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Liquidity Shocks and the Business Cycle

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Abstract

This paper studies the properties of an economy subject to random liquidity shocks. As in [Kiyotaki and Moore \[2008\]](#), liquidity shocks affect the ease with which equity can be used as to finance the down-payment for new investment projects.

We obtain a liquidity frontier which separates the state-space into two regions (liquidity constrained and unconstrained). In the unconstrained region, the economy behaves according to the dynamics of the standard real business cycle model. Below the frontier, liquidity shocks have the effects of investment shocks. In this region, investment is under-efficient and there is a wedge between the price of equity and the real cost of capital.

As with investment shocks, we argue that liquidity shocks are not an important source of business cycle fluctuations in absence of other frictions affecting the labor market.

Keywords: Business Cycle, Asset Pricing, Liquidity.

JEL Classification: E32, E44, D82.

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1 Overview

Motivation: This paper studies the role of liquidity shocks in a real business cycle framework. Liquidity shocks are shocks to the fraction of assets that may be sold as in previous work by [Kiyotaki and Moore \[2008\]](#) (henceforth KM). In conjunction with other financial frictions, these shocks are a potential source of business cycle fluctuations. Indeed, the recent financial crises has been attributed to a collapse in credit markets and, in particular, to a disruption in the use of existing assets as collateral.

Motivated by these events, this paper studies the full stochastic version of the KM model in order to provide further insights about liquidity shocks. Whereas most of the intuition in KM still holds in a stochastic environment, this version uncovers some characteristics that are not deduced immediately from the analysis around a non-stochastic steady-state. More precisely, we find that the state-space has two regions separated by a liquidity frontier. Each region has the properties that KM find for two distinct classes deterministic steady states.¹ The region above the liquidity frontier is governed by the dynamics of real business cycle model. In the region below the liquidity frontier, liquidity shocks (combined with limited enforcement constraints) play the role of shocks to the efficiency of investment providing an additional source of fluctuations.² This region is characterized by binding enforcement constraints which render the competitive allocation of resources to investment projects inefficient. Moreover, because in this region enforcement constraints are binding, this inefficiency shows up as a wedge between the replacement cost of capital and the price at which equity (backed by capital) is traded. We interpret this wedge as Tobin's q (henceforth, we refer to this wedge simply a q). By characterizing the liquidity frontier, we are able to show that liquidity shocks have stronger effects as the return to capital is high, either because the capital stock is low or because productivity is high.

Liquidity: In the rest of the paper, liquidity is interpreted as a property of an asset: an asset is liquid if gains from trade are sufficient to guarantee trade. Liquidity shocks are shocks to the fractions of assets which are liquid. The amount of liquidity is the fraction of liquid assets.³

The role of liquidity: In the model, investment has two characteristics that cause liquidity to become a source business cycle fluctuations. First, access to investment projects is limited to a fraction of the population. Second, investment is subject to moral-hazard so optimal financing requires a down-payment. The combination of these two characteristics *innus* gains from trading previously existing assets: entrepreneurs want to sell these assets to finance down-payments and, on the other hand, there is demand for these assets since not every entrepreneur is capable of investing. Shocks to liquidity interrupt trade when there are gains from trade.

The mechanism works in the following way. Due to moral hazard, in order to access

¹By classes of steady states refers we mean steady states under different parameterizations of the model.

²Liquidity shocks in the model here deliver the same dynamics as shocks to investment efficiency as in [Barro and King \[1984\]](#).

³Some authors use liquidity as a synonym of volume of trade. In the context of the model, the distinction between liquid assets and traded assets is important. For example, there could be states in which equity is entirely liquid but, on the other hand, it is not traded at all.

external financing, entrepreneurs are required to self-finance part of investment projects. To relax their external financing constraints, entrepreneurs can sell part of their assets. As in KM, liquidity shocks arrive exogenously and affect the amount of assets that can be sold. When liquidity is sufficiently low, it drives aggregate investment below efficient levels because it reduces the entrepreneurs' access to external financing. Because there are less assets sold (corresponding to projects in place), non-investing entrepreneurs are willing to supply more funds for new investment projects. On the other hand, because they are selling less assets (corresponding to older projects), investing entrepreneurs have less funds to finance their down-payments. With more supply for outside financing and less funds for internal financing, a wedge between the price of equity and the replacement cost of capital must occur to clear out the equity market respecting the limited enforcement constraints. Without this wedge, the market for equity of new projects would clear at a level where incentive constraints are not satisfied. This wedge causes Tobin's q to be different than one in some states. For example, after a strong liquidity shock, aggregate investment falls and q increases.

Through this channel, liquidity shocks play a potential role as a direct source and amplification mechanism of business cycle fluctuations. Natural questions are why and by how much? This paper is aimed at answering them.

Quantitative Findings: The main quantitative result in the paper is that liquidity shocks on their own may not explain strong recessions. In particular, we argue that one needs to introduce additional frictions on the labor market that interact with liquidity shocks in order to explain sizeable recessions. Our calibration exercise is purposely designed in such a way that the effects of liquidity shocks have the strongest possible effects. Nevertheless, when we study the impulse response to an extreme event in which all assets become illiquid we find that the response of output is a drop in 0.7% relative to the average output. The response of output to liquidity shocks is weak because liquidity shocks resemble investment shocks. Investment shocks have weak effects in neoclassical environments because output is a function of the capital stock which, in turn, moves very little in comparison to investment. This is the reason why the bulk of business cycle studies focus on total factor productivity shocks.⁴ Moreover, due to the non-linear nature of liquidity shocks, their effects on output are negligible if they are not close to a full market shutdown.

To affect output in a stronger way, liquidity shocks must also affect labor input decisions. One way to reproduce stronger effects is by introducing variable capital utilization as in Greenwood et al. [1988] into the model. With variable capital utilization, entrepreneurs face a trade-off between incrementing the utilization of capital and depreciating capital. When liquidity is tight, the economy is inefficient in allocating resources to investment. The opportunity cost of capital use increases. As a consequence, the utilization of capital and labor demand fall with a drop in liquidity. We incorporate this mechanism into the model to explain how the effects of liquidity on output may be much larger if it also affects the labor demand. The same calibration exercise with variable capital utilization has an effect close to 10 times as large.

We calibrate liquidity shocks in such a way that the effects are as large as possible. In

⁴This result is known in the literature at least since Barro and King [1984] and discussed recently in Justiniano et al. [2010].

the appendix, we derive the asset pricing properties of the model. When computing asset prices, we find that in order to obtain a reasonable mean and variance of the risk-free rate, liquidity shocks must fluctuate close to the liquidity frontier. If liquidity shocks fall too often into the unconstrained region, the variation in q is too low. On the other hand, if liquidity shocks fall deep inside the constrained region too often, risk free rates become excessively volatile. This finding reinforces our claim that if liquidity shocks are to explain an important part of the business cycle, they must also distort labor decisions.⁵

Related Papers. [He and Krishnamurthy \[2008\]](#) and [Brunnermeier and Sannikov \[2009\]](#) study environments in which agents are heterogenous because some are limited in their access to saving instruments (investment opportunities). Investment opportunities are also illiquid assets because of hidden action (gains from trade don't guarantee trade or intermediation). Thus, these papers focus on how a low relative wealth of agents with access to these opportunities distorts the allocation of other entrepreneurs from investing efficiently. Here we abstract from the importance of the relative wealth of agents but stress how the liquidity shocks to existing assets affect the liquidity of investment projects by reducing the amount available as collateral. Like us, these papers deliver regions of the state-space where constraints prevent efficient investment. These papers also stress the importance of global methods in understanding the non-linear dynamics of these financial frictions. This paper complements that work as a step towards understanding how changes in liquidity (rather than wealth) affect the allocation of resources to investment.

Another related paper is [Lorenzoni and Walentin \[2009\]](#). In that paper, a wedge between the cost of capital and the price of equity shows up as combination of limited access to investment opportunities (gains from trade) and limited enforcement (the inefficiency that prevents trade). Our papers share in common that investment is inefficient due to lack of commitment. In [Lorenzoni and Walentin \[2009\]](#) entrepreneurs may potentially default on debt whereas in KM, agents can default on equity generated by new projects. Thus, as in [He and Krishnamurthy \[2008\]](#) and [Brunnermeier and Sannikov \[2009\]](#), shocks propagate by affecting the relative wealth of agents carrying out investment opportunities which is something we abstract from. Our paper is related to theirs because both stress that the relation between q and investment is governed by two forces: shocks that increase the demand for investment (e.g., an increase in productivity) will induce a positive correlation between q and investment. The correlation moves in the opposite direction when the enforcement constraints are more binding tighter (e.g., with a fall in liquidity).

Finally, [del Negro et al. \[2010\]](#) is the closest to our work. This paper *in* *mus* nominal rigidities into the KM model. Nominal rigidities are an example of an amplification mechanism that our model is looking for in order to explain an important part of the business cycles. That paper corroborates our finding that without such an amplification mechanism, liquidity shocks on their own, cannot have important implications on output. Our papers are complementary as theirs tries to explain the recent financial crises as caused by a strong liquidity shock. The focus here is on studying liquidity shocks in an RBC. Other than that, our paper differs from theirs because it studies the behavior of the model globally whereas

⁵Similar conclusions are found in [Greenwood et al. \[2000\]](#) or [Justiniano et al. \[2010\]](#) when studying random investment specific shocks.

theirs is restricted to a log-linearized version of the model. We believe that the findings in both papers complement each other.

Organization: The first part of the paper describes KM's model and exploits an aggregation result to compute equilibria without keeping track of wealth distributions for a broad class of preferences. The following sections characterize the main properties of the model. We then discuss the business cycle implications and the effects of a strong liquidity dry-up episode. A later section *innus* variable capital utilization and discusses the main implications. The final section concludes the paper by proposing some challenges for future research. In the appendix of the paper we describe some extensions to the model and its asset pricing properties.

2 Kiyotaki and Moore's Model

The model is formulated in discrete time with an infinite horizon. There are two populations with unit measure, entrepreneurs and workers. Workers provide labor elastically and don't save. Entrepreneurs don't work but invest in physical capital which they use in privately owned firms. Each period, entrepreneurs are randomly assigned one of either of two types, investors and savers. We use superscripts i and s to refer to either type.

There are two aggregate shocks. A productivity shock $A \in \mathbb{A}$ where $\mathbb{A} \subset \mathbb{R}$ and a liquidity shock $\phi \in \Phi \subset [0, 1]$. The nature of these shocks will affect the ability to sell equity and will be clear soon. These shocks form Markov process that evolves according to stationary transition probability $\Pi : (\mathbb{A} \times \Phi) \times (\mathbb{A} \times \Phi) \rightarrow [0, 1]$. $\mathbb{A} \times \Phi$ and Π satisfy:

Assumption 1. \mathbb{A}, Φ are compact. Π has the Feller property.

It will be shown that the aggregate state for this economy is given by the aggregate capital stock, $K \in \mathbb{K}$ in addition to A and ϕ . \mathbb{K} is shown to be compact later in the paper. The aggregate state is summarized by a single vector $s = \{A, \phi, K\}$ and $s \in \mathbb{S} \equiv \mathbb{A} \times \Phi \times \mathbb{K}$.

2.1 Preferences of Entrepreneurs

We follow the exposition of Angeletos [2007] for the description of preferences which are of the class *innud* by Epstein and Zin [1989]. Preferences of entrepreneur of type j are given recursively by:

$$V(s) = U(c^j) + \beta \cdot U(\mathcal{CE}(U^{-1}(V^j(s'))))$$

where

$$\mathcal{CE} = \Upsilon^{-1}(E\Upsilon(\cdot))$$

where the expectation is taken over time t information.⁶

⁶Utility in Epstein and Zin [1989] is defined (equation 3.5) differently. This representation is just a monotone transformation of the specification in that paper. The specification here is obtained by applying U^{-1} to that equation .

The term \mathcal{CE} refers to the certainty equivalent utility with respect to the CRRA Υ transformation⁷. Υ and U are given by,

$$\Upsilon(c) = \frac{c^{1-\gamma}}{1-\gamma} \text{ and } U(c) = \frac{c^{1-1/\sigma}}{1-1/\sigma}$$

γ captures the risk-aversion of the agent whereas his elasticity of intertemporal substitution is captured by σ .

2.2 Production

Entrepreneurs manage their firms efficiently. Each firm is run by using an idiosyncratic capital endowment, $k \in [0, \infty]$. Entrepreneurs increase a capital endowment via investment projects. In addition, they purchase and sell equity from other entrepreneurs. Financial trading is explained below.

At the beginning of each period, entrepreneurs take the capital in their firms as given and choose labor inputs, l , optimally to maximize profits. Production is carried out according to a Cobb-Douglas production function $F(k, l) \equiv k^\alpha l^{(1-\alpha)}$, where α is the capital intensity. Because the production function is homogeneous, maximization of profits requires to maximize over the labor to capital ratio:

$$\max_{l/k} \left(\left[AF \left(1, \frac{l}{k} \right) \right] - w \frac{l}{k} \right) k$$

where w is the wage and A is the aggregate productivity shock.

2.3 Entrepreneur Types

Entrepreneurs are able to invest only upon the arrival of random investment opportunities. Investment opportunities are distributed i.i.d. across time and agents. An investment opportunity is available with probability π . Hence, each period, entrepreneurs are segmented into two groups, investors and savers, with masses π and $1 - \pi$ respectively. The entrepreneur's budget constraint is:

$$c_t + i_t^d + q_t \Delta e_{t+1}^+ = r_t n_t + q_t \Delta e_{t+1}^- \quad (1)$$

This budget is written in real terms. q_t is the price of equity in consumption units. The right hand side of 1 corresponds to the resources available to the entrepreneur. The first term is the return to equity holdings where r_t is the return on equity and n_t is the amount of equity held by the entrepreneur. The second term in the right is the value of sales of equity, Δe_{t+1}^- . This term is the difference between the next period's stock of equity e_{t+1}^- and the non-depreciated fraction of equity owned in the current period λe_t . The entrepreneur

⁷The transformation $\Upsilon(c)$ characterizes the relative risk aversion through $\gamma > 0$. The function $U(c)$ captures intertemporal substitution through $\sigma > 0$. When $\sigma\gamma < 1$, the second term in the utility function is convex in $V^j(s^j)$ and concave when the inequality is reversed. When $\gamma = \frac{1}{\sigma}$, one obtains the standard expected discounted. If these terms are further equalized to 1, the specification yields the log-utility representation.

uses these funds to consume c_t , to finance down payment for investment projects, i_t^d , and to purchase outside equity Δe_{t+1}^+ . Each unit of e_t^- entitles other entrepreneurs to rights over the revenues generated by the entrepreneurs capital and e_t^+ entitles the entrepreneur to revenues generated by other entrepreneurs. The net equity for each entrepreneur is therefore:

$$n_t = k_t + e_t^+ - e_t^- \quad (2)$$

The difference between saving and investing entrepreneurs is that the former are not able to invest directly. Thus, they are constrained to set $i_t^d = 0$. Outside equity and issued equity evolve according to

$$e_{t+1}^+ = \lambda e_t^+ + \Delta e_{t+1}^+ \quad (3)$$

and

$$e_{t+1}^- = \lambda e_t^- + \Delta e_{t+1}^- + i_t^s, \quad (4)$$

respectively. Notice that the stock of equity is augmented by sales of equity Δe_{t+1}^- and an amount i_t^s which is specified by the investment contract specified in the next section. Finally, the timing protocol is such that investment decisions are taken at the beginning of each period. That is, entrepreneurs choose consumption and a corporate structure before observing future shocks.

2.4 Investment, optimal financing and liquidity shocks

Investment opportunities and financing. When an investment opportunity is available, entrepreneurs choose a scale for an investment project, i_t . Projects increment the firm's capital stock one for one with the size of the project. Each project is funded by a combination of internal funding, i_t^d , and external funds i_t^f . External funds obtained by selling equity that entitles other entrepreneurs to the proceeds of the new project. Thus, $i_t = i_t^d + i_t^f$. In general, the ownership of capital created by this project may differ from the sources of funding. In particular, investing entrepreneurs are entitled to the proceeds of a fraction i_t^i of total investment, and the rest, i_t^s , entitles other entrepreneurs to those proceeds. Again, $i_t = i_t^i + i_t^s$.

Because the market for equity is competitive and equity is homogeneous, the rights to i_t^s are sold at the market price of equity q_t . Therefore, external financing satisfies $i_t^f = q_t i_t^s$. Notice that at the end of the period, the investing entrepreneur increases his equity in $i_t^i = i_t - i_t^s$ while he has contributed only $i_t - q_t i_t^s$. In addition, investment is subject to moral-hazard because entrepreneurs may divert funds from the project. By diverting funds, they are able to increment their equity stock up to a fraction $1 - \theta$ of the total investment without honoring the fraction of equity sold. There are no enforcement or commitment technologies available. The assumption that issues of new equity is subject to moral hazard as opposed to equity in place is tries to capture the idea that financial transactions on assets that are already in place are more easily enforced than those on assets that don't exist yet. In essence, we assume that funds from existing physical capital may not be diverted. On the other hand, one can interpret liquidity shocks also as stemming from a time varying version of

constraints on existing assets.

Utility will be shown to be an increasing function of equity only. The incentive compatibility condition for external financing is equivalent to:

$$(1 - \theta) i_t \leq i_t^i, \text{ or } i_t^s \leq \theta i_t \quad (5)$$

This condition states that the lender's stake in the project may not be higher than θ .⁸ Therefore, taking i_t^d as given, the entrepreneur solves the following problem when it decides how much to invest:

Problem 1 (Optimal Financing).

$$\max_{i_t^s > 0} i_t^i$$

taking $i_t^d > 0$ as given and subject to:

$$\begin{aligned} i_t^d + i_t^f &= i_t, \quad i_t^i + i_t^s = i_t, \quad q_t i_t^s = i_t^f \\ i_t^s &\leq \theta i_t \end{aligned}$$

Substituting out all the constraints, the problem may be rewritten in terms of i_t^d and i_t^s only.

Problem 2 (Optimal Financing Reduced Form).

$$\max_{i_t^s} i_t^d + (q_t - 1) i_t^s$$

taking i_t^d as given and subject to:

$$i_t^s \leq \theta i_t^d + \theta q_t i_t^s \quad (6)$$

The interpretation of this objective is clear. For every project, the investing entrepreneur increases his stock of equity $i_t^d + (q_t - 1) i_t^s$, which is the sum of the down payment plus the gains from selling equity corresponding to the new project, i_t^s . The constraint says that the amount of outside funding is limited by the incentive compatibility constraint. As q_t is lower, the constraint on external funding is tighter, because the investing entrepreneurs stake on the project is lower.

It is clear that the solution to this program depends on the value of $q_t \in (1, \frac{1}{\theta})$, the problem is maximized at the points where the incentive compatibility constraint binds. Therefore, at this price range, for every unit of investment i_t , the investing entrepreneurs finances the amount $(1 - \theta q_t)$ units of consumption and owns the fraction $(1 - \theta)$. This defines a new cost of equity,

$$q_t^R = \left[\frac{(1 - \theta q_t)}{(1 - \theta)} \right]$$

⁸The distinction between inside and outside equity makes this a q -theory of investment. The wedge occurs as a combination of two things. First, only a fraction π of agents have access to investment opportunities which generates a demand for outside equity. Limited enforcement causes the supply of outside equity to be limited by the incentive compatibility constraints. The value of q must adjust to equate demand with supply and this price may differ from 1.

q_t^R is less than 1, when $q_t > 1$ and equal to 1 when $q_t = 1$. When $q_t = 1$, the entrepreneur is indifferent on the scale of the project, so i^s is indeterminate within $[0, \theta i_t]$. The difference between q_t^R and q_t is a wedge between the cost of purchasing outside equity and the cost of generating inside equity. The physical capital run by the entrepreneur evolves according to

$$k_{t+1} = \lambda k_t + i_t$$

so using the definition of equity (2):

$$n_{t+1} = \lambda n_t + i_t - i_t^s + (\Delta e_{t+1}^+ - \Delta e_{t+1}^-) \quad (7)$$

Resellability Constraints. In addition to the borrowing constraint imposed by moral hazard, there is a constraint on the sales of equity created in previous periods. Resellability constraints impose a limit on sales of equity that may be sold at every period. These constraints depend on the liquidity shock ϕ_t :

$$\Delta e_{t+1}^- - \Delta e_{t+1}^+ \leq \lambda \phi_t n_t \quad (8)$$

Kiyotaki and Moore motivate these constraints by adverse selection in the equity market. [Bigio \[2009\]](#) and [Kurlat \[2009\]](#) show that such a constraint will follow from adverse selection stemming from private information on the quality of assets. There are multiple alternative explanations on why liquidity may vary over the business cycle. We discuss some alternative explanations in the concluding section. Here, what matters is that liquidity shocks, ϕ_t , prevent equity markets from materializing gains from trade.

Plugging in the resellability constraint and the incentive compatibility constraint into (7), an overall constraint is obtained:

$$n_{t+1} \geq (1 - \phi_t) \lambda n_t + (1 - \theta) i_t \quad (9)$$

Along the paper, this constraint will be referred to as a liquidity constraint. The constraint reads that equity holdings next period n_{t+1} are greater than or equal to the least amount of previously held equity the entrepreneur must keep for itself, $(1 - \phi_t) \lambda n_t$, plus the minimal amount of ownership over the new investment project such that the project is incentive compatible.

When $q_t > 1$, the cost of increasing equity by purchasing outside equity is larger than putting the same amount as down-payment and co-financing the rest ($q_t > q^R$). Moreover, when $q_t > 1$, by selling equity and using the amount as down-payment, the agent increases his equity by $q_t (q_t^R)^{-1} > 1$. Since when $q_t > 1$, investing entrepreneurs must set $\Delta e_{t+1}^+ = 0$ and (9) binds. Using these facts, the investing entrepreneur's budget constraint (1) may be rewritten in a convenient way by substituting (7) and $i_t^d = i_t - q_t i_t^s$. The budget constraint is reduced to:

$$c_t + q_t^R n_{t+1} = (r_t + q_t^i \lambda) n_t$$

where $q_t^i = q_t \phi_t + q^R (1 - \phi_t)$. When $q_t = 1$, this constraint is identical to the saving entrepreneur's budget constraint when the evolution of its equity is replaced into (1). Thus

without loss of generality, the entrepreneur's problem simplifies to:

Problem 3 (saver's problem).

$$V_t^s(w_t) = \max_{c_t, n_{t+1}} U(c_t) + \beta \cdot U(\mathcal{CE}_t(U^{-1}(V_{t+1}(w_{t+1}))))$$

subject to the following budget constraint:

$$c_t + q_t n_{t+1} = (r_t + \lambda q_{t+1}) n_t \equiv w_t^s$$

V_{t+1} represents the entrepreneur's future value. V_{t+1} does not include a type superscript since types are random and the expectation term is also over the type space also. w_{t+1} is the entrepreneur's virtual wealth which also depends on the type. The investing entrepreneur solves the following problem:

Problem 4 (investor's problem).

$$V_t^i(w_t) = \max_{i_t, c_t, n_{t+1}} U(c_t) + \beta \cdot U(\mathcal{CE}_t(U^{-1}(V_{t+1}(w_{t+1}))))$$

subject to following budget constraint:

$$c_t + q_t^R n_{t+1} = (r_t + q_t^i \lambda) n_t \equiv w_{t+1}^i$$

Finally, workers provide labor l_t to production in exchange for consumption goods c_t in order to maximize their static utility.

Problem 5 (Workers Problem).

$$U_t = \max_{c, l} \frac{1}{\left(1 - \frac{1}{\psi}\right)} \left[c - \frac{\varpi}{(1 + \nu)} (l)^{1+\nu} \right]^{1 - \frac{1}{\psi}}$$

and subject to the following budget constraint:

$$c = \omega_t l$$

Along the discussion we have indirectly shown the following Lemma:

Lemma 1. *In any equilibrium $q \in [1, \frac{1}{\theta}]$. In addition, when Tobin's q is $q > 1$, the liquidity constraint (9) binds for all investing entrepreneurs. When $q = 1$, policies for saving and investing entrepreneurs are identical and aggregate investment is obtained by market clearing in the equity market.*

q_t is never below 1 since capital is reversible. Models with adjustment costs would not have this property.⁹ The following assumption is imposed so that liquidity plays some role in the model:

⁹In Sargent [1980], investment is irreversible so q may be below one in equilibrium. This happens when, at $q=1$, the demand for consumption goods generates negative investment. q must adjust to below one such that the demand for investment is 0.

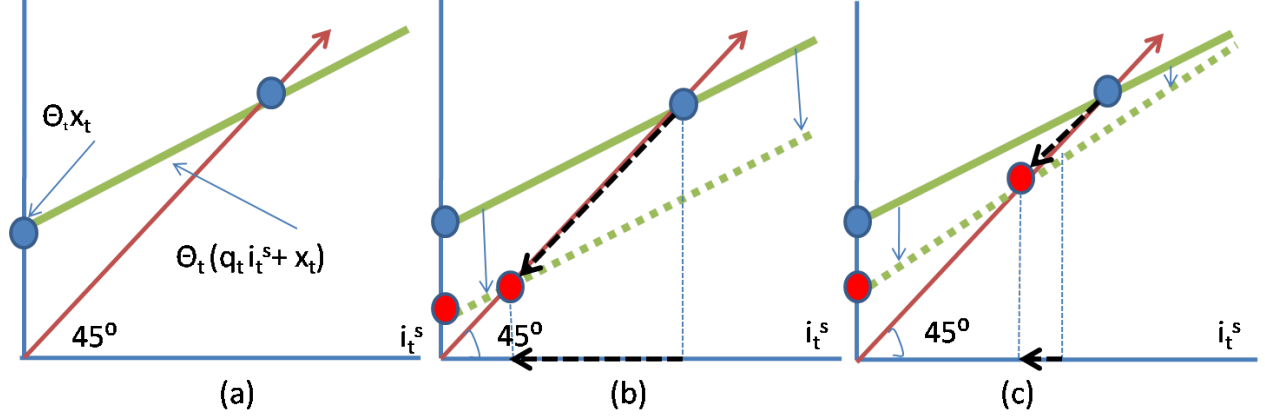


Figure 1: **Liquidity constraints** The left panel shows how borrowing constraints impose a cap on the amount of equity that can be sold to finance a down payment. The middle panel shows how liquidity shocks affect the amount of resources available as a down payment. The right panel shows the price effect of the shock which may or may not reinforce the liquidity shock.

Assumption 2. $\theta < 1 - \pi$

If this condition is not met, then the economy does not require liquidity at all in order to exploit investment opportunities efficiently. Before proceeding to the calibration of the model we provide two digressions: one about the intuition behind the amplification mechanism behind liquidity shocks and the other about a broader interpretation of the financial contracts in the model.

2.5 Intuition behind the effects

A brief digression on the role of ϕ_t is useful before studying the equilibrium in more detail. Whenever $q_t > 1$, external financing allows investing entrepreneurs to arbitrage. In such *nu*, the entrepreneur wants to finance investment projects as much as possible. We have already shown that when constraints are binding, the entrepreneur owns the fraction $(1 - \theta)$ of investment while he finances only $(1 - q_t\theta)$. If he uses less external financing he misses the opportunity to obtain more equity.

To gain intuition, let's abstract from the consumption decision and assume that the entrepreneur uses $x_t \equiv q_t\phi_t\lambda n_t$ to finance the down-payment. Thus, $i_t^d = x_t$. The constraints impose a restriction on the amount external equity that may be issued, $i_t^s \leq \theta i_t$. External financing is obtained by selling i_t^s equity at a price q_t so the amount of external funds for the project is $i_t^f = q_t i_t^s$. Since $i_t = i_t^f + i_t^d$, external financing satisfies,

$$i_t^f \leq q_t\theta(i_t^f + x_t). \quad (10)$$

Figure 1 describes the simple intuition behind the liquidity channel. The figure explains how changes in the amount x_t of sales of equity corresponding to older projects affects in-

vestment by restricting the amount of external financing for new projects. Panel (a) in Figure 1 plots the right and left hand sides of restriction on outside funding given by the inequality (10). Outside funding, i_t^f , is restricted to lie in the area where the affine function is above the 45-degree line. Since $q_t > 1$, the left panel shows that the liquidity constraint imposes a cap on the capacity to raise funds to finance the project. Panel (b) shows the effects of a decrease in x_t without considering any price effects. The fall in the down-payment reduces the intercept of the function defined by the right hand side of Figure 1. External funding, therefore, falls together with the whole scale of the project. Since investment falls, the price of q must rise such that the demand for saving instruments fall to match the fall in supply. The increase in the price of equity implies that the amount financed externally is higher. The effect on the price increases the slope and intercept which partially counterbalances the original effect. This effect is captured in the panel to the right of Figure 1.

2.6 Alternative Financial Contracts

There is a sense in which the operation of selling equity to finance the down-payment for new projects resembles Collateralized Debt Obligations (CDOs). In particular, selling equity is equivalent to signing a state contingent debt contract in which equity is used as collateral. In this case, if the principal of the debt contract is the future value of equity, and the interest is equal to the return of capital, then selling equity or issuing state contingent debt are equivalent for both parties. Thus, the financial contracts in the paper may be thought of as a primitive form of CDO with a very simple structure: the return is contingent on the assets return and there is a single trench. Moreover, there is no additional default risk because assets can be liquidated immediately. In reality, financial contracts based on CDOs are more complicated as they involve bundling assets and decomposing them into assets of different risk.

In terms of the model, selling equity (or securitizing) is more convenient for the investing entrepreneur than using the same amount of equity as collateral for the project. To see why, observe that the corresponding incentive compatibility constraint for a contract in which $\lambda\phi_t n_t$ assets are used as collateral, leads to the following incentive compatibility condition: $i_t^s \leq \theta(q_t i_t^s + i_t^d) + \lambda\phi_t n_t$. By incrementing a unit of collateral, the increment in external financing is $\frac{1}{1-q_t\theta}$. Therefore, the marginal value of one extra unit of collateral is $\frac{q_t-1}{1-q_t\theta}$. This quantity is smaller than $q_t (q_t^R)^{-1} = \frac{q_t-q_t\theta}{1-q_t\theta}$ since $q_t\theta < 1$ in equilibrium. One can also show that amount of equity supplied by investing entrepreneurs under these contracts is less than the amount if equity is sold even if the amount of external financing for this project is substantially larger. Thus, using equity as collateral instead of selling it leads to more inefficiencies. The intuition behind this is that investing entrepreneurs value equity less than saving entrepreneurs. At the point in which investing entrepreneur sell equity, they obtain the value of equity in terms of consumption units, which they convert once more, into equity by financing a larger portion of the project and raising external funds. This quantity is larger than the original amount of equity that would be used as collateral. Hence, by selling equity, the entrepreneur relaxes the incentive compatibility constraint further more.

Finally, the model abstracts from debt contracts with a fixed coupon. With a fixed coupon,

and using equity as collateral, the return for a saving entrepreneur would be $\min(R, (r_t + \lambda))$ for every unit of consumption loaned backed by a unit of equity. The model could be adapted by introducing this additional asset. The demand for this asset would be given by R and the solution to a portfolio problem and the supply given by the investing entrepreneurs willingness to obtain these loans.

3 Equilibrium

The right hand side of the entrepreneurs' budget constraints defines their corresponding wealth: $w_t^s = (r_t + q_t \lambda) n_t$ and $w_t^i = (r_t + q_t^i \lambda) n_t$. A stationary recursive equilibrium is defined by considering the distribution of this wealth vector.

Definition 1 (Stationary Recursive Competitive Equilibrium). *A recursive competitive equilibrium is a set price functions, $q, \omega, r : \mathbb{S} \rightarrow \mathbb{R}_+$ allocation functions, $n^{i,j}, c^j, l^j, i^j : \mathbb{S} \rightarrow \mathbb{R}_+$ for $j = i, s, w$, a sequence of distributions Λ_t of equity holdings, and a transition function, Ξ , for the aggregate nu such that:*

1. *Optimality of Policies: Given, q and ω , $c^j, n^{j'}, i^{j'}, i^{j'}$, $j = i, s, w$, solve the problems 3, 4 and 5.*
2. *Goods market clear at price 1.*
3. *Labor markets clear at price w .*
4. *Equity market clear at price q .*
5. *Firms are run efficiently and per unit of capital profits are equal to r .*
6. *Aggregate capital evolves according to $K'(s) = I(s) + \lambda K(s)$.*
7. *Λ_t and Ξ are consistent with the policy functions obtained from problems 3, 4 and 5.*

The equilibrium concept defined above is recursive for the aggregate state variables. What we mean by this is that the sequence of wealth distributions, Λ_t , is a relevant state variable determine individual quantities but not aggregate quantities.

Optimal Policies: Note that the problem for both entrepreneurs is similar. Both types choose consumption and equity holdings for next period but differ in the effective cost of equity that each of them faces. The following propositions describe the policy functions and show that these are linear functions of the wealth vector.

Proposition 1 (Savers policies). *Policy functions for saving entrepreneurs are given by:*

$$\begin{aligned} c_t^s &= (1 - \varsigma_t^s) w_t^s \\ q_t n_{t+1}^s &= \varsigma_t^s w_t^s \end{aligned}$$

The corresponding policies for investing entrepreneurs are,

Proposition 2 (investors policies). *Policy functions for investing entrepreneurs satisfy:*

(i) For all s_t such that $q_t > 1$, optimal policies are:

$$\begin{aligned} c_t^i &= (1 - \varsigma_t^i) w_t^i \\ q^R n_{t+1}^i &= \varsigma_t^i w_t^i \\ i_{t+1} &= \frac{n_{t+1} + (\phi_t - 1) \lambda n_t}{(1 - \theta)} \end{aligned}$$

and

(ii) For all s_t such that $q_t = 1$, policies are identical to the saving entrepreneurs and i_t is indeterminate at the individual level.

The marginal propensities to save of the previous proposition will depend only on the conditional expectation of conditional returns:

$$R_t^{ss} \equiv \frac{(r_t + q_{t+1}\lambda)}{q_t}, R_t^{is} \equiv \frac{(r_t + q_{t+1}\lambda)}{q_t^R}, R_t^{ii} \equiv \frac{(r_t + q_{t+1}^i\lambda)}{q_t^R} \text{ and } R_t^{is} \equiv \frac{(r_t + q_{t+1}^i\lambda)}{q_t} \quad (11)$$

The following proposition characterizes these marginal propensities.

Proposition 3 (Recursion). *The functions ς_t^i and ς_t^s , in Propositions 1 and 2 solve the following:*

$$(1 - \varsigma_t^i)^{-1} = 1 + \beta^\sigma \Omega_t^i \left((1 - \varsigma_{t+1}^i)^{\frac{1}{1-\sigma}}, (1 - \varsigma_{t+1}^s)^{\frac{1}{1-\sigma}} \right)^{\sigma-1} \quad (12)$$

$$(1 - \varsigma_t^s)^{-1} = 1 + \beta^\sigma \Omega_t^s \left((1 - \varsigma_{t+1}^i)^{\frac{1}{1-\sigma}}, (1 - \varsigma_{t+1}^s)^{\frac{1}{1-\sigma}} \right)^{\sigma-1} \quad (13)$$

and

$$\Omega_t^i(a_{t+1}^s, a_{t+1}^i) = \Upsilon^{-1} \mathbf{E}_t [(1 - \pi) \Upsilon (a_{t+1}^s R_{t+1}^{ss}) + \pi \Upsilon (a_{t+1}^i R_{t+1}^{si})]$$

and

$$\Omega_t^s(a_{t+1}^s, a_{t+1}^i) = \Upsilon^{-1} \mathbf{E}_t [(1 - \pi) \Upsilon (a_{t+1}^s R_{t+1}^{is}) + \pi \Upsilon (a_{t+1}^i R_{t+1}^{ii})]$$

When $(\sigma, \gamma) = (1, 1)$ then $\varsigma_t^s = \varsigma_t^i = \beta$.¹⁰

A proof for the three previous propositions is provided in the appendix. Because policy functions are linear functions of wealth, the economy admits aggregation (this is the well known *Gorman* aggregation result). The appendix also shows that the economy in KM, is identical to the one here, when the money supply in that paper is set to 0.

Labor Demand: Taking the physical capital k_t as given, firms are run efficiently. Using the first order conditions aggregate labor demand is obtained by integrating over the individual capital stock.

¹⁰We show in the appendix that (12) and (13) conform a contraction for preference parameters such that $1 - \gamma/\sigma - 1$ and returns satisfying $\beta^\sigma E_t [R^{ij}]$, $j \in \{i, s\} \geq 1$. On the other hand, as long as the entrepreneurs problem is defined uniquely, equilibria can be computed by checking that the iterations on (12) and (13). Upon convergence an equilibrium is found.

$$L_t^d = \left[\frac{A_t (1 - \alpha)}{\omega_t} \right]^{\frac{1}{\alpha}} K_t. \quad (14)$$

Equilibrium Employment: Workers consume their labor income so $c_t = \omega_t L_t^s(A_t, K_t)$. The solution to the workers problem defines a labor supply schedule. Solving for equilibrium employment pins down the average wage, $\omega_t = \bar{\omega}^{\frac{\alpha}{\alpha+\nu}} [A_t (1 - \alpha)]^{\frac{\nu}{\alpha+\nu}} K_t^{\frac{\nu\alpha}{\alpha+\nu}}$ and equilibrium employment

$$L_t^*(A_t, K_t) = \left[\frac{(1 - \alpha) A_t}{\bar{\omega}} \right]^{\frac{1}{\alpha+\nu}} [K_t]^{\frac{\alpha}{\alpha+\nu}}. \quad (15)$$

Returns to Equity: The return to capital owned by an entrepreneur is a function A_t and ω_t . From the firm's profit function and the equilibrium wage, return per unit of capital is:

$$r_t = \Gamma(\alpha) [A_t]^{\frac{\xi+1}{\alpha}} \bar{\omega}^{\frac{\xi}{\nu}} (K_t)^{\xi} \quad (16)$$

where $\Gamma(\alpha) \equiv [(1 - \alpha)]^{\frac{(\nu+1)}{(\alpha+\nu)}} \left[\frac{1}{(1-\alpha)} - 1 \right]$ and $\xi \equiv \frac{\nu(\alpha-1)}{\alpha+\nu} < 0$. ξ governs the elasticity of aggregate returns as a function of aggregate capital. The closer α is to 1, profits are more elastic to aggregate capital and the return is lower.

3.1 Characterization

The dynamics of the model are obtained by aggregating over the individual states using the policy functions described in the previous section.

Aggregate Output: Total output is the sum of the return to labor and the return to capital. The labor share of income is,

$$w_t L_t^s = \bar{\omega}^{\frac{\xi}{\nu}} [A_t (1 - \alpha)]^{\frac{\xi+1}{\alpha}} K_t^{\xi+1}.$$

The return to equity is (16). By integrating with respect to idiosyncratic capital endowment we obtain the capital share of income:

$$r_t(s_t) K_t = \Gamma(\alpha) [A_t]^{\frac{\xi+1}{\alpha}} \bar{\omega}^{\frac{\xi}{\nu}} K_t^{\xi+1}$$

Aggregate output is the sum of the two shares:

$$Y_t = \left[\frac{(1 - \alpha)}{\bar{\omega}} \right]^{\frac{1-\alpha}{\alpha+\nu}} A_t^{\frac{\xi+1}{\alpha}} K_t^{\xi+1} \quad (17)$$

Since, $0 < \xi + 1 < 1$, this ensures that this economy has decreasing returns to scale at the aggregate level. This ensures that \mathbb{K} is bounded. By consistency $K_t = \int n_t(w) \Lambda_{dw}$. Since investment opportunities are i.i.d, the fraction of equity owned by investing and saving entrepreneurs are πK_t and $(1 - \pi) K_t$ respectively. The aggregate consumption, C_t^s , and equity holdings, N_{t+1}^s , of saving entrepreneurs are:

$$\begin{aligned} C_t^s &= (1 - \zeta_t^s) (r_t + q_t \lambda) (1 - \pi) K_t \\ q_{t+1} N_{t+1}^s &= \zeta_t^s (r_t + q_t \lambda) (1 - \pi) K_t. \end{aligned}$$

When $q > 1$, these aggregate variables corresponding to investing entrepreneurs, C_t^i and N_{t+1}^i , are:

$$\begin{aligned} C_t^i &= (1 - \zeta_t^i) (r_t + \lambda q_t^i) \pi K_t \\ q_t^R N_{t+1}^i &= \zeta_t^i (r_t + \lambda q_t^i) \pi K_t \end{aligned}$$

The evolution of marginal propensities to save and portfolio weights are given by the solution to the fixed point problem in Proposition 3. In equilibrium, the end of the period fraction of aggregate investment owned by investing entrepreneurs $I_t^i(s_t)$ must satisfy the aggregate version of (9):

$$I_t^i(s_t) \leq \left(\frac{\zeta_t^i (r_t + \lambda q_t^i)}{q_t^R} - (1 - \phi_t) \lambda \right) \pi K_t. \quad (18)$$

$(1 - \phi_t) \pi \lambda K_t$ is the lowest possible amount of equity that remains in hands of investing entrepreneurs after they sell equity of older projects. $\frac{\zeta_t^i (r_t + \lambda q_t^i) \pi K_t}{q_t^R}$ is the equilibrium aggregate amount of equity holdings. The difference between these two quantities is the highest possible amount of equity they may hold, which corresponds to new investment projects. Similarly, for saving entrepreneurs, the amount of equity corresponding to new projects is,

$$I_t^s(s_t) \geq \left(\frac{\zeta_t^s (r_t + \lambda q_t) (1 - \pi)}{q_t} - \phi_t \pi \lambda - (1 - \pi) \lambda \right) K_t \quad (19)$$

In equilibrium, the incentive compatibility constraints (5) that hold at the individual level hold also at the aggregate level. Thus,

$$I_t^i(s_t) \leq \frac{(1 - \theta)}{\theta} I_t^s(s_t) \quad (20)$$

The above condition is characterized by a quadratic equation as a function of q_t . The following proposition is used to characterize market clearing in the equity market.

Proposition 4 (Market Clearing). *For (σ, γ) sufficiently close to $(1, 1)$, there exists a unique q_t that clears out the equity market and it is given by:*

$$q_t = \begin{cases} 1 & \text{if } 1 > x_2 > x_1 \\ x_2 & \text{if } x_2 > 1 > x_1 \\ 1 & \text{if otherwise} \end{cases}$$

where the terms x_2 and x_1 are continuous functions of $(\phi_t, \zeta_t^i, \zeta_t^s, r_t)$ and the parameters.

The explicit solution for (x_2, x_1) is provided in the appendix. The solution accounts for the incentive constraint. When $q > 1$, the constraints are binding for all entrepreneurs.

When, $q = 1$, investment at the individual level is not determined. At the aggregate level, without loss of generality, one can set the constraints above at equality when $q_t = 1$. By adding $I_t^i(s_t)$ and $I_t^s(s_t)$, one obtains aggregate investment. Because, q is continuous in ς^s and ς^i , the following is also true,

Proposition 5. $\varsigma_t^s, \varsigma_t^i, I_t$, and q_t are continuous functions of the aggregate state (s_t) .

The proof follows from the continuity of q_t given by Proposition 4. ς_t^s and ς_t^i are, in turn, also continuous when q_t is smooth. Continuous policy functions guarantee that I_t is also continuous. We use the continuity of the recursive equilibrium to establish a closed form for the liquidity frontier that separates the state-space into regions where constraints are and aren't binding.

3.2 The Liquidity Frontier

Substituting (18) and (19) into (20) at equality for $q_t = 1$, we obtain the minimum level of liquidity required such that constraints are not binding. Formally, we have:

Proposition 6 (Liquidity Frontier). $\exists!$, $\phi^* : \mathbb{A} \times \mathbb{K} \rightarrow \mathbb{R}$ defined by:

$$\phi^*(A, K) = \frac{[(1 - \pi)(1 - \theta) - \theta\pi]}{\lambda\pi} [\varsigma_t^s(r(A, K) + \lambda) - \lambda]$$

liquidity constraints (9) bind iff $\phi_t < \phi^*$.

Here, $r(A, K)$ is defined by equation (16). The proposition states that for any (K, A) selection of the state-space, \mathbb{S} , there is a threshold value ϕ^* such that if liquidity shocks fall below that value, the liquidity constraints bind. We call the function ϕ^* the *liquidity frontier*, as it separates the state space into two regions. \mathbb{S}^n is the set of points where the liquidity constraint binds so $\phi^* = \partial\mathbb{S}^n$. The interpretation of the liquidity frontier is simple. $[\varsigma_t^s(r(A, K) + \lambda) - \lambda]$ is the amount of entrepreneurs want to hold at $q = 1$. Since both types are identical when $q = 1$, then ς_t^s characterizes the demand for equity by both groups. By propositions 1 and 2, we know that $\varsigma_t^s(r(A, K) + \lambda)$ is the demand for equity stock per unit of wealth and λ the remaining stock of equity per unit of wealth. The difference between these quantities is the per-unit-of-capital demand for investment.

$[(1 - \pi)(1 - \theta) - \theta\pi]$ is the degree of enforcement constraint in this economy. As either the fraction of savers, $(1 - \pi)$, or the private benefit of diverting resources, $(1 - \theta)$, increase, the economy requires more liquid assets to finance a larger amount of down-payment in order to carry out investment efficiently. $\lambda\pi\phi$ is maximal supply of equity. Therefore, the liquidity frontier is the degree of liquidity that allows the largest supply of assets to equal the demand for investment project times the degree of enforcement exactly at the point where $q = 1$. There are several lessons obtained from this analysis.

Lessons: Proposition 6 shows that liquidity shocks have stronger effects as the return to capital $r(A, K)$ is larger. This function, is increasing in productivity, A , and decreasing in the capital stock, K (decreasing returns to scale). This is intuitive as the demand for investment is greater when returns are high.

The proposition is also important because it gives us a good idea of the magnitude that liquidity shocks must have in order to cause a disruption on efficient investment. Since $\varsigma_t^s \approx 1$, (because it is a marginal propensity to save wealth at a high frequency), then $[\varsigma_t^s (r(A, K) + \lambda) - \lambda] \approx r(A, K)$. In addition, if $\lambda \approx 1$, then,

$$\phi^*(A, K) \approx (1 - \theta) \left(\frac{(1 - \pi)}{\pi} - \frac{\theta}{(1 - \theta)} \right) r(A, K)$$

By assumption, $\theta \in (0, (1 - \pi))$. This implies that $\theta \rightarrow 0$, $\phi^*(A, K)$ will be close to $\frac{(1 - \pi)}{\pi}$, that the economy requires a very large amount of liquidity to run efficiently. On the other hand, as $\theta \rightarrow (1 - \pi)$, then $\phi^*(A, K)$ will be close to zero implying that the economy does not require liquidity at all. Moreover, we can show that for two values of θ there is almost-observational equivalence result.

Proposition 7 (Observational Equivalence). *Let ϱ be a recursive competitive equilibrium defined for parameters $x \equiv (\theta, \Pi, \Phi)$. Then there exists another recursive competitive equilibrium ϱ' for parameters $x' \equiv (\theta', \Pi', \Phi')$ which determines the same stochastic process for prices and allocations as in ϱ iff:*

$$\frac{(1 - \theta')(1 - \pi)}{\theta'\pi\lambda} \left(\frac{\varsigma_t^s (r_t + \lambda q_t)}{q_t} - \lambda \right) \in [0, 1] \quad (21)$$

for every $q_t, r_t, \varsigma_t^s \in \varrho$ when $q > 1$ and $\phi^*(A, K; \theta') \in [0, 1]$ for $q = 1$. If (21) holds then, the new triplet of parameters Π', Φ' is constructed in the following way: for every $\phi \in \Phi$, $\phi \geq \phi^*(A, K; \theta)$, assign to Φ' any value $\phi' \in (\phi^*(A, K; \theta'), 1)$. For every $\phi < \phi^*(A, K; \theta)$, assign the value given by (21). This procedure defines a map from Φ to Φ' . Finally, Π' should be consistent with Π .

We provide a sketch of the proof here. Condition (21) is obtained by substituting (18) and (19) into (20) at equality. The left hand side of the condition is the solution to a ϕ' such that for θ' , (20) also holds at equality. If this is the case, then the same q and investment allocations clear out the equity market. Therefore, if the condition required by the proposition holds, ϕ' as constructed, will yield the same allocations and prices as the original equilibrium. Since prices are the same, transition functions will be the same and so will the policy functions. If the $\phi^*(A, K; x') \notin [0, 1]$ then liquidity under the new parameters is insufficient. This shows the if part.

By contradiction, assume that the two equilibria are observationally equivalent. If (21) is violated for some $s \in \mathbb{S}$, then, it is impossible to find a liquidity shock between 0 and 1, such that (20) is solved at equality. For these states, $q_t > 1$ in the original equilibrium, but $q_t = 1$ in the equilibrium with the alternative parameters.

A final by-product of Proposition 6 is its use to compute an estimate of the amount of outside liquidity needed to run the economy efficiently. In order to guarantee efficient investment, an outside source of liquidity must be provided by fraction $(\phi^* - \phi_t) \lambda \pi$ of total capital stock every period. In this case, ϕ^* should be evaluated at $q_t = 1$ at all s_t . $(\phi^* - \phi_t)$ is a random variable. The stationary distribution of the state-space then can be used to compute the expected liquidity deficit for this economy. Several policy exercises can be computed using this analysis. For example, one can compute the amount of government subsidy required to run the economy efficiently.

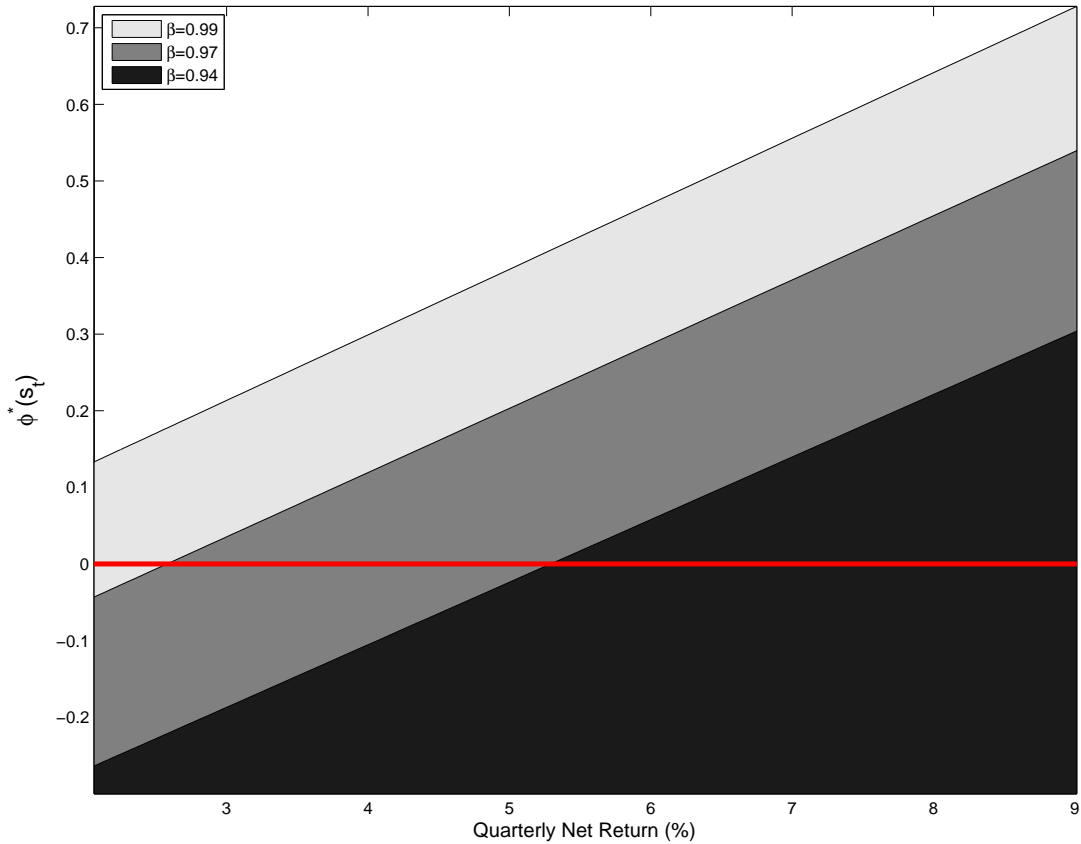


Figure 2: **Liquidity Frontier.** Numerical Examples.

Figure 2, plots the liquidity frontier as a function of returns $r(A, K)$, for three different values of β . Simulations show that the marginal propensities to save for values of $\theta = \{0.9, 0.7, 0.5\}$ are close to $\beta = \{0.99, 0.975, 0.945\}$ respectively. The frontier is increasing in the returns showing that weaker shocks will activate the constraints for higher returns. As the elasticity of intertemporal substitution increases, weaker shocks activate the liquidity constraints.

4 Results

4.1 Calibration

The nature of the calibration exercise is to make liquidity shocks have the largest effect possible in terms of output. Thus, we pick parameters within reasonable bounds for this purpose.

The model period is a quarter. The calibration of technology parameters is standard to

business cycle theory. Following the standard in the RBC literature, technology shocks are modeled such that their natural logarithm follows an AR(1) process. The persistence of the process, ρ_A , is set to 0.95 and the standard deviation of the innovations of this process, σ_A , is set to 0.016. The weight of capital in production α is set to 0.36¹¹. The depreciation of capital is set to $\lambda = 0.974$ so that annual depreciation is 10%.¹²

The probability of investment opportunities, π , is set to 0.1 so that it matches the plant level evidence of Cooper et al. [1999]. That data suggests that around 40% to 20% plants augment a considerable part of their physical capital stock. These figures vary according to a given plant's age. By setting π to 0.1, the arrival of investment opportunities is such that close to 30% of firms invest by the end of a year.

There is less consensus about parameters governing preferences. The key parameters for determining the allocation between consumption and investment are the time discount factor, β , and the elasticity of intertemporal substitution, σ . The discount factor, β , is set to 0.9762 based on a complete markets benchmark, as in Campanale et al. [2010]. For our base scenario, we set σ and γ to 1 to recover a log-preference structure. We check the robustness of the results by changing these values in an alternative calibration. Nevertheless, the marginal propensities to save ζ^i and ζ^s are roughly constant over the state-space. Picking a particular combination of these parameters is similar to modifying β and keeping the log-preference structure.¹³ Campanale et al. [2010] argue in favor of a low value for σ based on experimental evidence. Angeletos [2007] argues that empirical work based on asset pricing biases the estimate downwards and sets the parameter to 1. For an alternative calibration, we set this parameter to 0.9. The elasticity of intertemporal substitution σ affects the stationary distribution of the capital stock. Under, lower values of this parameter, the capital stock fluctuates around levels for which the return to capital is above 25% which seems large. The size of investment over the capital stock is also unreasonably large. In Campanale et al. [2010] this undesirable feature of a low value for σ in our model is corrected by changing the adjustment costs of investment. Adjustment costs are meant to capture some features of lumpy-investment observed in micro-level data. Since random investment opportunities are already capturing that feature, we do not include such costs. For the alternative scenario, we follow the estimates of Moskowitz and Vissing-Jørgensen [2002] suggesting a risk aversion parameter of around 10 for entrepreneurs.

The Frisch-elasticity is determined by the inverse of the parameter ν , which is set to 2. This parameter also shifts the curvature of the aggregate production function as a function of the aggregate capital stock. In other words, ν and α are crucial to determine the volatility of the returns to capital as the capital stock fluctuates around the stationary distribution. Volatility is a second order effect: in equilibrium, the distribution of capital will accommodate so that the average return to capital is consistent with the agent's marginal propensities to save (equations (24) and (25)). $\bar{\omega}$ is a normalization constant that plays no role in the

¹¹Acemoglu and Guerrieri [2008] have shown that the capital share of aggregate output has been roughly constant over the last 60 years.

¹²It is common to find calibrations where annual depreciation is 5%. This parameter is crucial in determining the magnitude of the response of output to liquidity shocks. In particular, this response is greater the larger the depreciation rate. We discuss this aspect later on.

¹³Nevertheless, asset prices change drastically when these parameters are modified.

model.

Two sets of parameters remain to be calibrated: θ , the parameter that governs the limited enforcement problem in investment and the parameters related to the liquidity shocks: Φ and Π . The limited enforcement parameter satisfies $0 \leq \theta < 1 - \pi = 0.9$. Thus, there is very large degree of freedom to calibrate θ . Moreover, there is almost an observational equivalence as shown in Proposition 7. Therefore, for comparison reasons only, we set $\theta = 0.15$, as this is the value chosen by del Negro et al. [2010]. The lower and upper bounds of Φ are calibrated by setting $(\phi_L, \phi_H) = (0.0, 0.3)$. This choice renders an unconditional mean of liquidity of 0.15 as in del Negro et al. [2010].

The meaning of θ should not be taken too literally. There are many ways to introduce financial frictions, the simplest of which is the limited enforcement problem presented here. In our view, θ should be understood as a parameter that captures the idea that external financing requires a down-payment to curtail incentives. Neither should $\frac{1}{\theta}$ be understood as a leverage ratio for the economy. The financial contracts described in the model as such that equity is sold to finance the down-payment. In reality, equity is sold and used as collateral. As described before, this distinction is irrelevant for the model but implies very different leverage ratios.

In the alternative calibration, we choose Φ using as a target a mean risk-free rate of roughly 1% and volatility of 1.7% which are roughly the values for the U.S. economy.¹⁴ We argued previously that θ and π determine the liquidity frontier. Φ and Π determine how often and how deeply will liquidity shocks enter the liquidity constrained region. Lower values θ and ϕ (on average) cause q to become increase. These parameters interplay to determine the volatility of the price of equity and the stochastic discount factor since they influence the effect on consumption after the entrepreneur switches from one type to another (equations (24) and (25)). Based on this consideration, we set $(\phi_L, \phi_H) = (0.17, 0.25)$ in the alternative scenario.

Finally, we describe the calibration strategy for the transition matrix Π . There are three desirable features that the transition probability Π should have: (1) That it induces (A_t, ϕ_t) to be correlated. Bigio [2009] and Kurlat [2009] study the relation between liquidity and aggregate productivity when liquidity is determined by the solution to a problem of asymmetric information. Both papers provide theoretical results in which the relation between productivity and liquidity is positive. (2) That A_t and ϕ_t are independent of ϕ_{t-1} . In principle, at least in quarterly frequency, liquidity should not explain aggregate productivity. We relax this assumption in the appendix where we discuss the effects of rare and persistent liquidity crises. (3) A_t has positive time correlation, as in the RBC-literature. We calibrate the transition to obtain these properties. In the computer \mathbb{A} is discrete. We let Π_A be the discrete approximation to an $AR(1)$ process with typical element $\pi_{a',a} = \Pr \{A_{t+1} = a' | A_t = a\}$. To construct the joint process for A_t and ϕ_t we first construct a monotone map from Φ to A and denote it by $A(\phi)$. This map is simply the order of elements in Φ . Let Π_u be a uniform transition matrix and let $\aleph \in [0, 1]$. We fill in Π by letting its typical element $\pi_{(a',\phi') \times (a,\phi)}$ be defined in the following way:

¹⁴We discuss the asset pricing properties of the model in the appendix

Calibration	Preferences					Technology				Aggregate Shocks					
	β	γ	σ	ν	ω	α	λ	π	θ	ρ_A	μ_A	σ_A	\aleph	ϕ_L	ϕ_H
Baseline	0.976	1	1	2	0.9	0.36	0.974	0.1	0.15	0.95	0	0.016	0.95	0	0.3
Alternative	0.976	0.9	10	2	0.9	0.36	0.974	0.1	0.15	0.95	0	0.016	0.95	0.17	0.25

Table 1: Calibration

$$\begin{aligned}
\pi_{(a',\phi') \times (a,\phi)} &\equiv \Pr \{A_{t+1} = a', \phi_{t+1} = \phi' | A_t = a, \phi_t = \phi\} \\
&= \pi_{a',a} \times (\aleph \cdot \pi_{A(\phi'),a} + (1 - \aleph) \cdot \Pi_u)
\end{aligned}$$

Approximated in this way, Π has all of the 3 properties we desire. The parametrization \aleph governs the correlation between A_t and ϕ_t . By construction (A, ϕ) are independent of ϕ , but A follows an AR(1) process. Under this assumption, the model requires 5 parameters to calibrate Π : (ρ_A, σ_A) the standard parameters of the AR(1) process for technology, (ϕ_L, ϕ_H) the lower and upper bounds of Φ and \aleph the degree of correlation between liquidity shocks and technology shocks.

When we study the asset pricing properties of the model, we find that the standard deviation of the risk-free rate approximates better the values found in the data when \aleph is close to 1. It is worth noting that when liquidity shocks are endogenous, as in [Bigio \[2009\]](#) or [Kurlat \[2009\]](#), a high autocorrelation, \aleph and (ϕ_L, ϕ_H) are delivered as outcomes of the model. The equity premium is higher for a higher values of the correlation. For these reasons we set $\aleph = 0.95$. The same risk free rate may be obtained by different combinations of Φ and a new parameter \aleph (which we discuss below) but at the expense of affecting its own volatility and the mean of the equity premium.

Alternative Calibration Strategy: An alternative calibration strategy could have followed from the work of [Chari et al. \[2007\]](#). These authors estimate an investment wedge for the U.S. economy, by reverse-engineering a wedge in a standard euler equation. The model presented here, delivers a similar wedge for each entrepreneur type. Since the model has two representative entrepreneurs (as well as hand-to-mouth workers), a direct mapping cannot be constructed from the wedges here to the wedges in that paper. An alternative calibration strategy may reverse-engineer the wedges in our paper, and then calibrate the liquidity shocks to deliver similar wedges as found in the data. We leave this task for a more detailed model.

4.2 Qualitative Results

In this section, we describe the main qualitative findings of the model obtained by computing the recursive competitive equilibria.¹⁵ These equilibria are computed using a grid size

¹⁵Fortran© and Matlab© codes are available upon request. The codes run in approximately 25 minutes and 2 hours respectively. For log-preferences, the computation time is reduced more than 100 times.

of 6 for \mathbb{A} and Φ . The grid size for the capital stock is 120. No important changes in the moments of the data generated by the model are obtained by increasing the grid size. Figure 3 plots q , q^i and q^R over the state-space. Each panel corresponds to a combination of an aggregate productivity shock and a liquidity shock. The aggregate capital state is plotted in the x-axis of each panel. Some plots show a threshold capital level for which q is above one and equal to 1 for capital above that level. Those points of the state-space coincide with points where the liquidity frontier is crossed. The figure helps explain how there are two forces in determining q in the model. By comparing the panels in the left from those in the right, we observe how q is lower in states where liquidity is higher. This happens because more liquidity relaxes the enforcement constraint at the aggregate level. To clear out the equity market when liquidity is tighter, q must increase to deter saving entrepreneurs from investing in projects that don't satisfy the incentive compatibility constraint. On the other hand, by comparing the upper row with the one on the bottom, we observe the effects of higher aggregate productivity. q increases when productivity is higher. This happens because agents demand more investment at these states since returns are predicted to be higher in the future. Inside the area below the liquidity frontier, the enforcement constraint is binding at an aggregate level. In these states, q must increase to make investing entrepreneurs invest more, and deter saving entrepreneurs from financing projects externally.

The corresponding plots for aggregate investment are depicted in Figure 4. Each panel plots 5 curves. These curves are the aggregate investment, investment in new projects owned by saving and investing entrepreneurs (the aggregate levels of i^s and i^i) and the net change in the equity position of both groups. The plots show that investment is close to being a linear function of capital, but this relation changes when the liquidity frontier is crossed. Why? Notice that there is a kink in aggregate investment when q rises above 1, which corresponds to points inside the liquidity frontier. Outside the liquidity constrained region, the lower the capital stock, the larger the demand to replenish the capital stock (returns to equity are higher). Inside the liquidity constrained region, the lower the capital stock, the lower the amount of liquid assets for the same liquidity shock. Because liquidity is lower, investing entrepreneurs have a smaller amount to use as down-payment: investment must be lower. This effect is explained in Section 2.5. In addition, there is an important wealth transfer from investing to saving entrepreneurs in the liquidity constrained region. Thus, the overall effect on investment is also influenced by a wealth effect.

The plots show that investment is greater as aggregate productivity increases but falls when there is less liquidity. Inside each plot, we observe that the fraction of investment owned by investing entrepreneurs is always around $(1 - \theta)$ which is 85% in our base calibration. The net increase in equity is much less though. Inside the model, investing entrepreneurs are financing new projects by selling older ones. This implies that they are selling a significant fraction of their old equity stock. Outside the liquidity constrained region, the increase in equity for both groups is exactly proportional to their relative populations. Below the liquidity frontier, the increase in equity by investing entrepreneurs increases because investing entrepreneurs sell older projects at a price q and their cost of generating equity is $(q^R)^{-1}$. Arbitrage opportunities translate into a wealth transfer from savers to investing entrepreneurs. Clearly, inside the liquidity constrained region, the volume of trade

in old projects falls as liquidity shocks approach to the liquidity frontier.

Marginal propensities to save are constant and equal to β for log-utility. When $\sigma < 1$, they satisfy the following characteristics: (1) in the constrained region, investing entrepreneurs have a lower marginal propensity to save which is by a strong governed by a wealth effect; (2) the marginal propensity to save of saving entrepreneurs is decreasing in q and the converse is true about investing entrepreneurs. These relations are reverted when $\sigma > 1$. Nevertheless, simulations shows that for different choices of σ , marginal propensities to save are not constant but don't move very much.

As in [He and Krishnamurthy \[2008\]](#) and [Brunnermeier and Sannikov \[2009\]](#), this model also highlights the importance of global methods in understanding the interaction between liquidity shocks and the constraints. Is linearizing the model a bad idea? Yes and no. No because, the behavior of inside either region seems to be very linear for our choice of parameters. This, in fact, is not true for other parameters. On the other hand, linearization may become problematic. The reason is that it may be highly likely that the liquidity frontier is crossed often because in stochastic environments, the capital stock is often above the desired level of capital. These states necessarily lie in the unconstrained region. [Bigio \[2009\]](#) shows that if illiquidity is caused by asymmetric information, liquidity shocks will fall close to the liquidity frontier. Moreover, in the appendix, we argue that in order to replicate a reasonable risk-free rate, liquidity shocks *must* fall close to the liquidity frontier. Because it's given in an analytical form (at least for log-preferences), the liquidity frontier may be used to asses how close the deterministic steady-states of an approximated model is to the liquidity frontier.

Figure 7 depicts the marginal unconditional distribution of the capital stock as well as the marginal distribution of conditional on constraints being binding. The next section describes the main quantitative properties of the model.

4.3 Moments

Statistics: Table 2 reports the standard RBC moments delivered by the model. The table is broken into the moments corresponding to the unconditional distribution and conditioned on the event that the economy is in the liquidity constrained region. Whenever they appear, the numbers in parenthesis correspond to the standard deviation corresponding to the average above them. All of the results are computed with respect to the model's stationary distribution of the aggregate state. The bottom row of the table reports the computed occupation time within the region below the liquidity frontier S^b (the region where liquidity constraints bind). For the baseline calibration, the occupation time within the liquidity constrained region is roughly 87%. The first row presents the average value of q , which is close to 1.5, meaning that the wedge between the cost of capital and the transformation rate is about 50%. In the constrained region, the wedge is on average 16% larger than its unconditional mean and 58% larger than in the unconstrained region. On the other hand, q is not very volatile. The standard deviation is about 0.5%. The second row reports the average unconditional investment/output ratio which is around 20%. This rate is slightly smaller in the constrained region because investment is inefficiently low in the constrained region. The average within the unconstrained region is considerably larger and equal to 30%. The

unconditional volatility of investment over the volatility of output is almost 4 times larger. This relative volatility is much lower within the unconstrained region. This figure is not surprising once one considers that workers are not smoothing consumption (on their own will). The fourth and fifth rows describe the average marginal propensities to consume of both entrepreneurs.¹⁶ The seventh column reports the relative size of output within each region. Output is on average 2.5% lower within the constrained region and 15% larger outside this region. Two things explain this large differences: (1) lower output is associated with a lower capital stock, which in turn, increases returns and the probability of binding constraints. (2) low liquidity is associated with low aggregate productivity shock. Similarly, worked hours are considerably larger within the unconstrained region.

The unconditional correlation between q and investment is negative and close to -95%. This happens because, most of the time, the economy is in the liquidity constrained region and the relation between these two variables is negative in this region (but theoretically undetermined). The fact that financing constraints can break the relation between q and investment is also found in prior work by [Lorenzoni and Walentin \[2009\]](#). In our model, the relation between q and investment breaks is explained by [Figure 4](#). The reason is that these correlations are driven by two forces that work in opposite directions: *ceteris paribus* liquidity shocks reduce these correlations and aggregate productivity shocks increase them. When liquidity shocks hit, the economy is more constrained, so q must rise to clear the market for new equity. This occurs together with a fall in investment. On the other hand, positive productivity shocks drive q as larger returns to equity are expected. q grows to clear out the equity market while investment is larger. The correlation computed under the baseline calibration is considerably lower than in the data. This suggests that the liquidity shocks calibrated here are extremely large, a feature that we openly acknowledge as we want to get as much as we can from the shocks.

Another interesting feature of the model is that it generates right skewness in the stationary distribution of the capital stock compared to a frictionless economy. The reason is that when the capital stock is lower, the liquidity constraints tend to be tighter. This feature is reflected into less investment than otherwise, which, in turn makes the capital stock recover at a slower pace. On the other hand, the average capital stock is lower in this economy than in a frictionless economy. One way to see this is that under log-preferences, the marginal propensities to save are constant, so all of the effect on investment is through a substitution effect. In the case of saving entrepreneurs, this effect is negative and dominates the increase on investment by investing entrepreneurs.

Results under the alternative calibration: [Table 3](#) reports the model's statistics for the alternative calibration. The calibration is such that the model delivers a reasonable risk-free rate, a feature that is discussed in the appendix. The important thing to note here is that the model requires liquidity shocks to be calibrated so that they fall in the region close to the liquidity frontier in order to obtain reasonable risk-free rates. Under this alternative calibration, the occupation time of liquidity constraints is 75% which is not very far from the occupation time in the baseline calibration. In contrast, the magnitude of liquidity shocks is much lower. This reflects itself in the average conditional and unconditional values of q , which

¹⁶In the baseline scenario, these are constants and equal to β .

is much lower and implies a lower wedge. The share of investment over output remains roughly at 20%. The volatility of investment over output falls considerably which reflects the smaller distortions on investment. In addition, the differences in output and hours are also considerably smaller. More interestingly, as opposed to the baseline scenario, average output and hours are larger in the liquidity constrained region. The reason is that liquidity constraints are driven much more with positive aggregate productivity (see discussion on the liquidity frontier), than by strong liquidity shocks. This aspect of the alternative calibration is also reflected in a larger and positive correlation between q and investment. In this scenario, the effects of technology shocks are more powerful than those corresponding to liquidity shocks, so the force towards a positive correlation dominates.

The analysis shows that the average marginal propensities to save for both entrepreneurs are close to each other. They vary in less than 3 decimal points along the whole state space and the standard deviation has an order of magnitude less than 5 decimal points. This suggests, as explained before, that the log utility is not a bad approximation if the discount factor is adjusted accordingly.¹⁷ Finally, the computations show that the unconditional distribution of capital is tighter, shifted to the right and less skewed.

Finally, the fact that asset prices and the relation between q and investment are closer to reality in the alternative calibration suggests that the size of liquidity shocks in the model are overstated. This reinforces the conclusions we obtain from the impulse response analysis: without labor frictions, liquidity shocks will not have an important effect on output.

Impulse Responses: Figure 5 plots the impulse response of several key variables after a drop in liquidity from the average level of 15% to 0%. The drop occurs only for one quarter after which liquidity follows its stochastic process. The responses are computed setting the economy at its unconditional average state. The dynamics implied by liquidity shocks resemble investment shocks. The plots show that investment falls contemporaneously with the drop in liquidity. The liquidity shock drives up q because there is less amount of liquidity to meet the supply of sources of external financing. An increase in q benefits investing entrepreneurs but at the aggregate level, induces a substitution effect towards consumption. Because the capital in place is fixed, output only responds 1 period after the shock is realized. All of the subsequent effect on output is driven by the fall in the capital stock which is caused by the fall in investment. Because output is constant during the period of the shock, overall consumption increases. Hours and wages co-move with the capital stock so both fall together in the periods after the liquidity shock is gone. The probability of binding constraints jumps in the period where the shock occurs, and remains slightly above the unconditional average as the returns to capital are larger.¹⁸ After the liquidity shock is gone, all of the variables in the model remain at lower levels as the capital stock slowly converges to its stationary level.

The calibration is purposely constructed in such a way that the overall effect on output is the largest possible. The peak drop on output is close to 0.7% of the unconditional mean of output, suggesting that the effects of an extreme fall in liquidity are not very large. We devote the next section to explain why. The dynamics under the alternative calibration are

¹⁷For example, for this scenario, one can choose a value of 0.971 for β and the results will be roughly similar.

¹⁸Recall that the liquidity frontier is higher when returns are larger

Variable	Moments		
	Unconditional	Constrained	Unconstrained
q	1.5 (0.499)	1.58 (0.492)	1 (0)
i/y	0.186	0.175	0.291
$\sigma(i) / \sigma(y)$	3.84	4.33	1.13
s_i	0.976 (0)	0.976 (0)	0.976 (0)
s_s	0.976 (0)	0.976 (0)	0.976 (0)
y/E[y]	1	0.977	1.15
hours/E[hours]	1	0.992	1.05
corr(q,i)	-0.945	-0.941	0
Occupation Time		0.871	0.129

Table 2: Key statistics under baseline calibration.

very similar. On the other hand, the effects of liquidity shocks are non-linear: a fall in liquidity from 15% to 10% delivers an overall effect 0.08% which is 10 times smaller than when liquidity falls from 15% to 0%.

4.4 Discussion: Why are the effects so small?

There is an intuitive explanation for why the liquidity shocks have a minor impact on output. For any reasonable calibration the ratio of investment to capital ratio will be small. Liquidity shocks introduce a friction on investment (when it is positive). The strongest possible effect of a liquidity shock is to prevent the economy from investing at all. Thus, at most, capital can fall by $1 - \lambda$. Without considering labor effects, any disruption acting on investment will drop output by no more than $(1 - \lambda^\alpha) \%$. In our calibration, this number is about 1.0%. In the model, the labor supply reacts positively to a fall in the capital stock so the effect is partially mitigated. On the other hand, liquidity shocks will not drive investment to 0 because investing entrepreneurs can invest by using internal funds. Thus, the fall in 0.7% in output should be expected.

Liquidity shocks also generate the wrong contemporaneous co-movement between output and consumption. This happens because, in the model, liquidity shocks don't affect any of the factors of production contemporaneously. Since output remains unchanged in the period of the shock, liquidity shocks distort allocations towards consumption. This result is originally pointed out in [Barro and King \[1984\]](#).

To generate stronger effects, liquidity shocks must interact with a distortion on labor. Models that introduce nominal rigidities are able to achieve a stronger amplification because a small change in the capital stock has important effects on output. Nominal rigidities prevent prices from adjusting so quantities fall. Although the original effect may be small, it propagates further more because a fall in aggregate demand causes output to contract fur-

Variable	Moments		
	Unconditional	Constrained	Unconstrained
q	1.02 (0.0298)	1.04 (0.0297)	1 (0)
i/y	0.199	0.201	0.191
$\sigma(i) / \sigma(y)$	1.48	1.45	1.51
s_i	0.971 (8.75e-005)	0.971 (7.95e-005)	0.971 (5.67e-005)
s_s	0.971 (6.76e-005)	0.971 (7.21e-005)	0.971 (5.67e-005)
y/E[y]	1	1.01	0.982
hours/E[hours]	1	1	0.997
corr(q,i)	0.135	0.039	0
Occupation Time		0.649	0.351

Table 3: Key statistics under alternative calibration.

ther more as prices cannot adjust. In [Justiniano et al. \[2010\]](#), investment shocks are combined with nominal rigidities so their effects are magnified and generate the right co-movement between output and consumption. For this same reason, liquidity shocks in [del Negro et al. \[2010\]](#), have important effects on output.

The finding that liquidity shocks have a muted impact is related to previous results found by [Kocherlakota \[2000\]](#) and [Cordoba and Ripoll \[2004\]](#) but for different reasons. [Kocherlakota \[2000\]](#) and [Cordoba and Ripoll \[2004\]](#) focus on borrowing limits that depend on asset values as an amplification mechanism based on the model of [Kiyotaki and Moore \[1997\]](#). Both papers agree that borrowing limits do not have much power as amplification mechanisms of productivity or monetary shocks under standard preferences. According, to [Liu et al. \[2010\]](#), the reason behind the muted response of output is that asset prices don't react much to the shock considered. The problem with collateral constraints is that even if one finds a shocks that reacts strongly, the effects on output comes from a large reallocation of a production input (capital or land) from the constrained sector to the unconstrained sector. Unfortunately, this explanation is unsatisfactory because we don't see large reallocations of capital or land during recessions. If there is some factor being reallocated during a crises it is more likely to be labor. Labor, off course, is rented, not bought, so it hardly has the feedback effect from prices highlighted by [Kiyotaki and Moore \[1997\]](#).

In summary, liquidity shocks must be combined with other mechanisms that act by affecting the labor input in order to have strong effects on output. The investment wedge in this paper cannot be directly mapped into the investment wedge in [Chari et al. \[2007\]](#) because the model presented here does not have a representative agent (it has two). Nevertheless, the findings here are consistent with theirs in suggesting that a labor wedges are more likely to be important in explaining the business cycle. The next section introduces variable capital utilization. This will introduce a wedge in labor demand that is a function of q. This provides an example of a mechanism that disrupts investment and labor decisions

together.

5 Extension: Variable Capital Utilization

We now modify the problem to introduce variable capital utilization as in [Greenwood et al. \[2000\]](#). With variable capital utilization, q has a negative effect on employment, so liquidity shocks also distort production. Again, we assume that the capital stock is fixed at the beginning of each period. The difference now is that in addition to choosing labor inputs, l , entrepreneurs also choose the fraction of the capital stock to be used in production, $h \in [0, 1]$. h is referred to as the capital utilization. Production is carried out according to a Cobb-Douglas production function that depends not on capital, but on the fraction of capital used in production: $F(hk, l) \equiv (hk)^\alpha l^{1-\alpha}$. The entrepreneur faces a trade-off for using a larger share of the capital stock. By using more of its capital stock, capital depreciates at a faster rate. Depreciation is modeled as a convex function of capital utilization $\delta(h) \equiv (1 - \lambda) - \frac{(h/b)^{(1+d)}}{(1+d)}$, where $d > 0$ is a parameter governing the convexity of the function and b is a normalization constant. The objective of the firm now takes into consideration the value of capital losses $q\delta(h)$ induced by the utilization. Under decentralized production, by arbitrage, the rental rate of capital must incorporate the loss in value of more utilization. Since we assume production is carried out efficiently by entrepreneurs, the allocations must equal decentralized production. Therefore, the objective of the firm is, as before, linear in the capital stock:

$$\max_{h, l/k} \left(\left[AF \left(h, \frac{l}{k} \right) \right] - w \frac{l}{k} - q\delta(h) \right) k$$

The derivation of the equilibrium conditions is similar as before. We jump straight to the results as nothing substantial changes.¹⁹

Main implications: We set d to 2 which is close to the values chosen by [Greenwood et al. \[2000\]](#). b is chosen to normalize utilization to be equal to 1 at the peak of the business cycle. λ is calibrated in order to keep the annual average depreciation rate at 10%. The rest of the parameters are kept constant. The average labor share of output is not affected (non-surprisingly as capital utilization is close to 1), so there is no need to modify α . We abstract from analyzing the statistics and focus on the response of output reported in [figure 6](#). The most striking finding is the fall in output which is almost 10 times larger than for the benchmark model. The mechanics work in the following way: when liquidity is tighter, as before, q increases. This increases the cost of capital utilization because the opportunity cost of depreciating capital is larger. Under the calibration, utilization falls in about 12% according to the dashed curve in the bottom right panel of the figure. Since, capital utilization is a complement of labor, the productivity of labor falls. Correspondingly, real wages and hours fall. In particular, the fall in worked hours is close to 2.5% which is significantly larger than the 0.3% found in the benchmark model. The reduction of two variable inputs of production, utilization and hours, reduces output contemporaneously. The magnitude of these effects

¹⁹The detail of equilibrium conditions for all of the extensions to the model is available by the author upon request.

crucially depends on the parameter d , which measures the elasticity of the response of utilization. Smaller values of these parameter increase the overall effect on output, but for a close set of parameters, the size of the effect on output does not vary much.

On the other hand, the response of consumption is negligible, but shows the same positive jump at the time of the liquidity shock. A decomposition of consumption by agent type, shows that it falls substantially for workers and investing entrepreneurs but, as before, increases for saving entrepreneurs due to a strong substitution between consumption and investment. At the aggregate level, the latter effect dominates. Following the period after the shock vanishes, the capital stock is lower and the rest of the variables recover slowly together lead by the evolution of the capital stock.

Alternative models with effects on labor: An alternative way in which liquidity shocks could affect the labor demand may be introduced by requiring part of the wage bill to be payed up-front. In such an environment, liquidity shocks affecting the working capital of the firm, would introduce a wedge between the marginal productivity of labor and wages. In particular, such constraints would make labor inefficiently low. The implications of variable capital utilization and working capital requirements are similar because in both environments, a variable input is affected by liquidity shocks. The same could be said of any other variable input such as intermediate goods for production.

Models with nominal rigidities have similar effects but for different reasons. After the occurrence of a liquidity shock, the aggregate demand for produced goods falls. In a frictionless this has now effects since prices adjust automatically. With nominal price rigidities and differentiated goods, the change in the relative demand of consumption to investment goods also shift demand away from the sector of firms stuck with lagged prices. This fall in demand the relative demand is met with a reduction in current output. Under nominal wage rigidities, this effect is further amplified. In comparison with [del Negro et al. \[2010\]](#), the same dramatic liquidity shock produces a similar effect as the model with variable capital utilization.²⁰

6 Concluding Remarks

This paper develops a framework that extends analysis of the model formulated by [Kiyotaki and Moore \[2008\]](#) into a fully stochastic environment. We show that two regimes coexist under which the economy behaves as in the standard RBC economy and regimes where the economy suffers liquidity shocks that have the effect of negative investment shocks. We conclude the paper providing several suggestions for future research.

Alternative amplification: Along the paper we stress that, in absence of frictions operating on the labor market, liquidity shocks may not explain sizeable changes in output. To have important effects on output, the model needs to be modified so that liquidity shocks also affect labor or intermediate good input decisions. We introduced variable capital utilization to highlight this point but perhaps the most natural way to do so is by imposing limited enforcement of labor contracts or for the purchase of intermediate goods. This aspect would

²⁰For the version of their model in which the Central Bank does not respond to the liquidity shock.

generate a working capital requirement for firms. With less liquidity, the amount of working capital is restricted and the allocation of labor inputs becomes inefficient.

An alternative way of affecting inputs is as by introducing nominal rigidities. [del Negro et al. \[2010\]](#) use Calvo pricing for this purpose. The drawback of this approach is that the same friction that amplifies liquidity shocks also induces a very large reduction in profits and labor income for not adjusting. In light of this, one wonders whether the same amplification is obtained in a model with state-dependent rigidities. Microeconomic evidence about pricing behavior during the crisis may be very useful to distinguish whether nominal rigidities are an important friction when shocks are large.²¹

Micro-foundations for liquidity shocks: The next step towards a better theory of liquidity is to explain what drives these shocks? We conceive three potential directions to explain large fluctuations in liquidity. First, adverse selection in the market for collateral may explain why liquidity constraints may be suddenly tighter. In a standard real business cycle model with idiosyncratic production risk, the distribution of these shocks don't matter. If entrepreneurs have private information on the quality of their capital, liquidity shocks may arise endogenously as a problem of asymmetric information at the time they sell older projects. [Eisfeldt \[2004\]](#) studies a problem with asymmetric in general equilibrium to understand risk-sharing. More recently, [Bigio \[2009\]](#) and [Kurlat \[2009\]](#) have independently developed such a theory based on asymmetric information on the quality of capital and equity respectively in the context of KM.

A second line of research should study the relation between liquidity and noisy information similar to some recent theories on demand shocks such as [Angeletos and La'O \[2009\]](#) or [Lorenzoni \[2009\]](#). For example, some assets may become illiquid if their are discrepancies in their valuations caused by noisy information.

A third approach should relate liquidity with an explicit financial system. [He and Krishnamurthy \[2008\]](#), [Brunnermeier and Sannikov \[2009\]](#), [Gertler and Karadi \[2009\]](#), [Gertler and Kiyotaki \[2010\]](#), and [Curdia and Woodford \[2009\]](#) study aspect when financial institutions are also subject to limited enforcement constraints, so changes in the net-worth of the financial system have effects on their role as intermediaries. [Bigio \[2010\]](#) takes a different approach. That paper stresses that the net-worth of the financial system is key in determining the equilibrium in markets with asymmetric information. When the financial system's position gets worse, it can bare less risk so pooling contracts are worse and liquidity drops.

Other assets: There also remains a technical challenge for introducing money or other assets in this economy in a fully stochastic environment. A second asset in this environment will feature a discontinuity in its excess demand at the liquidity frontier. This aspect may be problematic to compute equilibria of the model. Perhaps this difficulty may be resolved by introducing a second degree of heterogeneity but there is no immediate answer to do this.

Introducing other assets may be important for policy and positive analysis. This extension may be a useful to explain the effects of monetary policy and other unconventional policies in different points of the cycle. For example, it would be useful to understand if

²¹Moreover, the policy implication are very different when inefficiencies in labor input choices are explained by nominal rigidities instead of working capital requirements. The former suggests that policies targeted at reducing nominal rigidities may be equally effective in reducing the effects of liquidity shocks.

debt financing of government deficits may cause a crowding-out effect that worsens the lack of liquidity.

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7 Proof of Optimal Policies (Propositions 1,2,3)

This section provides a proof of the optimal policies described in Section 3. The strategy is guess and verify. For *saving* entrepreneurs, the guess is: $V(w; t) = U(a_t^s w)$, $c^s(w; t) = (1 - \varsigma_t^s) w$, $n_{t+1}^s = \frac{\varsigma_t^s w}{q_t}$ and for the *investing* entrepreneur, the guess is: $V^i(w_t^i) = U(a_t^i w_t^i)$, $c^i(w_t^i; t) = (1 - \varsigma_t^i) w_t^i$, $n_{t+1}^i(w_t^i) = \frac{\varsigma_t^i w_t^i}{q_t^i}$.

7.1 Step 1: First Order Conditions

Using this guess, the first order conditions for ς_t^s is:

$$(\varsigma_t^s) : q_t U'(c_t) = \beta \left(E_t \left[\pi \Upsilon(a_t^s w_{t+1}^s) + (1 - \pi) \Upsilon(a_t^i w_{t+1}^i) \right] \right)^{\frac{\gamma-1/\sigma}{1-\gamma}} \dots \quad (22)$$

$$\left((1 - \pi) \Upsilon'(a_{t+1}^s w_{t+1}^s) a_{t+1}^s R_{t+1}^{ss} + \pi \Upsilon'(a_{t+1}^i w_{t+1}^i) a_{t+1}^i R_{t+1}^{is} \right)$$

The last term follows from independence in investment opportunities. Observe also that wealth is mapped into future wealth by the following identities: $w_{t+1}^s = R_{t+1}^{ss} \varsigma_t^s w_t^s$, $w_{t+1}^i = R_{t+1}^{is} \varsigma_t^i w_t^i$, $w_{t+1}^s = R_{t+1}^{si} \varsigma_t^i w_t^s$. Using these definitions w_t^s is factored out from (22) to obtain:

$$(n^{',s}) : U'(c_t) = \beta (\varsigma_t^s w_t)^{-1/\sigma} \left(E_t \left[(1 - \pi) \Upsilon(a_{t+1}^s R_{t+1}^{ss}) + \pi \Upsilon(a_{t+1}^i R_{t+1}^{is}) \right] \right)^{\frac{\gamma-1/\sigma}{1-\gamma}} \dots \quad (23)$$

$$E_t \left[(1 - \pi) \Upsilon'(a_{t+1}^s R_{t+1}^{ss}) a_{t+1}^s R_{t+1}^{ss} + \pi \Upsilon'(a_{t+1}^i R_{t+1}^{is}) a_{t+1}^i R_{t+1}^{is} \right]$$

By replacing the guess for consumption and clearing out wealth, an Euler equation in terms of the marginal propensities to save is obtained:

$$(1 - \varsigma_t^s)^{-1/\sigma} = \beta (\varsigma_t^s)^{-1/\sigma} \left(E_t \left[(1 - \pi) \Upsilon(a_{t+1}^s R_{t+1}^{ss}) + \pi \Upsilon(a_{t+1}^i R_{t+1}^{is}) \right] \right)^{\frac{\gamma-1/\sigma}{1-\gamma}} \dots \quad (24)$$

$$E_t \left[(1 - \pi) \Upsilon'(a_{t+1}^s R_{t+1}^{ss}) a_{t+1}^s R_{t+1}^{ss} + \pi \Upsilon'(a_{t+1}^i R_{t+1}^{is}) a_{t+1}^i R_{t+1}^{is} \right]$$

An identical condition may be obtained for the investing entrepreneur by following the same steps:

$$(1 - \varsigma_t^i)^{-1/\sigma} = \beta (\varsigma_t^i)^{-1/\sigma} \left(E_t \left[(1 - \pi) \Upsilon(a_{t+1}^s R_{t+1}^{si}) + \pi \Upsilon(a_{t+1}^i R_{t+1}^{ii}) \right] \right)^{\frac{\gamma-1/\sigma}{1-\gamma}} \dots \quad (25)$$

$$E_t \left[(1 - \pi) \Upsilon'(a_{t+1}^s R_{t+1}^{si}) a_{t+1}^s R_{t+1}^{si} + \pi \Upsilon'(a_{t+1}^i R_{t+1}^{ii}) a_{t+1}^i R_{t+1}^{ii} \right]$$

7.2 Step 2: Verification of the Value function

Replacing our guess and using the envelope theorem we obtain $U'((1 - \varsigma_t^s) w_t) = V'_w(a_t^s w_t) a_t^s$. Thus, $(a_t^s)^{1-\frac{1}{\sigma}} = (1 - \varsigma_t^s)^{-\frac{1}{\sigma}}$ and similarly, for the investing entrepreneur $(a_t^i)^{1-\frac{1}{\sigma}} = (1 - \varsigma_t^i)^{-\frac{1}{\sigma}}$.

Rewriting both relations gives us or $a_t^j = (1 - \zeta_t^j)^{\frac{1}{1-\sigma}}$, $j = i, s$. The certainty equivalent of a unit of wealth tomorrow in the saving state is given by the function:

$$\Omega^s (a_{t+1}^s, a_{t+1}^i) = \Upsilon^{-1} E_t [(1 - \pi) \Upsilon (a_{t+1}^s R_{t+1}^{ss}) + \pi \Upsilon (a_{t+1}^i R_{t+1}^{si})]$$

which is homogeneous of degree 1. The same is true about:

$$\Omega^i (a_{t+1}^s, a_{t+1}^i) = \Upsilon^{-1} E_t [(1 - \pi) \Upsilon (a_{t+1}^s R_{t+1}^{is}) + \pi \Upsilon (a_{t+1}^i R_{t+1}^{ii})]$$

Equations (24) and (25) may be written in terms of this and we obtain:

$$(1 - \zeta_t^s)^{-\frac{1}{\sigma}} = \beta (\zeta_t^s)^{-1/\sigma} \Omega^s (a_{t+1}^s, a_{t+1}^i)^{1-1/\sigma}$$

and

$$(1 - \zeta_t^i)^{-\frac{1}{\sigma}} = \beta (\zeta_t^i)^{-1/\sigma} \Omega^i (a_{t+1}^s, a_{t+1}^i)^{1-1/\sigma}$$

Clearing out ζ_t^s from the right hand side, and adding 1 to both sides yields:

$$(1 - \zeta_t^s)^{-1} = 1 + \beta \Omega^s (a_{t+1}^s, a_{t+1}^i)^{\sigma-1}$$

and

$$(1 - \zeta_t^i)^{-1} = 1 + \beta \Omega^i (a_{t+1}^s, a_{t+1}^i)^{\sigma-1}$$

Note that any value function satisfies the envelope condition so any pair of functions (ζ_t^s, ζ_t^i) satisfying this recursion guarantee that the functional form guessed for V^i and V^s satisfy the Bellman equation. This operator appears in the statement of Proposition 3. Propositions 1 and 2 follow immediately. Finally, we show conditions under which 3 is a contraction.

7.3 Step 3: Uniqueness of policy functions

Assume that $\beta^\sigma E_t [R^{sj}(s)^{(\sigma-1)}] < \beta^\sigma E_t [R^{ij}(s)^{(\sigma-1)}] < 1$, where the expectation is with respect to the future state s and the agents type. To simplify notation let $x(s) = (1 - \zeta^i(s))^{-1}$ and $y(s) = (1 - \zeta^s)^{-1}$ so that recursions (24) and (25) become:

$$\begin{aligned} x(s) &= 1 + (\beta)^\sigma \Omega_t^s \left(x(s)^{\frac{1}{\sigma-1}}, y(s)^{\frac{1}{\sigma-1}} \right)^{\sigma-1} \\ y(s) &= 1 + (\beta)^\sigma \Omega_t^i \left(x(s)^{\frac{1}{\sigma-1}}, y(s)^{\frac{1}{\sigma-1}} \right)^{\sigma-1} \end{aligned}$$

where Ω_t^s and Ω_t^i are defined in the previous step. These equations define an operator $T : \mathbb{S}^{[1, \infty]^2} \rightarrow \mathbb{S}^{[1, \infty]^2}$. That is, the operator maps continuous bounded functions from the state \mathbb{S} to $[1, \infty]$ into itself. Since \mathbb{S} is compact, then $\mathbb{S}^{\mathbb{R}^2_+}$ is a complete metric space with respect to the sup-norm for the product space. Thus, T defines a recursion:

$$(x_{t+1}, y_{t+1}) = T(x_t, y_t)$$

We check that the recursion satisfies Blackwell's sufficient conditions that guarantee that T is a contraction.

Monotonicity. Whenever $\sigma < 1$, the power function $z^{\frac{1}{\sigma-1}}$ is decreasing. Thus, for any $x > x'$ we have: $x'^{\frac{1}{\sigma-1}} < x^{\frac{1}{\sigma-1}}$. Therefore, the certainty equivalent with respect to Υ for (x', y') is larger than that of the terms (x, y) . Hence, $\Omega_t^s \left(x(s)^{\frac{1}{\sigma-1}}, y(s)^{\frac{1}{\sigma-1}} \right) < \Omega_t^s \left(x'(s)^{\frac{1}{\sigma-1}}, y(s)^{\frac{1}{\sigma-1}} \right)$. Since $\sigma - 1$ is negative, then $\Omega_t^s \left(x(s)^{\frac{1}{\sigma-1}}, y(s)^{\frac{1}{\sigma-1}} \right)^{\sigma-1} > \Omega_t^s \left(x'(s)^{\frac{1}{\sigma-1}}, y(s)^{\frac{1}{\sigma-1}} \right)^{\sigma-1}$. If $\sigma > 1$, the argument is the same except that everything is monotone increasing. The analysis is identical for Ω_t^i . This shows the monotonicity of T .

Discounting 1. Assume $\frac{1-\gamma}{\sigma-1} \geq 1$. $\Omega_t^i \left(x(s)^{\frac{1}{\sigma-1}}, y(s)^{\frac{1}{\sigma-1}} \right)^{\sigma-1}$ can be transformed into a form of a certainty equivalent of a function $z^{\frac{1-\gamma}{\sigma-1}}$ where under a new probability measure. Using the definition of Ω_t^s , we have that,

$$\begin{aligned} & \left(\Upsilon^{-1} E_t \left[(1-\pi) \Upsilon \left((x(s) + \psi)^{\frac{1}{\sigma-1}} R^{ss}(s) \right) + \pi \Upsilon \left((y(s) + \psi)^{\frac{1}{\sigma-1}} R^{si}(s) \right) \right] \right)^{\sigma-1} \\ = & \left(\Upsilon^{-1} E_t \left[\frac{(1-\pi) \Upsilon \left((x(s) + \psi)^{\frac{1}{\sigma-1}} R^{ss}(s) \right) + \pi \Upsilon \left((y(s) + \psi)^{\frac{1}{\sigma-1}} R^{si}(s) \right)}{E_t [R^{sj}(s)^{1-\gamma}]} \right] \right)^{\sigma-1} \\ & \cdot E_t [R^{sj}(s)^{(1-\gamma)(\sigma-1)}]^{-\frac{1}{1-\gamma}} \end{aligned}$$

By dividing and multiplying by $E_t [R^{sj}(s)^{1-\gamma}]$ we are transforming the term inside the bracket into an expectation under a new probability measure where the probabilities are now weighted by the function $R^{sj}(s)^{1-\gamma} / E_t [R^{sj}(s)^{1-\gamma}]$. Thus, we have transformed the equation above into the certainty equivalent of a random variable under a convex function. When $\frac{1-\gamma}{\sigma-1} > 1$, then $z^{\frac{1-\gamma}{\sigma-1}}$ is convex. Observe that any random variable x and constant ψ , the certainty equivalent of $(x + \psi)$ under a convex function is smaller than ψ plus the certainty equivalent of x . Therefore, we have:

$$\begin{aligned} & 1 + (\beta)^\sigma \Omega_t^s \left((x(s) + \psi)^{\frac{1}{\sigma-1}}, (y(s) + \psi)^{\frac{1}{\sigma-1}} \right)^{\sigma-1} \\ \leq & 1 + (\beta)^\sigma \Omega_t^s \left((x(s))^{\frac{1}{\sigma-1}}, (y(s))^{\frac{1}{\sigma-1}} \right)^{\sigma-1} + \psi \beta^\sigma E_t [R^{sj}(s)^{(1-\gamma)(\sigma-1)}]^{-\frac{1}{1-\gamma}} \\ \leq & 1 + (\beta)^\sigma \Omega_t^s \left((x(s))^{\frac{1}{\sigma-1}}, (y(s))^{\frac{1}{\sigma-1}} \right)^{\sigma-1} + \psi \beta^\sigma E_t [R^{sj}(s)^{(\sigma-1)}] \end{aligned}$$

where the second inequality follows from the certainty equivalent under a concave function. Hence, $\beta^\sigma E_t [R^{sj}(s)^{(\sigma-1)}] < 1$ is sufficient to satisfy discounting. In particular it will hold for σ sufficiently close to 1 and R^{sj} bounded. The same argument can be shown to hold for the recursion on $y(s)$.

Final Step. To complete the proof one needs to modify Theorem 3.3 in [Stokey et al. \[1989\]](#) from single-valued to two-valued functions. This is done by using a new norm by taking

the max over the sup of each component. The rest of the proof is identical.

7.4 CRRA Preferences

One can follow the same steps as before to derive the operator for constant relative risk aversion preferences (CRRA). The recursion for the CRRA case is:

$$(1 - \zeta_t^s)^{-1} = 1 + \beta^\sigma E_t \left[(1 - \pi) (1 - \zeta_{t+1}^s)^{\frac{1}{\sigma}} (R_t^{ss})^{1 - \frac{1}{\sigma}} + \pi (1 - \zeta_{t+1}^i)^{\frac{1}{\sigma}} (R_t^{is})^{1 - \frac{1}{\sigma}} \right]^\sigma$$

and

$$(1 - \zeta_t^i)^{-1} = 1 + \beta^\sigma E_t \left[(1 - \pi) (1 - \zeta_{t+1}^s)^{\frac{1}{\sigma}} (R_t^{si})^{1 - \frac{1}{\sigma}} + \pi (1 - \zeta_{t+1}^i)^{\frac{1}{\sigma}} (R_t^{ii})^{1 - \frac{1}{\sigma}} \right]^\sigma$$

7.5 Logarithmic Preferences

Provided that $1/\sigma = \gamma = 1$, preferences are log, as in KM. Replacing the solution for the constrained guy, guessing and verifying: $\zeta_t^s = \zeta_t^i = \beta$ one may pull out ζ_t^i

$$\begin{aligned} (1 - \zeta_t^s)^{-1} &= 1 + \beta^\sigma (1 - \zeta_{t+1}^s)^{-1} \Omega(1, 1)^{\sigma-1} \\ &= \sum_{j=0}^{\infty} [\beta^\sigma \Omega(1, 1)^{\sigma-1}]^j \\ &= \frac{1}{1 - \beta^\sigma \Omega(1, 1)^{\sigma-1}} \end{aligned}$$

which satisfies

$$\zeta_t^s = \beta$$

for $\sigma = 1$. A similar argument guarantees the conjecture for ζ_t^i :

$$\zeta_t^s = \zeta_t^i = \beta$$

8 Proof of Equity Market Clearing

We use the market clearing condition for out $I^s(s)$ and $I^d(s)$ and substitute in the policy functions (1) and (2). We have that a solution at equality q must satisfy:

$$\frac{(1 - \pi) (\zeta^s (r + q\lambda) / qK - \lambda K) - \pi \varphi K}{\theta} \leq \frac{\pi [\zeta^i (r + q^i \lambda K) / q^R - (1 - \phi) K]}{(1 - \theta)}$$

Clearing out K from both sides and using the definition of q^i this expression becomes,

$$\begin{aligned} \frac{(1-\pi)\zeta^s r - ((1-\pi)(1-\zeta^s)\lambda + \pi\phi)q}{\theta q} &\leq \frac{\pi[\zeta^i(r + q(1-\phi)) - (1-\zeta^i)(1-\phi)q^R]}{(1-\theta)q^R} \\ &= \frac{\pi\zeta^i(r + q(1-\phi))}{(1-\theta q)} - \frac{\pi(1-\zeta^i)}{(1-\theta)}(1-\phi) \end{aligned}$$

We get rid of the denominators and rearrange terms to obtain,

$$\begin{aligned} &(1-\pi)\zeta^s r(1-\theta q) - ((1-\pi)(1-\zeta^s)\lambda + \pi\phi)q(1-\theta q) \\ &\leq \theta q \pi \zeta^i (r + q\phi) - \theta q(1-\theta q) \pi \frac{(1-\zeta^i)}{(1-\theta)}(1-\phi) \end{aligned}$$

From this equation we obtain a quadratic expression for q

$$\begin{aligned} &(1-\pi)\zeta^s r - q\theta(1-\pi)\zeta^s r - ((1-\pi)(1-\zeta^s)\lambda + \pi(1-\phi))q \\ &+ \theta q^2((1-\pi)(1-\zeta^s)\lambda + \pi(1-\phi)) \\ &\leq q\theta\pi\zeta^i r + q^2\theta\pi\zeta^i(1-\phi) - \theta q \frac{(1-\zeta^i)}{(1-\theta)}\phi + q^2\theta^2 \frac{(1-\zeta^i)}{(1-\theta)}\pi\phi \end{aligned}$$

This inequality is characterized by a quadratic equation in q^* ,

$$(q^*)^2 A + q^* B + C = 0 \tag{26}$$

with coefficients,

$$\begin{aligned} A &= -\theta \left(\left[(1-\pi)(1-\zeta^s) - \frac{\pi(1-\zeta^i)}{(1-\theta)}\theta \right] \lambda + \frac{\pi(1-\zeta^i)}{(1-\theta)}\phi \right) \\ B &= \theta r((1-\pi)\zeta^s + \pi\zeta^i) + ((1-\pi)(1-\zeta^s)\lambda + \pi\phi) - \theta \frac{(1-\zeta^i)}{(1-\theta)}(1-\phi)\pi \\ C &= -(1-\pi)\zeta^s r \end{aligned}$$

It is clear that C is negative. A pointwise inspection of A shows that this terms is also negative provided that $(1-\zeta^i)$ is arbitrarily close to $(1-\zeta^s)$ to each other. By Assumption 2, $(1-\pi) > \frac{\theta\pi}{(1-\theta)}$ and by Proposition 3, for σ sufficiently close to 1, $(1-\zeta^i)$ is arbitrarily close to $(1-\zeta^s)$. This means that the first term in the brackets is positive, so A will be negative. For σ sufficiently close to 1 again, B is also positive because,

$$(1-\pi)(1-\zeta^s)\lambda - \theta \frac{(1-\zeta^i)(1-\phi)}{(1-\theta)}\pi > (1-\pi)(1-\zeta^s)(\lambda - (1-\phi)) \geq 0$$

Evaluated at 0, (26) is negative. It reaches a maximum at $-\frac{B}{2A} > 0$ and then will diverge to $-\infty$. The discriminant (26) is also positive. Observe that,

$$B = B_1 + B_2$$

where

$$B_1 = ((1 - \pi)(1 - \zeta^s)\lambda + \pi\phi) - \theta \frac{(1 - \zeta^i)}{(1 - \theta)} (1 - \phi)\pi$$

and

$$B_2 = \theta r ((1 - \pi)\zeta^s + \pi\zeta^i)$$

So then, $B_2 > \theta(1 - \pi)\zeta^s r$ and $B_1 > \left((1 - \pi)(1 - \zeta^s)\lambda + \pi(1 - \zeta^i)\phi - \theta \frac{(1 - \zeta^i)(1 - \phi)}{(1 - \theta)} \pi \right)$. Thus, $B^2 - 4AC > B^2 - 4B_1B_2 = (B_1 - B_2)^2 > 0$. Thus, the roots (q_1, q_2) of (26) are both positive. (20) is satisfied when (26) is negative. We let q_2 be the greatest amongst these roots. There are three cases:

Case 1: If $1 > q_1$, then $q = 1$ the aggregate incentive compatibility constraint is not binding at $q = 1$.

Case 2: If $1 \in (q_1, q_2)$, then, $q = 1$ violates the aggregate incentive compatibility constraint. Since for $q > 1$, constraints hold with equality, it must be the case that $q = q_2$ is the unique value of q satisfying the constraint (20) at equality.

Case 3: If $1 < q_1$, (20) does not bind and $q = 1$ once more.

9 Additional Extensions

9.1 Asset Pricing Properties

Liquidity shocks deliver interesting asset pricing properties. It is known from the asset pricing literature that in order to obtain reasonable asset prices, the neoclassical growth model needs significant variation in q to generate a non-negligible equity premium. Liquidity shocks cause variation in q without resorting to adjustment costs as in [Jermann \[1998\]](#). Liquidity shocks also behave as a source of uninsurable idiosyncratic risk. From [Constantinides and Duffie \[1996\]](#), we know these source of variation may explain part of the equity premium. [Tallarini \[2000\]](#), shows that preferences that distinguish the intertemporal elasticity of substitution from risk aversion help to explain the equity premium puzzle. Since we extend the framework in KM beyond log-utility the model is placed in good footing. Recent asset pricing models combine production adjustment costs with EZ preferences and are capable of explaining asset prices(see for example [Campanale et al. \[2010\]](#) and [Croce \[2009\]](#)). Since liquidity shocks have the potential to explain some asset pricing features, we use asset prices to infer more about liquidity shocks.²² Finally, the model also delivers a state-dependent liquidity premium which is interesting on its own.

As a benchmark, consider the price of riskless bond in zero net supply. Note saving entrepreneurs determine the price of this asset. This follows since saving and investing entrepreneurs cannot have their euler equations satisfied with equality at the same time when $q > 1$. Since the return to private equity for investing entrepreneurs is higher, only saving entrepreneurs will purchase a bond. Thus, the euler equation of the later should determine pricing kernels.

The first order condition yields a pricing formula for any asset with return $r_{t+1}^{s,n}$.

$$1 = \beta \left[\frac{\varsigma_t^s}{(1 - \varsigma_t^s)} \right]^{-\frac{1}{\theta}} E_t \left[\left[\mathcal{C}\mathcal{E}_t \left((1 - \varsigma_{t+1}^j)^{\frac{1}{1-\theta}} R_{t+1}^j \right) \right]^{\gamma - \frac{1}{\theta}} (1 - \varsigma_{t+1}^j)^{\frac{(1-\gamma)}{1-\theta}} (R_{t+1}^j)^{-\gamma} r_{t+1}^{s,n} \right]$$

where R^s is the gross return of the portfolio of saving entrepreneurs. The stochastic discount factor here is expressed as a function of marginal propensities to consume and the return to the average portfolio. Hence the stochastic discount factor is given by:

$$\mu_t(s_{t+1}) = \beta \left[\frac{\varsigma_t^s}{(1 - \varsigma_t^s)} \right]^{-\frac{1}{\theta}} \left[\mathcal{C}\mathcal{E}_t \left((1 - \varsigma_{t+1}^j)^{\frac{1}{1-\theta}} R_{t+1}^j \right) \right]^{\gamma - \frac{1}{\theta}} \mathcal{E}_t \left[(1 - \varsigma_{t+1}^j)^{\frac{(1-\gamma)}{1-\theta}} (R_{t+1}^j)^{-\gamma} \right]$$

where

$$\mathcal{E}_t \left[(1 - \varsigma_{t+1}^j)^{\frac{(1-\gamma)}{1-\theta}} (R_{t+1}^j)^{-\gamma} \right] = \left[\pi (1 - \varsigma_{t+1}^i)^{\frac{(1-\gamma)}{1-\theta}} (R_{t+1}^i)^{-\gamma} + (1 - \pi) (1 - \varsigma_{t+1}^s)^{\frac{(1-\gamma)}{1-\theta}} (R_{t+1}^s)^{-\gamma} \right]$$

²²Since investment opportunities are random here, the model attempts to mimic lumpy investment. We don't introduce adjustment costs for this reason.

The return to a riskless bond in zero net supply is given by $R^b = E[\mu_t]^{-1}$. On the other hand, the return to equity for the saving entrepreneur defined in equation (11) can be decomposed in the following way:

$$\begin{aligned} R_{t+1}^s &= \pi \frac{r_t + \lambda q_{t+1}^i}{q_t} + (1 - \pi) \frac{r_t + \lambda q_{t+1}}{q_t} \\ &= \frac{r_t + \lambda q_{t+1}}{q_t} - \lambda \pi \left[(1 - \phi_{t+1}) \frac{(q_{t+1} - q_{t+1}^R)}{q_t} \right] \end{aligned}$$

This expression is convenient because it allows us to decompose the return on equity. The first term is the standard return on equity considering the depreciation. The second term reduces the return on equity by a liquidity component. With probability π a fraction $(1 - \phi_{t+1})$ of the remaining equity λ , becomes illiquid. This fraction reduces the return by the wedge $(q_{t+1} - q_{t+1}^R)$. Notice that this term disappears when $\phi_{t+1} = 1$.

Combining the euler equations corresponding to equity and a riskless bond the following condition is obtained:

$$E[\mu(s') (R^s(s') - R^b(s'))] = 0 \quad (27)$$

The excess returns on equity is decomposed in the following way:

$$\begin{aligned} E[R^s(s')] - R^b(s') &= - \frac{\text{cov}(\mu(s'), R^s(s'))}{E(\mu(s'))} \\ &= - \underbrace{\frac{\text{cov}(\mu(s'), (r(s') + q(s') \lambda))}{q(s) E(\mu(s'))}}_{\text{Pure Risk Premium}} \\ &\quad + \underbrace{\pi \lambda \frac{\text{cov}(\mu(s'), (1 - \phi_{t+1})(q(s') - q^R(s')))}{q(s) E(\mu(s'))}}_{\text{Liquidity Premium}} \end{aligned}$$

Hence, the private equity premium is obtained by the composition of two terms. These are, first a standard risk adjustment component: $\frac{\text{cov}(\mu(s'), (r(s') + q(s') \lambda))}{E(\mu(s'))}$ and second, a liquidity component that is given by the effects covariance between the wedge $(q_t - q_{t+1}^R)$ scaled by $(1 - \phi_{t+1})$ and the stochastic discount factor. When liquidity constraints don't bind, the second term vanishes. The lower the probability that constraints bind in the future, the liquidity component will be smaller.

Liquidity Spectrum: The decomposition allows us to obtain a relation between pure risk and liquidity. If particular asset's return (in particular, in zero net supply) is correlated with the returns to equity but is purely liquid, it would be $E[R^{Liquid}]$ have an expected payoff equivalent to $E[R^s(s')]$ minus the liquidity premium. Thus, we have a relation for the expected returns of assets with different liquidity:

Calibration	Risk Free Rate	Equity Premium	Liquidity Premium
Baseline	1.1069 (2.14)	0.22419 (0.542)	0.022063 (0.625)

Table 4: Key asset prices computed from unconditional distribution.

Calibration	Risk Free Rate	Equity Premium	Liquidity Premium
Baseline	1.02 (2.74)	0.23 (0.142)	0.0226 (0.0147)

Table 5: Key asset prices computed from distribution conditional on liquidity constrained region.

$$R^b(s') < E[R^{Liquid}(s')] < E[R^s(s')]$$

Calibration Results: We assume, although not modeled explicitly, that the economy experiences exogenous growth in labor productivity. For this reason, asset pricing formulas are computed using a different value for β equal 0.991. A similar calibration is used in [Campanale et al. \[2010\]](#). Tables 4 and 5 describe the asset pricing properties of the model corresponding to the unconditional distribution and the distribution conditional on binding constraints. The first column plots the expected returns of a risk free bond in zero net supply. The baseline scenario predicts returns above 1.1% which is close to the average in the U.S.. In the constrained region, the risk free rate is slightly lower but also close to 1%. These figures suggest that this rate does not vary much across regions. The volatility of the risk-free rate is close to 2% which is slightly above the average for the U.S. economy. On the other hand, the equity premium is 0.22% which is far below the premium found in the data, 7%. Finally, we find that the liquidity premium of equity is about 10% of the equity premium.

When liquidity shocks are more severe, that is, if we shift the support of Φ to the left, the average risk-free rate falls. This suggests the behavior of the risk-free rate is tied to the stationary distribution of the returns to equity. When liquidity shocks are stronger, the stationary distribution of capital shifts to the left, making the return to equity larger but more volatile at the same time. Risk-aversion makes the risk-free rates fall. This happens because q is on average higher, so there is more variation in consumption. The volatility of returns to equity also explains why \aleph needs to be large. With a low-correlation between aggregate productivity and liquidity, the return to equity will fluctuate due to changes in prices. By increasing \aleph , we are essentially lowering the conditional volatility of equity. Unfortunately, the effect on the equity-premium operate in the opposite direction, so this explains why the equity premium must be low, if the risk-free rate is close to 1. As opposed to [Tallarini \[2000\]](#), the model presented entrepreneurs face a lot of idiosyncratic risk, although conditionally the conditional volatility (on the agents type) of consumption is low.

9.2 Production Externality

We modify the model by introducing a production externality as in [Brunnermeier and Sannikov \[2009\]](#). We assume that labor is more efficient as the aggregate capital stock grows. In particular we assume that effective labor inputs are given by $\varphi(K)^{\frac{1}{1-\alpha}} l$. We assume that $\varphi(K) = \left(\frac{K_t}{\bar{K}}\right)^\delta$, where δ is a parameters that measures the degree of decreasing returns to scale on the production externality. \bar{K} is a parameters that allows to normalize the production externality to 1 when $K_t = \bar{K}$. The larger δ , the larger the externality of aggregate capital. At the beginning of each period, entrepreneurs take their own firms capital and the aggregate capital stock as given. They then, choose labor inputs optimally. Production is carried out according to a Cobb-Douglas under a new production function $F(k, l) \equiv k^\alpha (\varphi(K) l)^{(1-\alpha)}$. By homogeneity, maximization requires:

$$\max_{l/k} \left(\left[A \varphi(K) F \left(1, \frac{l}{k} \right) \right] - w \frac{l}{k} \right) k$$

Pursuing the same steps as before, we obtain that per firm return to capital is:

$$r_t = \Gamma(\alpha) [\varphi(K) A_t]^{\frac{\xi+1}{\alpha}} \bar{\omega}^{\frac{\xi}{\nu}} K_t^\xi$$

whereas the aggregate equilibrium labor input is:

$$L_t^*(A_t, K_t) = \left(\frac{\varphi(K) A_t (1-\alpha)}{\bar{\omega}} \right)^{\frac{1}{\alpha+\nu}} K_t^{\frac{\alpha}{\alpha+\nu}}$$

The terms $\Gamma(\alpha)$ and ξ where defined before. The last two equations are identical to equations (15) and (16) except for the fact that wherever A_t showed up, no the term is interchanged by $\varphi(K) A_t$. This implies that the total factor productivity is affected by the aggregate capital stock. The following assumption ensures that the capital stock is bounded.

Assumption 3. $\delta \leq 1$ and $\nu > 1$

This assumption is sufficient to ensure that $\xi + 1 \in (0, 1)$ so that returns to aggregate capital suffer diminishing returns to scale. The assumption that $\nu > 1$ is equivalent to saying that the labor supply responds to less than 1-to-1 with respect to wage increments. $\delta \leq 1$ states that there are no increasing returns to scale in the production externality.

Main Implications The main implication of the production externality are its effect on hours in the subsequent periods after the shock. Effectively, this externality works as if the capital share of output where larger. The additional effect on output follows from a direct effect on the externality and the indirect effect on labor. The magnitude of the fall depends on δ . The results aren't substantially affected for the same calibration choice as in [Brunnermeier and Sannikov \[2009\]](#).

9.3 Rare Liquidity Shocks

In this section we study a rare liquidity dry-up event in the model. We study the effects of assume that a liquidity may dry-up entirely with a very low probability. To study these

shocks, we adapt the Markov chain described in our baseline calibration in the following way: we stack a 0 to the original liquidity shock space. We assume that there is a crisis state x_t such that liquidity shocks follow their regular process when $x_t = 1$ and are set to 0 when $x_t = 0$. In addition, the binary Markov matrix corresponding to x_t is given by:

$$\Pi_x = \begin{bmatrix} 0.99 & 0.75 \\ 0.01 & 0.25 \end{bmatrix}$$

With these values, severe liquidity crises occur as rare and mildly persistent events. Thus, we define the matrix $\tilde{\pi}_{(a',\phi') \times (a,\phi)}$ conditional on $x_t = 0$. The entries are such that:

$$\begin{aligned} \tilde{\pi}_{(a',\phi') \times (a,\phi)} &\equiv \Pr \{a_{t+1} = a', \phi_{t+1} = \phi' | a_t = a, \phi_t = \phi, x_t = 0\} \\ &= \begin{cases} \pi_{a',a} & \text{if } \phi' = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The transition matrix for the new process is:

$$\Pi_L = \begin{bmatrix} 0.99 \otimes \Pi & 0.75 \otimes \Pi \\ 0.01 \otimes \bar{\Pi} & 0.25 \otimes \bar{\Pi} \end{bmatrix}$$

where Π is the same as in the baseline calibration.

Main Implications The main implication of rare liquidity event is that introduces a more prolonged effect on output as the capital stock takes longer to recover. In particular, q and investment would react more slowly but the rest of the dynamics would mostly be the same as in the baseline scenario.

10 Figures

Value of q

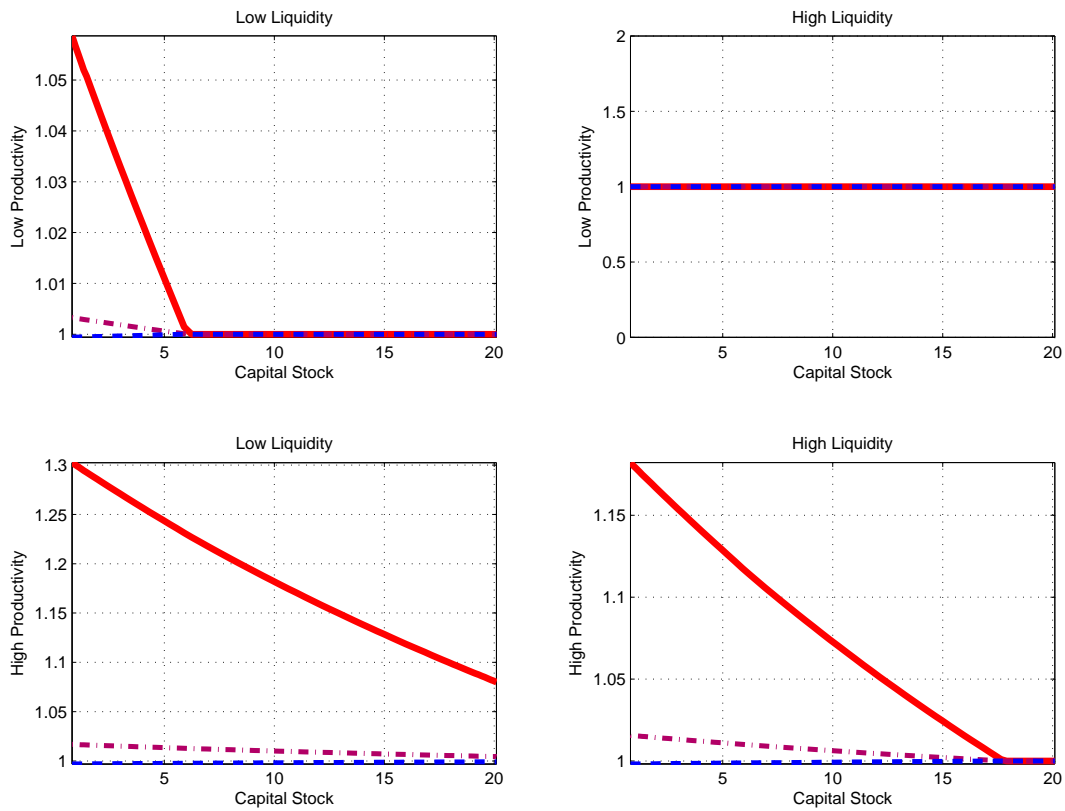


Figure 3: **Value of Tobin's Q along the state-space.** The figure shows the value of q (solid), q^R (---) and q^i (---) as a functions of aggregate capita states for 4 combinations of aggregate productivity and liquidity shocks. The figures show that at wedge exists the higher the aggregate productivity shock or the tighter the liquidity shock. For all four cases, the wedge is greater the lower the capital stock, reflecting that liquidity matters more, when returns are higher.

Investment Decomposition (Including Purchases)

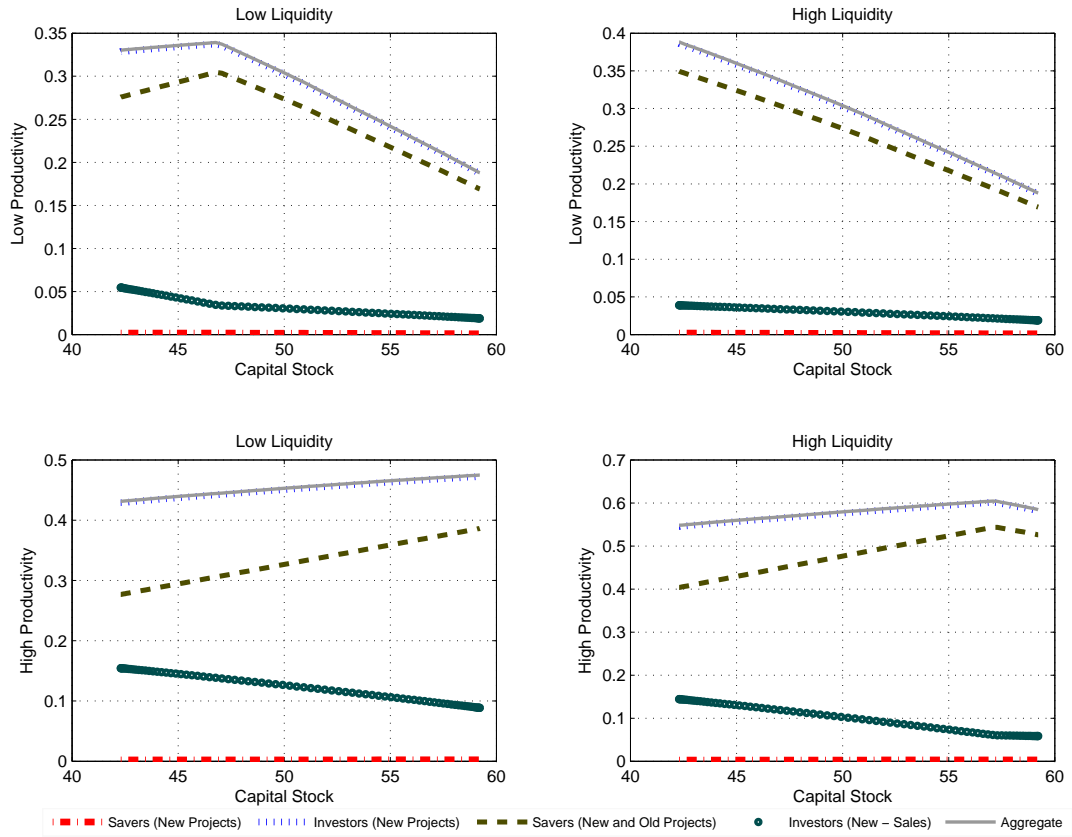


Figure 4: **Investment along the state-space.** The figure shows the value of the functions i_t and i_t^c as functions of aggregate capita states for 4 combinations of aggregate productivity and liquidity shocks. The figures show that liquidity constraints bind the higher the aggregate productivity shock or the tighter the liquidity shock. For all four cases, constraints bind the lower the capital stock, when returns are higher.

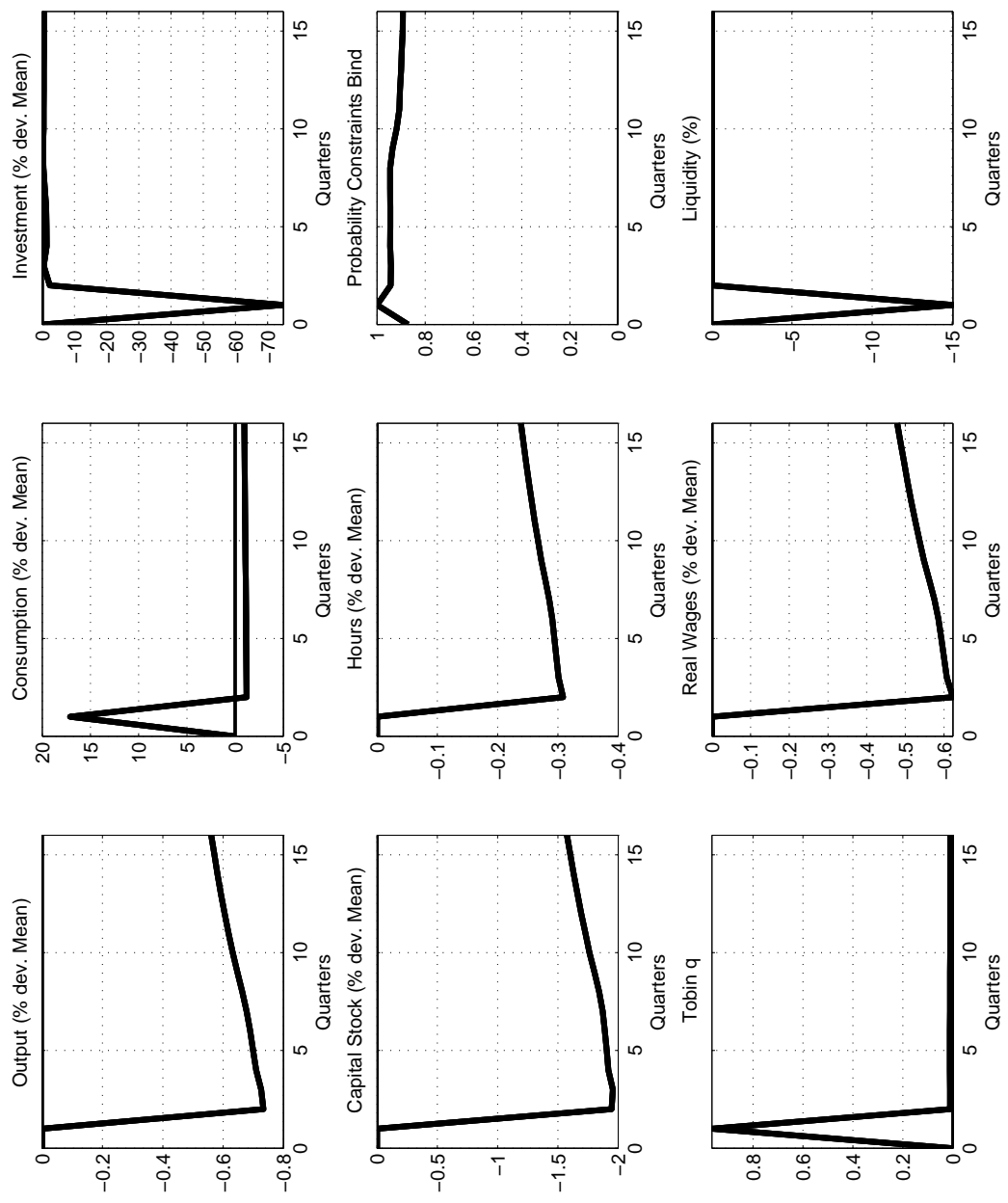


Figure 5: **Liquidity Shock Impulse Response.** The figure plots the response of variables to a fall in the liquidity shock from 15% to a full shutdown. The figure is computed by computing 500000 Montecarlo simulations starting from the unconditional expectation of the state.

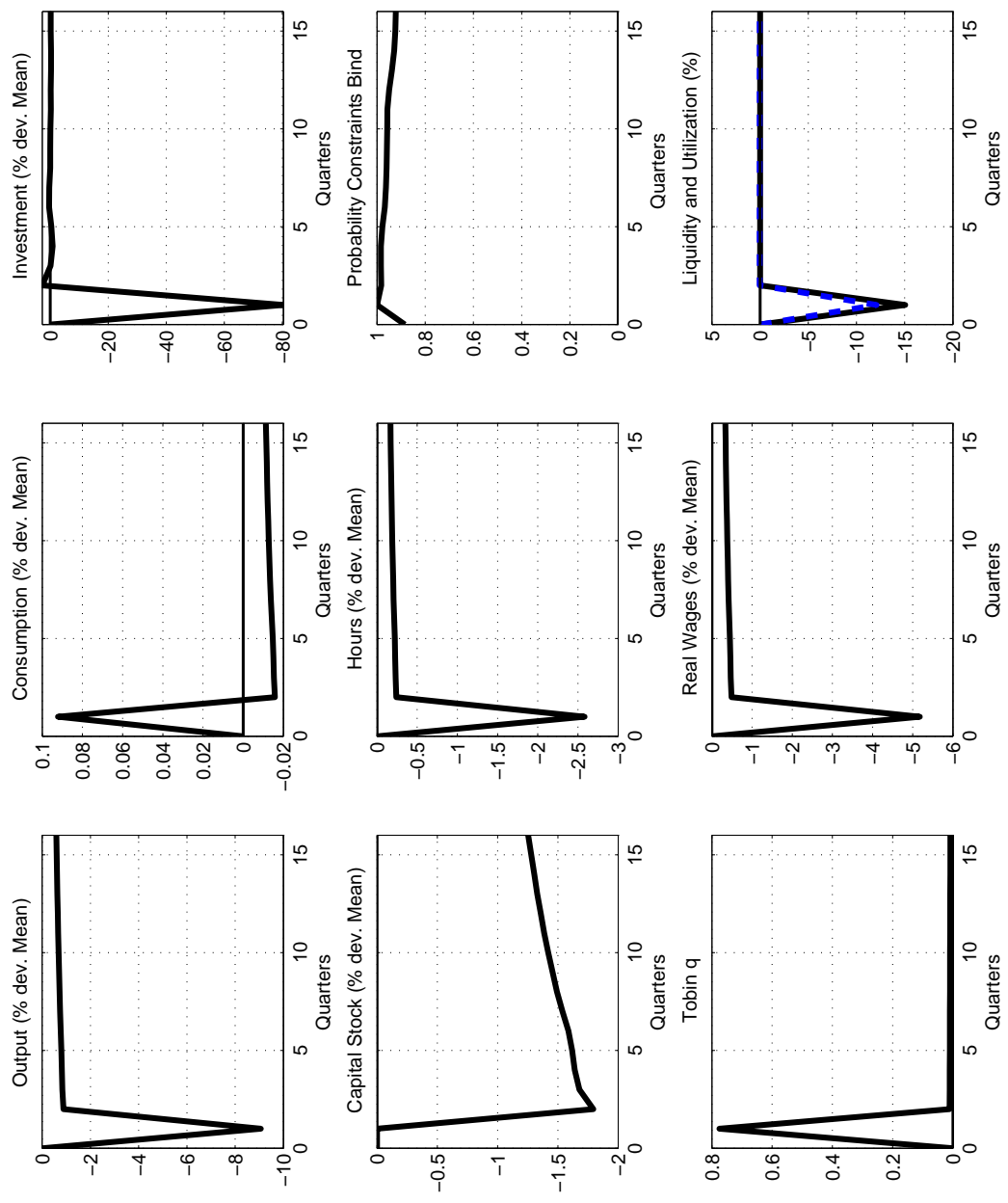


Figure 6: **Liquidity Shock Impulse Response with Variable Capital Utilization.** The figure plots the response of variables to a fall in the liquidity shock from 15% to 5% for the model with variable capital utilization. Capital utilization is depicted in the dashed curve in the bottom right panel. The figure is computed by computing 500000 Monte Carlo simulations starting from the unconditional expectation of the state.

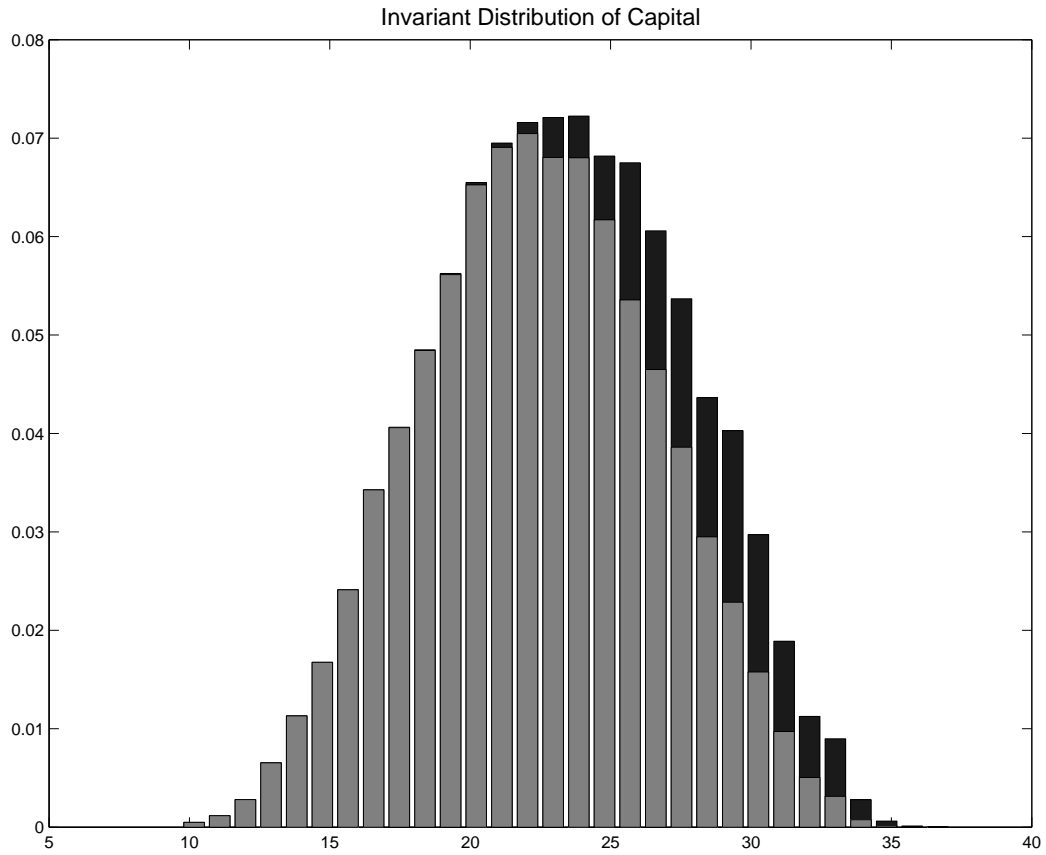


Figure 7: **Invariant Distributions.** The figure shows the invariant distribution of capital the capital stock.