

Who Matters in Coordination Problems?*

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Abstract

We consider a common investment project that is vulnerable to a self-fulfilling coordination failure and hence is strategically risky. Based on their private information, agents – who have heterogeneous investment incentives – form expectations or “sentiments” about the project’s outcome. We find that the sum of these sentiments is constant across different strategy profiles and it is independent of the distribution of incentives. As a result, we can think of sentiment as a scarce resource divided up among the different payoff types. Applying this finding, we show that agents who benefit little from the project’s success have a large impact on the coordination process. The agents with small benefits invest only if their sentiment towards the project is large per unit investment cost. As the average sentiment is constant, a subsidy decreasing the investment costs of these agents will “free up” a large amount of sentiment, provoking a large impact on the whole economy. Intuitively, these agents, insensitive to the project’s outcome and hence to the actions of others, are influential because they modify their equilibrium behavior only if the others change theirs substantially.

JEL classification: C7, D8, O12.

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1 Introduction

Coordination failures can be phenomenally costly to society. Perhaps the most important examples are missed opportunities for economic development.¹ Indeed, Debraj Ray (2000) sets out his survey of development economics based on the following observation:

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¹Coordination problems also arise in bank runs (Diamond and Dybvig, 1983), currency attacks (Obstfeld, 1996), law enforcement (Sah, 1991), standard setting (Farrell and Saloner, 1985), technology adoption (Katz and Shapiro, 1986), the use of fiat money (Kiyotaki and Wright, 1989) etc.

“Paul Rosenstein-Rodan (1943) and Albert Hirschman (1958) argued that [lack of] economic development could be thought of as a massive coordination failure, in which several investments do not occur simply because other complementary investments are not made. . .”

The risk of such a damaging coordination failure is an obvious cause for public intervention. As intervention is costly, the policy-maker will want to target it on the agents who have the largest impact in the coordination process. In this paper, we lay down the foundations of how to identify these.

Which agents are most influential in the development process? Those from towns or those from the countryside? The skilled ones or the unskilled ones? All these groups differ in their investment costs and in their benefits from development. Should the subsidies be targeted on those with large or low benefits? On those with low or high investment costs? Distinguishing the groups with a large influence on the coordination process is a complex task, as in the presence of externalities the investment incentives of any group affect the behavior of all the other groups. Thus, without a formal model, it is hard to identify the natural target for intervention.

In our model, each agent from a large population simultaneously decides between two courses of action, say, whether to invest in a “project” or not. If the project succeeds, the investors enjoy private benefits which exceed the incurred investment costs. Agents who do not invest do not receive the benefits, so free riding is not an issue. The project succeeds if and only if the proportion of investing agents exceeds a critical level. The agents receive noisy private signals about the critical level of investment, which leads to strategic uncertainty and possibly to a coordination failure.

To assess the influence of different payoff types on the coordination outcome, we examine a *heterogeneous* population, which consists of homogeneous groups, g , with corresponding benefits b_g and costs c_g . For the moment, let us assume that the project’s outcome does not depend on the investors’ group identities. For example, in the context of technology adoption this means that network externalities depend only on the measure of adopting agents and not on the incurred adoption costs.

When the noise is small, we are able to express the coordination outcome of the heterogeneous population as the coordination outcome of a (virtual) homogeneous population endowed with a representative payoff function. The representative payoffs turn out to be a weighted average of the groups’ payoffs. The – endogenously determined – weights in this aggregation exercise provide a good measure of the groups’ influence on the coordination outcome. We find that *the group that benefits least from the project’s success has the largest aggregation weight*; it has an excessive (relative to its population share) impact on the coordination outcome. Therefore, a subsidy that decreases investment costs will have the largest effect when aimed at the group with the lowest benefit from the project.

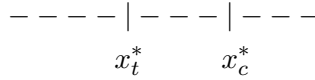


Figure 1: Critical signals differ across the groups. (Higher signals are associated with higher payoffs so that types above the critical signal of their group invest.)

We will derive the result by an analysis of *investors' sentiments*. When the project's outcome is binary,² we define sentiment as the probability assigned to the project's success. While the sentiment of each group depends on the entire profile of investment incentives, the aggregate sentiment turns out to be fixed: the across-the-groups average of sentiments is independent of the investment incentives. This constraint on the sentiments is the key to our equilibrium analysis.

To explain the notion of sentiments and its implications we need to discuss the details of strategic uncertainty. Players receive private, noisy signals, x^i , where higher signals indicate lower level of investment necessary for the successful outcome and thus, in monotone equilibria,³ higher expected returns. A monotone equilibrium is characterized by a tuple of critical signals, $(x_g^*)_g$, one for each group, such that each player invests if (and only if) her private signal, x^i , exceeds the critical signal, $x_{g^i}^*$, of her group, g^i . Generically, when the investment incentives, (b_g, c_g) , differ across groups, the critical signals differ as well and thus the the critical types from different groups have substantially different expectations about the aggregate investment. To see this, consider a population consisting of two distinct groups, say country folk and townsfolk. The group with smaller c_g/b_g ratio (say, townsfolk) is willing to accept more risk associated with investment and thus its critical signal will be lower than the critical signal of the group with the higher ratio (country folk). See Figure 1.

The townsfolk's critical type, x_t^* , has a *pessimistic* belief about the aggregate investment. To see why, note that she believes that other players receive private signals similar to her signal. As country folk invest only if their signal exceeds $x_c^* > x_t^*$, conditional on her signal, x_t^* , she finds it likely that only few country folk will invest. Symmetrically, the country folk's critical type, x_c^* , has an *optimistic* belief about the aggregate investment, as townsfolk invest already at signal x_t^* , which is below her signal, x_c^* .

It is not an accident that the beliefs of the two critical types are of an opposite nature. In fact, this observation complies with a general principle that we identify, and which holds for an arbitrary number of groups, independently of the assumed error distributions or payoffs. Normalizing the population size to 1, let us define the *critical belief* of a group as the belief

²Below we consider projects with a continuum of possible outcomes, in which case the sentiment is defined as expectation over the outcome.

³As it turns out, essentially all equilibria are monotone.

of the group's critical type about aggregate investment l : a probability density $\lambda_g(l)$ on $[0, 1]$. Our first result states that the (point-wise) average of those densities is constant.

The Belief Constraint. *The (population-weighted) across-the-groups average of the critical beliefs is the uniform density on $[0, 1]$.*

The beliefs of the critical types depend on their relative positions as well as on the error distributions and may be quite complicated. The belief constraint identifies a simple and robust aggregate statistic. This simplifies the equilibrium analysis, as it avoids the complex calculation of the critical beliefs of every group.

Importantly, the belief constraint also leads to a conceptual innovation: it allows us to treat the optimism (or sentiment) about aggregate investment as a virtual *resource* which is available in a fixed amount and is distributed among the critical types. If the critical belief of one group is optimistic then, to comply with the constraint, the other groups' critical beliefs must be pessimistic on average. Thinking of optimism as a finite resource is useful because it provides an intuition for the excessive influence of the groups with low benefits, b_g .

The expected payoff from investment of the critical type from group g is

$$b_g \times p_g - c_g,$$

where p_g is the probability of the project's success as evaluated by the critical type x_g^* . In equilibrium, the critical type must be indifferent between investing and not investing, so $b_g \times p_g - c_g = 0$. If benefit b_g is low then the agents from the group are not too sensitive to a variation in their sentiments. Hence, to keep the critical type indifferent, any intervention that changes the group's incentives must be offset by a relatively large change in the sentiment p_g . As the average sentiment is constant, the sentiments of other groups change substantially as well. For example, a subsidy decreasing investment cost c_g of the insensitive group g decreases substantially the sentiment $p_g = \frac{c_g}{b_g}$, and so the sentiment available to other groups substantially increases. Note that to achieve the same change of sentiment by subsidizing other groups with higher benefits, would have required a higher subsidy.

Our paper belongs to the global game literature originated by Hans Carlsson and Eric van Damme (1993). The most common applied global game setup is the one reviewed by Stephen Morris and Hyun Shin (2003), which consists of an incomplete information game with strategic complementarities played by many players who share a *common* payoff function.⁴ In that setup every player follows the same threshold strategy. This greatly simplifies the analysis because the critical type turns out to have a very simple belief about

⁴It has been used to study currency attacks (Morris and Shin, 1998), bankruptcies (Morris and Shin, 2004), bank runs (Goldstein and Pauzner, 2005), debt crises (Morris and Shin 2004), political revolutions (Edmond 2008), and other coordination problems.

the aggregate investment, which is independent of the assumed error distributions: she believes that the aggregate investment is distributed uniformly. This well-known property is a special case of our belief constraint when applied to the setup with only one group.

The analysis of global games with heterogeneous payoffs is harder. We briefly review the existing literature here. David Frankel, Stephen Morris and Ady Pauzner (2003) prove equilibrium uniqueness in a large class of games with strategic complementarities, also allowing for payoff-heterogeneity. However, their general framework provides only a partial characterization of the unique equilibrium, and hence it is not directly applicable. Solutions of heterogeneous global games are known for a few particular setups. Stephen Morris and Hyun Shin (2003) note that heterogeneity in the quality of private signals has no consequence in a two-action global game in the absence of payoff heterogeneity. Giancarlo Corsetti et al. (2004) characterize the impact of a large trader on a population of small ones. Bernardo Guimaraes and Stephen Morris (2007) allow for payoff heterogeneity in a model of currency attacks. The standard intuition suggests that a few risk-neutral agents would suffice to make the whole economy appear risk-neutral, because they can provide hedge to the risk-averse agents. However, risk averse speculators have a non-negligible influence on the attack in the Guimaraes-Morris model. The high influence of the risk-averse players conforms to our result. Risk aversion decreases the utility benefit from a successful attack, and therefore the risk-averse agents have a large impact on the coordination outcome.⁵

Finally, Sergei Izmalkov and Mehmet Yildiz (forthcoming) provide a formal definition of investor sentiments. Their concept of sentiment is essentially the belief of the (unique) critical type about aggregate investment, which in their model is derived from exogenous heterogeneous prior beliefs. Sentiment in our model also represents the belief of a critical type about aggregate investment, but it is an endogenous equilibrium phenomenon, and it differs across groups even under common priors.

The remainder of the paper is organized in a number of short sections. We start by specifying the information structure and formulating the belief constraint, in Section 2. In Section 3 we describe details of the investment project and derive its relationship with the aggregate critical sentiment “supplied” to the critical types. In Section 4 we specify the investment incentives and using the indifference of critical types, we find the “demand” for critical sentiment. Section 5 provides the solution to the coordination problem, by matching “demand” to “supply”. In Section 6 we find a representative payoff function that aggregates payoffs of individual groups. The impact of individual groups is naturally measured by their endogenous aggregation weights. We further elaborate the intuition behind our results in Section 7. We discuss the robustness of our assumptions in Section 8, followed by the proof of the belief constraint. Finally, Section 10 concludes. The proofs not presented in the main

⁵Bernardo Guimaraes’s and Stephen Morris’s method of dealing with payoff heterogeneity applies only to projects with binary outcomes. For their setup, the equilibrium threshold of a heterogenous population is a simple average of sub-population thresholds. This simple aggregation result does not generalize to setups with richer outcome spaces.

body of the paper are in the Appendix.

2 The Information Structure and the Belief Constraint

A continuum of players of measure 1, indexed by $i \in [0, 1]$ simultaneously decide whether to invest a unit amount into a common project. The returns to investment depend on an uncertain *fundamental*, θ , and on the aggregate investment. The players are heterogeneous in two aspects. First, they differ in their information about θ . Second, the population is divided into G groups, and the investment incentives differ across these groups. The population share of group g is denoted by m_g . The analysis of this section is independent of the details of the project and of the investment incentives, and hence we introduce them only later, in Sections 3 and 4, respectively. In this section we focus on the information structure.

Each player i is endowed with a *type*, which comprises of a pair of numbers (x^i, g^i) . The first number is a noisy private signal about the fundamental,

$$x^i = \theta + \sigma\eta^i,$$

where σ is a scaling parameter and η^i is a random error. The second number is a label of group $g^i \in \{1, \dots, G\}$ to which player i belongs. The support of the fundamental, θ , is an interval $[\underline{\theta}, \bar{\theta}]$ with $\underline{\theta} < 0$ and $\bar{\theta} > 1$. We will vary σ in our analysis and we will be interested in setups with σ small. In particular, we assume $\sigma < \bar{\theta} - 1$ and $\sigma < -\underline{\theta}$.

The probability distributions are as follows. The fundamental, θ , is distributed uniformly on its support.⁶ The combination of a player's error and her group identity, (η^i, g^i) , is a two-dimensional random variable with support in $[-1, 1] \times \{1, \dots, G\}$; it is i.i.d. across players, and independent of θ . We assume no aggregate uncertainty and hence the marginal probability, $\Pr(g^i = g)$, equals the population share, m_g , of group g . We do not require the random variables η^i and g^i to be independent. The c.d.f. of the conditional error $\eta^i \mid (g^i = g)$ is denoted by $F_g(\cdot)$ and has a closed support contained in $[-1, 1]$; the associated p.d.f. $f_g(\cdot)$ is assumed to be continuous. This information structure accommodates setups in which players from different groups draw errors from different distributions F_g . In the special case, when η^i and g^i are independent, the errors are i.i.d. across players from all the groups.

We examine monotone strategy profiles defined by a G -tuple of critical signals $\mathbf{x}^* = (x_g^*)_g$, such that each type (x^i, g^i) invests if (and only if) $x^i \geq x_{g^i}^*$. We denote such a strategy by $a(x^i, g^i)$:

$$a(x^i, g^i) = \begin{cases} \text{Invest} & \text{if } x^i \geq x_{g^i}^*, \\ \text{Not Invest} & \text{if } x^i < x_{g^i}^*. \end{cases} \quad (1)$$

⁶The uniformity of the prior becomes unimportant in the limit of precise signals, $\sigma \rightarrow 0$, as discussed in Section 8.

For the equilibrium analysis below, only tuples with all critical types distant from the boundaries of the support of θ will be relevant. In particular, without loss of generality of the equilibrium analysis below, we restrict \mathbf{x}^* to the set $X = [-\sigma, 1 + \sigma]^G$.

The aggregate investment, l , is defined as

$$l = \int_0^1 a^i di, \quad (2)$$

where $a^i \in \{0, 1\}$ is the investment decision of player i . The assumption, that the investment of any group has the same impact on the aggregate investment is just a normalization: If players of group g were deciding between investing 0 and $w(g)$ units, the aggregate investment would be $\int_0^1 w(g^i) a^i di$. However, the unequal investors' weights could be accommodated by modifying group sizes m_g to $m_g w(g)$ while keeping the budgets and aggregation weights equal to 1.

Under a monotone strategy profile defined by a tuple of critical signals $\mathbf{x}^* \in X$, the aggregate investment is a non-decreasing function of the realized fundamental θ :

$$l = \ell(\theta; \mathbf{x}^*) = \Pr \left(\{(x, g) : x \geq x_g^*\} \mid \theta \right).$$

We will mostly omit the dependence of $\ell(\theta)$ on \mathbf{x}^* from the notation.

In the next proposition we examine the beliefs about $\ell(\theta)$ formed by the critical types, the *critical beliefs*. Let $\lambda_g : [0, 1] \rightarrow \mathbb{R}_+$ denote the p.d.f. of the conditional random variable

$$\ell(\theta) \mid (x_g^*, g).$$

Thus, $\lambda_g(l)$ is the probability density assigned to investment level l by a player from group g who observed the critical signal x_g^* . Again, we omit the dependence of $\lambda_g(l)$ on \mathbf{x}^* from the notation.

Proposition 1 (The Belief Constraint). *For any tuple of critical signals $\mathbf{x}^* \in X$, the average critical belief is the uniform belief on $[0, 1]$:*

$$\sum_{g=1}^G m_g \lambda_g(l) \equiv 1.$$

See Figure 2. The proof, relegated to Section 9, builds on known results for homogeneous populations. Let us state here the known belief characterization for the homogeneous population as a corollary of Proposition 1:

Corollary 1 (Laplacian Property, (Morris and Shin, 2003)). *Suppose that the population of players is homogeneous, $G = 1$, and players follow a symmetric monotone strategy profile with a critical signal $x^* \in [-\sigma, 1 + \sigma]$. Then the player who receives the critical signal believes that the aggregate investment is distributed uniformly on $[0, 1]$.*

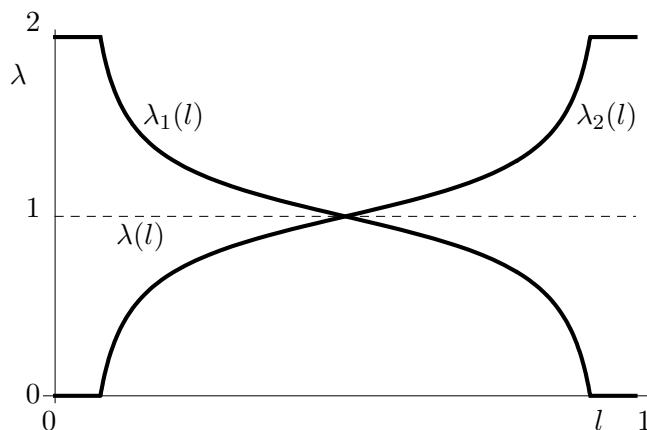


Figure 2: Illustration of the belief constraint, Proposition 1. The average belief $\lambda(l) = m_1\lambda_1(l) + m_2\lambda_2(l)$ is the uniform belief on $[0, 1]$.

The corollary, known in the global game literature as *Laplacian property*, has an intuitive explanation. We paraphrase Stephen Morris and Hyun Shin (2003): The critical type, x^* , constitutes a boundary in between the investing and non-investing types. She is uncertain about the realized proportions of types above and below her. These proportions are determined by the rank of her (critical) signal within the realized population of players' signals. The only information the critical type receives is her own private signal, which is entirely uninformative about the rank of her signal and consequently about the aggregate investment.

The belief constraint establishes that, albeit in the heterogeneous population the Laplacian property does not hold for the critical type of any particular group, it holds on average across the groups.

Let us illustrate the belief constraint with an example in which the distance between critical signals is large and, consequently, the critical types know whether players from other groups invest. Consider two groups, $m_1 = m_2 = 1/2$, and $x_1^* + 2\sigma < x_2^*$. In this case, type $(x_1^*, 1)$ knows that no player from group 2 invests because, according to her information, the signals of all players satisfy $x^i \leq \theta + \sigma \leq x_1^* + 2\sigma < x_2^*$. Additionally, the critical type $(x_1^*, 1)$ believes that the measure of investing players from her group is distributed uniformly on $[0, 1/2]$ because she does not know the rank of her critical signal in the population of her own group. Hence the critical type $(x_1^*, 1)$ believes that the aggregate investment from the two groups is distributed uniformly on $[0, 1/2]$. The critical type $(x_2^*, 2)$ knows that all players from group 1 invest, as, according to her information, the signals of all players satisfy $x^i \geq \theta - \sigma \geq x_2^* - 2\sigma > x_1^*$. Again, she believes that investment from her group is distributed uniformly on $[0, 1/2]$. Hence she believes that the total investment is distributed uniformly on $[1/2, 1]$. The average of the two critical beliefs is the uniform belief on $[0, 1]$. In the general case, when the critical signals are close to each other, the critical types will be uncertain about the investment from other groups, and their beliefs, $\lambda_g(l)$, may be complex.

Remarkably, however, the average of these complex beliefs is always the simple, uniform belief, see Figure 2.

3 The Project and the “Supply” of Optimism

We now specify the defining characteristic of the common investment project – its outcome rule. We then use the belief constraint from the previous section to find a restriction on the expectations about the project’s outcome formed by the critical types.

The project’s outcome

$$\pi(\theta, l) = \begin{cases} o(\theta, l) & \text{if } l \geq 1 - \theta, \\ 0 & \text{if } l < 1 - \theta, \end{cases}$$

is a real number which measures the degree of its success. The project fails if investment l does not reach $1 - \theta$. When $l \geq 1 - \theta$, the project succeeds and $o(\theta, l)$ measures the extent of the success. A specification often used in literature lets $o \equiv 1$, in which case the project’s outcome is binary and no shades of success are distinguished.

We impose the following assumptions on the function o :

A1 $o(\theta, l)$ is strictly positive,

A2 $o(\theta, l)$ is non-decreasing in both arguments,

A3 $o(\theta, l)$ is continuous in θ .

The assumptions imply that the project’s outcome, $\pi(\theta, l)$, is non-decreasing in θ and l . For negative realizations of θ the project fails regardless of the players’ actions, and it always succeeds for $\theta > 1$. The aim of our analysis is to predict the outcome for intermediate values of θ in $[0, 1]$.

For each group g , we denote by $p_g(\mathbf{x}^*)$ the expectation about the outcome formed by the critical type x_g^* . The mapping $p_g : X \rightarrow [0, 1]$ is defined as

$$p_g(\mathbf{x}^*) = E \left[\pi(\theta, \ell(\theta; \mathbf{x}^*)) \mid (x_g^*, g) \right],$$

where the source of uncertainty is the realization of θ , unknown to player i . We will refer to $p_g(\mathbf{x}^*)$ as the *critical sentiment of group g* under the tuple \mathbf{x}^* . If $o = 1$, so that the outcome of the project is binary, the sentiment $p_g(\mathbf{x}^*)$ is simply the probability that the critical type of group g assigns to success. We also introduce notation for the *aggregate critical sentiment*:

$$p(\mathbf{x}^*) = \sum_{g=1}^G m_g p_g(\mathbf{x}^*).$$

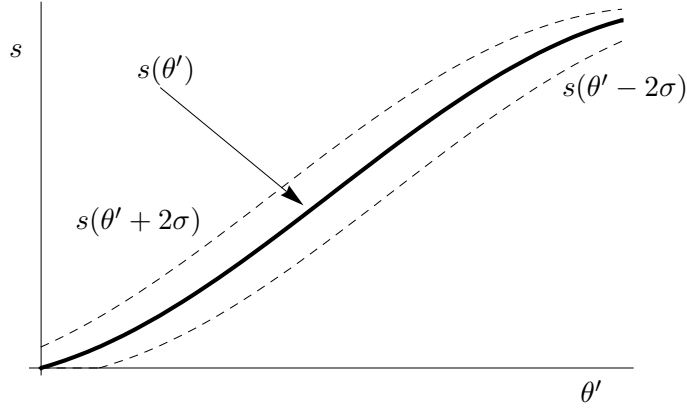


Figure 3: If all the critical signals lie close to θ' then the aggregate critical sentiment $p(\mathbf{x}^*) = \sum_g m_g p_g(\mathbf{x}^*)$ lies close to $s(\theta')$, see Corollary 2.

In the equilibrium analysis below, only tuples of critical types close to each other will be relevant. Accordingly, we will consider here only tuples that satisfy the following proximity condition:

$$\text{There exists } \theta' \in [0, 1] \text{ such that } |x_g^* - \theta'| \leq \sigma \text{ for each } g. \quad (3)$$

If (3) holds and σ is small then all the critical types agree that the realized fundamental lies close to θ' . Yet, the critical types disagree over the outcome because their beliefs, $\lambda_g(l)$, about the aggregate investment differ. Still, the critical sentiments, $p_g(\mathbf{x}^*)$, are not unrelated across the groups. The belief constraint implies that the aggregate critical sentiment, $p(\mathbf{x}^*)$, approximately equals the outcome expectation based on the uniform belief over l :

$$p(\mathbf{x}^*) \approx \sum_g m_g \int_0^1 \pi(\theta', l) \lambda_g(l) dl = \int_0^1 \pi(\theta', l) \sum_g m_g \lambda_g(l) dl = \int_0^1 \pi(\theta', l) dl. \quad (4)$$

We denote the last expression by $s(\theta')$ and refer to it as the “supply” of sentiment. The function s is continuous, $s(\theta) = 0$ for negative θ , and strictly increasing for $\theta > 0$.

The following corollary of the belief constraint states the approximate relation (4) formally.

Corollary 2. *If the proximity condition (3) is satisfied, then the aggregate critical sentiment is near $s(\theta')$:*

$$p(\mathbf{x}^*) \in [s(\theta' - 2\sigma), s(\theta' + 2\sigma)].$$

See Figure 3 for illustration. The proof of the corollary is in the Appendix.

4 Investment Incentives and the “Demand” for Optimism

In this section we specify the payoff functions and derive another restriction on the sentiments formed by the critical types.

The payoff for not investing is normalized to 0 for all players, while player i 's payoff for investing is

$$u(\theta, l, g^i) = b(\theta, g^i)\pi(\theta, l) - c(\theta, g^i).$$

The payoff functions, together with the information structure from Section 2 and the project's outcome function from Section 3 define a Bayesian game among the players.

We assume that the groups differ in their investment costs $c(\theta, g)$ and in their benefits from the success of the project $b(\theta, g)$. We impose the following assumptions on b and c :

A4 $b(\theta, g)$ is strictly positive.

A5 $c(\theta, g)$ is strictly positive, and $b(\theta, g)o(\theta, l) > c(\theta, g)$ for $\theta, l > 0$.

A6 $b(\theta, g)$ is non-decreasing, $c(\theta, g)$ is non-increasing in θ .

A7 Both $b(\theta, g)$ and $c(\theta, g)$ are continuous in θ .

Assumption A4, together with the success rule, assures strategic complementarity: the incentive to invest increases with the investment activity of the opponents. Assumption A5 implies that if the player knew that the project will fail she would strictly prefer not to invest, and if she knew the project succeeds she would strictly prefer to invest because the benefit of investment to a successful project always exceeds the investment cost, regardless of the shade of success o . Assumption A6 together with A1 implies that the payoff for investing increases, *ceteris paribus*, with θ and hence θ can be interpreted as the “quality” of the project. A simple example of benefit and cost functions are payoffs b_g and c_g , which depend only on g but not on the fundamental, θ .

Having described the investment incentives, we can derive another endogenous restriction on the critical sentiments. Let us again consider a small σ and a tuple \mathbf{x}^* with all critical types in the proximity of some value of the fundamental θ' . Again, all the critical types agree that the realized fundamental $\theta \approx \theta'$ and hence they know that $b(\theta, g) \approx b(\theta', g)$, and $c(\theta, g) \approx c(\theta', g)$. In equilibrium, all the critical types must be indifferent between investing, which pays approximately $b(\theta', g) \times p_g(\mathbf{x}^*) - c(\theta', g)$, and not investing, which pays 0. Therefore, the critical sentiment

$$p_g(\mathbf{x}^*) \approx \frac{c(\theta', g)}{b(\theta', g)},$$

for each group g . In equilibrium, a group with high cost/benefit ratio must have optimistic critical sentiment and vice versa.

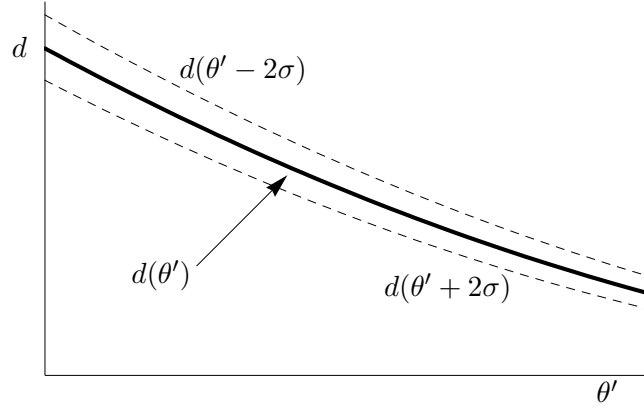


Figure 4: If all the critical signals lie close to θ' then the aggregate critical sentiment $p = \sum_g m_g p_g$ lies close to $d(\theta')$, see Lemma 1.

Let us define the function

$$d(\theta') = \sum_{g=1}^G m_g \frac{c(\theta', g)}{b(\theta', g)}.$$

We refer to $d(\theta')$ as to the “demand” for sentiment. It is the (approximate) aggregate amount of critical sentiment “needed” to make all the critical types indifferent between investing and not investing. Assumptions A4-7 assure that d is continuous, non-increasing, and strictly positive.

Lemma 1. *Suppose that the strategy profile defined by the tuple of critical types \mathbf{x}^* constitutes a Bayes-Nash equilibrium. Further, suppose that \mathbf{x}^* satisfies the proximity condition (3). Then the aggregate critical sentiment is near $d(\theta')$:*

$$p(\mathbf{x}^*) \in [d(\theta' + 2\sigma), d(\theta' - 2\sigma)].$$

See Figure 4 for illustration. The proof is in the Appendix.

5 The Equilibrium Outcome

For any tuple of critical signals \mathbf{x}^* the aggregate investment $\ell(\theta, \mathbf{x}^*)$ is an increasing function of θ and hence the project succeeds if and only if the fundamental exceeds the unique solution⁷ of

$$\ell(\theta; \mathbf{x}^*) = 1 - \theta.$$

We call this solution the *critical* fundamental and denote it by $\theta^*(\mathbf{x}^*)$. We will often suppress its dependence on \mathbf{x}^* in the notation.

⁷The solution exists because $\ell(\theta) < 1 - \theta$ for $\theta < 0$, $\ell(\theta) > 1 - \theta$ for $\theta > 1$, and ℓ is continuous. The solution is unique because $\ell(\theta)$ is nondecreasing.

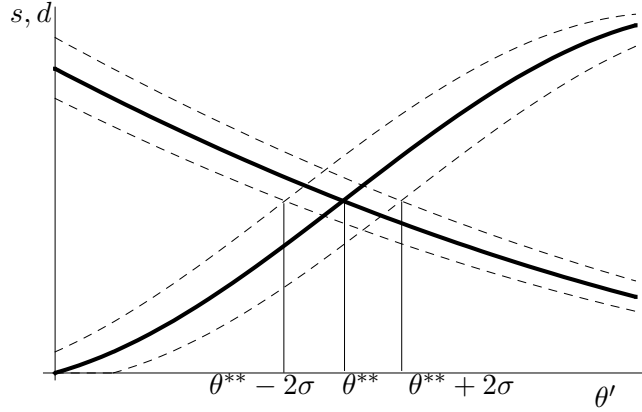


Figure 5: The project succeeds for all realizations of $\theta \geq \theta^{**} + 2\sigma$ and fails for all $\theta \leq \theta^{**} - 2\sigma$; see Proposition 2.

In equilibrium, the critical signals of all groups lie close to the critical fundamental θ^* . To see this, assume that $x_g^* > \theta^* + \sigma$ for some group g . Then, because of the bounded errors, the critical type (x_g^*, g) knows that $\theta > \theta^*$ and thus knows that the project will succeed. Assumption A5 assures that the payoff for investment is strictly positive whenever the project succeeds and so the critical type violates her indifference condition. Similarly, if $x_g^* < \theta^* - \sigma$, the critical type would know that the project will fail, and hence she would strictly prefer not to invest. The following lemma summarizes.

Lemma 2 (proximity of critical signals). *For any equilibrium tuple of critical signals \mathbf{x}^* :*

$$|x_g^* - \theta^*(\mathbf{x}^*)| \leq \sigma,$$

for each g .

Once we know that the critical signals satisfy the proximity condition (3) with θ^* , we can use Corollary 2 and Lemma 1. These two results imply that the aggregate sentiment $p(\mathbf{x}^*)$ approximates both the supply $s(\theta^*(\mathbf{x}^*))$ and demand $d(\theta^*(\mathbf{x}^*))$ of sentiment. Therefore,

$$s(\theta^*) \approx d(\theta^*),$$

and thus θ^* is close to the solution of

$$s(\theta') = d(\theta'). \tag{5}$$

We denote the solution of equation (5) by θ^{**} . The solution exists because, for $\theta' < 0$, $s(\theta') = 0 < d(\theta')$ and for $\theta' > 1$ $s(\theta) = \int o(\theta', l) dl \geq o(\theta', 0) > d(\theta')$ and both functions s and d are continuous. The solution is unique because s is strictly increasing and d non-increasing. The following proposition formally states the result:

Proposition 2. *Suppose that the players use rationalizable strategies.⁸ Then*

1. *the project succeeds whenever $\theta > \theta^{**} + 2\sigma$ and fails whenever $\theta < \theta^{**} - 2\sigma$,*
2. *each player i invests whenever $x^i > \theta^{**} + 3\sigma$, and she does not invest whenever $x^i < \theta^{**} - 3\sigma$,*

*where θ^{**} is the unique solution of $s(\theta') = d(\theta')$, and independent of the assumed error distributions.*

The proof is in the Appendix.

We say that θ^{**} is the *solution* of the coordination problem, where the problem is defined by the population $(m_g, b(\theta, g), c(\theta, g))_{g=1}^G$ and by the outcome function $\pi(\theta, l)$. The solution θ^{**} is relevant for predicting the coordination outcome both from the *ex ante* and the *ex post* perspective. From the *ex post* point of view, an observer who observes the realized fundamental $\theta > \theta^{**}$ knows that the project succeeds if σ is sufficiently small. From the *ex ante* perspective, the probability of success decreases with θ^{**} when σ is small.

The equilibrium characterization is based on an analogy to aggregate supply and demand. The advantage of the analogy is that it identifies two independent parts of the analysis. We defined the “supply” of aggregate sentiment using only the belief constraint and the outcome rule, but *not* the investors’ incentives. As the belief constraint does not depend on the investment incentives, the “supply” side is defined solely by the project’s characteristics. The “demand” for the aggregate sentiment was defined based on the indifference conditions of the critical types and hence it is a function of investment benefits and costs but it is independent of the project’s characteristics. Distinguishing the two sides will be useful below, where we examine how the variation of incentives affects the critical fundamental θ^{**} . The critical fundamental plays the role of the “clearing price” that equates “supply” and “demand” of sentiment. As the “supply” side is independent of the incentives, we can focus solely on the “demand” side which greatly simplifies the analysis.

6 Who Matters in Coordination Problems?

In this section we provide an aggregation result that characterizes the critical fundamental of the heterogeneous population in terms of a representative payoff. For a heterogeneous population with group-dependent payoffs $u(\theta, l, g)$, we identify a homogeneous population, with representative payoff function $\tilde{u}(\theta, l)$, that has the same critical fundamental θ^{**} as the solution to the original heterogeneous problem. This representative payoff turns out to be an endogenously weighted average of the group-wise payoffs, *with the low-benefit groups having large weights*.

⁸Assuming rationalizable instead of equilibrium behavior makes the result stronger; see the proof in the Appendix.

Corollary 3 (Representative Payoffs). *The solution of the coordination problem with a heterogeneous population $(m_g, u(\theta, l, g))_{g=1}^G$ is identical to the solution of a problem with a homogeneous population with payoff function*

$$\tilde{u}(\theta, l) = \sum_g \tilde{m}(\theta, g) u(\theta, l, g),$$

where the aggregation weights of group g ,

$$\tilde{m}(\theta, g) = \frac{m_g/b(\theta, g)}{\sum_{h=1}^G m_h/b(\theta, h)},$$

are decreasing with $b(\theta, g)$.

We only sketch the proof: Note that the game with the representative payoff is a special case of our set-up and therefore the coordination problem of the homogeneous population with the representative payoff has a well-defined, unique solution. The corollary then can be verified by noting that the “demand” and “supply” functions resulting for the heterogeneous population coincide with the ones resulting for the representative homogeneous population.

We have found that the players who are less sensitive to the project’s outcome have a larger impact on coordination than the sensitive players. The excessive influence of the insensitive group makes them a natural target of policy interventions. For example, consider that the policy maker has committed s funds per capita for a subsidy scheme, and therefore she can credibly promise subsidy s/m_g per capita to the members of the targeted group g . By our results, the heterogeneous population coordinates as a homogeneous population with representative payoffs $\tilde{u}(\theta, l) = \sum_g \tilde{m}_g u(\theta, l, g)$ where the weights \tilde{m}_g are proportional to $\frac{m_g}{b_g}$. The subsidy increases $u(\theta, l, g)$ in the targeted group by s/m_g . As a result, the representative payoff increases by $\tilde{m}_g s/m_g$, which is proportional to $\frac{m_g}{b_g} s/m_g = s/b_g$. Hence, the effect of subsidization is largest when it is aimed at the group with the lowest benefit, b_g , independently of the costs, c_g , and of the distribution of group sizes, m_g .

Remark: The aggregation result can be easily extended to the case where groups differ in their impact on the project. If aggregate investment is $l = \int_0^1 w(g^i) a^i di$, then the aggregation weights are given by

$$\tilde{m}(\theta, g) = \frac{m_g w(g)/b(\theta, g)}{\sum_{h=1}^G m_h w(h)/b(\theta, h)},$$

and hence the most influential group is the one with the highest w/b ratio.

In the next section we discuss the intuition behind the excessive influence of the insensitive players.

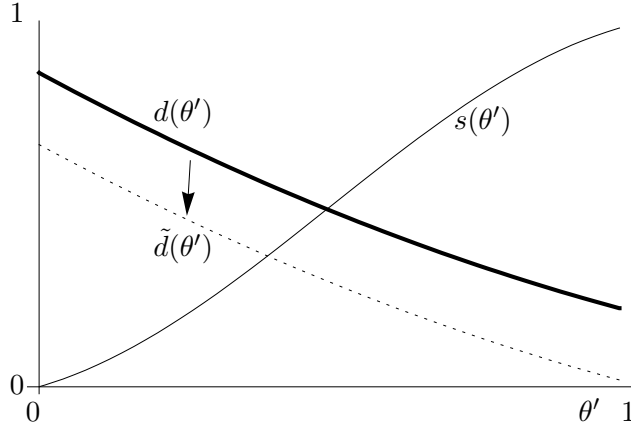


Figure 6: An improvement of investment incentives of a group causes a downward shift of the “demand” function. The shift is large when $b(\cdot, g)$ is small.

7 Intuition

Why do the players with low benefits have large impact on the critical fundamental? To develop an intuition, we will consider a policy intervention targeted on a single group that slightly modifies the group’s incentives. As we discussed at the end of Section 5, variation of incentives affects the “demand” for sentiment but not the “supply”, and thus we only need to examine the “demand” shift.

To quantify the impact of the intervention on the “demand” for sentiment, we compute the rate of substitution between the investment incentives b , c and the sentiment p^i . First, when σ is small, so that signals x^i are very precise, player i knows that her benefit and cost are approximately $b(x^i, g^i)$ and $c(x^i, g^i)$. Letting sentiment p^i denote the player’s expectation about the project’s outcome, her expected net investment return is close to

$$b(x^i, g^i) \times p^i - c(x^i, g^i).$$

If her benefit $b(\cdot, g^i)$ increases by db then, to keep the expected return constant, p^i must decrease by $\frac{p^i}{b(x^i, g^i)} db$. Similarly, if $c(\cdot, g^i)$ decreases by dc then p^i must decrease by $\frac{1}{b(x^i, g^i)} dc$. In both cases the rate of substitution is proportional to $\frac{1}{b(x^i, g^i)}$.

The concept of rate of substitution turns out to be naturally useful when applied to the critical types because their expected returns are automatically kept constant, equal to 0, by the indifference conditions. Consider a small intervention affecting incentives, $b(\cdot, g)$ or $c(\cdot, g)$, of a particular group g . The sentiment p_g that keeps the critical type of group g indifferent necessarily adjusts and hence, the whole “demand” curve, $d(\theta')$, for the aggregate level of sentiment adjusts. The adjustment is large when the rate of substitution between the sentiment and the investment incentives of the targeted group is large, which is the case when the benefit is small. As illustrated on Figure 6 the shift of the “demand” curve will

cause a shift in the intersection θ^{**} and this will be large when the benefit of the targeted group is small. Technically speaking, the group with a small benefit has a large impact because derivative of demand

$$d(\theta') = \sum_{g=1}^G m_g \frac{c(\theta', g)}{b(\theta', g)}.$$

with respect to $b(\theta', g)$ or $c(\theta', g)$ is large when $b(\theta', g)$ is small.

The above intuition can be complemented by drawing an analogy with mixed strategy equilibria. In a mixed equilibrium, players $-i$ must make i indifferent. If i 's payoff parameters change then $-i$ must modify their strategies to keep i indifferent. If player i is quite insensitive to $-i$'s actions then $-i$ have to modify their strategy substantially. Hence, a change in i 's payoff has the larger impact on the strategy profile the less sensitive i is to the opponents' behavior. The monotone strategy equilibria in our model are pure but, as in the mixed equilibrium analogy, the equilibrium is determined by indifference conditions. In order to keep the critical types of insensitive groups indifferent, others have to modify their behavior substantially.

8 Robustness Check

In order to gauge the generality of our results, let us discuss the importance of our major assumptions.

Our most general result is the belief constraint, Proposition 1. This result does not require any restrictions on the payoff functions; it holds under any monotone strategy profile defined by a tuple of critical signals and under the standard global-game information structure: errors must be independent across players and of θ and the prior must be uninformative. The latter assumption is not too restrictive as any well-behaved prior belief about θ becomes approximately uninformative when signals are sufficiently precise.⁹ Hence, for small σ , the belief constraint remains approximately valid under any prior.

The belief constraint becomes useful in games that have monotone threshold equilibria. The existence of such equilibria is assured in our game because it satisfies global game assumptions: state monotonicity, the existence of dominance regions, and strategic complementarity. As discussed in Morris and Shin (2003), the last assumption can be relaxed. The monotonicity of the project's outcome $\pi(\theta, l)$ with respect to the aggregate investment l can be replaced by single-crossing in l . If the error functions satisfy monotone likelihood ratio property then the characterization of monotone equilibria remains valid, though other, non-monotone equilibria may emerge.

In fact, we impose a further restriction on the payoff structure. The discontinuity in the project's outcome guarantees that all the critical signals lie in the proximity of the critical

⁹See David Frankel, Stephen Morris and Ady Pauzner (2003). They show that for any prior $\phi(\theta)$ the posterior beliefs converge to the posteriors formed under the uniform prior, as $\sigma \rightarrow 0$.

fundamental θ^{**} , the lowest at which the project succeeds. Hence, when the signals are very precise, the critical types approximately agree on the value of θ and the analysis can solely focus on the differences in the beliefs over aggregate investment. However, even in the absence of the outcome discontinuity, the critical types often lie in the proximity of each other, and the results carry over. Let us discuss the details in a setup with two groups.

Assume that the function $\pi(\theta, l)$ measuring the level of the project's outcome is continuous and increasing in both arguments, and that it has dominance regions. As before, let the payoff for investing be $u(\theta, l, g) = b(\theta, g)\pi(\theta, l) - c(\theta, g)$. Let us consider a population consisting of two groups. Let $\bar{\theta}_g$ be the unique solution of

$$\int_0^{m_g} u(\theta, l, g) \frac{dl}{m_g} = 0,$$

and $\underline{\theta}_g$ be the unique solution of

$$\int_{1-m_g}^1 u(\theta, l, g) \frac{dl}{m_g} = 0.$$

The variables $\bar{\theta}_g, \underline{\theta}_g$ are natural bounds on the critical signal x_g^* . The variable $\bar{\theta}_g$ would be the critical signal of population g if group $-g$ never invested, and $\underline{\theta}_g$ would be the critical signal of the group g if group $-g$ always invested.

The following proposition states that our results carry over whenever the bounds on the critical signals overlap across the two groups.

Proposition 3. *Suppose players use rationalizable strategies. Then each player i invests whenever $x^i > \theta_g^{**} + 3\sigma$, and she does not invest whenever $x^i < \theta_g^{**} - 3\sigma$, where θ_g^{**} are as follows:*

1. *If $\bigcap_g [\underline{\theta}_g, \bar{\theta}_g] \neq \emptyset$ then $\theta_1^{**} = \theta_2^{**}$ is the unique solution of $s(\theta') = d(\theta')$. Moreover, the aggregation result in Corollary 3 applies.*
2. *If $\bigcap_g [\underline{\theta}_g, \bar{\theta}_g] = \emptyset$ then $\theta_1^{**} = \bar{\theta}_1$ and $\theta_2^{**} = \underline{\theta}_2$, where the labels g are such that $\bar{\theta}_1 < \underline{\theta}_2$.*

The proof is in the Appendix.

In the case when the bounds do not overlap, the critical signals lie far from each other, and the critical beliefs about the investment from the other group are trivial. The lower critical type knows that players from the other group do not invest, whereas the higher critical type knows that the players from the other group invest. Consequently, whenever our aggregation result does not apply, aggregation is actually not needed; the interaction between different groups is trivial.

Finally, let us discuss the time structure of the model. We have set up our model as a simultaneous move game. However, in many applications players from different groups move sequentially. For example, in the case of industrialization, the townsfolk may receive

information about new technologies before the country folk, who may learn about the technology only by observing the aggregate action of the townsfolk. Such sequential interaction can be studied within our setup. Assume that first the townsfolk receive private signals, $x_t^i = \theta + \sigma\eta_t^i$, with $\eta_t^i \sim F_t$ with a support on $[-1, 1]$, upon which the townsfolk decide (simultaneously) whether to invest or not. Second, the country players observe private signals, $x_c^i = y + \eta_c^i$, where y is some monotone transformation of the aggregate townsfolk investment, l_t . Consider monotone equilibria with critical signals x_t^* , x_c^* . The aggregate investment of the townsfolk is $l_t = 1 - F_t\left(\frac{x_t^* - \theta}{\sigma}\right)$ if $\theta \in [x_t^* - \sigma, x_t^* + \sigma]$ and $l_t = 0$ or 1 otherwise. As l_t , and hence y , is a monotone function of θ , the country folk can deduce information about θ from their noisy observation of y . If we assume that the observed aggregate statistic is $y = -F_t^{-1}(1 - l_t)$ then the model becomes analytically tractable. To see this, notice that a linear transformation of the country folk's private signals $\sigma x_c^i + x_t^*$ equals $\theta + \sigma\eta_c^i$. Thus, if we restrict attention only to monotone threshold equilibria, the sequential game has the same equilibria as the simultaneous game with the same error distributions.¹⁰

9 The Proof of the Belief Constraint

As the belief constraint is our central result, we wish to include its demonstration in the main body of the paper. It consists of finding a virtual homogeneous problem in which the critical belief is related to the critical beliefs of the original heterogeneous problem. This provides a useful characterisation of the original heterogeneous problem because the critical belief in the virtual homogeneous problem is well understood.¹¹

Let us define the mapping $\Delta(x^i, g^i) = x^i - x_{g^i}^*$ that reduces the two-dimensional type (x^i, g^i) to a one-dimensional signal, which we denote by $\tilde{x}^i = \Delta(x^i, g^i)$ and we call it a *virtual signal*. Notice that the strategy

$$a(x^i, g^i) = \begin{cases} \text{Invest} & \text{if } x^i \geq x_{g^i}^*, \\ \text{Not Invest} & \text{if } x^i < x_{g^i}^* \end{cases}$$

depends on the type (x^i, g^i) only via the virtual signal \tilde{x}^i : $a(x^i, g^i) \equiv \tilde{a}(\Delta(x^i, g^i))$, where

$$\tilde{a}(\tilde{x}^i) = \begin{cases} \text{Invest} & \text{if } \tilde{x}^i \geq \tilde{x}^*, \\ \text{Not Invest} & \text{if } \tilde{x}^i < \tilde{x}^*, \end{cases}$$

with the virtual threshold set to $\tilde{x}^* = 0$.

¹⁰This modeling approach to sequential global games has been first used in Dasgupta (2007). It has been widely used in the emerging dynamic global games literature as it conveniently reduces dynamic problems to static ones. This approach assumes away informational externalities and other details. Hence it is useful in cases where the details of social learning are not the main focus of the analysis.

¹¹A similar proof strategy is used by Eugen Kováč and Jakub Steiner (2008) in a different context. They study a dynamic global game and reduce the complex critical belief in the dynamic environment to a simple, well understood belief in a virtual static global game.

Next, we analyze the random variable $\ell(\theta) | (\tilde{x}^i = \tilde{x}^*)$. (It can be interpreted as a belief about l of a player who knows that her virtual signal is $\tilde{x}^i = \tilde{x}^*$, but does not know her original type (x^i, g^i) .) Recall that the investment level $l = \ell(\theta)$ where $\ell(\theta)$ was defined as

$$\ell(\theta) = \Pr \left(\{(x, g) : x \geq x_g^*\} | \theta \right).$$

Notice that $\ell(\theta)$ also satisfies

$$\ell(\theta) = \Pr \left(\tilde{x}^i \geq \tilde{x}^* | \theta \right).$$

All players use the identical virtual critical signal \tilde{x}^* and all are identical at the *ex ante* stage. In this symmetric environment, the belief $\ell(\theta) | (\tilde{x}^i = \tilde{x}^*)$ is already well understood. A player receiving the critical virtual signal has no information about the aggregate investment:

Lemma 3 (Laplacian property, Morris and Shin, 2003). *The belief about the measure of aggregate investment conditional on the critical virtual signal, $\ell(\theta) | (\tilde{x}^i = \tilde{x}^*)$, is uniformly distributed on $[0, 1]$.*

For convenience, we include the proof in the Appendix.

Finally, we relate the belief of the virtual critical type \tilde{x}^* to the beliefs of the original critical types (x_g^*, g) . We observe that the virtual signal is entirely uninformative about player i 's original group identity, g^i :

Lemma 4.

$$\Pr \left((x^i, g^i) = (x_g^*, g) | \tilde{x}^i = \tilde{x}^* \right) = m_g,$$

for all groups g .

The proof is in the Appendix.

The result in Lemma 4 is intuitive. Consider a player who is told only that her virtual signal is critical, $\tilde{x}^i = \tilde{x}^*$, but receives no additional information about her original type (x^i, g^i) . She knows that she is one of the original critical types (x_g^*, g) because $\tilde{x}^i = x^i - x_{g^i}^* = \tilde{x}^* = 0$. Yet, she learns nothing about her original group identity: In the proof of Lemma 4 we show that the virtual signal, \tilde{x}^i , is uniformly distributed¹² and hence the observation of \tilde{x}^i does not contain any information and the posterior belief $g^i | (\tilde{x}^i = 0)$ equals the prior belief.

Lemma 4 implies that the belief of the virtual critical type is the following compound lottery:

¹²Apart from close to boundaries of the support of θ .

$$\ell(\theta) | (\tilde{x}^i = \tilde{x}^*) = \begin{array}{l} \nearrow^{m_1} \ell(\theta) | ((x^i, g^i) = (x_1^*, 1)) \\ \dots \\ \rightarrow \dots \\ \dots \\ \searrow^{m_G} \ell(\theta) | ((x^i, g^i) = (x_G^*, G)) \end{array}$$

Hence, denoting the p.d.f. of $\ell(\theta) | (\tilde{x}^i = \tilde{x}^*)$ by $\lambda(l)$, we get

$$1 = \lambda(l) = \sum_g m_g \lambda_g(l),$$

where we used Lemma 3 in the first equality.

Q.E.D.

10 Conclusion

Economic agents that are relatively insensitive to others' actions have a large impact on the whole economy during coordination processes, because a large change in the others' behavior is required to motivate them to change their own behavior. We formalize this intuition in a global game model with heterogeneous payoffs and information. The analysis focuses on the beliefs about aggregate investment held by critical types – the players who based on their private information are indifferent between the two available courses of action. These critical beliefs turn out to be interrelated by a simple constraint: their average across all groups is a uniform belief.

The belief constraint implies that groups that are relatively insensitive to the project's outcome have relatively large influence in the coordination process. Suppose, for example, that the investment cost of a particular group decreases. To keep the critical type of the group indifferent, she must become less optimistic about the aggregate investment. Then, in order to satisfy the belief constraint, the critical types from other groups have to become more optimistic. If the members of the directly affected group are not too sensitive to the project's outcome then the changes of beliefs induced by the initial change in costs are large and they have large consequences on the equilibrium coordination outcome of the whole economy.

Appendix

Proof of Corollary 2. A player who has received the critical signal, x_g^* , knows that $\theta \leq x_g^* + \sigma \leq \theta' + 2\sigma$. Using the monotonicity of $\pi(\theta, l)$ we get

$$p_g(\mathbf{x}^*) \leq E[\pi(\theta' + 2\sigma, \ell(\theta)) \mid (x^i, g^i) = (x_g^*, g)] = \int_0^1 \pi(\theta' + 2\sigma, l) \lambda_g(l) dl.$$

Summing up over g we obtain

$$\begin{aligned} p(\mathbf{x}^*) &\leq \sum_g m_g \int_0^1 \pi(\theta' + 2\sigma, l) \lambda_g(l) dl = \int_0^1 \pi(\theta' + 2\sigma, l) \sum_g m_g \lambda_g(l) dl \\ &= \int_0^1 \pi(\theta' + 2\sigma, l) dl = s(\theta' + 2\sigma). \end{aligned}$$

A symmetric argument establishes the lower bound. \square

Proof of Lemma 1. The expected payoff conditional on being the critical type, (x_g^*, g) , is 0. As in the previous proof, a player who has received the critical signal, x_g^* , knows that $\theta \leq x_g^* + \sigma \leq \theta' + 2\sigma$. Using the monotonicity of functions b and c with respect to θ we get $0 \leq b(\theta' + 2\sigma, g) \times p_g(\mathbf{x}^*) - c(\theta' + 2\sigma, g)$. By rearranging, we obtain

$$\frac{c(\theta' + 2\sigma, g)}{b(\theta' + 2\sigma, g)} \leq p_g(\mathbf{x}^*).$$

Summing up over g gives $d(\theta' + 2\sigma, g) \leq p(\mathbf{x}^*)$. The upper bound follows from the symmetric argument. \square

Proof of Proposition 2. Let us first analyze monotone Bayes-Nash equilibria defined by a tuple of critical signals \mathbf{x}^* .

By Lemma 2 the proximity condition, $|x_g^* - \theta'| \leq \sigma$ for each g , is satisfied with $\theta' = \theta^*(\mathbf{x}^*)$. Therefore we can use Corollary 2 and Lemma 1. The corollary implies that $p(\mathbf{x}^*) \leq s(\theta^*(\mathbf{x}^*) + 2\sigma)$ and the lemma implies $d(\theta^*(\mathbf{x}^*) + 2\sigma) \leq p(\mathbf{x}^*)$ and therefore

$$d(\theta^*(\mathbf{x}^*) + 2\sigma) \leq s(\theta^*(\mathbf{x}^*) + 2\sigma).$$

The function d is non-increasing and s is strictly increasing. Therefore $\theta^*(\mathbf{x}^*) + 2\sigma$ lies above the unique solution of $d(\theta') = s(\theta')$:

$$\theta^*(\mathbf{x}^*) + 2\sigma \geq \theta^{**}.$$

Rearranging gives $\theta^*(\mathbf{x}^*) \geq \theta^{**} - 2\sigma$. The symmetric argument establishes the upper bound $\theta^*(\mathbf{x}^*) \leq \theta^{**} + 2\sigma$. The bounds $\theta^{**} - 3\sigma \leq x_g^* \leq \theta^{**} + 3\sigma$ follow from the bounds on $\theta^*(\mathbf{x}^*)$ and from Lemma 2.

We have proved that statements 1. and 2. of Proposition 2 hold in any monotone symmetric equilibrium.¹³ Moreover these two results hold under any *rationalizable* strategy profile: The analyzed game is monotone supermodular and symmetric. Hence by Milgrom and Roberts (1990) and Van Zandt and Vives (2006), all rationalizable strategies are bounded by the greatest and the least Bayesian equilibria which are monotone and symmetric. Applied to our game, this implies that the greatest and least equilibrium are defined by some tuples of critical signals $\underline{\mathbf{x}}^*$ and $\bar{\mathbf{x}}^*$, respectively. The statements 1. and 2. of Proposition 2 hold for the greatest and the least rationalizable strategy profile and hence they hold for any rationalizable strategy profile. \square

Proof of Proposition 3. Statement 1: We will show that, for sufficiently small σ , the proximity result from Lemma 2 applies: there exists θ' such that $|x_g^* - \theta'| < \sigma$ for both critical signals in any monotone strategy profile. When the proximity result holds then the proof of Proposition 2 applies.

Suppose for contradiction that $x_g^* < x_{-g}^* - 2\sigma$. Then type (x_g^*, g) knows that no player from the group $-g$ invests. Therefore, type (x_g^*, g) believes that l is distributed uniformly on $[0, m_g]$ and so

$$0 = E [u(\theta, l, g) | (x_g^*, g)] \leq \int_0^{m_g} u(x_g^* + \sigma, l, g) \frac{dl}{m_g}.$$

Therefore

$$\bar{\theta}_g \leq x_g^* + \sigma. \quad (6)$$

Similarly, type $(x_{-g}^*, -g)$ knows that all players from the group g invest. Therefore, type $(x_{-g}^*, -g)$ believes that l is distributed uniformly on $[m_g, 1]$ and so

$$0 = E [u(\theta, l, -g) | (x_{-g}^*, -g)] \geq \int_{1-m_g}^1 u(x_{-g}^* - \sigma, l, -g) \frac{dl}{m_g}.$$

Therefore

$$\underline{\theta}_{-g} \geq x_{-g}^* - \sigma. \quad (7)$$

$\bigcap_g [\underline{\theta}_g, \bar{\theta}_g] \neq \emptyset$ implies that $\underline{\theta}_{-g} \leq \bar{\theta}_g$. This, combined with inequalities (6) and (7) leads to $x_g^* \geq x_{-g}^* - 2\sigma$, which contradicts the hypothesis. Thus $x_g^* \geq x_{-g}^* - 2\sigma$ and symmetrically $x_{-g}^* \geq x_g^* - 2\sigma$, as needed for the proximity condition.

Statement 2: Consider any equilibrium in threshold strategies. Type $(x_1^*, 1)$ can satisfy the indifference condition only if $x_1^* \leq \bar{\theta}_1 + \sigma$. Similarly $x_2^* \geq \underline{\theta}_2 - \sigma$. We use labeling in which $\bar{\theta}_1 < \underline{\theta}_2$. Therefore, for sufficiently small σ , $x_1^* < x_2^* - 2\sigma$. Then the type $(x_1^*, 1)$ knows that no player from group 2 invests. Hence her expected payoff for investment is bounded in the interval $\left[\int_0^{m_1} u(x_1^* - \sigma, l, 1) \frac{dl}{m_1}, \int_0^{m_1} u(x_1^* + \sigma, l, 1) \frac{dl}{m_1} \right]$, and the indifference

¹³Players from different groups may play different actions in a *symmetric* equilibrium because a strategy is a mapping from type to action and the two-dimensional types include the group identity.

condition implies $|x_1^* - \bar{\theta}_1| \leq \sigma$. Similarly $|x_2^* - \theta_2| \leq \sigma$.

We have proved that player i invests whenever $x^i > \theta_g^{**} + \sigma$, and she does not invest whenever $x^i < \theta_g^{**} - \sigma$ in any monotone symmetric equilibrium.¹⁴ The extension to all rationalizable strategy profiles is the same as at the end of the proof of Proposition 2. \square

Proof of Lemma 3 (based on Morris and Shin 2003). The virtual error defined as $\tilde{\eta}^i = \frac{\tilde{x}^i - \theta}{\sigma}$ is a compound lottery

$$\left(\eta_1^i + \frac{x_1^*}{\sigma}, \eta_2^i + \frac{x_2^*}{\sigma}, \dots, \eta_G^i + \frac{x_G^*}{\sigma}; m_1, m_2, \dots, m_G \right).$$

Therefore $\tilde{\eta}^i$ is i.i.d. across players, and also independent from θ . Let \tilde{F} denote the c.d.f. of $\tilde{\eta}^i$. (The mean of $\tilde{\eta}^i$ is not 0, so that the virtual signal \tilde{x}^i has a bias, but that is not relevant for the Laplacian Property.)

Note that if critical signals x_g^* are not sufficiently close to each other then the support $S_{\tilde{\eta}}$ of $\tilde{\eta}$ may not be connected. In that case, the inverse of \tilde{F} may not be uniquely defined at the points of discontinuity of the support. We uniquely specify the inverse function $\tilde{F}^{-1}(p)$ as $\min_{\eta \in S_{\tilde{\eta}}} \{\tilde{\eta} : \tilde{F}(\eta) = p\}$.

The computation below shows that the c.d.f. of $\ell(\theta) \mid (\tilde{x}^i = x_1^*)$ is indeed that of the uniform distribution on $[0, 1]$. In the first equality we use that $\ell(\theta) = \Pr(\tilde{x}^j > \tilde{x}^* \mid \theta)$.

$$\begin{aligned} & \Pr\left(\ell(\theta) < z \mid \tilde{x}^i = \tilde{x}^*\right) = \\ & \Pr\left(\Pr(\tilde{x}^j \geq \tilde{x}^* \mid \theta) < z \mid \tilde{x}^i = \tilde{x}^*\right) = \\ & \Pr\left(\Pr\left(\tilde{\eta}^j \geq \frac{\tilde{x}^* - \theta}{\sigma}\right) < z \mid \tilde{x}^i = \tilde{x}^*\right) = \\ & \Pr\left(1 - \tilde{F}\left(\frac{\tilde{x}^* - \theta}{\sigma}\right) < z \mid \tilde{x}^i = \tilde{x}^*\right) = \\ & \Pr\left(1 - \tilde{F}(\tilde{\eta}^i) < z \mid \tilde{x}^i = \tilde{x}^*\right) = \\ & \Pr\left(\tilde{\eta}^i > \tilde{F}^{-1}(1 - z)\right) = \\ & 1 - \tilde{F}\left(\tilde{F}^{-1}(1 - z)\right) = z. \end{aligned}$$

\square

Proof of Lemma 4. We fix player i and omit her index. Let us write capital letter for a random variable, and small letter for its realization. To distinguish it from the number of groups G , we denote the random variable describing player i 's group by G' .

¹⁴To unify the formulation of Proposition 3, we stated in both statements that player i invests whenever $x^i > \theta_g^{**} + 3\sigma$, and she does not invest whenever $x^i < \theta_g^{**} - 3\sigma$.

Let (Θ, X, G') be a random variable describing the fundamental and the type of player i . The probability density of (Θ, X, G') is

$$p_{\Theta, X, G'}(\theta, x, g) = \frac{1}{\bar{\theta} - \underline{\theta}} f_g \left(\frac{x - \theta}{\sigma} \right) \frac{1}{\sigma} m_g,$$

where $f_g(\cdot)$ is the p.d.f. of the conditional random variable $\eta^i \mid (g^i = g)$.

The marginal probability density of (X, G') is

$$p_{X, G'}(x, g) = \int_{x-\sigma}^{x+\sigma} \frac{1}{\bar{\theta} - \underline{\theta}} f_g \left(\frac{x - \theta}{\sigma} \right) m_g \frac{d\theta}{\sigma} = \frac{m_g}{\bar{\theta} - \underline{\theta}},$$

for all $x \in [\underline{\theta} + \sigma, \bar{\theta} - \sigma]$. The marginal probability density of the virtual signal $\tilde{x} = x - x_g^*$ is

$$p_{\tilde{X}}(\tilde{x}) = \sum_g p_{X, G'}(\tilde{x} + x_g^*, g) = \sum_g \frac{m_g}{\bar{\theta} - \underline{\theta}} = \frac{1}{\bar{\theta} - \underline{\theta}},$$

if

$$\tilde{x} + x_g^* \in [\underline{\theta} + \sigma, \bar{\theta} - \sigma] \text{ for each } g. \quad (8)$$

Finally, if \tilde{x} satisfies (8) then

$$\Pr(G' = g \mid \tilde{X} = \tilde{x}) = \frac{p_{X, G'}(\tilde{x} + x_g^*, g)}{p_{\tilde{X}}(\tilde{x})} = m_g.$$

Notice that $\tilde{x} = \tilde{x}^* = 0$ satisfies (8) because we consider only critical signals $x_g^* \in [\underline{\theta} + \sigma, \bar{\theta} - \sigma]$ (which is without loss of generality because if player receives signal outside of $[\underline{\theta} + \sigma, \bar{\theta} - \sigma]$ then she knows that θ lies in a dominance regions. Hence signals outside of $[\underline{\theta} + \sigma, \bar{\theta} - \sigma]$ cannot satisfy the indifference condition.) \square

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