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for Binary Rare Events Data: an
Application to Credit Defaults**

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Generalized Extreme Value Regression for Binary Rare Events Data: an Application to Credit Defaults

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Abstract

The most used regression model with binary dependent variable is the logistic regression model. When the dependent variable represents a rare event, the logistic regression model shows relevant drawbacks. In order to overcome these drawbacks we propose the Generalized Extreme Value (GEV) regression model. In particular, in a Generalized Linear Model (GLM) with binary dependent variable we suggest the quantile function of the GEV distribution as link function, so our attention is focused on the tail of the response curve for values close to one. The estimation procedure is the maximum likelihood method. This model accommodates skewness and it presents a generalization of GLMs with log-log link function. In credit risk analysis a pivotal topic is the default probability estimation. Since defaults are rare events, we apply the GEV regression to empirical data on Italian Small and Medium Enterprises (SMEs) to model their default probabilities.

1 Introduction

Many of the most significant event in several areas are rare events - in economics and finance, in medicine and epidemiology, in meteorology and natural science and in international relations. In economics and finance, some pivotal applications of the extreme value theory and the rare event methodology are credit risk, Value At

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Risk and financial strategy of risk management (Embrechts et al., 1997; Dahan and Mendelson, 2001; Finkenstdt and Holger, 2003; Barro, 2009). In the natural science and in epidemiology the rare events as natural disasters and the epidemics occur infrequently but they are considered of great importance (Frei and Schar, 1998; Roberts, 2000). In international relations, revolutions, massive economic depression and economic shocks are rare events (King and Zeng, 2001). Methodology for modelling occurrence of a rare event is well established. The Poisson distribution is generally used to model the frequency of rare events (see Falk, et al., 2010). In Generalized Linear Models (GLMs) literature the log-linear model is commonly used to model independent Poisson counts (McCullagh and Nelder, 1989).

We consider binary rare events data, i.e. binary dependent variables with a very small number of ones. In GLM literature (see McCullagh and Nelder, 1989; Dobson and Barnett, 2008) several models for binary response variable have been proposed by considering different link functions: logit, probit, log-log and complementary log-log models. However, the most used model for binary variables is the logistic regression. The logistic regression shows some important drawbacks in rare events studies: the probability of rare event is underestimated and the logit link is a symmetric function, so the response curve approaches zero as the same rate it approaches one. Moreover, commonly used data collection strategies are inefficient for rare event data (King Zeng, 2001). The bias of the maximum likelihood estimators of logistic regression parameters in small sample sizes, that has been well analysed in literature (McCullagh and Nelder, 1989; Mansky and Lerman 1977; Hsieh Mansky and McFadden, 1985), is amplified in the rare event study. Most of these problems are relatively unexplored by literature (King and Zeng, 2001).

The main aim of this paper is to overcome the drawbacks of the logistic regression in rare events studies by proposing a new model for binary dependent data with an asymmetric link function given by the quantile function of the Generalized Extreme Value (GEV) random variable. In the extreme value theory, the GEV distribution is used to model the tail of a distribution (Kotz Nadarajah, 2000; Coles, 2004). Since we focus our attention on the tail of the response curve for the values close to 1, we have chosen the GEV distribution. In GLMs (Agresti, 2002) the log-log and the complementary log-log link functions are used since they are asymmetric functions. In particular, the log-log link function is the quantile function of the Gumbel random variable. The inverse function of the complementary log-log is one minus the cumulative distribution function of the Gumbel random variable.

The present paper is organized as follows. The next section explains the main drawbacks of the logistic regression model for rare events data. In Section 3 the GEV model for binary rare events data is proposed. The subsection 3.1 presents the Weibull regression model as a particular case of the GEV model. Finally, in Section 4 we apply our proposal to empirical data to estimate the default probability in credit risk analysis. In particular, the first subsection describes the dataset of Italian Small and Medium Enterprises (SMEs) and the second subsection shows the estimation results by applying the GEV model to these data. In the following subsection, the predictive accuracies of the logistic regression model and the GEV model are compared for different sample percentages of rare events. Finally, the last section is

devoted to conclusions. In appendix, we report the score functions and the Fisher information matrix of the parameters of the GEV model.

2 The main drawbacks of the logistic regression for rare events data

Let $Y_1, Y_2, \dots, Y_i, \dots, Y_n$ be Bernoulli independent random variables that are equal to one with probability π_i and zero with probability $(1 - \pi_i)$ for $i = 1, 2, \dots, n$. A Generalized Linear Model (GLM) considers a monotonic and twice differentiable function $g(\cdot)$, called *link function*, and a covariate vector \mathbf{x}_i such that

$$g(\pi) = \beta' \mathbf{x}_i.$$

By applying the inverse function of $g(\cdot)$, it results that

$$\pi_i = g^{-1}(\beta' \mathbf{x}_i).$$

In the logistic regression model the probability π_i is a logistic cumulative distribution function

$$\pi(\mathbf{x}_i) = \frac{\exp(\beta' \mathbf{x}_i)}{1 + \exp(\beta' \mathbf{x}_i)} \quad (1)$$

with

$$\beta' = [\beta_0, \beta_1, \dots, \beta_k] \quad \mathbf{x}' = [1, x_1, \dots, x_k].$$

By applying the inverse function to the equation (1), the logistic link function is the transformation used for the linearity

$$\text{logit}(\pi(\mathbf{x}_i)) = \ln \left(\frac{\pi(\mathbf{x}_i)}{1 - \pi(\mathbf{x}_i)} \right) = \beta' \mathbf{x}_i$$

The maximum likelihood method is usually used to estimate the parameters vector β .

The logistic regression shows important drawbacks when we study rare events data. Firstly, when the dependent variable represents a rare event, the logistic regression could underestimate the probability of occurrence of the rare event. Secondly, commonly used data collection strategies are inefficient for rare event data (King and Zeng, 2001). In order to overcome this drawback the choice-based or endogenous stratified sampling (case-control design) is used. The strategy is to select on Y by collecting observations for which $Y = 1$ and a random selection of observations for which $Y = 0$. This sampling method is usually supplemented with a prior correction of the bias of MLE estimators. An alternative procedure is the weighting the data to compensate the differences in the sample and population fractions of ones induced by choice-based sampling by the weighted exogenous sampling maximum estimator. Manski and Lerman, (1977), McCullagh and Nelder (1989) show an analytical

approximation for the bias in the MLE estimates to account for finite sample. This bias is amplified in application with rare events. Thirdly, the logit link is symmetric about 0.5

$$\text{logit}(\pi(\mathbf{x}_i)) = \ln\left(\frac{\pi(\mathbf{x}_i)}{1 - \pi(\mathbf{x}_i)}\right) = -\text{logit}(\pi(\mathbf{x}_i)) = -\ln\left(\frac{\pi(\mathbf{x}_i)}{1 - \pi(\mathbf{x}_i)}\right).$$

This means that the response curve for $\pi(\mathbf{x}_i)$ approaches zero at the same rate it approaches one. If the dependent variable represents a rare event, a symmetric link function is not appropriated. Since a counting rare event is usually modelled by a Poisson distribution, which has positive skewness, it is coherent to choose an asymmetric link function in order to obtain a response curve that approaches zero at a different rate it approaches one.

In rare events data values one of the dependent variable are more informative than zero, this follows by the variance matrix

$$V(\hat{\beta}) = \left[\sum_{i=1}^n \pi_i(1 - \pi_i) \mathbf{x}'_i \mathbf{x}_i \right]^{-1}.$$

The part of this matrix affected by rare events is the factor $\pi_i(1 - \pi_i)$. Most rare events applications yield small estimates of $P\{Y_i = 1 | \mathbf{x}_i\} = \pi_i$ for all observations. However, if the logit model has some explanatory power, the estimate of π_i among observations for which rare events are observed (i.e. for which $Y_i = 1$) will usually be larger, and closer to 0.5, because probabilities in rare events studies are normally very small, than among observations for which $Y_i = 0$. The result is that $\pi_i(1 - \pi_i)$ will usually be larger for ones than zeros and so the variance will be smaller. In this situation, additional ones will cause the variance to drop more and hence are more informative than additional zeros.

For this reason, in this paper, we focus our attention on the tail of the response curve for the values closed to one.

3 The Generalized Extreme Value (GEV) regression model

Extreme value theory is a robust framework to analyse the tail behaviour of distributions. Extreme value theory has been applied extensively in hydrology, climatology and also in the insurance industry (Embrechts et al., 1997). Embrechts (1999, 2000) considers the potential and limitations of extreme value theory for risk management. Without being exhaustive here, De Haan et al. (1994) and Danielsson and de Vries (1997) study quantile estimation. Bali (2003) uses the GEV distribution to model the empirical distribution of returns. Mc Neil (1999) and Dowd (2002) give an extensive overview of extreme value theory for risk management.

Unlike the normal distribution that arises from the use of the central limit theorem on sample average, the extreme value distribution arises from the limit theorem of

Fisher and Tippet (1928) on extreme values or maxima in sample data. The class of GEV distributions is very flexible with the tail shape parameter τ controlling the shape and size of the tails of the three different families of distributions subsumed under it. The three families of extreme value distributions can be nested into a single parametric representation, as shown by Jenkinson (1955) and von Mises (1936). This representation is known as the Generalized Extreme Value (GEV) distribution and its cumulative distribution function is given by

$$F_X(x) = \exp \left\{ - \left[1 + \tau \left(\frac{x - \mu}{\sigma} \right) \right]^{-\frac{1}{\tau}} \right\} \quad -\infty < \tau < \infty, \quad -\infty < \mu < +\infty \quad \sigma > 0 \quad (2)$$

defined on $S_X = \{x : 1 + \tau(x - \mu)/\sigma > 0\}$. The parameter τ is a shape parameter, while μ and $\sigma (> 0)$ are location and scale parameters respectively.

The Type II (Fréchet-type distribution) and the Type III (Weibull-type distribution) classes of the extreme value distribution correspond respectively to the case $\tau > 0$ and $\tau < 0$, while the Type I class (Gumbel-type distribution) arises in the limit as $\tau \rightarrow 0$. The corresponding distributions of $(-X)$ are also called extreme value distributions. We underline that Fréchet and Weibull distributions are related by a change of sign.

In this paper we propose a generalization of the log-log model by using the quantile function of the GEV distribution as link function. For this reason we call this proposal Generalized Extreme Value (GEV) regression model.

For a binary response variable Y_i and the vector of explanatory variables \mathbf{x}_i , let $\pi(\mathbf{x}_i) = P\{Y_i = 1 | \mathbf{x}_i\}$. Since we consider the class of GLMs, we suggest the GEV cumulative distribution function as the response curve

$$\pi(\mathbf{x}_i) = \exp\{-[1 + \tau(\beta' \mathbf{x}_i)]^{-1/\tau}\}. \quad (3)$$

with

$$\beta' = [\beta_0, \beta_1, \dots, \beta_k] \quad \mathbf{x}' = [1, x_1, \dots, x_k].$$

For $\tau \rightarrow 0$ the previous model (3) becomes the response curve of the log-log model and for $\tau > 0$ it becomes the Weibull response curve, a particular case of the GEV one.

The link function of the GEV model is given by

$$\frac{[-\ln \pi(x)]^{-\tau} - 1}{\tau} = \beta' \mathbf{x} \quad (4)$$

that represents a noncanonical link function.

For the interpretation of the parameters β and τ , we suppose that the value of the j -th regressor (with $j = 1, 2, \dots, k$) is increased by one unit and all the other independent variables remain unchanged. Let \mathbf{x}^* the new covariate values, whereas \mathbf{x} denotes the original covariate values. From the equation (4) we deduce that $\beta_j = g(\pi(\mathbf{x}^*)) - g(\pi(\mathbf{x}))$ with $j = 1, 2, \dots, k$. This means that if the parameter β_j (with $j = 1, 2, \dots, k$) is positive and all the other parameters are fixed, by increasing the j -th regressor

the estimate $\pi(\mathbf{x})$ decreases. Otherwise, if β_j is negative, by increasing the j -th regressor the estimate $\pi(\mathbf{x})$ of the GEV model also increases.

Moreover, we analyse the parameter β_0 : for all fixed values of τ and for a null independent variable, β_0 have a positive monotonic relationship with the estimate of $\pi(\mathbf{x})$. Finally, we analyse the influence of the τ parameter on $\pi(\mathbf{x})$. We find that for $\beta_0 = 0$ and by considering null values for all the covariates, from the GEV model we obtain an estimate $\pi(\mathbf{x})$ that is about equal to e^{-1} for all the values of τ . This means that $\pi(\mathbf{x})$ variations depend on the covariate variations and not on τ variations.

We propose to estimate the parameters of these models by the maximum likelihood method. Let $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$ a simple random sample of size n from Y , the log-likelihood function is

$$l(\beta, \tau) = \sum_{i=1}^n \left\{ -y_i [1 + \tau(\beta' \mathbf{x}_i)]^{-1/\tau} + (1 - y_i) \ln[1 - \exp\{-[1 + \tau(\beta' \mathbf{x}_i)]^{-1/\tau}\}] \right\}. \quad (5)$$

Some simulation studies are developed to verify the existence of maximum of the likelihood function, considered as a function of only one parameter for fixed values of the other parameters (likelihood profile function).

The score functions, obtained by differentiating the log-likelihood function with respect to the known parameters β and τ (see Appendix) are give by

$$\frac{\partial l(\beta, \tau)}{\partial \beta_j} = - \sum_{i=1}^n x_{ij} \frac{\ln[\pi(\mathbf{x}_i)]}{1 + \tau \beta' \mathbf{x}_i} \frac{y_i - \pi(\mathbf{x}_i)}{1 - \pi(\mathbf{x}_i)} \quad j = 0, 1, \dots, k, \quad (6)$$

$$\frac{\partial l(\beta, \tau)}{\partial \tau} = \sum_{i=1}^n \left[\frac{1}{\tau^2} \ln(1 + \tau \beta' \mathbf{x}_i) - \frac{\beta' \mathbf{x}_i}{\tau(1 + \tau \beta' \mathbf{x}_i)} \right] \frac{y_i - \pi(\mathbf{x}_i)}{1 - \pi(\mathbf{x}_i)} \ln[\pi(\mathbf{x}_i)]. \quad (7)$$

Since the inverse of the link function (3) is a cumulative distribution function only for the values $\{\mathbf{x}_i : 1 + \tau \mathbf{x}_i > 0\}$, in order to identify the maximum likelihood estimates, we apply a constrained optimization with $\{\mathbf{x}_i : 1 + \tau \beta' \mathbf{x}_i > 0\}$. The asymptotic standard errors of the maximum likelihood estimators of the parameters in the models are given by the Fisher's information matrix (see Appendix). Since the Fisher information matrix is not a diagonal matrix (see Appendix), the maximum likelihood estimators of the parameters β and τ are dependent and they cannot be computed separately.

Since the score functions do not have closed-form, the maximum likelihood estimators need to be obtained by numerically maximizing the log-likelihood function using a constrained nonlinear optimization algorithm. The optimization algorithms require the specification of initial values to be used in iterative scheme.

Our suggestion is to use as initial point estimate for τ a value closed to zero. For this value the GEV model becomes the log-log model. Hence, in order to obtain the initial point estimate for β , we analyse the log-log or Gumbell regression model (see Agresti, 2002) with the response curve

$$\pi(\mathbf{x}_i) = \exp(-\exp(\beta' \mathbf{x}_i)). \quad (8)$$

We compute the log-likelihood function of the Gumbel regression

$$\begin{aligned}
 l(\beta) &= \sum_{i=1}^n \{y_i \ln[\pi(\mathbf{x}_i)] + (1 - y_i) \ln[1 - \pi(\mathbf{x}_i)]\} \\
 &= \sum_{i=1}^n \{y_i \ln[\exp[-\exp(\beta' \mathbf{x}_i)]] + (1 - y_i) \ln[1 - \exp[-\exp(\beta' \mathbf{x}_i)]]\} \\
 &= \sum_{i=1}^n \{y_i [-\exp(\beta' \mathbf{x}_i)] + (1 - y_i) \ln[1 - \exp[-\exp(\beta' \mathbf{x}_i)]]\}.
 \end{aligned} \tag{9}$$

The score functions are given by

$$\frac{\partial l(\beta)}{\partial \beta_j} = \sum_{i=1}^n x_{ij} \ln[\pi(\mathbf{x}_i)] \frac{y_i - \pi(\mathbf{x}_i)}{1 - \pi(\mathbf{x}_i)} \quad j = 0, 1, \dots, k. \tag{10}$$

To identify the initial values for β , we choose $\beta_j^* = 0$ for $j = 1, \dots, k$. By substituting $\beta_j^* = 0$ for $j = 1, \dots, k$ in equation (9) we obtain

$$\beta_0^* = \ln[-\ln(\bar{y})].$$

We use the initial values proposed for the log-log model in order to identify the initial values β^* for the GEV regression model. In particular, we propose to use $\tau^* \simeq 0$, $\beta_j^* = 0$ for $j = 1, \dots, k$ and $\beta_0^* = \ln[-\ln(\bar{y})]$. Afterwards, by substituting the initial values for the parameter β in the equation (6) for $j = 0$ we obtain the estimate of τ for the first step of the iterative procedure. By using such estimate of τ in the equation (6), we obtain the estimates of β_j with $j = 0, 1, \dots, k$ for the first step in the GEV regression.

3.1 Weibull regression for binary data

A particular case of the GEV cumulative distribution function (2) for $\tau > 0$ is the Weibull cumulative distribution function

$$F(x) = \exp \left\{ - \left[- \frac{x - \mu}{\sigma} \right]^k \right\} \quad x > \mu \quad -\infty < \mu < +\infty \quad \sigma > 0 \quad k > 0, \tag{11}$$

where μ and $\sigma (> 0)$ are, respectively, a location and a scale parameters and k is a shape parameter.

By considering the Weibull cumulative distribution function (11) in the GLM for binary dependent variable, the response curve of the Weibull regression model is

$$\pi(\mathbf{x}_i) = \exp[-(\beta' \mathbf{x}_i)^k], \tag{12}$$

where $k > 0$. The response curve of the Weibull regression model (12) is a particular case of the GEV response curve (3) for $\tau > 0$. On the one hand, the Weibull response curve is an asymmetric function, analogously to the response curve (8) of the Gumbel regression model. On the other hand, unlike the Gumbel response curve (8), the $\pi(\mathbf{x}_i)$ in the Weibull model (12) approaches 1 sharply and approaches 0 slowly. In particular, the behaviour of Weibull response curve depends on k : if k increases $\pi(\mathbf{x}_i)$ approaches sharper both 0 and 1. If the value of the j -th regressor (with $j = 1, 2, \dots, k$) is increased and all the other independent variables remain unchanged, the Weibull response curve (12) decreases when $\beta_j > 0$ and increases when $\beta_j < 0$. The link function of the Weibull regression model

$$\left[\ln \frac{1}{\pi(\mathbf{x}_i)} \right]^{1/k} = \beta' \mathbf{x}_i. \quad (13)$$

is a noncanonical link function.

We compute the log-likelihood function of the Weibull regression

$$\begin{aligned} l(\beta, k) &= \sum_{i=1}^n \{y_i \ln[\pi(\mathbf{x}_i)] + (1 - y_i) \ln[1 - \pi(\mathbf{x}_i)]\} \\ &= \sum_{i=1}^n \{-y_i (\beta' \mathbf{x}_i)^k + (1 - y_i) \ln[1 - \exp(-(\beta' \mathbf{x}_i)^k)]\}. \end{aligned} \quad (14)$$

The score functions are given by

$$\frac{\partial l(\beta, k)}{\partial \beta_j} = -k \sum_{i=1}^n x_{ij} \frac{\ln[\pi(\mathbf{x}_i)]}{\beta' \mathbf{x}_i} \frac{y_i - \pi(\mathbf{x}_i)}{1 - \pi(\mathbf{x}_i)} \quad j = 0, 1, \dots, k, \quad (15)$$

$$\frac{\partial l(\beta, k, \mathbf{y})}{\partial k} = -k \sum_{i=1}^n \ln[\pi(\mathbf{x}_i)] \ln[\beta' \mathbf{x}_i] \frac{y_i - \pi(\mathbf{x}_i)}{1 - \pi(\mathbf{x}_i)}. \quad (16)$$

In order to apply an iterative algorithm, we need to identify the initial values β^* and k^* for the parameters. Our suggestion is to use $k^* = 1$, $\beta_j^* = 0$ for $j = 1, \dots, k$ and

$$\beta_0^* = \ln \left[1 - \frac{n}{\bar{y}} \right]. \quad (17)$$

We obtain the initial value (17) by substituting $\beta_j^* = 0$ for $j = 1, \dots, k$ and $k^* = 1$ in (15) for $j = 0$. We highlight that the Weibull regression with $k = 1$ is a log-linear model whose response curve is the cumulative distribution function of an exponential random variable (McCullagh and Nelder, 1989).

4 An Application to Credit Default

Credit risk forecasting is one of the leading topics in modern finance, as the bank regulation has made increasing use of external and internal credit ratings (Basel Committee on Banking Supervision, 2005). Statistical credit scoring models try to predict the probability that a loan applicant or existing borrower will default over a given time-horizon, usually of one year. According to the Basel Committee on Banking Supervision (2004), banks are required to measure the one year default probability for the calculation of the equity exposure of loans. In this framework, banks adopting the Internal-Rating-Based (IRB) approach are allowed to use their own estimates of PDs. Moreover, Basel II requires these banks to build a rating systems where the PD is the main input. For that reason, in many credit risk models such as CreditMetrics (Gupton et al., 1997), CreditRisk+ (Credit Suisse Financial Products, 1997) or CreditPortfolioView (Wilson, 1998), default probabilities are essential input parameters.

Altman (1968) was the first to use a statistical model to predict default probabilities of firms, calculating his well known Z-Score using a standard discriminant model. Almost a decade later Altman et al. (1977) modified the Z-Score by extending the data-set to larger-sized and distressed firms. Besides this basic method, more accurate ones such as logistic regression, neural networks, smoothing nonparametric methods and expert systems have been developed and are now widely used for practical and theoretical purposes in the field of credit risk measurement (Hand and Henley 1997a, b).

SMEs play a very important role in the economic system of many countries and particularly in Italy (about 90% of Italian firms are SMEs (Vozzella, Gabbi 2010). Furthermore, a large part of the literature (Altman, Sabato 2006; Ansell and al. 2009; Ciampi, Gordini 2008; Vozzella, Gabbi 2010) has focused on the special character of small business lending and the importance of relationships banking for solving information asymmetries. The informative asymmetries puzzle affects particularly SMEs for their difficulty to estimate and make known their fair value. Therefore, the lending to SMEs is riskier than to large corporates (Altman, Sabato 2006; Dietsch, Petey 2004; Saurina, Trucharte 2004). As a consequence, Basel II (BCBS, 2004) establishes that banks should develop credit risk models specifically addressed to SMEs. Only a few studies consider SMEs (Andreeva et al., 2011; Altman and Sabato, 2007; Altman et al. 2010; Hu and Ansell, 2007) since the gathering of SMEs data is quite difficult. Discriminant analysis and logistic regression have been the most widely used methods for constructing scoring systems for SMEs (e.g. Hand and Henley, 1997a, b; Hand and Niall, 2000).

In this paper we propose the GEV regression model in order to overcome the drawbacks of the logistic regression for rare events. Since defaults in credit risk analysis are rare events, we apply the GEV model to empirical data on Italian SMEs to model the default probability. Compliant to Basel II, the default probability is one year forecasted. Therefore, let Y_t be a binary r.v. such that

$$Y_t = \begin{cases} 1, & \text{if a firm is default at time } t; \\ 0, & \text{otherwise.} \end{cases}$$

and let \mathbf{x}_{t-1} be the covariate vector at time $t - 1$. In this application we aim at estimating the conditional probability of default

$$\pi(\mathbf{x}_{t-1}) = P(Y_t = 1 | \mathbf{x}_{t-1})$$

by applying and comparing the GEV and the logistic regression models.

4.1 The data set

Data used in our analysis comes from AIDA-Bureau van Dijk, a large Italian financial and balance sheets information provider. We consider defaulted and non defaulted SMEs over the years 2005 – 2009. In particular, since the default probability is one year forecasted, the covariates concern the period of time 2004 – 2008. The database contains accounting data of around 210,000 Italian firms with total asset below 10 millions euro (Vozzella and Gabbi, 2010). From the sample we exclude the firms without the necessary information on the covariates.

Often default definitions for credit risk models concern single loan defaults of a company versus a bank, as also emerges from the Basel II instructions. This is the case for banks building models based on their portfolio data, that is relying on single loans data which are reserved (e.g., Altman and Sabato (2005) develop a logit model for Italian SMEs based on the portfolio of a large Italian bank). However, traditional structural models (i.e. Merton, 1974) refer to a firm-based definition of default: a firm defaults when the value of the assets is lower than the value of the liabilities, that is when equity is negative. In this work default is intended as the end of the firms activity, i.e. the status, where the firm needs to liquidate its assets for the benefit of its creditors. In practice, we consider a default occurred when a specific firm enters a bankruptcy procedure as defined by the Italian law. The reason for this choice lies in the data availability.

In according with Altman and Sabato (2006) on this dataset we apply a choice-based or endogenous stratified sampling. In this sampling scheme data are stratified by the values of the response variable. We draw randomly the observations within each stratum defined by the two categories of the dependent variable (1=default, 0=non-default) and we consider all the defaulted firms. Then, we select a random sample of non-defaulted firms over the same year of defaults in order to obtain a percentage of defaults in our sample as close as possible to the default percentage (5 %) for Italian SMEs (Cerved Group, 2011). In order to analyze the properties of our model for different probabilities of the rare event $P\{Y = 1\}$, we consider also a default percentage of 1%.

By applying the choice-based sampling, the observations are dependent. Since the sample sizes of this application are high, according to the superpopulation theory

(Prentice, 1986) we can consider all the examined samples as simple random samples.

4.2 The estimation results

We apply the GEV regression model proposed in this work to the AIDA database. This application is interesting since it concerns SMEs, on which the availability of data is very difficult, in the Italian credit market, which could be different from other countries.

In order to model the default event, we choose the independent variables that represent financial and economic characteristics of firms according to the recent literature (Vozzella and Gabbi, 2010; Ciampi and Gordini, 2008; Altman et al., 2006). These covariates cover the most relevant aspects of firm's operations: leverage, liquidity and profitability.

Firstly, we consider 16 covariates: liquidity ratio, current ratio, leverage, solvency ratio, debt/EBITDA, return on equity, return on investment, turnover per employee, added value per employee, cash flow, banks/turnover, debt/equity ratio, return on solvency, EBITDA/turnover, total personnel costs/added value, cash flow/turnover. Secondly, we examine the multicollinearity and we remove the variables with a Variance Inflation Factor higher than 5 (Greene, 2000, p.257-258). Thirdly, by applying the GEV model 7 variables are significant at the level of 5% for the PD forecast:

- *Solvency ratio*: the ratio of a company's income over the firm's total debt obligations;
- *Return on investment*: the ratio of the returns of a company's investments over the costs of the investment;
- *Turnover per employee*: the ratio of sales divided by the number of employees;
- *Added value per employee*: the enhancement added to a product or service by a company divided by the number of employees;
- *Cash flow*: the amount of cash generated and used by a company in a given period;
- *Bank loans over turnover*: short and long term debts with banks over sales volume net of all discounts and sales taxes;
- *Total personnel costs over added value*: the ratio of a company's labour costs divided by the enhancement added to a product or service by a company.

In order to avoid the overfitting, data are randomly divided into two parts: a sample on which the regression models are estimated and a control sample on which we evaluate the predictive accuracy of the models. The Table 1 reports the parameter estimates obtaining by applying the GEV model to the sample of 1485 defaulters and 29700 non-defaulters over the years 2005 – 2008.

In section 3 we explain the interpretation of the parameters of the GEV model. According to these interpretations we can analyse the influence of each variable in Table 1 on the PD estimate.

<i>Parameter</i>	<i>Estimate</i>
τ	$1.1173e^{-1}$
Intercept	1.1139
Solvency ratio	$8.1916e^{-4}$
Return on Equity	$7.5574e^{-4}$
Turnover per employee	$-1.7804e^{-3}$
Added value per employee	$1.8200e^{-4}$
Cash flow	$-7.2060e^{-7}$
Bank loans over turnover	$3.8413e^{-4}$
Total personnel costs divided added value	$5.8071e^{-4}$

Table 1 Parameter estimates using the sample of Italian SMEs (1485 defaulters and 29700 non-defaulters) over the years 2005 – 2008.

At first, Ansell et al. (2009) explain that the solvency ratio should have an inverse relationship with the PD estimate, coherently with our result but in contrast with Ansell et al.'s (2009) result. The return on equity and the added value per employee show the same kind of relationship with the PD, the first result coincides but the second one is in contrast with Ciampi and Gordini (2008). We highlight that our result for the added value per employee coincides with the expectations.

On the contrary, the turnover per employee and the cash flow show a direct relationship with PD, coherently with Altman and Sabato (2006), Ciampi and Gordini (2008). The last two results in Table 1 are in contrast with the expectations: bank loans divided turnover and total personnel costs divided added value show an inverse relationship with the PD estimate. For this reason we analyse the results obtained in literature for these two variables. Ciampi and Gordini (2008) obtain a direct relationship of bank loans divided turnover with the PD estimate. Alternatively, Altman and Sabato (2006) consider the short term debt over equity book value to model the PD and they show that this variable has an inverse relationship with the PD estimate, analogously to our result. On the contrary, Fantazzini and Figini (2009) show that the short term debt has a direct influence on PD, coherently with the expectation. Ciampi and Gordini (2008) consider also total personnel costs divided added value in their regression model but their analysis shows a result coherent with the expectations and in contrast with the one showed in Table 1. Fantazzini and Figini (2009) consider also labour costs in their model. In particular, they introduce the personnel expenses over sales in the regression model and this variable shows a direct influence with the PD estimate.

4.3 Forecasting accuracy

We compare the predictive accuracy of the GEV regression model here proposed with the one of logistic regression model. The predictive accuracy of these models is assessed using two performance measures: the Mean Square Error (MSE) and the Mean Absolute Error (MAE), defined as

$$MSE = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2 \quad MAE = \frac{1}{n} \sum_{i=1}^n |y_i - \hat{y}_i|. \quad (18)$$

where y_i and \hat{y}_i are the actual and the predicted dependent variable on loan i , respectively. Models with lower MSE and MAE forecast the dependent variable more accurately.

The identification of defaulters is a pivotal aim for bank internal models. The reason is that it is much more costly to classify a SME as non-defaulter when it is a defaulter than to classify a SME as defaulter when it is a defaulter. In particular, when a defaulted firm is classified as non-defaulter by scoring model, banks give it a loan. If the borrower becomes defaulter, the bank may lose the whole or a part of the credit exposure. On the contrary, when a non-defaulter is classified as defaulter, bank loses only interest on loans.

For this reason, we focus our attention on the tail of the response curve for values of the dependent variable equal to one that represent defaults. Since for banks the underestimation of the PD could be very risky, the main aim of this subsection is to show that the GEV model overcomes the drawback of the logistic regression in the underestimation of rare events. For all these reasons we compare the two models by computing MAE and MSE only for the defaulters. This means that in the equations (18) we consider only the positive errors $y_i - \hat{y}_i > 0$ and n is the number of defaulters. We denote these errors by MAE^+ and MSE^+ , respectively.

Since the developed models may overfit the data, resulting in over-optimistic estimates of the predictive accuracy, the MSE and the MAE must be assessed on a sample which is different from that used in estimating the model parameters. We choose a control sample size of 10% of the sample size used in estimating the model parameters. In particular, models are validated by using out-of-sample and out-of-time tests. In the first approach the model is fitted using data from a sample (*Sample 1*) and tested on a different sample (*Out-of-sample* sample). On the contrary, in the out-of-time test the model is fitted using data from one time period (*Sample 2*) and tested on a subsequent period (*Out-of-time* sample). In particular in the second approach we estimate the model by using defaulters and non defaulters over the years 2005 – 2008 and we test the model by using the observations from 2009. The out-of-sample and out-of-time sample sizes are reported in Table 2.

We compare the sample and the control sample predictive accuracy of our proposal

	<i>Sample 1</i>		<i>Out-of-sample</i>		<i>Sample 2</i>		<i>Out-of-time</i>	
	1%	5%	1%	5%	1%	5%	1%	5%
<i>Defaulters</i>	1,485	1,485	165	165	1,650	1,650	64	64
<i>Non-defaulters</i>	133,650	29,700	14,850	148,500	33,000	3,300	5,760	1,280

Table 2 The out-of-sample and out-of-time samples.

with the logistic regression model on AIDA data for different sample percentages of the rare event $\{Y = 1\}$. As above-mentioned, we consider both the probabilities of the rare event 0.05 and 0.01.

Table 3 reports the MAE and the MSE for each model and for each sample frequency of the rare event on the sample (denoted by the subscript “c”) and on the out-of-sample (denoted by the subscript “cs”). In order to evaluate the weights of the errors MSE^+ and MAE^+ on the respective (total) errors MSE and MAE, we compute the ratios (MSE^+/MSE) and (MAE^+/MAE) weighted by sample relative frequencies of the rare events, equal to 0.05 and 0.01. We report their values between round brackets in Table 3.

From the results reported in Table 3, our proposal exhibits both the MAE and the MSE lower than the respective errors of the logistic regression model for both the sample and control sample and for both the sample percentages of rare events. By comparing the percentages in round brackets, we deduce that for the logistic regression model the weights of positive errors are relevant. On the contrary, for our proposal the weights of these positive errors $y_i - \hat{y}_i > 0$ are negligible. This means that the GEV model overcomes the drawback of the logistic regression in the underestimation of rare events.

Since the errors for the sample and the control sample are similar, the covariates are significant for the default discrimination of both the regression models. This means that both the models are well-explained. Moreover, by comparing the errors for different sample percentages of rare events, our model improves its accuracy by reducing the occurrence probability of the rare event. On the contrary, the logistic regression model shows worse performance. Since the main aim for banks is the

Sample percentage of $\{Y = 1\}$	Error	Models	
		GEV regression	Logistic regression
5%	MAE_s^+	0.4248(4.06%)	0.8829(52.49%)
	MSE_s^+	0.2080(3.40%)	0.8171(97.50%)
	MAE_{cs}^+	0.4171 (3.99%)	0.8702 (51.86%)
	MSE_{cs}^+	0.1967 (3.22%)	0.8067 (97.43%)
1%	MAE_s^+	0.3331(0.55%)	0.9502(45.46%)
	MSE_s^+	0.1399 (0.34%)	0.9270(89.13%)
	MAE_{cs}^+	0.3234 (0.53%)	0.9320(45.46%)
	MSE_{cs}^+	0.1301 (0.32%)	0.9084(89.05%)

Table 3 Forecasting accuracy measures of different models over different sample percentage of rare event $\{Y = 1\}$ on the sample (denoted by the subscript “c”) and the out-of-sample (denoted by the subscript “cs”).

forecasting of default probability, the two models are validated on a subsequent period. This means that the two models are fitted on data referring the period of time 2004-2008 and the out-of-time predictive accuracy is evaluated on the sample referring to 2009. The sample sizes used in the out-of-time test are reported in Table 2. We compare the predictive accuracy of our model with the logistic one by computing the MAE^+ and MSE^+ on the sample (denoted by the subscript “c”) and out-of-time sample (denoted by the subscript “cs”). The results in Table 4 show both the MAE^+ and MSE^+ lower than the respective errors of the logistic model. The results obtained from the out-of-time test are coherent with the ones of the out-of-sample

test: in Table 4 the percentages in round brackets exhibit relevant weights of the positive errors $y_i - \hat{y}_i > 0$ for the logistic regression and irrelevant positive errors for our model. Moreover, by comparing the errors for different sample percentages of rare events the logistic model shows a worst performance for lower sample percentage of rare events: the values of the MAE^+ and MSE^+ increase by reducing the percentage of rare events. On the contrary, the GEV model improves its accuracy. From these results the GEV model can be considered a suitable regression model for rare events.

Finally, in order to analyse the robustness of the GEV model we estimate the coef-

Sample percentage of $\{Y = 1\}$	Error	Models	
		GEV regression	Logistic regression
5%	MAE_s^+	0.4282(4.14%)	0.8815(52.53%)
	MSE_s^+	0.2103(3.50%)	0.8161(97.61)
	MAE_{cs}^+	0.4375 (3.96%)	0.9100 (48.35%)
	MSE_{cs}^+	0.2119 (3.19%)	0.8489 (94.74%)
1%	MAE_s^+	0.3509(0.58%)	0.9478(45.53%)
	MSE_s^+	0.1541 (0.39%)	0.9246(88.90%)
	MAE_{cs}^+	0.3393 (0.55%)	0.9797(47.56%)
	MSE_{cs}^+	0.1369 (0.33%)	0.9601(88.90%)

Table 4 Forecasting accuracy measures of different models over different sample percentage of rare event $\{Y = 1\}$ on the sample (denoted by the subscript “c”) and the out-of-time (denoted by the subscript “cs”).

ficients of both the regression models on a sample with a given sample percentage of rare events and we evaluate the accuracy on a sample with a different sample percentage of defaulters. By computing the MAE and the MSE on all the SMEs, the errors of our model do not change in comparison with the respective errors above-mentioned on a sample with the same percentage of defaulters¹. This means that our model is robust for different sample percentages of defaulters. On the contrary, for the logistic regression models the same errors are significantly different.

Conclusions remarks

In this work we aim at proposing a new GLM regression model with a flexible asymmetric link function for binary response data in rare event studies. At first, we analyse the main drawbacks of the logistic regression model for rare events data. As well known, the GEV distribution is a suitable function for modelling extreme values and rare event data. For this reason we propose the quantile function of the GEV distribution as link function. The GEV model depends on the regression parameters and on the shape parameter of the GEV distribution. Since the score functions do

¹ Form these results the GEV model can be considered a suitable regression model for rare events.

not have closed-form, we obtain the maximum likelihood estimators by maximizing the log-likelihood function using an iterative algorithm. We specify initial values of parameters for this iterative algorithm and the Fisher information matrix. The main advantage of the GEV model is its excellent performance to identify the rare events. Thanks to this characteristic, the drawback with the logistic regression model in underestimating the probability of rare events is overcome.

In order to evaluate the performance of our methodological proposal we apply the GEV model to empirical data on Italian Small Medium Enterprises (SMEs). Since defaults are events, we model the default probability for Italian SMEs over the years 2005-2009 by considering financial and economic covariates of SMEs. We test this model by comparing the predictive accuracy of our model with the logistic one. The application shows the substantial underestimation of the default probability by applying the logistic regression model. By reducing the sample frequencies of rare events (defaults), the predictive performance of the logistic regression model to identify the rare events becomes worse. On the contrary, the GEV model overcomes the underestimation problem and its accuracy to identify the rare events improves by reducing the sample percentage of rare events. Finally, we show that the GEV model is a robust model, unlike the logistic regression model.

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5 Appendix

In this appendix we obtain the score functions and the Fisher information matrix for β and τ of the GEV regression model. The notation used here is defined in the section 3. At first, in order to compute the score functions we consider the following equations

$$\frac{\partial l_i(\beta, \tau, y_i)}{\partial \beta_j} = \frac{\partial l_i(\pi(\mathbf{x}_i))}{\partial \pi(\mathbf{x}_i)} \frac{\partial \pi(\mathbf{x}_i)}{\partial \beta_j} \quad \frac{\partial l_i(\beta, \tau, y_i)}{\partial \tau} = \frac{\partial l_i(\pi(\mathbf{x}_i))}{\partial \pi(\mathbf{x}_i)} \frac{\partial \pi(\mathbf{x}_i)}{\partial \tau} \quad (19)$$

with $j = 1, 2, \dots, k$ and $i = 1, 2, \dots, n$. From equations (3) and (5) we obtain that

$$\begin{aligned} \frac{\partial l_i(\pi(\mathbf{x}_i))}{\partial \pi(\mathbf{x}_i)} &= \frac{y_i}{\pi(\mathbf{x}_i)} - \frac{1 - y_i}{1 - \pi(\mathbf{x}_i)} \\ \frac{\partial \pi(\mathbf{x}_i)}{\partial \beta_j} &= -x_{ij}(1 + \tau \beta' \mathbf{x}_i)^{-\left(\frac{1}{\tau} + 1\right)} \exp \left[-(1 + \tau \beta' \mathbf{x}_i)^{-\frac{1}{\tau}} \right] \\ \frac{\partial \pi(\mathbf{x}_i)}{\partial \tau} &= -(1 + \tau \beta' \mathbf{x}_i)^{-\frac{1}{\tau}} \left[\frac{1}{\tau^2} \ln(1 + \tau \beta' \mathbf{x}_i) - \frac{\beta' \mathbf{x}_i}{\tau(1 + \tau \beta' \mathbf{x}_i)} \right] \exp \left[-(1 + \tau \beta' \mathbf{x}_i)^{-\frac{1}{\tau}} \right] \end{aligned}$$

Substituting the former results in equations (19), the score functions (6) and (7) are obtained.

The second order partial derivatives of the log-likelihood function with respect to parameters (β, τ) are

$$\begin{aligned} \frac{\partial^2 l_i(\beta, \tau, y_i)}{\partial^2 \beta_j} &= \frac{\partial^2 l_i(\pi(\mathbf{x}_i))}{\partial^2 \pi(\mathbf{x}_i)} \left[\frac{\partial \pi(\mathbf{x}_i)}{\partial \beta_j} \right]^2 + \frac{\partial l_i(\pi(\mathbf{x}_i))}{\partial \pi(\mathbf{x}_i)} \frac{\partial^2 (\pi(\mathbf{x}_i))}{\partial^2 \beta_j} \\ \frac{\partial^2 l_i(\beta, \tau, y_i)}{\partial^2 \tau} &= \frac{\partial^2 l_i(\pi(\mathbf{x}_i))}{\partial^2 \pi(\mathbf{x}_i)} \left[\frac{\partial \pi(\mathbf{x}_i)}{\partial \tau} \right]^2 + \frac{\partial l_i(\pi(\mathbf{x}_i))}{\partial \pi(\mathbf{x}_i)} \frac{\partial^2 (\pi(\mathbf{x}_i))}{\partial^2 \tau} \\ \frac{\partial^2 l_i(\beta, \tau, y_i)}{\partial \beta_j \partial \beta_k} &= \frac{\partial^2 l_i(\pi(\mathbf{x}_i))}{\partial^2 \pi(\mathbf{x}_i)} \frac{\partial \pi(\mathbf{x}_i)}{\partial \beta_j} \frac{\partial \pi(\mathbf{x}_i)}{\partial \beta_k} + \frac{\partial l_i(\pi(\mathbf{x}_i))}{\partial \pi(\mathbf{x}_i)} \frac{\partial^2 (\pi(\mathbf{x}_i))}{\partial \beta_j \partial \beta_k} \end{aligned}$$

$$\frac{\partial^2 l_i(\beta, \tau, y_i)}{\partial \beta_j \tau} = \frac{\partial}{\partial \beta_j} \left[\frac{\partial l_i(\beta, \tau, y_i)}{\partial \tau} \right]$$

where

$$\begin{aligned} \frac{\partial^2 l_i(\pi(\mathbf{x}_i))}{\partial^2 \pi(\mathbf{x}_i)} &= -\frac{y_i}{[\pi(\mathbf{x}_i)]^2} - \frac{1-y_i}{[1-\pi(\mathbf{x}_i)]^2} \\ \frac{\partial^2(\pi(\mathbf{x}_i))}{\partial^2 \beta_j} &= x_{ij}^2 (1 + \tau \beta' \mathbf{x}_i)^{-\frac{1}{\tau}-1} \exp -(1 + \tau \beta' \mathbf{x}_i)^{-\frac{1}{\tau}} \left[\frac{\tau \beta' \mathbf{x}_i - \tau}{1 + \tau \beta' \mathbf{x}_i} \right] \\ \frac{\partial^2(\pi(\mathbf{x}_i))}{\partial^2 \tau} &= \pi(\mathbf{x}_i) [-\ln(\pi \mathbf{x}_i)] \left[\frac{1}{\tau^2} \ln(1 + \tau \beta' \mathbf{x}_i) - \frac{\beta' \mathbf{x}_i}{\tau(1 + \tau \beta' \mathbf{x}_i)} \right]^2 [-\ln(\pi \mathbf{x}_i) - 1] \\ &\quad - \left[-\frac{2}{\tau^3} \ln(1 + \tau \beta' \mathbf{x}_i) + \frac{2\beta' \mathbf{x}_i + \tau(\beta' \mathbf{x})^2}{\tau^2(1 + \tau \beta' \mathbf{x}_i)} \right] \\ \frac{\partial^2(\pi(\mathbf{x}_i))}{\partial \beta_j \partial \beta_k} &= x_{ij} x_{ik} (1 + \tau \beta' \mathbf{x})^{-\frac{1}{\tau}-1} \exp -(1 + \tau \beta' \mathbf{x}_i)^{-\frac{1}{\tau}} \left[\frac{\tau \beta' \mathbf{x}_i - \tau}{1 + \tau \beta' \mathbf{x}_i} \right] \end{aligned}$$

The Fisher information is the negative of the expectation of the second derivatives of the log-likelihood with respect to the parameters β and τ

$$\begin{aligned} -E \left(\frac{\partial^2 l_i(\beta, \tau, y_i)}{\partial \beta_j \partial \beta_k} \right) &= -\frac{\partial \pi(\mathbf{x}_i)}{\partial \beta_j} \frac{\partial \pi(\mathbf{x}_i)}{\partial \beta_k} \\ -E \left(\frac{\partial^2 l_i(\beta, \tau, y_i)}{\partial^2 \pi(\mathbf{x}_i)} \right) &= -\frac{\partial \pi(\mathbf{x}_i)}{\partial \beta_j} \frac{\partial \pi(\mathbf{x}_i)}{\partial \beta_k} \frac{1}{\pi(\mathbf{x}_i)[1-\pi(\mathbf{x}_i)]} \\ -E \left(\frac{\partial^2 l_i(\beta, \tau, y_i)}{\partial \beta_j \partial \tau} \right) &= -x_{ij} \frac{\ln^2[\pi(\mathbf{x}_i)] \pi(\mathbf{x}_i)}{(1 + \tau \beta' \mathbf{x})[1 + \pi(\mathbf{x}_i)]} \left[\frac{1}{\tau^2} \ln(1 + \tau \beta' \mathbf{x}_i) - \frac{\beta' \mathbf{x}_i}{\tau(1 + \tau \beta' \mathbf{x}_i)} \right] \\ -E \left(\frac{\partial^2 l_i(\beta, \tau, y_i)}{\partial^2 \tau} \right) &= -\frac{\partial^2 \pi(\mathbf{x}_i)}{\partial^2 \tau} \frac{1}{\pi(\mathbf{x}_i)[1-\pi(\mathbf{x}_i)]} \end{aligned} \quad (20)$$

since $E \left(\frac{\partial l_i(\pi(\mathbf{x}_i))}{\partial \pi(\mathbf{x}_i)} \right) = 0$.

By substituting the previous results in the equations (20) the Fisher information matrix is obtained.