned}
E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\sum_{k=0}^{\tau_i}\beta^k\Delta^{(m)}_{ik}\right] &\leq \sum_{k=1}^{\infty}\beta^k E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\Delta^{(m)}_{ik}\right] \\
&= \sum_{k=1}^{\infty}\beta^k\left(E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\Delta^{(m)}_{ik}\mathbf{1}_{\neg Z^{(m)}_k}\right] + E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\Delta^{(m)}_{ik}\mathbf{1}_{Z^{(m)}_k}\right]\right) \\
&\leq \epsilon C + \sum_{k=1}^{\infty}\beta^k E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\Delta^{(m)}_{ik}\mathbf{1}_{Z^{(m)}_k}\right], \qquad \text{(D.8)}
\end{aligned}$$

for some constant $C > 0$. The last inequality follows from Lemmas D.3 and D.6.

Note that $\Delta^{(m)}_{ik} \leq 2\sup_{f\in\mathcal{S}_1}\pi_m(x_{ik}, f, \tilde{n}^{(m)}_k)$. Hence,

$$\begin{aligned}
\sum_{k=1}^{\infty}\beta^k E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[\Delta^{(m)}_{ik}\mathbf{1}_{Z^{(m)}_k}\right] &\leq \sum_{k=1}^{\infty}\beta^k E_{\hat{\mu}^{(m)},\mu^{(m)},\lambda^{(m)}}\left[2\sup_{f\in\mathcal{S}_1}\pi_m(x_{ik}, f, \tilde{n}^{(m)}_k)\mathbf{1}_{Z^{(m)}_k}\right] \\
&\leq 2\sup_{\mu\in\mathcal{M}_{ns}}\sum_{k=1}^{\infty}\beta^k E_{\mu,\mu^{(m)},\lambda^{(m)}}\left[\sup_{f\in\mathcal{S}_1}\pi_m(x_{ik}, f, \tilde{n}^{(m)}_k)\mathbf{1}_{Z^{(m)}_k}\right] \\
&= 2\sup_{\mu\in\mathcal{M}_{ns}}\sum_{k=1}^{\infty}\beta^k E_{\mu}\left[\sup_{f\in\mathcal{S}_1}\pi_m(x_{ik}, f, \tilde{n}^{(m)}_k)\right]\mathcal{P}[Z^{(m)}_k], \qquad \text{(D.9)}
\end{aligned}$$

because $\sup_{\mu\in\mathcal{M}_{ns}}$ is attained by an oblivious strategy, so $f^{(m)}_{-i,k}$ evolves independently from x_{ik}. Assumption 6.1.3 together with Proposition 6.1 implies

$$\sup_{\mu\in\mathcal{M}_{ns}} E_{\mu}\left[\sum_{k=1}^{\infty}\beta^k\sup_{f\in\mathcal{S}_1}\pi_m(x_{ik}, f, \tilde{n}^{(m)}_k)\mathcal{P}[Z^{(m)}_k]\right] \leq \sup_{\mu\in\mathcal{M}} E_{\mu}\left[\sum_{k=1}^{\infty}\beta^k\overline{\pi}(x_{ik})\mathcal{P}[Z^{(m)}_k]\right].$$

By the second part of Assumption 6.1.3, $\sup_{\mu\in\mathcal{M}} E_{\mu}\left[\sum_{k=1}^{\infty}\beta^k\overline{\pi}(x_{ik})\right] < \infty$. Moreover, by Proposition 6.2, for all $k \geq 1$, $\mathcal{P}(Z^{(m)}_k) \to 0$ as $m \to \infty$. Then by the dominated convergence theorem (D.9)$\to_m 0$. Finally, ϵ in (D.8) is arbitrary. The result follows. □

Proof of Theorem 6.1. Let $\mu^{*(m)}$ be an optimal (non-oblivious) best response to $(\mu^{(m)}, \lambda^{(m)})$; in particular,

$$V^{(m)}_0(x, s^{(m)}_0|\mu^{*(m)}, \mu^{(m)}, \lambda^{(m)}) = \sup_{\mu\in\mathcal{M}_{ns}} V^{(m)}_0(x, s^{(m)}_0|\mu, \mu^{(m)}, \lambda^{(m)}).$$

Let

$$\hat{V}^{(m)}(x) = V^{(m)}_0(x, s^{(m)}_0|\mu^{*(m)}, \mu^{(m)}, \lambda^{(m)}) - V^{(m)}_0(x, s^{(m)}_0|\mu^{(m)}, \lambda^{(m)}) \geq 0.$$

The ANME property, which we set out to establish, asserts that for all $x \in \mathbb{N}$, $\lim_{m\to\infty}\hat{V}^{(m)}(x) = 0$.

For any m, because $\mu^{(m)}$ and $\lambda^{(m)}$ attain a NOE, for all x,

$$\tilde{V}_0^{(m)}(x|\mu^{(m)}, \lambda^{(m)}, s_0^{(m)}) = \sup_{\tilde{\mu} \in \tilde{\mathcal{M}}_{ns}} \tilde{V}_0^{(m)}(x|\tilde{\mu}, \mu^{(m)}, \lambda^{(m)}, s_0^{(m)}) = \sup_{\tilde{\mu} \in \mathcal{M}_{ns}} \tilde{V}_0^{(m)}(x|\tilde{\mu}, \mu^{(m)}, \lambda^{(m)} s_0^{(m)}),$$

where the last equation follows because there will always be an optimal nonstationary oblivious strategy when optimizing a nonstationary oblivious value function even if we consider more general strategies. It follows that

$$\begin{aligned}
\hat{V}^{(m)}(x) &= \Big(V_0^{(m)}(x, s_0^{(m)}|\mu^{*(m)}, \mu^{(m)}, \lambda^{(m)}) - \tilde{V}_0^{(m)}(x|\mu^{(m)}, \lambda^{(m)}, s_0^{(m)})\Big) \\
&\quad + \Big(\tilde{V}_0^{(m)}(x|\mu^{(m)}, \lambda^{(m)}, s_0^{(m)}) - V_0^{(m)}(x, s_0^{(m)}|\mu^{(m)}, \lambda^{(m)})\Big) \\
&\leq \Big(V_0^{(m)}(x, s_0^{(m)}|\mu^{*(m)}, \mu^{(m)}, \lambda^{(m)}) - \tilde{V}_0^{(m)}(x|\mu^{*(m)}, \mu^{(m)}, \lambda^{(m)} s_0^{(m)})\Big) \\
&\quad + \Big(\tilde{V}_0^{(m)}(x|\mu^{(m)}, \lambda^{(m)} s_0^{(m)}) - V_0^{(m)}(x, s_0^{(m)}|\mu^{(m)}, \lambda^{(m)})\Big) \\
&\equiv A^{(m)}(x) + B^{(m)}(x).
\end{aligned}$$

To complete the proof, we will establish that $A^{(m)}(x)$ and $B^{(m)}(x)$ converge to zero.

Let $\Delta_{it}^{(m)} = |\pi_m(x_{it}, s_{-i,t}^{(m)}) - \pi_m(x_{it}, \tilde{s}_t^{(m)})|$. It is easy to see that

$$\begin{aligned}
A^{(m)}(x) &\leq E_{\mu^{*(m)}, \mu^{(m)}, \lambda^{(m)}} \left[\sum_{k=0}^{\tau_i} \beta^k \Delta_{ik}^{(m)} \,\middle|\, x_{i0} = x, s_{-i,0}^{(m)} = s_0^{(m)} \right] \\
B^{(m)}(x) &\leq E_{\mu^{(m)}, \lambda^{(m)}} \left[\sum_{k=0}^{\tau_i} \beta^k \Delta_{ik}^{(m)} \,\middle|\, x_{i0} = x, s_{-i,0}^{(m)} = s_0^{(m)} \right],
\end{aligned}$$

By the triangle inequality,

$$\Delta_{ik}^{(m)} \leq |\pi_m(x_{ik}, s_{-i,k}^{(m)}) - \pi_m(x_{ik}, f_{-i,k}^{(m)}, \tilde{n}_k^{(m)})| + |\pi_m(x_{ik}, f_{-i,k}^{(m)}, \tilde{n}_k^{(m)}) - \pi_m(x_{ik}, \tilde{s}_k^{(m)})|.$$

The result therefore follows from Lemmas D.5 and D.7. □

E Figures

Figure 1: NOE evolution from original OE to new OE after a shock to profits.

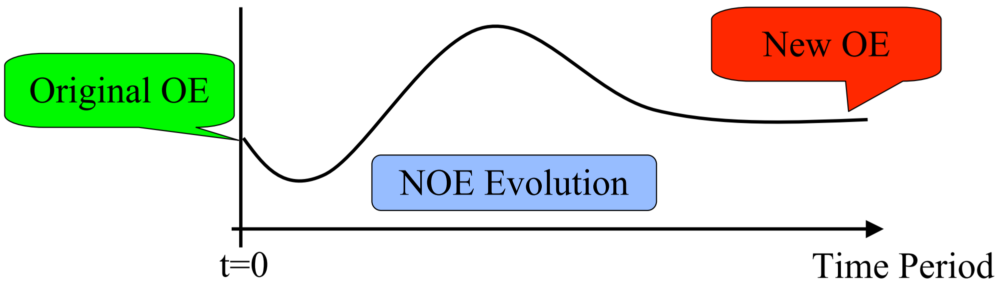


Figure 2: Percentage approximation error bound for different market sizes.

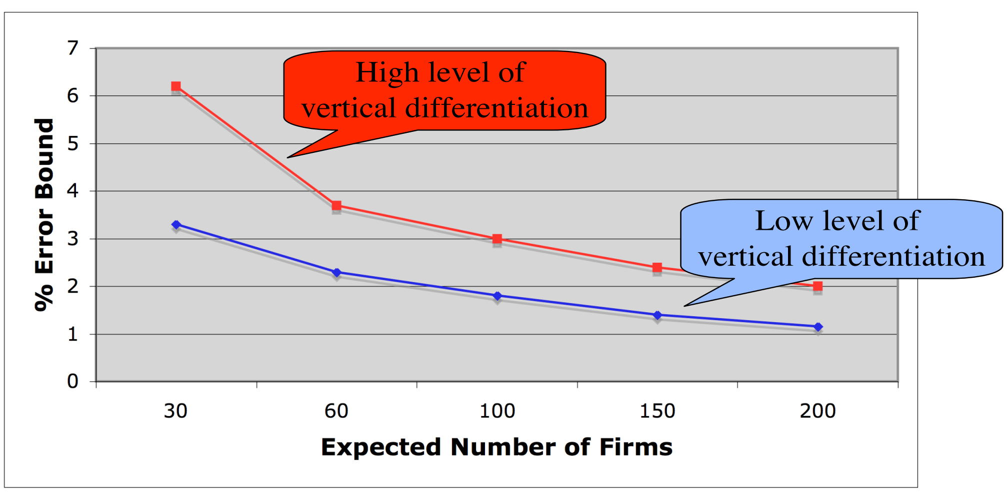


Figure 3: NOE total number of firms and entry rate as a function of the time period for low level of vertical differentiation.

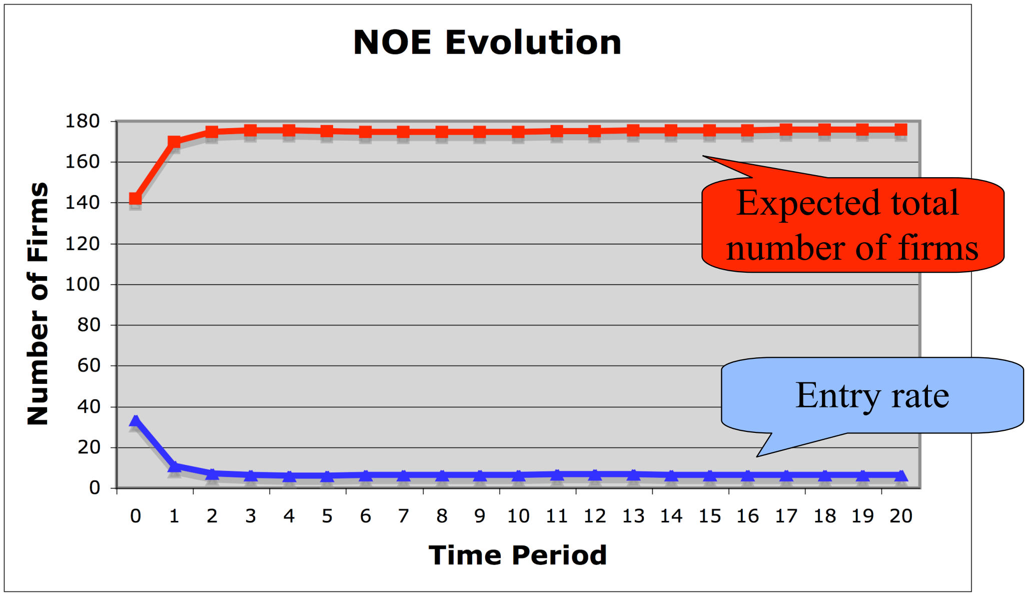
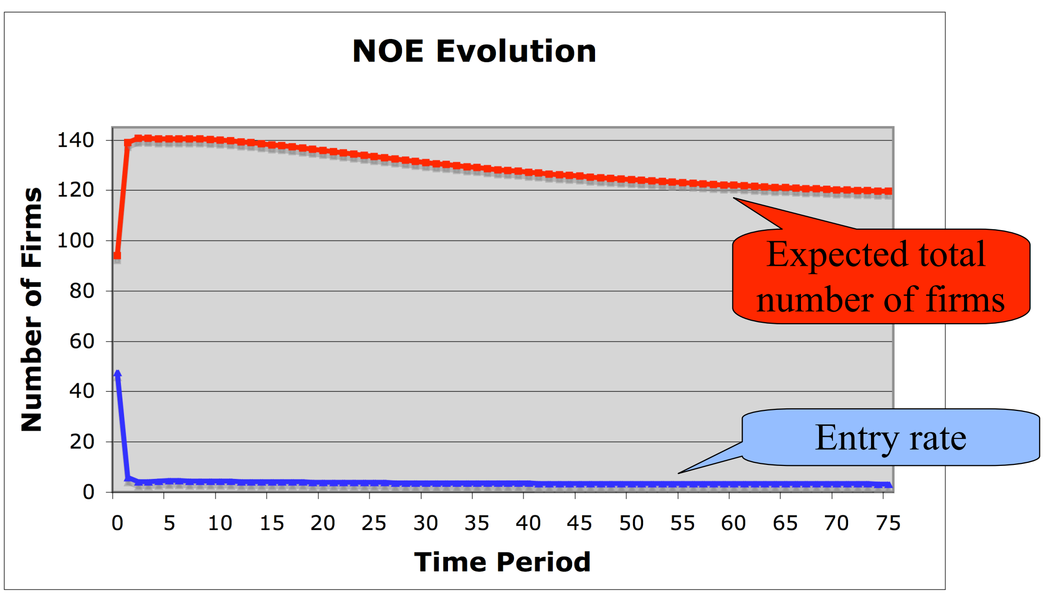


Figure 4: NOE total number of firms and entry rate as a function of the time period for high level of vertical differentiation.



References

Aliprantis, C. D. and K. C. Border (2006). *Infinite Dimensional Analysis* (Third ed.). Springer.

Bertsekas, D. P. (2001). *Dynamic Programming and Optimal Control, Vol. 2* (Second ed.). Athena Scientific.

Doraszelski, U. and M. Satterthwaite (2007). Computable markov-perfect industry dynamics: Existence, purification, and multiplicity. Working Paper, Harvard University.

Ericson, R. and A. Pakes (1995). Markov-perfect industry dynamics: A framework for empirical work. *Review of Economic Studies 62*(1), 53 – 82.

Marsden, J. E. and M. J. Hoffman (1993). *Elementary Classical Analysis* (Second ed.). W. H. Freeman and Company.

Royden, H. (1988). *Real Analysis* (Third ed.). Pearson Education.

Weintraub, G. Y., C. L. Benkard, and B. Van Roy (2007). Extensions to oblivious equilibrium. Working Paper, Stanford University.

Weintraub, G. Y., C. L. Benkard, and B. Van Roy (2008a). Computational methods for oblivious equilibrium. Working Paper, Stanford University.

Weintraub, G. Y., C. L. Benkard, and B. Van Roy (2008b). Markov perfect industry dynamics with many firms. *Econometrica 76*(6), 1375–1411.