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Essays in Behavioral Microeconomic Theory

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Proefschrift

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Contents

Chapter 1

Introduction

It is widely accepted that the traditional microfoundations of economics, the so-called *homo economicus*, have some severe flaws. The elegant framework put forward by Bernoulli, von Neumann, Morgenstern and many others, while powerful and flexible to construct most of the economic literature, from health economics to finance, from environmental economics to game theory, fails to agree with simple behavioral observations that experimental economics and other fields have established.

In the Expected Utility Model the utility of a sequence of possible outcomes is given by the sum of the utility of each possible outcome (which are therefore assumed to be separable and additive), across time periods and states of nature, weighted with its probability of happening. When different time periods are involved, each period is usually also weighted with a time discount, which has a constant discount factor between equally distant periods. Savage (1954) proposes an extension of this model to the cases where the probability of each state of nature is not given, using only simple rationality postulates. This is the so-called Subjective Expected Utility.

Deviations from this standard framework include for instance the fact that individuals seem to have *loss aversion* (see Kahneman and Tversky (1979)), that is, the marginal utility of a gain from the status quo is considerably lower than the (absolute) marginal utility of a loss. This affects their choices under risk, when gains and losses are possible, in a way not explained by risk aversion. Moreover, there is evidence that individuals do not discount the same time intervals at a constant rate (see Ainslie (1991)). This implies that there may be preference reversals as time passes.

While some departures from the standard theory may be considered simple psychological anecdotes, others have serious implications both for microeconomics and macroeconomics. And while experimental economics and behavioral economics became well established fields inside economics in the last 40 years, much of its accumulated knowledge has not been used completely in other fields.

It is the purpose of this thesis to contribute to this connection and to study the impact of some of the behavioral economics ideas in other fields. To be specific, three behavioral departures from the Expected Utility Model are considered: ambiguity aversion, time inconsistent preferences and imperfect recall in games. The first one is applied in auction theory, the second one to savings decisions, and the last one in a price competition model.

Chapter 2 presents the outcome of a dynamic price-descending auction when the distribution of the private values is uncertain and bidders exhibit ambiguity aversion. In contrast to sealed-bid auctions, in open auctions the bidders get information about the other bidders' private values and may therefore update their beliefs on the distribution of the values. The bidders have smooth ambiguity preferences and update their priors using consequentialist Bayesian updating.

It is shown that ambiguity aversion usually affects bidding behavior the same way risk aversion does, but the main result is that this is not the case for continuous price descending auctions. This is new among a few theoretical cases where ambiguity aversion does not reinforce the risk aversion implications.

Chapter 3 focuses on the behavior of a decision maker whose preferences are dynamically inconsistent, when that inconsistency is acknowledged by the individual. This chapter proposes a new model on this issue, inspired by the model of staggered prices from Calvo (1983). Individuals are modeled as lacking self-control and being prone to present-biased impulses. In some random periods they decide according to a constant discount rate but in the other periods they follow their present-biased impulses. In the former they recognize that inconsistent actions may be chosen, but in the latter they naively believe that their momentary optimal plan will be followed in the next periods and make up for it. The possible sequences of naive and consistent decisions form a tree, where the upper decisions dominate the lower ones, composing a socially structured game (see Herings, van der Laan, and Talman (2007)). It is shown that this model solves some of the puzzling results of other theoretical frameworks.

Furthermore, aggregating the possible trajectories according to their probability, leads to a unique outcome. It is suggested that this outcome can be interpreted as the the behavior of a representative agent of a macroeconomic model. Some examples of consumption and savings decisions are discussed.

Chapter 4[1.1] studies the consequence of an imprecise recall of the price by the consumers in the Bertrand price competition model for a homogeneous good. It is shown that this creates room for firms to be able to charge prices above the competitive price, the markup increasing with the size of the recall errors. Moreover, firm with higher costs may still persist in the market. They will however have a higher equilibrium price, so that price dispersion arises.

Furthermore, if bigger recall errors happen then both consumers and firms on the aggregate level may be worse off, when the cost difference between firms is big enough. Thus, there are situations where the protection of a monopolist against entrants is a welfare maximizing policy. The introduction of more firms in the market does not have a significant impact on the prices.

[1.1] This chapter is based on Carvalho (2009).

Chapter 2

Static and Dynamic Ambiguous Auctions

2.1 Introduction

In auction theory it is assumed that bidders know the distribution from which the private values are drawn. If this distribution is uncertain, a subjective distribution of possible distributions is still needed for modeling purposes, which can then be reduced to a single distribution.

This is not the case under ambiguity aversion, the case where a decision maker is averse to uncertainty about the risk. Ambiguity aversion is portrayed by the seminal experiment in Ellsberg (1961), where decision makers prefer to bet on lotteries with known probabilities, instead of unknown, even if a priori their expected payoff is the same.

This chapter studies the consequences of relaxing the assumption of knowledge of the distribution of private values, on equilibrium bidding behavior. Ambiguity averse preferences are modeled using the smooth ambiguity model developed in Klibanoff, Marinacci, and Mukerji (2005). In the first-price sealed-bid auctions, ambiguity aversion leads to higher bids even if bidders are risk neutral, whereas ambiguity has no consequence on dynamic auctions, either price ascending or descending, if the price changes continuously. This later result is independent of the risk attitude of bidders, and it is a new qualitative result on ambiguity aversion.

This chapter is structured as following. Section 2.2 describes the evolution of the literature and some of its issues, Section 2.3 presents and explains the basics, Section 2.4 discusses static auctions under ambiguity aversion, Section 2.5 goes through a dynamic auction, and Section 2.6 concludes.

2.2 Literature

Knight (1921) makes apparently the first distinction between risk and ambiguity, calling the latter uncertainty, reason for which the terms *ambiguity*, *uncertainty* and *Knightian uncertainty* are used interchangeably in the literature. Knight refers to "measurable uncertainty" as *risk*, whereas *uncertainty* should be restricted to cases not "susceptible of measurement". Ellsberg (1961) provides on the other hand the first formal definition of ambiguity, through some experiments that violate Savage's Subjective Expected Utility Axioms. In these experiments, later called the Ellsberg paradox, subjects tend to prefer unambiguous lotteries in a way that cannot be reproduced by risk aversion.

The Ellsberg's paradox consists in an experiment with an urn with 30 red balls and 60 being either black or yellow with unknown distribution. Define lotteries as the vector (r_R, r_B, r_Y) which pays r_i, $i \in \{R, B, Y\}$, if a ball of color i is drawn. Subjects are to make two choices, first between lottery $(1, 0, 0)$ and lottery $(0, 1, 0)$, second between lottery $(1, 0, 1)$ and lottery $(0, 1, 1)$. Typically subjects prefer $(1, 0, 0)$ over $(0, 1, 0)$ implying that their subjective probability for red is higher than that for black. However they tend to prefer lottery $(0, 1, 1)$ over $(1, 0, 1)$ which implies the opposite, their subjective probability for red is lower than that for black. This paradox is independent of the risk aversion of the subjects. Intuitively subjects have a preference towards known risks, i.e. unambiguous lotteries. Ellsberg's results have been replicated by other experiments, see Camerer and Weber (1992) for a survey.

Schmeidler (1989) suggests that individuals act as if their subjective probability for ambiguous events were lower than for objective equivalent ones. That is, the subjective probability attached to black in the experiment, is lower than that for red. This leads to non-additive probabilities, i.e. subjective probabilities that do not add up to one. Taking this to calculate the expected utility using the usual Riemann Integral with a probability measure leads to inconsistencies like discontinuities in the integrand and violation of monotonicity (see Chapter 16 in Gilboa (2009)). Schmeidler (1989) uses therefore capacities, generalized probabilities. The expected utility of an act using capacities is given by the Choquet Integral, from which this model derives its name, Choquet Expected Utility. Taking v to be the capacity (probability), the Choquet Expected Utility of a given act (a mapping from the states of nature to outcomes) f, with $f(\omega) \geq 0$ for all $\omega \in \Omega$, is given by

$$V(f) = (C) \int_{\Omega} f dv \equiv \int_{0}^{\infty} v(f \geq t) dt,$$

where $(C)\int$ stands for the Choquet Integral and Ω is the state space. If the capacity of event A, $v(A)$, is interpreted as the worth of coalition A in a Transferable Utility Cooperative Game, the Choquet Integral can be written in a more intuitive way. Given the non-additivity of $v(\cdot)$ and its ambiguity aversion interpretation given above, $v(\cdot)$ should be convex - some authors take this convexity as the definition of ambiguity aversion (for a discussion on the formal definition of Ambiguity Aversion see Epstein (1999)). If it is convex, then the corresponding TU game has a non-empty core $Core(v)$. Schmeidler (1986) shows that in this case, the above Choquet integral can be written as

$$(C) \int_{\Omega} f dv = \min_{p \in Core(v)} \int_{\Omega} f dp. \tag{2.1}$$

As in the core of a TU game where the allocation of a player should not only be checked against its value alone but also against all coalitions she may belong to, the probability of a state should not enter directly but checked over all the capacities of subsets of the states of nature to which it belongs.

The Multiple Priors or Maxmin Expected Utility model proposed by Gilboa and Schmeidler (1989), while derived from independent axioms, has an intuition which is related to expression (2.1). It assumes that the individual acts as if she had multiple (additive) priors for the subjective probability. The expected utility of an act is the minimum expected utility across the priors. In the transferable utility game interpretation, this minimum is then the socially stable core (defined in Herings, van der Laan, and Talman (2007)) where the least favorable outcomes have higher power. The individuals then proceed to maximize across these minima, therefore the name Maxmin Utility. Utility of act f over the set of priors $\mathcal{P}$ is given by

$$V(f) = \min_{p \in \mathcal{P}} E_p[f].$$

This model coincides with the Choquet Expected Utility if the set of priors $\mathcal{P}$ equals the core of some capacity v. As Gilboa (2009) points out the set of priors should not be interpreted as the set of all possible (given the available information) probability distributions, which would be too broad, but as implicit subjective probabilities in line with Savage's Subjective Probability Framework.

Bewley (2002) (originally from 1986) proposes another multiple priors model, where act f is preferred over act g if its expected utility is higher for all priors.

Ghirardato, Maccheroni, and Marinacci (2004) suggest that ambiguity, i.e. uncertainty on the probabilities of the states of nature, and ambiguity attitude, i.e. the way agents react to ambiguity, should be separated in the utility functionals. They propose axiomatically the α-Maxmin Expected Utility where the utility of act f is given by

$$V(f) = \alpha \min_{p \in \mathcal{P}} E_p[f] + (1 - \alpha) \max_{p \in \mathcal{P}} E_p[f],$$

where α is a parameter that captures the ambiguity attitude of the agent. For $\alpha = 1$ the agent will be ambiguity averse as in the Maxmin model.

Variational Preferences were proposed by Maccheroni, Marinacci, and Rustichini (2006), inspired on the Multiplier Preferences from Hansen and Sargent (2001) which draws from Robust Control, where different priors p are weighted through an *ambiguity index* $c(p)$, whose value increases with the ambiguity level of the prior,

$$V(f) = \min_{p \in \Delta(\Omega)} \left(\int_\Omega u(f) dp + c(p) \right),$$

$u(\cdot)$ being the usual Bernoulli utility function and $\Delta(\Omega)$ being the set of distributions over the state space Ω. Notice that the minimization is carried out over all possible priors.

A further set of models also weights priors, in a similar way that outcomes are weighted with their probability of occurring in the Expected Utility Model. This class is called Recursive Expected Utility or Second Order Beliefs, because each prior is assigned a (second-order) probability of being the correct one, these second-order being distributed with probability measure μ. Usually priors are indexed through some parameter $\theta \in \Theta$ and p_θ is the probability distribution for prior θ. The utility of an act will be calculated by aggregating the certainty equivalent of each probability prior, across all priors. The most common model in this category, the Smooth Ambiguity Preferences, is proposed in Klibanoff, Marinacci, and Mukerji (2005) where this aggregation is a concave (convex for ambiguity loving preferences) functional $\phi(\cdot)$ which could be interpreted as a second order Bernoulli function. Its concavity represents the aversion to uncertainty on the correct prior. The utility of act f is defined as

$$V(f) = \int_{\Theta} \phi \left(\int_{\Omega} u(f) dp_{\theta} \right) d\mu.$$

While clearly routed in the multiple priors model, the smooth ambiguity preferences have a straightforward intuition. In terms of attitude towards risk, a concave Bernoulli utility function performs the task of assigning lower weight to high outcomes and higher weight to low ones when adding the outcomes up, so that a risk averse individual focuses more on the bad results. With ambiguity, an ambiguity averse individual with a concave $\phi(\cdot)$ will analogously stress those priors, i.e. those possible probability distributions, that yield the worst scenarios in terms of expected outcome.

Ambiguity aversion and dynamics, i.e. preference updates as new information is gathered, have been two concepts difficult to be reconciled. The main issue can be discussed using a dynamic version of the Ellsberg paradox proposed by Epstein and Schneider (2003). Consider the same experiment but with an additional step after the ball is taken from the urn, where the individual gets to know whether the ball is yellow or not. Initially an ambiguity averse individual prefers lottery $(0, 1, 1)$ over $(1, 0, 1)$. After the ball is drawn, she will have $(0, 1, 1) \sim (1, 0, 1)$ if the ball is yellow. In the other case, if she bayesianly updates the priors for the remaining balls, she shall have $(1, 0, 1) \succ (0, 1, 1)$. Take for instance the Maxmin Expected Utility model with the following set of priors $\mathcal{P} = \{(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}), (\frac{1}{3}, \frac{1}{6}, \frac{1}{2})\}$. Conditional on not being yellow these priors become $\{(\frac{2}{5}, \frac{3}{5}, 0), (\frac{2}{3}, \frac{1}{3}, 0)\}$ using Bayes rule. So the maxmin expected utility for $(0, 1, 1)$ is initially $\frac{2}{3}$ and then $\frac{1}{3}$, while for $(1, 0, 1)$ it decreases only from $\frac{1}{2}$ to $\frac{2}{5}$. Thus, the individual does not keep his preferences in none of the intermediate states, that is the preferences do not satisfy dynamic consistency, which is loosely defined as the non-reversal of preferences from period t to $t + 1$ between two acts which are equal until t, but one is preferred for every possible prior in $t + 1$.

Different solutions have been followed in the literature. These are to enforce dynamic consistency through the choice of the time aggregating functional (as in Klibanoff, Marinacci, and Mukerji (2009) for the Smooth Ambiguity Model), backward induction like the sophisticated agents in Pollak (1968) (as in Siniscalchi (2010)), the imposition of consistency conditions on the priors (as in Epstein and Schneider (2003)), or discretionary priors update rules which depend on the preferences, the events and the choice problem (as in Klibanoff and Hanany (2007) and Hanany and Klibanoff (2009)). In the above example, a dynamically consistent ambiguity averse individual would then compulsory prefer $(1, 0, 1)$ over $(0, 1, 1)$ in the beginning.

Another approach is to discard dynamic consistency, accept the above apparent preference change and impose consequentialism, which states that the decision maker is indifferent between two acts which yield the same payoffs for all priors, irrespectively of what happened in the previous periods. To understand the implication of this assumption in the above example, consider the preference on $(1, 0, 0)$ vs. $(0, 1, 0)$. The intermediate step bares no change, either the ball is yellow and payoff is zero in both or it is not yellow and the two available lotteries are still between red and black, so an ambiguity averse individual should prefer the first over the second in both periods. If now compared to the preferences $(0,1,1) \succ (1,0,1)$ in the first period, consequentialism then states that the ambiguity averse decision maker should switch its preference in the intermediate step if the ball is not yellow, because $(1,0,1)$ and $(0,1,1)$ coincide with $(1,0,0)$ and $(0,1,0)$, respectively, in the remaining nodes. Consequentalism is satisfied if the decision maker follows a Bayesian update rule for the priors.

Consequentialist priors update rules have been axiomatized according to different requirements. Gilboa and Schmeidler (1993) axiomatize the Dempster-Shafer update rule for the Multiple Priors Model. As new information becomes available for the decision maker, she picks those priors that assign maximum likelihood to the information and updates them with Bayes rule. They also show this coincides with Bayesian updating for capacities, provided that the Choquet and Maxmin preferences coincide. Pires (2002) axiomatizes a different Bayesian update rule where all priors are kept and all are updated according to Bayes rule.

Ozdenoren and Peck (2008) further suggests that dynamic inconsistent behavior of ambiguity averse individuals can be interpreted as consistent subgame perfect equilibrium strategies in a game against nature, which influences ambiguous outcomes.

There is also a rich empirical, applied and experimental literature on Ambiguity Aversion.

Hey, Lotito, and Maffioletti (2007) use an inventive device to simulate ambiguity in the lab. Subjects can see a bingo blaster and estimate the number of balls with different colors. Through a series of binary tests, the authors conclude that Choquet Expected Utility fits the date best, but also claim that the decisions vary a lot across individuals.

In a portfolio choice application, Dow and Werlang (1992) show that an agent with Maxmin Expected Utility has a price range for which she chooses not to buy and not to sell an asset. This behavior is not due to some status quo bias (as in the Bewley (2002) model) but as a safe allocation consideration.

Epstein and Schneider (2003) claim that ambiguity aversion may explain the home bias that investors exhibit. Ju and Miao (2009) use ambiguity aversion in an asset pricing model to show that it can explain the equity premium and its volatility.

This is not to say that this literature is consensual. For instance, the experiments in Halevy (2007) show that there is a significant positive correlation between displaying ambiguity aversion and violation of reduction of compound objective lotteries. See Al-Najjar and Weinstein (2009) for further criticism.

Dominiak, Dürsch, and Lefort (2009) test the dynamic version of the Ellsberg experiment and find that most subjects tend to follow consequentialism, meaning that they are not acting in a dynamically consistent way.

For more comprehensive reviews on the literature see Etner, Jeleva, and Tallon (2009).

Few work has been put forward assessing the impact of ambiguity aversion on auctions. Using Choquet Expected Utility, Salo and Weber (1995) show that ambiguity aversion may explain the (beyond risk aversion) overbidding in first-price sealed-bid auctions when the distribution of private values or the number of bidders is ambiguous. Lo (1998) examines first and second-price sealed-bid auctions when both the bidders' and the auctioneer's preferences follow the Maxmin model, indicating that the effects of ambiguity attitudes are similar, but not equal, to those of risk in terms of bidding and revenue. Bose, Ozdenoren, and Pape (2006) study the optimal static auction mechanism with ambiguity. Chen, Katuscak, and Ozdenoren (2007) test experimentally the bidding behavior in first-price sealed-bid auctions and get lower over-bidding in the ambiguity treatment. Bose and Daripa (2009) is the first analyzing dynamic auctions with ambiguity (bidders choose strategies from backward induction), but from the optimal auction point of view. They show that with ambiguity, modeled with Maxmin preferences, the auctioneer can extract almost all surplus, in contrast to the unambiguous case.

The experiments in Armantier and Treich (2009) indicate that probabilistic bias are the main drive of overbidding in first-price sealed-bid auctions. Some experimental literature use compound lotteries to simulate ambiguity. While theoretically they are very different concepts, most ambiguity aversion models can also have a bad reduction of compound lotteries interpretation. Moreover as mentioned above, there seems to be a high correlation between individuals exhibiting one and the other behavior. Liu and Colman (2009) compare decisions between single-choice and repeated-choice Ellsberg urn choices. In the latter, decision makers tend to pick the ambiguous option more frequently. Kocher and Trautmann (2011) run an experiment where subjects can choose to participate in a risky or in an ambiguous first-price sealed-bid auction. While the equilibrium price is the same in both, bidders tend to avoid the ambiguous auction.

2.3 Framework

In conventional Auction Theory the bidders (and the auctioneer) have limited information of each other. They are not aware of the value that the auctioned object represents for the other players and therefore they do not know the other players' payoffs. For any results to be established one must clearly make quantitative assumptions, so it is assumed that the probabilistic distribution of these values is common knowledge. While the assumption of perfect information on the probabilistic distribution may be too strong, any more elaborate assumptions end up to be equivalent through compound lottery reduction. It is known that, risk aversion aside, individuals display aversion to risky choices where the probability distribution of the outcomes is not perfectly known, i.e. they display Ambiguity Aversion. A popular method to generalize Expected Utility Theory to allow for these preferences to be included, is the Smooth Ambiguity Model from Klibanoff, Marinacci, and Mukerji (2005). Instead of using a single distribution of the unknown parameters, ambiguity is introduced through multiple possible distributions.

Formally there are multiple prior probability measures π_θ, where $\theta \in \Theta$ indexes the priors, over the possible states of nature ω, with $\omega \in \Omega$. Particular to this ambiguity model is the assumption of a probability measure over the different priors, represented by μ defined from 2^Θ to $[0,1]$. Ambiguity Aversion is then modeled in a similar way as Risk Aversion, that is using a concave function $\phi(\cdot)$ to aggregate the (certainty equivalent) outcomes of act f over all priors with μ, that is aggregating $\int_\Omega u(f(\omega))d\pi_\theta$ over θ. Act f maps a state of nature $\omega \in \Omega$ to an outcome $f(\omega)$ yielding utility $u(f(\omega))$, where the utility function $u(\cdot)$ is taken to be (weakly) concave to represent risk (neutrality) aversion. The utility of f in the smooth ambiguity model is given by

$$U(f) = \int_\Theta \phi\left(\int_\Omega u(f(\omega))d\pi_\theta\right) d\mu. \quad (2.2)$$

This model is chosen for several reasons. It is a smooth model, meaning that differentiable functionals may be used so that the utility itself is differentiable, in opposition to most Ambiguity Aversion models. Moreover the model allows to distinguish between the consequences of different levels of ambiguity, given by the spread of the prior, and those of idiosyncratic ambiguity aversion, given by the shape of $\phi(\cdot)$. A further reason is related to dynamic decisions under ambiguity, namely the update of priors as new information is received. Having a probability measure on the priors allows to put more weight on priors that seem to be more credible with the new information[2.1].

In all the basic auctions being considered here an indivisible good is being auctioned. The private values of the good to the n bidders are randomly drawn from distribution F_θ with support $[0,1]$, with $\theta \in \Theta$. Private values are assumed to be independently drawn across agents. The probability of each possible distribution F_θ is given by the measure μ on 2^Θ.

To enable a comparison with the unambiguous case, an equivalent subjective probability distribution F_U will be defined, satisfying

$$\int F_\theta^{n-1}(x)d\mu = F_U^{n-1}(x),\ \forall x \in [0,1]. \quad (2.3)$$

F_U can be interpreted as the reduced probability distribution that an ambiguity neutral bidder considers. Let $G_\theta(x) = F_\theta^{n-1}(x)$ and similarly for $G_U(x)$,

$$\int G_\theta(x)d\mu = G_U(x),\ \forall x \in [0,1].$$

Notice that this implies

$$\int \frac{d}{dx}G_\theta(x)d\mu = \frac{d}{dx}G_U(x),\ \forall x \in [0,1].$$

Moreover it is assumed that all priors θ, $\theta \in \Theta$, are such that an auction with F_θ as the value distribution has a unique monotonic equilibrium pricing strategy.

It should be underlined that these priors are the same across all bidders and they represent the beliefs that the bidders have after learning their own value. Otherwise, given their own value they would update their second order beliefs μ according to it.

2.4 Static ambiguous auctions

The two most common types of static auctions are considered, the first-price sealed-bid auction and the second-price sealed-bid auction.

[2.1] For a critic on the Smooth Ambiguity Model see Epstein (2010).

2.4.1 First-price sealed-bid auction

In the first-price sealed-bid auction, all bidders submit one bid at the same time. The good is then given to the bidder with the highest bid, for which she pays the offered price.

Ambiguity neutrality

Consider the case of ambiguity neutral bidders with F_θ for priors and μ the measure on the priors. The first-price sealed-bid auction will be equivalent to the unambiguous case where values follow the F_U distribution defined in equation (2.3). This follows directly from the usual reduction of compound lotteries, or mathematically as the combination of the two integrals (2.2) to a single measure. With ambiguity neutrality, that is with $\phi(y) = y$, any expectation becomes simply

$$\begin{aligned}
U(f) &= \int_\Theta \phi\left(\int_\Omega f(\omega)dF_\theta\right)d\mu \\
&= \int_\Theta \int_\Omega f(\omega)dF_\theta d\mu \\
&= \int_\Omega f(\omega)dF_U,
\end{aligned}$$

which is the ambiguity neutrality case.

Ambiguity aversion

If bidders have ambiguity aversion modeled as in (2.2), the priors cannot be reduced to a single distribution. For a given increasing differentiable strategy for the first-price sealed-bid auction $\beta_1(\cdot)$, where the index 1 stands for first-price, followed by the $n-1$ opponents, a bidder with value v who chooses to bid as if she had value z, will win the auction with probability $G_\theta(z)$, yielding in that case a utility of $u(v - \beta_1(z))$, according to prior $\theta \in \Theta$. The certainty equivalent of this choice is then, still according to prior θ, $G_\theta(z)u(v-\beta_1(z))$. To compute the expected utility one has to aggregate over all priors, which leads to the expected utility

$$\int \phi\left(G_\theta(z)u(v - \beta_1(z))\right)d\mu.$$

The best response for the strategy $\beta_1(\cdot)$ will therefore solve

$$\max_z \int \phi\left(G_\theta(z)u(v - \beta_1(z))\right)d\mu.$$

First order condition yields

$$\begin{aligned}
&\int \phi'\left(G_\theta(z)u(v - \beta_1(z))\right) \times \\
&[G'_\theta(z)u(v-\beta_1(z)) - G_\theta(z)u'(v-\beta_1(z))\beta'_1(z)]\,d\mu = 0. \qquad (2.4)
\end{aligned}$$

The term in the second bracket is the optimum condition for the unambiguous case for each prior θ. The equilibrium can be seen as a weighted mean, the

$\phi'(\cdot)$ terms being the weights. Introducing ambiguity aversion renders $\phi'(\cdot)$ decreasing, stressing those terms in the integral where $G_\theta(z)$ is lower.

In equilibrium the bidders bid according to their value, i.e. $z = v$, hence the above equation may be rewritten as

$$\beta_1'(v) = \frac{\int \phi'\left(G_\theta(v)u(v-\beta_1(v))\right) G_\theta'(v)d\mu}{\int \phi'\left(G_\theta(v)u(v-\beta_1(v))\right) G_\theta(v)d\mu} \times \frac{u(v-\beta_1(v))}{u'(v-\beta_1(v))}.$$

Assume for this section that $\phi(\cdot)$ is such that $\phi'(ab) = \phi'(a)\phi'(b)$, for example with the usual exponential form, $\phi(h) = \frac{1}{\alpha}h^\alpha$, for some $\alpha \in (0,1)$, this simplifies to

$$\beta_1'(v) = \frac{\int \phi'\left(G_\theta(v)\right) G_\theta'(v)d\mu}{\int \phi'\left(G_\theta(v)\right) G_\theta(v)d\mu} \times \frac{u(v-\beta_1(v))}{u'(v-\beta_1(v))}. \tag{2.5}$$

Suppose now that all priors are such that they can be ordered in the following way, $F_{\theta_1}(x) < F_{\theta_2}(x)$ for any $x > 0$ if $\theta_1 < \theta_2$. This implies that $G_{\theta_1}(x) < G_{\theta_2}(x)$ for any $x > 0$. Thus for higher θ, the term $\phi'(G_\theta(v))$ will be lower for the same $v > 0$. Following this assumption on the ordering of the cumulative distribution functions, it is also assumed[2.2] that for the hazard rate

$$\frac{F_{\theta_1}'(x)}{F_{\theta_1}(x)} > \frac{F_{\theta_2}'(x)}{F_{\theta_2}(x)} \quad \forall x > 0, \text{ if } \theta_1 < \theta_2.$$

Following the definition of $G_\theta(\cdot)$, its derivative $G_\theta'(x)$ equals $(n-1)F_\theta^{n-2}(x)F_\theta'(x)$ so that

$$\frac{G_\theta'(x)}{G_\theta(x)} = (n-1)\frac{F_\theta'(x)}{F_\theta(x)}.$$

Using the last assumption this implies that

$$\frac{G_{\theta_1}'(x)}{G_{\theta_1}(x)} > \frac{G_{\theta_2}'(x)}{G_{\theta_2}(x)}.$$

See below for some examples.

Now, it is easy to see that the expression $\frac{a_{-i}+ca_i}{b_{-i}+cb_i}$ moves monotonously from $\frac{a_{-i}}{b_{-i}}$ to $\frac{a_i}{b_i}$ as c goes from 0 to ∞. Therefore in the first fraction of expression (2.5) the terms of priors with lower θs will have a higher weight as ambiguity aversion increases. Given that lower θs have a higher $\frac{G_\theta'(x)}{G_\theta(x)}$ ratio, the first fraction in (2.5) will be higher for higher ambiguity aversion. Therefore the concavity of $\phi(\cdot)$ implies

$$\frac{\int \phi'\left(G_\theta(v)\right) G_\theta'(v)d\mu}{\int \phi'\left(G_\theta(v)\right) G_\theta(v)d\mu} > \frac{\int G_\theta'(v)d\mu}{\int G_\theta(v)d\mu}, \tag{2.6}$$

and the ratio on the left-hand side is decreasing with the ambiguity aversion parameter α, i.e. increasing with ambiguity aversion. The ratio in the right-hand side is the ratio that appears in the differential equation defining the ambiguity neutral bidding equilibrium strategy, $\beta_{1,N}(\cdot)$, where the index N stands for Neutrality, that is the one in case of linear $\phi(\cdot)$,

$$\beta_{1,N}'(v) = \frac{\int G_\theta'(v)d\mu}{\int G_\theta(v)d\mu} \times \frac{u(v-\beta_1(v))}{u'(v-\beta_1(v))}.$$

[2.2]The second assumption while independent from the first, is not a strong one. To see this notice that the numerators are ordered in an increasing way.

Now if $\beta_1(v) < \beta_{1,N}(v)$ then $\frac{u(v-\beta_1(v))}{u'(v-\beta_1(v))} > \frac{u(v-\beta_{1,N}(v))}{u'(v-\beta_{1,N}(v))}$, and given (2.6) one gets $\beta_1'(v) > \beta_{1,N}'(v)$. But at $v = 0$ it is easy to see that $\beta_1(0) = \beta_{1,N}(0) = 0$. One can therefore not have $\beta_1(v) < \beta_{1,N}(v)$ for any $v > 0$ because that would imply $\beta_1'(v) > \beta_{1,N}'(v)$, a contradiction. Thus it must be that $\beta_1'(v)$ is higher than $\beta_{1,N}'(v)$ for any $v > 0$. This implies the following result.

Lemma 2.1 *In the First-Price Sealed-Bid Auction with Smooth Ambiguity the equilibrium bid increases as ambiguity aversion arises.*

The following examples illustrate the lemma.

Ambiguous order with linear priors

Consider a set of priors in $[0, 1]$ where values are drawn from distributions with the following probability density functions $F_\theta'(x) = (1+\theta)-2\theta x$, with $\theta \in [-1, 1]$. For $\theta_1 < \theta_2$ it holds that $F_{\theta_1}(x) < F_{\theta_2}(x)$ and

$$\frac{F_{\theta_1}'(x)}{F_{\theta_1}(x)} > \frac{F_{\theta_2}'(x)}{F_{\theta_2}(x)},$$

because $\frac{F_\theta'(x)}{F_\theta(x)} = \frac{1}{x} - \frac{1}{1/\theta+1-x}$ for any x.

Recall that the ambiguity aversion term $\phi'(F_\theta(x))$ stresses those priors with lower $F_\theta(x)$, i.e. those with lower θ. Take for instance $\theta = -1$. According to this prior, the value of the opponent will be drawn from $F_{-1}(x) = x^2$, meaning that there is higher probability of confronting a bidder with a higher value, in comparison to the other extreme case $\theta = 1$, when $F_1(x) = 2x - x^2$ for example. The ambiguity averse bidder will therefore choose to place a higher bid in equilibrium.

Ambiguous order with exponential priors

Consider the priors $F_\theta(x) = x^\theta$ for $0 \le x \le 1$ with $\theta > 0$. The hazard rate will be

$$\frac{F_\theta'(x)}{F_\theta(x)} = \frac{\theta}{x}.$$

The assumptions are clearly satisfied (in reverse order though), i.e. $F_{\theta_1}(x) > F_{\theta_2}(x)$ and $\frac{F_{\theta_1}'(x)}{F_{\theta_1}(x)} < \frac{F_{\theta_2}'(x)}{F_{\theta_2}(x)}$ for any x if $\theta_1 < \theta_2$.

Ambiguous mean

Consider the case with two equally likely priors $\theta = 1, 2$ with uniform distribution of length $a < 1$, whose total support is $[0, 1]$. These priors create the following conceptual problem to a bidder whose private value v is not included in the support of all priors, for instance if $v = 0.1$ and there are two priors with support $[0, 0.8]$ and $[0.2, 1]$. This bidder will reject the second prior from the start, so that the ambiguity is not the same across bidders.

It is therefore assumed that the prior distributions are of the following type for some $0 < \epsilon < \frac{1}{1-a}$,

$$F_\theta'(x) = \begin{cases} a^{-1} - \frac{1-a}{a}\epsilon & \text{if } x \in [0, a] \text{ for } \theta = 1 \text{ or if } x \in [1-a, 1] \text{ for } \theta = 2, \\ \epsilon & \text{otherwise.} \end{cases}$$

As $\epsilon \to 0$, some of the fractions $\frac{F'_\theta(x)}{F_\theta(x)}$ become undetermined. Using $\phi(h) = \frac{1}{\alpha}h^\alpha$, $\alpha \in (0,1)$, it can still be proved that

$$\frac{\sum_\theta \phi'\left(G_\theta(v)\right) G'_\theta(v)}{\sum_\theta \phi'\left(G_\theta(v)\right) G_\theta(v)}$$

weakly decreases with α. See the appendix.

Closed-form solutions

One can get an explicit solution for the equilibrium bidding strategies if the priors are chosen appropriately. Take n risk neutral bidders and a finite set of priors $\mathcal{P} = \{F_1, \ldots, F_m\}$, all equally probable (i.e., $\mu_i = \frac{1}{m}$ for all $i = 1, \ldots, m$), such that $\frac{1}{m}\sum_{i=1}^m F'_i(x) = 1$ for all $x \in [0,1]$. Such set of priors satisfies $\frac{1}{m}\sum_{i=1}^m F_i(x) = x$, meaning that for an ambiguous neutral bidder with only one opponent ($n = 2$), these priors correspond to a uniform distribution. For $n > 2$ and $x \in (0,1)$ one has that $\frac{1}{m}\sum_{i=1}^m F_i^{n-1}(x) \geq x^{n-1}$ or $F_U(x)^{n-1} \geq x^{n-1}$, with strict inequality if there are at least two priors with different values.

In words, with this set of priors $\mathcal{P}$ the reduced cumulative distribution of the opponents, F_U, has a higher value for any value x than a uniform distribution with $n-1$ opponents would have. That is for any value v that the bidder may have, there is here a lower probability of having opponents with higher values than it would happen with a uniform distribution. In an auction with ambiguous neutral bidders, the equilibrium bidding strategy would therefore assign lower bids for each value than the corresponding bid in an auction with uniform distribution.

Choosing the ambiguity aversion parameter $\alpha = \frac{1}{n-1}$, simplifies the equilibrium conditions considerably,

$$\begin{aligned}
\beta'_1(v) &= \frac{\int \phi'\left(G_i(v)\right) G'_i(v) d\mu}{\int \phi'\left(G_i(v)\right) G_i(v) d\mu} \times \frac{u(v-\beta_1(v))}{u'(v-\beta_1(v))} \\
&= (n-1)\frac{\sum_{i=1}^m F_i(v)^{(\alpha-1)(n-1)} F_i(v)^{n-2} F'_i(v)}{\sum_{i=1}^m F_i(v)^{(\alpha-1)(n-1)} F_i(v)^{n-1}}(v-\beta_1(v)) \\
&= (n-1)\frac{\sum_{i=1}^m F_i(v)^{\alpha(n-1)-1} F'_i(v)}{\sum_{i=1}^m F_i(v)^{\alpha(n-1)}}(v-\beta_1(v)) \\
&= (n-1)\frac{\sum_{i=1}^m F'_i(v)}{\sum_{i=1}^m F_i(v)}(v-\beta_1(v)) \\
&= \frac{n-1}{n}(v-\beta_1(v)).
\end{aligned}$$

The equilibrium bid is thus the same as the basic non-ambiguous with uniformly distributed values, $\beta_1(v) = \frac{n-1}{n}v$, even if there are less opponents with higher values. Like risk aversion, aversion to ambiguity pushes the bidders to play a safer strategy which increases their chance to win at the expense of lower payoffs.

Take for instance the set of equally probable priors $\mathcal{P} = \{F_1, F_2\}$ with $F_1(x) = x^a$ and $F_2(x) = 2x - x^a$, where $0 \leq x \leq 1$, $a \in [1,2]$ and $n = 3$. At $a = 1$ the priors are both the uniform distribution so there is no ambiguity and the usual equilibrium arises. At $a > 1$, however, the reduced distribution with which an ambiguous neutral bidder ($\alpha = 1$) calculates her expected payoff is different. For $a = 2$ it will be $F_U^2(x) = \frac{1}{2}(x^4 + (2x - x^2)^2) = x^2(1 + (1-x)^2) > x^2$ for any $x > 0$. Now for $\alpha = \frac{1}{2}$ and for any $a \in [1, 2]$, the ambiguity averse bidders have as equilibrium strategy the usual $\beta_1(v) = \frac{2}{3}v$. Notice that increasing the parameter a increases the probability of a low value of opponents but increases the ambiguity, and has no effect in this solution because the two effects cancel out.

2.4.2 Second-price sealed-bid auction

Lemma 2.2 *In the ambiguous Second-Price Sealed-Bid Auction with ambiguity averse bidders with smooth ambiguity preferences, bidding their own value, i.e. $\beta_2(v) = v$, is an equilibrium.*

Proof. The proof is straightforward as in the ambiguity neutral and risk neutral case. Provided that other bidders play according to $\beta_2(v)$, bidding less than v decreases the probability of winning the auction without yielding higher payments, and bidding more than v increases the number of chances in which the auction is won, but all of which will yield negative payoffs. ■

This result is confirmed experimentally in Chen, Katuscak, and Ozdenoren (2007).

2.5 Dynamic ambiguous auction

Dynamics and Ambiguity Aversion have been difficult to stitch together in the literature, as it was remarked in Section 2.2. Different approaches yield quite different forecasts. In this chapter a consequentialist Bayesian update[2.3] rule is adopted for various reasons. First, the only empirical evidence available indicates that subjects follow consequentialist update rules in the simple dynamic Ellsberg experiment, see Dominiak, Dürsch, and Lefort (2009). Second, models with dynamically consistent preferences use recursive update rules. In a price-descending auction where the price decreases continuously it is not clear how this recursive rule should be applied. And if a discrete process is considered, the size of the price decrease in each period would have an important impact on the outcome of these models[2.4].

The setting in an open price descending auction bidders is much richer, since bidders can collect information as the auction runs. When the distributions are not ambiguous, as the auction price descends and no bid is placed, there is only one type of information that bidders learn, namely they learn that there are no opponents with values above some given threshold.

[2.3]Updating is arguably not the best term given that strictly speaking there is no new information. Put differently, in the beginning of the auction bidders can infer what will be their beliefs at some future point, provided that that point is reached.

[2.4]It could still be argued that forward looking decision makers could recognize their changing preferences and choose suboptimal bidding strategies, i.e. stopping earlier, to prevent the predicted outcome if that would maximize their expected payoff, in line with Siniscalchi (2010). While proving that that cannot be the case is beyond the scope of this chapter, all numerical simulations that were conducted show that at no point the bidders prefer to bid at the current price instead of the equilibrium one - except obviously for the equilibrium price bid.

But that is not the case with ambiguity. Consider the case where bidders have two priors on the distribution of the opponents. One indicates a higher probability of higher values, and the other of lower values. As the price descends and bidders exclude the possibility of having opponents with the highest possible values, the first prior starts to look less likely than in the beginning, since the first prior decrees that there is a stronger possibility of the auction ending with a high bid. As the auction goes on, bidders take the second prior to be more believable and evaluate their strategies according to this update believe. Conditional on the fact that no bidder stopped the auction until price p, the prior beliefs, both F_θ, $\theta \in \Theta$, and μ, will be 'updated'.

Let the conditional Bayesian beliefs, conditional on the fact that $x \leq y$ for some given y, $0 \leq y \leq 1$, be represented by $F_{\theta,y}(x)$, i.e.,

$$F_{\theta,y}(x) = \frac{F_\theta(x)}{F_\theta(y)}, \quad x \leq y, \theta \in \Theta.$$

The probability measure on the priors is also updated to μ_y. For given y, $0 \leq y \leq 1$, it is defined by

$$\mu_y(A) = \frac{\int_A F_\theta^{n-1}(y)d\mu}{\int_\Theta F_\theta^{n-1}(y)d\mu} = \frac{\int_A G_\theta(y)d\mu}{\int_\Theta G_\theta(y)d\mu},\ A \in 2^\Theta.$$

Ambiguity neutrality

When individuals are ambiguity neutral, the existence of ambiguity should not affect the equilibrium, even if their probability measure μ is updated. In this section it is shown that indeed ambiguity does not affect the equilibrium outcome.

Take $\beta_{D,N}(v)$ to be the monotonous equilibrium bidding strategy for a bidder with value v, D standing for Dutch auctioneer. Suppose the $n-1$ opponents are playing this strategy and the descending price reaches level p, implying that the values of the opponents are smaller than $\beta_{D,N}^{-1}(p)$. For a given own private value v, the bidder may bid the good at p receiving

$$\int v - pd\mu_z = v - p = v - \beta_{D,N}(z),$$

where z is the private value for which p is the optimal bid, $z = \beta_{D,N}^{-1}(p)$. The bidder may consider to bid as a lower type $y < z$, whose bid wins with probability (according to the updated priors) $G_{\theta,z}(y) = F_{\theta,z}^{n-1}(y)$, receiving

$$\int G_{\theta,z}(y)(v - \beta_{D,N}(y))d\mu_z.$$

Let y be marginally smaller than z, $y = z - \Delta$, and let Δ go to zero. The marginal gain from Δ will be

$$
\begin{aligned}
& \int G_{\theta,z}(z)\left(v-\beta_{D,N}(z)\right) - \\
& \Delta\left(G^{'}_{\theta,z}(z)(v-\beta_{D,N}(z)) - G_{\theta,z}(z)\beta'_{D,N}(z)\right)d\mu_z - (v-\beta_{D,N}(z)) \\
= & \int (v-\beta_{D,N}(z)) - \\
& \Delta\left(G^{'}_{\theta,z}(z)(v-\beta_{D,N}(z)) - \beta'_{D,N}(z)\right)d\mu_z - (v-\beta_{D,N}(z)) \\
= & \int -\Delta\left(G^{'}_{\theta,z}(z)(v-\beta_{D,N}(z)) - \beta'_{D,N}(z)\right)d\mu_z,
\end{aligned}
$$

where $G_{\theta,z}(z) = 1$ for any θ is used. In equilibrium the optimal response has $v = z$ such that the marginal gain is zero,

$$
\begin{aligned}
\beta'_{D,N}(v) - (v-\beta_{D,N}(v))\int G^{'}_{\theta,v}(v)d\mu_v &= 0, \\
\beta'_{D,N}(v) - (v-\beta_{D,N}(v))\int \frac{G^{'}_{\theta}(v)}{G_{\theta}(v)}\frac{G_{\theta}(v)}{\int G_{\vartheta}(v)d\mu}d\mu &= 0, \\
\beta'_{D,N}(v) - (v-\beta_{D,N}(v))\int \frac{G^{'}_{\theta}(v)}{G_U(v)}d\mu &= 0, \\
\beta'_{D,N}(v) &= (v-\beta_{D,N}(v))\frac{G^{'}_U(v)}{G_U(v)}.
\end{aligned}
$$

The best response satisfies the same condition as the optimal bid in the static auction. The equilibrium conditions for both auctions are therefore equivalent.

Ambiguity aversion

Let $\beta_D(v)$ be the equilibrium bid in an Open Price Descending Auction. The gains from delaying Δ are now

$$
\begin{aligned}
& \int \phi\left(G_{\theta,z}(z-\Delta)u(v-\beta_D(z-\Delta))\right)d\mu_z - \phi\left(u(v-\beta_D(z))\right) \\
\approx & \int \phi\left(G_{\theta,z}(z)u(v-\beta_D(z))\right) - \Delta\phi'\left(G_{\theta,z}(z)u(v-\beta_D(z))\right) \\
& \left(G^{'}_{\theta,z}(z)u(v-\beta_D(z)) - G_{\theta,z}(z)u'(v-\beta_D(z))\beta'_D(z)\right) \\
& -\phi\left(u(v-\beta_D(z))\right)d\mu_z \\
= & -\Delta\phi'\left(u(v-\beta_D(z))\right)\int G^{'}_{\theta,z}(z)u(v-\beta_D(z)) - u'(v-\beta_D(z))\beta'_D(z)d\mu_z.
\end{aligned}
$$

As $\Delta \to 0$, in equilibrium the marginal gain should be zero at $z = v$,

$$\phi'\left(u(v-\beta_D(v))\right)\left[u'(v-\beta_D(v))\beta_D'(v) - \int G_{\theta,v}^{'}(z)u(v-\beta_D(v))d\mu_v\right] = 0,$$

$$u'(v-\beta_D(v))\beta_D'(v) - u(v-\beta_D(v))\int G_{\theta,z}^{'}(v)d\mu_z = 0,$$

$$u'(v-\beta_D(v))\beta_D'(v) - u(v-\beta_D(v))\int \frac{G_\theta^{'}(v)}{G_\theta(v)}\frac{G_\theta(v)}{\int G_\vartheta(v)d\mu}d\mu = 0,$$

$$u'(v-\beta_D(v))\beta_D'(v) - u(v-\beta_D(v))\int \frac{G_\theta^{'}(v)}{G_U(v)}d\mu = 0,$$

$$\beta_D'(v) = \frac{u(v-\beta_D(v))}{u'(v-\beta_D(v))}\frac{G_U^{'}(v)}{G_U(v)}. \quad (2.7)$$

This result holds for any differentiable $\phi(\cdot)$, implying that in the dynamic auction, the optimal strategy does not depend on the ambiguity aversion level of the bidders.

Lemma 2.3 *In a Dutch Auction with Smooth Ambiguity the equilibrium bidding strategy is independent of the Ambiguity Attitude of the bidders, i.e.* $\beta_D = \beta_{D,N}$.

Proof. Above. ■

Lemma 2.4 *Expected utility, given by smooth ambiguity preferences, from an ambiguous Dutch auction is lower than that of the equivalent unambiguous one.*

Proof Given the concavity of ϕ, it follows that

$$\begin{aligned}\int \phi\left(G_\theta(v)u(v-\beta_D(v))\right)d\mu &< \phi\left(\int G_\theta(v)u(v-\beta_D(v))d\mu\right) \\ &= \phi\left(\int G_\theta(v)u(v-\beta_{D,N}(v))d\mu\right). \blacksquare\end{aligned}$$

One important corollary follows from the previous results.

Corollary 2.1 *If there is any participation cost in the Dutch Auction, less bidders will choose to participate in an ambiguous auction than in the equivalent unambiguous one.*

These results also show that first-price sealed-bid auctions are not equivalent to open price descending auctions when ambiguity and ambiguity aversion are present. Karni (1988) points out that that equivalence only holds necessarily with expected utility maximizing agents. Moreover, this bidding difference which cannot be explained by risk aversion, is in agreement with the experimental literature (see e.g. Kagel and Roth (1995)) which shows that first-price sealed-bid auctions have bids and revenues which are higher than those of the risk neutral Nash Equilibrium and of the Dutch auctions.

Anticipating consequentalism

As discussed in the introduction of this section, it is not clear how dynamic ambiguity should be modeled. It is possible however to see that even if the bidder anticipates his consequentialist and therefore possibly dynamic inconsistent beliefs, she still chooses to play the same equilibrium bidding strategy - provided that the others do the same.

A bidder who evaluates her equilibrium strategy before the bidding price arrives, that is with previous priors, may find the equilibrium strategy to be suboptimal. That is the case at the beginning, where the bidder would rather behave as in the first-price sealed-bid auction. Given that there is no a priori way of setting the bid in a dynamic auction, the bidder can only choose to bid immediately instead of bidding at the equilibrium strategy. So one should compare the certain payoff at a higher bid b with the expected payoff of waiting until the equilibrium, using for this the priors updated until then.

Given that closed form solutions are needed to make this comparison it is impossible to establish a general result, but some examples indicate that the bidders opt for playing the equilibrium strategy defined above. Take for instance the set of equally probable ($\mu_1 = \mu_2 = \frac{1}{2}$) priors $\mathcal{P} = \{F_1, F_2\}$ with $F_1(x) = x^{\frac{m}{n-1}}$ and $F_2(x) = (2x^{n-1} - x^m)^{\frac{1}{n-1}}$, with m chosen appropriately (guaranteeing that F_1 and F_2 are non-decreasing and with codomain $[0,1]$), risk neutrality and $\phi(h) = \frac{1}{\alpha}h^\alpha$. The reduced distribution will be $G_U(x) = x^{n-1}$ so that the equilibrium strategy is $\beta_D(v) = \frac{n-1}{n}v$. The updated priors conditional on the maximum value of bidders having values lower than y, $0 \leq y \leq 1$, will be

$$F_{i,y}(x) = \frac{F_i(x)}{F_i(y)}, \qquad \mu_i(y) = \frac{\frac{1}{2}F_i^{n-1}(y)}{\frac{1}{2}\left(F_1^{n-1}(y) + F_2^{n-1}(y)\right)} = \frac{F_i^{n-1}(y)}{y^{n-1}}, \quad i = 1, 2.$$

At bid b the bidder with value v compares the payoff of stopping, $\frac{1}{\alpha}(v-b)^\alpha$ with that of waiting until the equilibrium bid $\beta_D(v)$,

$$\sum_{j=1,2} \mu_j(y) \left[(F_{j,y}(v))^{n-1} \left(v - \frac{n-1}{n}v \right) \right]^\alpha,$$

where $y = \min\{1, \frac{n}{n-1}b\}$. Notice that for $b > \frac{n-1}{n}$ and assuming that all bidders play the equilibrium strategy, there is still no value that can be discarded because b is higher than any equilibrium bid. There is therefore no update of the priors.

Let $m = 4$, $n = 3$ and $\alpha = \frac{1}{2}$. At the beginning of the auction, stopping at a future b yields the expected utility displayed in Figure 2.1 for a bidder with value $v = \frac{3}{4}$. This is the problem that the bidder faces in the first-price sealed-bid auction, the maximum payoff occurs thus at a bid higher than the equilibrium strategy in the open price descending auction, $\frac{n-1}{n}v = \frac{1}{2}$. There is a slight kink at $b = \frac{2}{3}$, which is the equilibrium bid of the bidder with the highest value. The probability of winning has therefore a kink because it goes below 1 for $b < \frac{2}{3}$.

Figure 2.2 represents the expected utility of two possible strategies, equilibrium strategy β_D and stopping at the current bid b, at different timings of the descending auction, more precisely at bid b. Contrary to Figure 2.1, here only the probability of winning is changing with b. For $b \in [\frac{2}{3}, 1]$ there is no type of opponent that can be discarded, there is thus no update of the priors and the

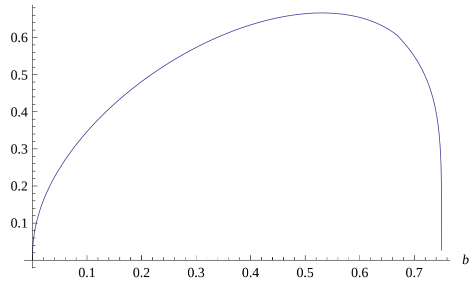

Figure 2.1: Expected utility as anticipated at the beginning of the auction, as a function of the bid b, for a bidder with $v = \frac{3}{4}$.

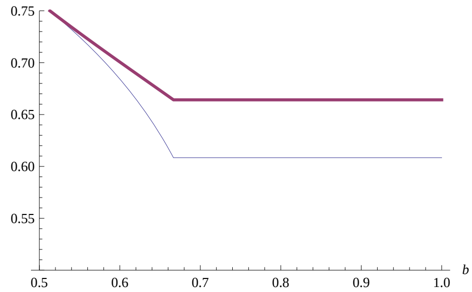

Figure 2.2: Expected utility as anticipated as bid b is reached, of playing the equilibrium bid strategy β_D (thick) and of accepting the momentary price b, for a bidder with $v = \frac{3}{4}$.

probabilities are fixed. The important aspect of this graph is to show that the equilibrium strategy β_D (even if not being the optimal bid for any point in time with $b > \beta_D$) always outperforms the only possibility that the bidder at ongoing bid b has, to stop at b. At each point the bidder that anticipates his changing preferences, cannot do better than wait and play β_D.

2.6 Conclusion

In Auction Theory one of the basic assumptions is that of common knowledge of the distribution of the values of the bidders, that is each bidder knows the distribution from which the values of her opponents are drawn. This chapter relaxes this assumption in the spirit of the literature in Ambiguity Aversion with multiple priors and derives the equilibrium bids in basic single-good auctions.

It is shown that ambiguity aversion increases the bid in the first-price sealed-bid auction, but ambiguity has no impact in open price descending auctions. While the first result is intuitive, the second result follows from the fact that as the auction occurs and the price descends, the bidders learn about the distribution of the values of their opponents, eroding thus the ambiguity that was present in the beginning.

This entails two important results. The first concerns Auction Theory, it indicates that first-price sealed-bid auctions and open price descending need not to be theoretically equivalent. This implies that, in the presence of ambiguity, there is no revenue equivalence between those auctions.

The second is a significant result in Ambiguity Aversion, because the chapter provides a new example where ambiguity aversion and risk aversion do not have the same qualitative effect on the outcomes of a model. For instance, Gollier (2009) in a portfolio choice model, shows that ambiguity aversion may lead to an increased demand of a risky or ambiguous asset. The present chapter sustains that ambiguity aversion has the same qualitative consequence on static auctions as risk aversion, but that it is not the case for dynamic auctions.

2.7 Appendix

2.7.1 Ambiguous mean

Here it will be proven that for the example with ambiguous mean

$$\frac{\sum_\theta \phi'\left(G_\theta(v)\right) G'_\theta(v)}{\sum_\theta \phi'\left(G_\theta(v)\right) G_\theta(v)}$$

weakly decreases with α. For any v with $v \leq 1-a$, the fraction is

$$\begin{aligned}
\frac{\sum_\theta \phi'\left(G_\theta(v)\right) G'_\theta(v)}{\sum_\theta \phi'\left(G_\theta(v)\right) G_\theta(v)} &= \frac{\sum_\theta F_\theta^{(n-1)(\alpha-1)}(v)\cdot(n-1)F_\theta^{n-2}(v)F'_\theta(v)}{\sum_\theta F_\theta^{(n-1)(\alpha-1)}(v)\cdot F_\theta^{n-1}(v)} \\
&= (n-1)\frac{\frac{1}{a}\left(\frac{v}{a}\right)^{\alpha(n-1)-1}+0}{\left(\frac{v}{a}\right)^{\alpha(n-1)}+0} \\
&= (n-1)\frac{1}{v},
\end{aligned}$$

which is independent of α.

For any v with $1-a < v \leq a$, the fraction is

$$\begin{aligned}
\frac{\sum_\theta \phi'\left(G_\theta(v)\right) G'_\theta(v)}{\sum_\theta \phi'\left(G_\theta(v)\right) G_\theta(v)} &= \frac{\sum_\theta F_\theta^{(n-1)(\alpha-1)}(v)\cdot(n-1)F_\theta^{n-2}(v)F'_\theta(v)}{\sum_\theta F_\theta^{(n-1)(\alpha-1)}(v)\cdot F_\theta^{n-1}(v)} \\
&= (n-1)\frac{\frac{1}{a}\left(\frac{v}{a}\right)^{\alpha(n-1)-1}+\frac{1}{a}\left(\frac{v-(1-a)}{a}\right)^{\alpha(n-1)-1}}{\left(\frac{v}{a}\right)^{\alpha(n-1)}+\left(\frac{v-(1-a)}{a}\right)^{\alpha(n-1)}} \\
&= (n-1)\frac{1+\left(\frac{v-(1-a)}{v}\right)^{\alpha(n-1)-1}}{v+\left(\frac{v-(1-a)}{v}\right)^{\alpha(n-1)}} \\
&= \frac{n-1}{v}\frac{1+\left(1-\frac{1-a}{v}\right)^{\alpha(n-1)-1}}{1+\left(1-\frac{1-a}{v}\right)^{\alpha(n-1)}}.
\end{aligned} \quad (2.8)$$

The following derivative

$$\frac{\partial}{\partial q}\frac{1+y^{q-1}}{1+y^q} = \frac{(1-y)y^{q-1}\ln y}{(1+y^q)^2},$$

with $y \in (0,1)$ and $q > 0$ is negative. Substituting $y = 1-\frac{1-a}{v}$ and $q = \alpha(n-1)$, it is concluded that (2.8) is decreasing in α for any $v \in (1-a, a]$.

For any v with $v > a$,

$$\begin{aligned}
\frac{\sum_\theta \phi'\left(G_\theta(v)\right) G'_\theta(v)}{\sum_\theta \phi'\left(G_\theta(v)\right) G_\theta(v)} &= (n-1)\frac{0+\frac{1}{a}\left(\frac{v-(1-a)}{a}\right)^{\alpha(n-1)-1}}{1+\left(\frac{v-(1-a)}{a}\right)^{\alpha(n-1)}} \\
&= (n-1)\frac{1}{a\left(\frac{v-(1-a)}{a}\right)^{1-\alpha(n-1)}+(v-(1-a))}.
\end{aligned}$$

Given that $a < v \leq 1$ it follows $0 < \frac{v-(1-a)}{a} \leq 1$, so the above fraction is decreasing in α.

Chapter 3

Staggered Time Consistency and Impulses

3.1 Introduction

One of the aspects of the Expected Utility Model that has been largely criticized has been the constant rate of time discounting for the additive utility levels. Strotz (1956) already pointed out from introspection that closer time gaps are more discounted than distant ones, that is the discount from today to tomorrow is bigger than between two consecutive days in the far future. This obviously raises the issue of time inconsistency, meaning that the optimal trade-off between date t_0 and date t_1 depends strongly on the date when that consideration is taken. An individual may prefer 'work' to 'beach', but when the 'work' period comes closer that preference may reverse.

Strotz (1956) proposes three ways to mathematically model the decision making with inconsistent preferences. The individual may be unaware of the inconsistency and continuously discard previous optimal plans engaging in new ones, she may recognize it and follow a strategy of precommitment always following a previous plan, or recognize it and choose a consistent plan, which is "*the best plan among those that he will actually follow*". The first one, the so-called *naive* behavior, is problematic because it implies that individuals simply do not recognize that they are not following their own planned actions, in particular some of those involving immediate costs and delayed rewards. The second one implies the existence of some point in time where all decisions were taken (except for those dependent on unexpected events) and that those decisions are followed even if considered far from optimal when reconsidered at some future point.

The second possibility was only modeled by Pollak (1968). The *sophisticated* individual's behavior is the outcome of a game among its different *selves*, the decision makers of each decision period. The self of period t knows how the next self will respond to its current decisions, and both know how period $t+2$ acts, and so forth. By taking all those future strategies into account, the present self maximizes its own discounted utility. It is therefore "*the best plan among those that he will actually follow*". Notice that this subgame perfect equilibrium coincides with the Expected Utility Model model if preferences are time consistent, given that all the selves agree. This concept yields interesting results in complex frameworks (see Laibson (1997) where voluntary precommitment arises in equilibrium). It comes however as a disappointment in simple ones. In O'Donoghue and Rabin (1999) one individual is to choose one movie out of four that come in an increasing quality sequence. The sophisticate happens to choose the first and worst one because the first self anticipates that the second and the third selves would not wait until the best movie[3.1]. Moreover the sophisticates end up behaving as the time-consistent individuals in various settings, which clearly reduces its added value.

O'Donoghue and Rabin (2001) present the first model where both concepts (naivete and sophistication) are blended. They use the expression *partial naivete* to refer to a behavior with self-control problems, where individuals only partially recognize their inconsistency. Formally, individuals have preferences with (β, δ) quasi-hyperbolic discounting $(1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots)$ and recognize their time inconsistency, but think their present-biased preferences are given by $(\bar{\beta}, \delta)$ with $1 > \bar{\beta} > \beta$. In other words they underestimate their present-bias.

DellaVigna and Malmendier (2006) show in their empirical analysis of consumer decisions in the health club industry that people choose annual contracts, rather than pay-per-visit fees, apparently as a commitment device. This is a sophisticated behavior, for they recognize that in the future they will choose a lower attendance due to present-biased preferences. But gym users also underestimate their actual attendance which is evidence for some naivete. As Frederick, Loewenstein, and O'Donoghue (2002) put it *'casual observation and introspection suggest that people are somewhere in between these two extremes'*. McClure, Laibson, Loewenstein, and Cohen (2004) present a neurological study showing that immediate and delayed monetary rewards are processed by separate neural systems. This means that the distinction is stronger than one might think in the first place.

Ariely and Wertenbroch (2002) run three experiments on the willingness to have costly commitments and on its success. Subjects (students) have to complete several tasks (real coursework assignments). Some are given the possibility of self-imposing earlier deadlines, which are costly because later deadlines would give more flexibility. Still, students do choose earlier deadlines for the first assignments, which shows preference for self-control mechanisms. Moreover, in these tasks there are penalties for delays. Surprisingly those subjects with externally imposed deadlines have less delays than those with self-imposed deadlines. This indicates that individuals are not able to choose the best commitment device. Furthermore, the subjects with self-imposed deadlines have less delays than those with a simple global end deadline. Given that there was no external influence on both of these types, that is both could have chosen to follow the same working schedule, the fact that those without a self-imposition of deadlines had more delays indicates that the delays were not only caused by unforeseen causes but by lack of self-control.

It is therefore clear that individuals have present-biased impulses, that they do recognize them and are thus willing to have costly commitment devices, but these commitment devices are not fully successful. That is, due to the lack of self-control which shows up in the impulses, the outcome is not what a priori would be considered optimal. In other words, individuals seem to both display

[3.1] As a comparison the naive watches the third movie with the same parameters.

sophisticated and naive features at the same time.

In this chapter a new model of lack of self-control is proposed. It is inspired by model of Calvo (1983) on sticky prices, the so-called staggered prices. In this model firms cannot adjust their prices to the current optimum in every period. They are rather able to do so with some probability in every period. When given that opportunity they not only consider the current optimum but also future optima. Intertemporal decision making with time inconsistent preferences seems to follow a similar pattern: Individuals cannot tell in advance whether they will act rationally or follow a present-biased impulse in a given period. Whenever they are able to think it through, they take possible future deviations into account. This model seems to be more in line with the neurological separation reported by McClure, Laibson, Loewenstein, and Cohen (2004).

There are three main characteristics of the present model, it does not use intricate internal decision models for each period (see Ainslie (2010) for a criticism of that approach), it is able to capture both the desire for commitment devices as well as random impulsive behavior, and it incorporates quasi-hyperbolic discounting, a stylized fact from the experimental literature within a self-control model.

Section 3.2 puts this chapter in the literature context, the model is formalized in Section 3.3, Section 3.4 entails a discussion on simple applications and compares the results with the other quasi-hyperbolic models, Section 3.5 proposes a macroeconomic aggregate interpretation of the decision model, Section 3.6 discusses some possible extensions and concludes.

3.2 Literature

Strotz (1956) is the first paper in the literature focusing on the issue of dynamic inconsistency of preferences, mainly driven by introspection. The author notes that time gaps closer to the present are discounted more heavily (in terms of the aggregation of additive utility) than those in the far future. He then discusses how the individuals might cope with contradictory preferences over time. Pollak (1968) proposes a subgame perfect equilibrium played by the subsequent selves of the individual. This elegant solution leads to a dynamically consistent plan, meaning that no self will deviate from the equilibrium, created out of dynamically inconsistent preferences. Ainslie (1991) provides an early summary of experimental and psychological evidence on intertemporal inconsistent present-bias, proposing a generalized hyperbola as the best fit for the time discounting implicit in the decisions in the experimental literature, hence the name *hyperbolic discounting.*

O'Donoghue and Rabin (1999) discuss the recognition of the contradiction by the individuals, indicating with simple examples that both naive and sophisticates have serious drawbacks. Moreover they show an instance where "*sophisticates have even worse self-control problems*" than naives. O'Donoghue and Rabin (2001) propose a model with *partially naive* individuals, where they do realize their inconsistency playing sophisticate, but assume a smaller present-bias than the actual one. O'Donoghue and Rabin (2003) further propose another intermediate model, one where individuals act as sophisticates but only perform the backward induction reasoning for a few future periods.

Gul and Pesendorfer (2001) and Gul and Pesendorfer (2004) are the seminal papers of a parallel literature which explains the demand for commitment devices by individuals, without using time inconsistent preferences. The authors postulate that individuals exert self-control to resist temptation. This self-control has a cost, defined as the difference between the utility of the optimum and that of the temptation alternative. Closely related is the *dual-self* literature, initiated by Thaler and Shefrin (1981), which has many extensions such as Fudenberg and Levine (2006). Decisions of an individual are modeled as an agency problem with a principal and an agent. The planner (principal) and the doer (agent) have the same preferences, but the latter one only values present utility. The planner in order to optimize intertemporally, will restrict the set of options available to the doer, under some cost. In the spirit of the current chapter, Chatterjee and Krishna (2009) have a dual-self model where the doer may randomly have an alter-ego with a different utility function. Occasionally the doer will therefore pick options which are seen as inferior by the planner.

Ainslie (2010) provides a comprehensive and critical analysis of the literature.

3.3 Random lack of self-control

3.3.1 Motivation

It is clear that individuals have time inconsistent preferences (which does is not the same as having people acting time inconsistently). It is not as clear, but rather accepted, that individuals recognize this problem. There are innumerous examples of people using costly precommitment devices (annual contract in health clubs, keeping less money in the wallet[3.2]) which indicates that people are willing to solve the problem. But it is also clear that once in a while individuals take decisions whose implicit present-bias is at odds with the previous plans. Take the "*I am going on a diet*" case: People are able to battle against their present-bias by not eating chocolates, but sometimes they follow a quick impulse and when doing so they seem to believe they will make up for it in the future.

3.3.2 Model

For the sake of simplicity, this chapter only focuses on the simplified version of the hyperbolic discounting, the so-called quasi-hyperbolic discounting proposed by Phelps and Pollak (1968) and Laibson (1996). It is assumed there is a random process where with probability p, $0 < p < 1$, the individual acts naively with (β, δ) quasi-hyperbolic preferences, with $\beta, \delta \in (0, 1]$, and with probability $1 - p$ she acts consistently with time consistent preferences with δ and forecasting possible deviations.

Formally in each period t with probability p the individual has a naive im-

[3.2]The author's favorite is an official Google plugin for the Chrome internet browser called Chrome Nanny. With it the user can lock the internet access to given webpages, for a given period of time.

pulsive behavior maximizing

$$\max_{\{x_t,\ldots,x_T\}} E[u(x_t, A_t) + \beta \sum_{s=t+1}^{T} \delta^{s-t} u(x_s, A_s(x_{s-1}))] \tag{3.1}$$
$$\text{subject to constraints on } A_t \text{ and } x_s,\ s = t, \ldots, T,$$

where $u(\cdot)$ is the instantaneous Bernoulli utility function, T the time horizon, and A_s represents the state variable(s) at the beginning of period s and is therefore a function of x_{s-1}, $A_s(x_{s-1})$. This is the custom non-recursive intertemporal maximization with the addition of the present-bias parameter β. Given that only the present plan x_t of a naive self will be performed, the future plans x_s, $s > t$, are of no significance and it will be sufficient to denote the first term of the above solution by $x_t^N(A_t)$, $t = 0, \ldots, T$.

With probability $1-p$, in period t the individual acts consistently, maximizing

$$\max_{x_t} E[u(x_t, A_t) + \delta \mathcal{V}_{t+1}(A_{t+1}(x_t))] \tag{3.2}$$
$$\text{subject to constraints on } A_t \text{ and } x_t,$$

where

$$\begin{aligned} \mathcal{V}_s(A_s) = p\left[u(x_s^N(A_s), A_s) + \delta \mathcal{V}_{s+1}(A_{s+1}(x_s^N(A_s)))\right] + \\ \max_{x_s}(1-p)\left[u(x_s, A_s) + \delta \mathcal{V}_{s+1}(A_{s+1}(x_s))\right] \end{aligned}$$
$$\text{subject to constraints on } A_s \text{ and } x_s,\ s = 0, \ldots, T,$$

and $\mathcal{V}_{T+1} = 0$. Notice that $\mathcal{V}_{t+1}$ depends on x_t through A_{t+1}. This is a recursive maximization where the consistent self takes into account that in the next period there is the possibility of either having a naive impulse or acting consistently. Each one of them has different consequences for the following periods as denoted in the definition of $\mathcal{V}_s(A_s)$. Let the choice of the consistent self in period t be denoted by $x_t^C(A_t)$, $t = 0, \ldots, T$.

To understand what the previous definitions mean consider a case with three periods, where the constraints are omitted for simplicity. The reasoning must be done recursively as following. In the last period there is no intertemporal decision to make so naive and consistent selves have a common choice $x_3^N(A_3) = x_3^C(A_3) = x_3(A_3)$. In period 2 the naive self maximizes $u(x_2) + \beta\delta u(x_3)$ and the consistent self maximizes

$$\begin{aligned} u(x_2) + \delta \mathcal{V}_3(A_3(x_2)) &= u(x_2) + \delta\left[p\ u(x_3^N(A_3(x_2))) + (1-p)u(x_3^C(A_3(x_2)))\right] \\ &= u(x_2) + \delta u(x_3(A_3(x_2))). \end{aligned}$$

For now only the discount factor changes, but in the first period the consistent self takes into account that the individual may act naively in period 2. So she maximizes

$$u(x_1) + \delta p\left[u(x_2^N(x_1)) + \delta u(x_3^N(x_1))\right] + \delta(1-p)\left[u(x_2^C(x_1)) + \delta u(x_3^C(x_1))\right],$$

where $x_2(\cdot)$ is a function of x_1 through $A_2(x_1)$, and $x_3(\cdot)$ is a function of x_1 through $A_2(x_1)$ and $A_3(x_2)$. The naive simply solves for the maximum of $u(x_1) + \beta[\delta u(x_2) + \delta^2 u(x_3)]$.

3.4 Applications

3.4.1 When to go to the movies

O'Donoghue and Rabin (1999) consider a problem of an individual that has to choose which movie to go to. She can only choose one out of the four movies that will come up in consecutive weekends, and that come in an increasing sequence of quality. The utility levels she assigns to them is 3, 5, 8 and 13, where the first is the worst and the last the best. The authors take $\beta = \frac{1}{2}$ and $\delta = 1$.

The naive individual initially chooses the last movie ($13\beta > 8\beta, 5\beta, 3$), in the second period sticks with the same decision but in the third period her present-bias pushes her to the theater ($8 > 13\beta$).

But the result of the sophisticate is the striking one. The sophisticate recognizes that her present-bias impulse in the third period would lead her to the movie, so that there is no hope in waiting for the last movie. But then she also recognizes that she would recognize it in the second period so that she would go to the movies immediately in the second period ($5 > 8\beta$). So going to the third is also impossible. Now the choice of the sophisticate in the first period is between the first and the second movie, and due to her present-bias she will actually go to the first (and worst) one.

Following the proposed model of staggered consistency, in the third period the consistent self will wait ($13 > 8$) but the naive one will not ($8 > 13\beta$). So if the individual gets to the third weekend there is probability p of watching the third one, and $1-p$ of watching the last one. In the second period the naive self does not recognize the possible inconsistency in the following period so among 5, 8β and 13β she prefers 13β, that is she prefers to wait for the last movie. The consistent compares 5 and $8p+(1-p)13$, so she also chooses to wait. Same thing for both selves in the first period. Concluding this individual watches the third movie with probability p and the last one with probability $1-p$.

3.4.2 When to do a report

Another example from O'Donoghue and Rabin (1999) is as follows. Imagine that the individual from the previous example may go to all the movies, except for one because she has to write a report in one of the weekends. The utility of the report is constant over time and is only perceived in the far future, so that its value is irrelevant. Its cost is the disutility from not watching the movie. Put simple the choice now is: What movie not to go to?

The naive postpones the report in the first weekend ($-3 < -5\beta$), in the second ($-5 < -8\beta$), and in the third ($-8 < -13\beta$). She ends up doing it in the last possible weekend loosing the best movie.

The sophisticate recognizes that she will not do the report on the third weekend (as above), so the self of the second period chooses to do it already because $-5 > -13\beta$. The first self recognizes this and goes to the movies because $-3 < -5\beta$. Conclusion, the sophisticate recognizes the postponing problem and does the report in the second weekend.

The staggered consistent individual knows that as a naive she will always postpone. If the report is not ready in the third period the consistent self will do it immediately. So the individual will work on it with probability $1-p$ and postpone it with p. In the second period the consistent compares -5 (doing it now) and $-13p-8(1-p)$ (leaving it for later), so she chooses to do it immediately. Same reasoning for period 1. Conclusion: The probability of doing it in the first period is $1-p$, in the second period it is $p(1-p)$, in the third period it is $p^2(1-p)$ and in the last period it is p^3. For small p it is most likely to have the report done in the first week, then in the second. For large p, the highest likelihood for last period and then the third.

3.4.3 Consuming and saving

Consider now the usual problem of an individual receiving a deterministic income flow who is to decide how much to consume and how much to save. To solve for the individual with staggered consistency one needs once again to solve the problem with backward induction. Notice that the naive and the consistent (constant discount) cases are just particular cases of the general framework by setting $p = 1$ and $p = 0$. Moreover when one uses logarithmic utility, the sophisticate acts as a consistent individual, due to unitary elasticity of substitution.

Formally, the individual receives an income flow y_t, where t is the period, which is known in advance. The individual decides how much to consume, c_t, consumption c yielding utility $u(c)$, and how much to save. Savings are applied in an asset A that yields interest r in the following period. It is further assumed that the individual is liquidity-constrained in the sense that she cannot borrow, that is $A_t \geq 0$ in all periods. The individual has a (β, δ) quasi-hyperbolic discount with probability p and an exponential discount with δ with probability $(1-p)$.

Consider a three period problem. In the last period, both selves consume all the available wealth $c_3^N(A_2) = c_3^C(A_2) = y_3 + (1+r)A_2$, where c_3^N and c_3^C are the consumption choices in period 3 of the naive and the consistent selves, respectively. Both yield utility $\mathcal{V}_3(A_2) = u(c_3^N(A_2)) = u(c_3^C(A_2))$.

In the period before the selves maximize $u(c_2) + \beta\delta\mathcal{V}_3(A_2)$, with $\beta = 1$ for the consistent, subject to $c_2 \leq y_2 + (1+r)A_1$. Denote the solutions by $c_2^N(A_1)$ and $c_2^C(A_1)$.

In the first period the naive self maximizes

$$u(c_1) + \beta\delta u(c_2(A_1)) + \beta\delta^2 u(c_1(A_2)),$$

with the corresponding constraints. The consistent self however takes the two different possible paths into account, maximizing therefore

$$u(c_1) + \delta\mathcal{V}_2(y_1 - c_1),$$

with

$$\begin{aligned}\mathcal{V}_2(A_1) = p\left[u(c_2^N(A_1)) + \delta\mathcal{V}_3(A_2(c_2^N(A_1)))\right] + \\ (1-p)\left[u(c_2^C(A_1)) + \delta\mathcal{V}_3(A_2(c_2^C))\right],\end{aligned}$$

subject to the budget constraints $c_{i+1} \leq y_{i+1} + (1+r)A_i$.

Take the income flow to be $(y_1, y_2, y_3) = (15, 10, 10)$, utility to be logarithmic, $u(c) = \ln(c)$, and the following parameters are used, $\beta = 0.8$, $\delta = 0.98$, $r = 0.05$ and $p = 0.5$.

Figure 3.1 shows the eight possible consumptions paths (the selves have the same behavior in the last period so there are four coincidences in the paths). Notice that the consistent self always chooses to consume more in the next period, and the opposite for the naive.

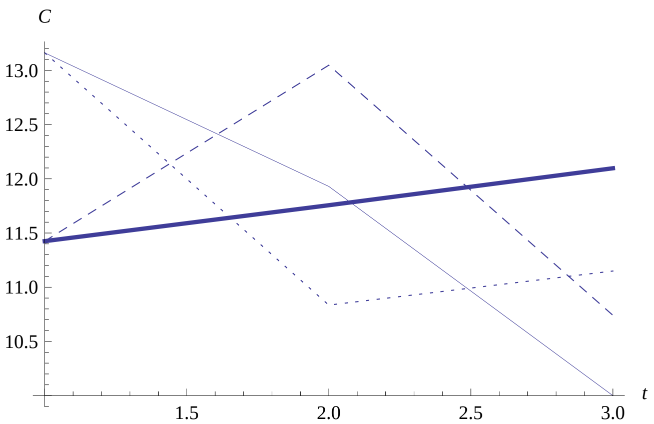

Figure 3.1: Path of consumption for $\{c_1^C, c_2^C\}$ (thick line), $\{c_1^C, c_2^N\}$ (dashed line), $\{c_1^N, c_2^C\}$ (dotted line) and $\{c_1^N, c_2^N\}$ (thin line).

3.5 Macroeconomic interpretation

While the staggered consistent model, when one applies it to a single individual, delivers a distribution of outcomes instead of just one outcome, it has quite interesting applications to behavioral macroeconomics. Interpreting the model from a macroeconomic point of view, one can consider the existence of many agents confronted with the same problem. It is then possible to aggregate all the agents by taking a weighted average of all possible outcomes of the random process. The result is then unique and can be thought as the behavior of a representative agent, as well as the (a priori) expected utility for each individual.

3.5.1 Consuming and saving

Consider again the example in Section 3.4.3. The macroeconomic representative agent follows the average of the paths, weighted according to their probability. For instance the path with (naive, consistent, naive) occurs with probability $p(1-p)p$. The result of the aggregate behavior is depicted in Figure 3.2.

3.5.2 Long-run asset

Long-run assets are particularly interesting when investigating preference dynamics, because it is a device that individuals may use in a given period to influence the sets of possible actions in future periods, possibly distant ones. Here an extension from the previous example is worked out.

The individual receives a given same income flow, y_t. She has now two possibilities to save, a short-run asset A that yields interest r in the following period and a long-run one A_L whose interest r_L is paid after two periods. It is assumed that the individual cannot trade its holdings of the long-run asset. To make things interesting it is further assumed that $(1+r)^2 > (1+r_L)^2 > 1$, implying that holding a positive amount of A_L is a costly action because holding twice the same amount of short-run asset would yield a higher interest payment. Moreover, it is assumed that the individual is liquidity-constrained in the sense

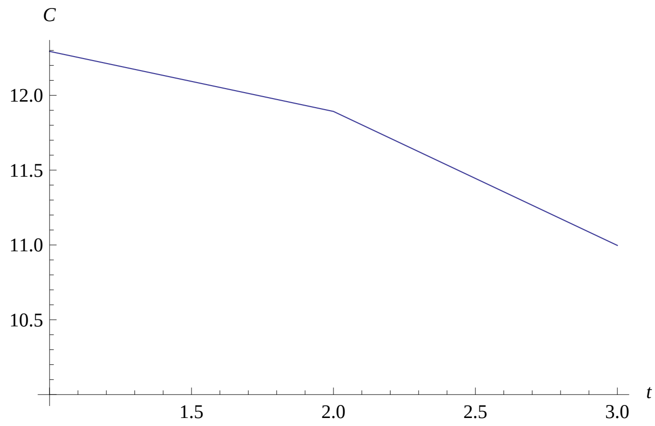

Figure 3.2: Averaged consumption choices.

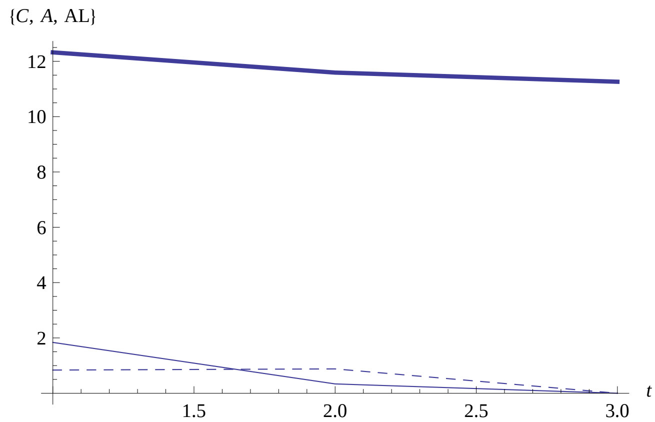

Figure 3.3: Path of consumption (thick line) and asset holdings, long-run (straight) and short-run (dashed).

that she cannot borrow, that is $A, A_L \geq 0$. The first period income is taken to be bigger than the following ones so that the first period decision (the only one where it may be optimal for the individual to hold some amount of the long-run asset) has a particular role.

The algebra is simple but lengthy and of no relevance, thus only the solution with the above parameter choices is shown. The long-run interest rate is chosen to be $r_L = 0.04$. Recall that $r = 0.05$.

Given these values, in the first period the consistent self chooses to consume $c^C = 11.48$, and keep $A = 1.68$ and $A_L = 1.84$, the naive self chooses $c^N = 13.16$, and keep $A = 1.84$ and $A_L = 0$. The consistent self recognizes the possibility of overconsuming in the following period, choosing therefore to costly hold the long-run asset. The naive on the other hand, thinks she will not undersave in the next period, choosing therefore $A_L = 0$. Figure 3.3 shows the path of the three variables over the periods aggregated over all possible paths according to their probability as described above.

The long-run asset is a costly commitment device and it may be optimal or

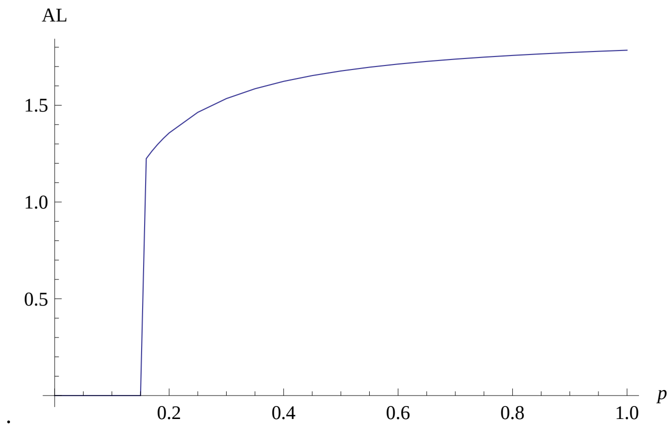

Figure 3.4: Long-run asset demand A_L in the first period by the consistent self, as a function of probability of occurring a present-bias impulse p.

not to hold it, depending on how important it is to make the commitment. If the probability of deviation is low, the commitment device is of lower interest. This is depicted in Figure 3.4 which shows the optimal level of A_L, for the consistent self in the first period, depending on the probability p of acting naively. As expected A_L grows monotonically with p, but there is a jump at $p = 0.15$ below which the individual simply picks $A_L = 0$.

The cost of the commitment device is also an important variable in this choice. Figure 3.5 depicts the demand for the long-run asset A_L (as well for the short-run asset in the first period) as a function of r_L. As r_L approaches r, i.e. as the cost of commitment decreases, the consistent self chooses to hold more of it. There is a discontinuity at $r_L = 0.018$ below which another type of commitment device yields higher pay-offs. The liquidity constraint of second period naive self becomes active, which forces its consumptions choice to be closer to that of the consistent self. Note how costly it is to hold assets at this r_L that yield $(1 + r_L)^2 = 1.0816$ when holding twice the short-run asset would yield $(1 + r)^2 = 1.1025$.

3.6 Discussion

The empirical literature on intertemporal decision has mainly focused on consistency and discount rates. How an individual with inconsistent preferences actually behaves is still unclear. The theoretical models until now have intuitive assumptions, even if some fit the data in a broad class of problems. It is also clear that models with partial naivete are the ones closer to reality, but the possibilities are innumerous. The chapter proposes a quite intuitive and flexible model of partial naivete. Individuals are assumed to recognize their inconsistency and acting according to that knowledge, but once in a while skip the optimal plan by following a present-biased impulse. Some clear examples where this model seem to be realistic is the attitude towards shopping, starting a diet, doing an assignment, etc. People try to prevent present-bias and they are indeed successful most of the time, but not always.

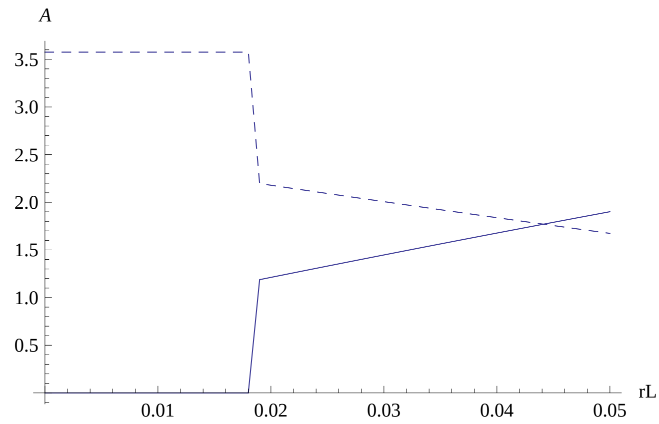

Figure 3.5: Long-run asset demand A_L (normal line) and short-run asset demand A (dashed line) in the first period by the consistent self, as a function of its interest rate r_L.

The model may also give an explanation for apparent inconsistencies in empirical evidence. Depending on the perception of the probability of the impulses the same individual with fixed time preferences may exhibit totally different intertemporal trade-offs.

This leads to a possible extension of the model. In line with the partial naivete model from O'Donoghue and Rabin (2001), the perception of the probability of the present-bias may be an underestimation, or even an overestimation, of the actual probability. Other extension would be considering naive and sophisticated selves instead of naive and consistent selves. However this would be a rather complex model and, as mentioned above, sophisticated behavior has some theoretical and intuitive problems.

Comparing to the other existing model of partial naivete in O'Donoghue and Rabin (2001), this one seems more intuitive and more powerful in explaining why individuals choose to incur great costs to precommit. In O'Donoghue and Rabin (2001) people are not totally aware of their inconsistency, so they do not see much of a point in registering for swimming classes instead of a paying a cheaper pay-per-visit fee in the swimming pool. But if people do realize it but lack total control in all periods, they are willing to incur that cost.

Chapter 4

Vague Price Recall and Price Competition

4.1 Introduction

There is hardly someone that knows the precise price of a given consumption good in the closest stores. Take a retail store, one may have a reasonable idea on the average price level at different stores but one does not recall the precise price of golden apples. The textbook model of price competition for homogeneous goods assumes however that consumers are fully informed about the prices posted by the firms. Everything else equal this means that not all consumers head to the store actually having the lowest prices, as commonly assumed.

While the issue is absent from the Industrial Organization literature, the Marketing literature has handled it for decades. Monroe and Lee (1999) present a summary on price-awareness research, stating that previous studies have found that the average absolute recall error ranges from 6% to 19.45% of the correct price[4.1]. Neglecting this bounded rationality of consumers cannot be seen as a merely theoretical simplification, for it implies that in the basic price competition model with a homogeneous good firms charge a markup over the cost and that profits are therefore non-zero. Furthermore equilibrium prices will depend on the number of firms and do not equal marginal costs.

In this chapter a model for the pricing behavior of firms in a Bertrand setting facing consumers with imperfect recall is proposed, abstracting from other (classical) deviations as heterogeneous goods, search costs, spatial competition, product differentiation, price discrimination, etc. The imperfect recall on prices is modeled as a random shock (with mean zero) that is added to the real price. Consumers decide where to shop following their wrongly recalled prices, but at the store the demanded quantity is a function of the real price. It should be understood as a price competition model between retail stores that sell goods with low prices, such as supermarkets or alike.

Both exogenous and endogenous shocks are considered. In the latter consumers are aware of their limitations and can choose to make a higher costly

[4.1] The authors argue in the mentioned paper that consumers do recall more than what they explicitly acknowledge, but there is no doubt that price recall is not perfect.

effort in order to improve the accuracy of their price recall. The firms are however fully rational and maximize their profits anticipating the errors of the consumers.

While the proposed model is a static game, it can be interpreted as the (constant) outcome of a repeated game where there is no learning process by the consumers.

It is shown that firms charge a markup, following a pricing strategy equivalent to that of a monopolist facing a demand with higher price elasticity. The markup increases with the incorrect recall of prices by consumers. Once the full awareness of the price is dropped, price dispersion becomes a possibility because consumers do not fully react to the price differences. In the present model it arises due to cost differentials, which also means that firms with higher costs are not driven out of the market. Hence, instead of mixed strategies or random strategy pricing equilibria, this model proposes the inability of consumers to screen between high and low cost firms as an explanation for the existence of price dispersion on a homogeneous good. The monopoly analogy is still robust for different costs. Intuitively one might foresee that larger recall errors would imply lower equilibrium price gaps because of diminishing incentives to price differentiation among firms. In reality the low cost firm will choose to make its price advantage more salient, increasing price dispersion.

Introducing more firms has a weak competition pressure on prices, which does not lead to marginal cost pricing even with infinitely many firms. In fact increasing the number of firms can at most have the same effect as reducing the standard deviation of the price recall errors of consumers to half. It is shown that the equilibrium price dispersion has a sensitive dependence on the cost structure of the firms in the market.

The chapter is structured as follows. Section 4.2 reviews the literature in the field, Section 4.3 introduces the basic model and discusses the main implications, Section 4.4 examines the introduction of more firms and the importance of the price structure, Section 4.5 endogenizes the price recall error committed by the consumers, Section 4.6 compares the results of the present chapter with closely related models in the literature, and Section 4.7 concludes.

4.2 Related literature

There is a closely related literature to this chapter on pricing with boundedly rational consumers. Hehenkamp (2002) proposes an evolutionary game where consumers only receive information about the prices of the firms with some given probability. Sellers on the other hand have a probability of learning about the other sellers' prices and profits, mimicking the one with higher profit. Depending on the level of sluggishness, i.e. frequency with which they receive new information, the equilibrium price will fall between the marginal cost and monopoly pricing regimes. Chen, Iyer, and Pazgal (2005) use the limited memory model of Dow (1991), where consumers do not recall the exact price but only a price range to which it belongs, the price ranges being optimally chosen. These consumers constitute a fraction of all consumers in a price competition setting, the remaining being either fully informed or fully uninformed. In equilibrium they choose to have finer partitions in the low prices range, and firms choose to have a degenerate random price strategy, where the number of possible prices equals the number of memory partitions. In Gabaix and Laibson (2004) and Gabaix, Laibson, and Li (2005) consumers make errors when evaluating their inherent value of a product. Having firms competing on an homogeneous good leads in this case to a markup. The authors show that increasing competition, i.e. more firms in the market, has almost no consequence in terms of a markup decrease. If allowed to increase noise of the product evaluation, e.g. through confusing characteristics of the good, they choose to do so in an inefficient way.

Another branch of pricing models where boundedly rational consumers are exploited is add-on pricing. In Gabaix and Laibson (2006) consumers are unable to fully take into account the add-on charge, so that firms have positive profits. Ellison (2005) again shows that competition does not always eliminate positive profits which arise in equilibrium in a hidden add-on price model, with vertical and horizontal differentiation.

Search costs are a further source of imperfect competition leading to similar outcomes. The seminal paper by Diamond (1971) assumes that consumers do not know the prices of firms, having to visit different stores, only purchasing when a price below a given cutoff price is found. Prices will be adjusted to a unique equilibrium price in finite time, namely the monopoly price. Stahl (1989) shows that assuming two types of consumers (zero and positive search costs) leads to intermediate results, i.e. equilibrium prices between the marginal cost and the monopoly pricing.

Starting with Varian (1980) there is a literature mainly interested in price dispersion, where it is assumed that firms choose a random pricing strategy, in opposition to a fixed price. That is, the strategy space is a set of probability distributions, not the the positive real numbers. In Varian (1980) a fraction of the consumers is persistently uniformed about the prices, but having a reservation price. If stores are allowed to choose a random price distribution, they choose to do so in equilibrium balancing the probability of having the lowest price (and therefore getting the informed consumers) and maximizing profits with the uninformed consumers. Spiegler (2006) comes to a similar conclusion when all consumers are unable to take the random pricing strategy into account, and thus sample the prices in the stores taking thereafter that sample as the final price and picking the lowest price. In this context bounded rationality can also be attributed to firms as in Baye and Morgan (2004). Consumers are fully informed and rational whereas the firms choose random pricing strategies playing either Nash equilibrium, quantal-response equilibrium or ε-equilibrium. It is shown that the last two are closer to the results obtained in experiments (where subjects choose prices and rational consumers are played by the computer). In Alos-Ferrer, Ania, and Schenk-Hoppe (2000) firms play a pricing oligopoly evolutionary game, following a simple behavior of imitation and experimentation.

While formally close to the present chapter, the product differentiation paper by Perloff and Salop (1985) has a different motivation. Their paper suggests a model with differentiated goods where consumers have heterogeneous preferences over the available products. Firms exploit that by charging a markup, which is increasing in the variance of the preferences. In the limit case of fixed demand that model is formally equivalent to the models where consumers make mistakes about the value of the good (as in Gabaix and Laibson (2004) and Gabaix, Laibson, and Li (2005)) and to the present model, where consumers have imperfect recall of prices. This is discussed in Section 4.6.1.

On the empirical side Monroe and Lee (1999) show that consumers do not perfectly recall the prices when explicitly asked to do so. Baye and Morgan (2004) and Pan, Ratchford, and Shankar (2004) indicate that price dispersion exists in settings which are very close to the textbook Bertrand competition. In the experiment of Kalayci and Potters (2009) individuals playing the firms choose to make product comparisons more complex to the individuals acting as consumers, so that they do not choose optimally allowing firms to charge a markup. This markup is increasing in the confusion caused in consumers.

While having a different motivation, the present model can be thought of one of horizontal differentiation with fixed positions as a first intuition. The value of the good is the same across consumers, but each one is biased towards one of them. Therefore having a higher price does not imply zero demand. Firms compete for the indifferent consumer (here the one recalling prices as being equal) at the margin, etc. But as it shall be seen the results differ, because the 'gap' between one consumer and one firm, in the horizontal differentiation models, decreases the value of the good to the consumer either due to transportation costs or preferences in the product space. That is not the case for imperfect price recall. This is discussed thoroughly in Section 4.6.2.

4.3 Model

4.3.1 Basic setup

Consider two risk neutral firms, A and B, selling one homogeneous good whose cost of production is zero. Firms announce their price simultaneously, p_A by firm A and p_B by firm B.

Consumer $\alpha \in [0,1]$ recalls prices $p_i'^{\alpha} = p_i + \epsilon_i^{\alpha}$, for $i = A, B$, where ϵ_A^{α} and ϵ_B^{α} are independently and identically distributed shocks for each α with non-degenerate probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$ with an expected value of zero[4.2]. The consumers then do their shopping at the firm with the lowest recalled price[4.3] (it can be assumed that they randomize in case of a tie, which happens with probability zero). At the store they learn the real price so the demand curve is therefore given by the real and not the recalled price. The intuition is that the consumer adapts its demand when confronted with the real price, in the same that consumers react to price promotions that they see in a store.

It is assumed that transportation costs between the two firms are high enough, in the sense that the consumer does not visit the second store if she learns that the real price of that firm is higher than the recalled price of the other firm. This may be pictured as having the consumer at home wondering where to buy a product without accurately remembering the prices. When the consumer gets to the chosen store and learns the real price, her cost of going

[4.2] For a more detailed model of memory related bounded rationality see Mullainathan (2002).

[4.3] An equivalent interpretation of the same setting is to consider that the consumer visits the firms sequentially. On the arrival at the second firm, she does not correctly recall the first price p_1 but recalls it plus a shock given by $p_1 + \epsilon_1 - \epsilon_2$. If she considers the second price as lower, she buys there instantly, otherwise she goes back to the first firm. This interpretation is very close to the model in Chen, Iyer, and Pazgal (2005), but it relies on the following assumption. It must be assumed that in those cases where the consumer decides to return to the first firm, she never returns to the second if she learns that she made a mistake. In other words the transportation costs of the two first trips are negligible, but the third trip is infinitely costly.

to the other store and checking its price as well is higher than the (possible) expected gain. Again, the model pretends to capture the price competition on small goods with low prices[4.4].

Firms anticipate this behavior and play best response to their competitor's strategy, resulting in a Nash Equilibrium in pure strategies. Notice that for a degenerate distribution shock with zero variance and zero mean this model reduces to the basic Bertrand model.

Let $D(p)$ be the demand curve of each consumer, where p is the real price of the firm at which the consumer is buying the good. Firms are assumed to be aware of this demand function. The probability that at prices p_i of firm i and p_j of firm j, $i, j = A, B$, $i \neq j$, consumer α recalls that firm i has a lower price than firm j is given by

$$\begin{aligned} P\left(p_i'^{\alpha} < p_j'^{\alpha}\right) &= P\left(p_i + \epsilon_i^{\alpha} < p_j + \epsilon_j^{\alpha}\right) = P\left(\epsilon_i^{\alpha} < \epsilon_j^{\alpha} + p_j - p_i\right) \\ &= \int_{-\infty}^{\infty} f(y)F(y + p_j - p_i)dy \\ &= g(p_j - p_i), \end{aligned}$$

where $g(x)$ is the market share of a firm whose price is lower than its competitor's price by x. It is given by $g(x) = \int_{-\infty}^{\infty} f(y)F(y+x)dy$. Notice that $g'(x) = \int_{-\infty}^{\infty} f(y)f(y+x)dy$. Moreover $g(x)$ is independent of α because all consumers have the same price recall shock distribution. Now given the behavior of consumers, firm i, $i = A, B$, maximizes expected profits $\Pi_i(p_i, p_j) = \mu p_i g(p_j - p_i) D(p_i)$ over $p_i \geq 0$, where μ is the number of consumers. Without loss of generality, in terms of price setting, it is assumed that $\mu = 1$. The first order condition for a maximum is

$$\begin{aligned} \frac{\partial \Pi_i}{\partial p_i} = g(p_j - p_i)D(p_i) - p_i D(p_i) g'(p_j - p_i) + p_i g(p_j - p_i) D'(p_i) = 0 \\ \Leftrightarrow -p_i \frac{D'(p_i)}{D(p_i)} + p_i \frac{g'(p_j - p_i)}{g(p_j - p_i)} = 1 \\ \Leftrightarrow \varepsilon(p_i) - p_i \frac{d}{dx} \ln g(p_j - p_i) = -1, \quad (4.1) \end{aligned}$$

where $\varepsilon(p) \equiv \frac{d \ln D(p)}{d \ln p}$ is the price elasticity of demand. Notice that this equation can be rewritten as

$$\varepsilon(p_i) + \varepsilon_i^g(p_i, p_j) = -1 \qquad (4.1')$$

with $\varepsilon_i^g(p_i, p_j) \equiv p_i \frac{\partial}{\partial p_i} \ln g(p_j - p_i) = -p_i \frac{d}{dx} \ln g(p_j - p_i)$ being the own price elasticity of the market share of firm i. This is just the usual result of profit maximization but it will help giving some insight to the model later on in this section. Now further assumptions on functional forms are stated.

Assumption 4.1 *The random shocks ϵ_A and ϵ_B are iid with mean 0, variance σ^2 and full support on the real line with probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$. Moreover the distribution of ϵ_A and ϵ_B is such that $g(\cdot)$ is logconcave, i.e. $\frac{d}{dx} \ln g(x)$ is non-increasing in x.*

[4.4] Dropping this assumption would create an upper bound on the price gap that firms might consider. While this does not affect the symmetric equilibria, it would affect asymmetric equilibria that arise in the next section.

Assumption 4.2 *The price elasticity $\varepsilon(p)$ of demand is non-increasing, continuous and $-1 \leq \varepsilon(0) \leq 0$.*

Notice that Assumption 4.2 is rather weak, linear demand is an example that satisfies it. Also compare it with the necessary conditions for the classic monopolistic price setting model to be well-defined, i.e. having a unique profit maximizing price, namely that $\varepsilon(p)$ takes the value -1 for one and only one $\hat{p}$ and that it is bigger (smaller) than -1 before (after) $\hat{p}$[4.5]. Assuming $\varepsilon(0) \geq -1$ guarantees that the equilibrium will not be a corner solution, having firms outside the market. The following results will be used.

Lemma 4.1 *For each $p_j \geq 0$, $j \neq i$, $i = A, B$, equation (4.1) has a unique positive solution.*

Proof
First it is shown that the left-hand side (LHS) of (4.1) is decreasing in p_i. The first term, $\varepsilon(p_i)$, is non-increasing according to Assumption 4.2. In the second term the part $\frac{d}{dx} \ln g(p_j - p_i)$ is positive, because

$$\frac{d}{dx} \ln g(p_j - p_i) = \frac{g'(p_j - p_i)}{g(p_j - p_i)} > 0.$$

From Assumption 4.1 it is non-decreasing in p_i since

$$\frac{\partial}{\partial p_i} \left[\frac{d}{dx} \ln g(p_j - p_i) \right] = -\frac{d^2}{dx^2} \ln g(p_j - p_i) \geq 0,$$

where $\frac{\partial x}{\partial p_i} = -1$ was used. Because firm i only considers $p_i \geq 0$, p_i is obviously positive and increasing in p_i. Therefore minus their product, i.e. the second term on the LHS is negative and decreasing in p_i. Hence, the LHS is decreasing in p_i.
Now, for $p_i = 0$ the LHS equals $\varepsilon(0) \geq -1$ because $\frac{\partial}{\partial p_i} \ln g(p_j - p_i)|_{p_i=0}$ is finite due to the random variables' full support. For $p_i > 0$ the first term of the LHS, $\varepsilon(p_i)$, is negative and non-increasing in p_i. In the second term $\lim_{p_i \to \infty} \frac{d}{dx} \ln g(p_j - p_i)$ is strictly positive because $\frac{d}{dx} \ln g(p_j - p_i)$ is strictly positive and non-decreasing in p_i. This part of the second term is multiplied by p_i, therefore the second term goes to $-\infty$ as $p_i \to \infty$. This implies that the limit of the LHS is $-\infty$ as $p_i \to \infty$. Hence the LHS must equal -1 for a unique p_i, for any given $p_j \geq 0$. ∎

The sufficiency of the condition for a maximum is implied by the above mentioned uniqueness.

Lemma 4.2 *The first order conditions are sufficient for the profit maximization problem.*

Proof
It follows from the proof of Lemma 4.1 that the solution for $\frac{\partial \Pi_i}{\partial p_i} = 0$ for $p_i \geq 0$ is unique and that $\frac{\partial \Pi_i}{\partial p_i}$ switches its sign, from positive to negative, at that point.

[4.5] These conditions guarantee that the derivative of the profit function equals zero for one unique price, and that the profit function is quasi-concave.

These properties imply that $\Pi_i(p_i, p_j)$ is quasi-concave on p_i for any p_j, so that the FOC are sufficient. ■

The next step is to see that this game has a unique Nash equilibrium. It was shown in the proof of Lemma 4.2 that the reaction curves are properly defined, the next step is to prove that they cross. The symmetry of the problem indicates the possible existence of a symmetric equilibrium. It is known that for each p_j there is a unique $p_i(p_j)$ solving (4.1), so it can be checked in (4.1) whether some $p_i = p_j = p$ is a possible solution. For firm i, $i = A, B$, (4.1) becomes

$$\varepsilon(p) - p\frac{g'(0)}{g(0)} = -1.$$

Since $g(0) = \frac{1}{2}$,

$$2pg'(0) = 1 + \varepsilon(p). \tag{4.2}$$

From Assumption 4.2 which implies that $\varepsilon(0) \geq -1$ and $\varepsilon(p)$ is non-increasing, and the fact that $g'(0)$ is positive it is concluded that equation (4.2) has just one solution $p^* \geq 0$.

Lemma 4.3 *The reaction curves implied by the first order conditions yield a unique (subgame perfect) equilibrium price p^* for both prices.*

Proof
From (4.1) it is possible to prove that the slope of the reaction curve $p_i(p_j)$ lies between 0 and 1. To see this consider an increase Δ in $p_j \geq 0$, which increases the LHS of (4.1). An increase in p_i (to be precise in $p_i(p_j)$) of Δ would offset the decrease of $g(\cdot)$ (because g contains the term $p_j - p_i$) but would increase the factor by which $\frac{\partial}{\partial p_i} \ln g(p_j - p_i)$ is multiplied, that is p_i, and decrease $\varepsilon(p_i)$, yielding a total decrease in the LHS. Because the LHS is strictly monotonous in p_i, the price p_i implicitly defined by (4.1) (that is the best reaction of firm i) must increase less than Δ. In other words the slope of the reaction curve satisfies $\frac{\partial p_i(p_j)}{\partial p_j} < 1$ for all $p_j \geq 0$, and by symmetry $\frac{\partial p_j(p_i)}{\partial p_i} < 1$ for all $p_i \geq 0$. This means that in the (p_j, p_i)-plane, $p_i(p_j)$ is always flatter than $p_j(p_i)$, implying that they only cross once. Notice that they must cross at least once given that $p_i(0) \geq 0$ and $p_j(0) > 0$.There is therefore no other equilibrium besides $p_A = p_B = p^*$.■

Proposition 4.1 *In the basic model where consumers suffer a price recall shock and firms face zero costs, the firms are able to charge a nonnegative price in equilibrium p^* satisfying*

$$2p^*g'(0) = 1 + \varepsilon(p^*). \tag{4.3}$$

Moreover they charge $p^* = 0$ if and only if a monopolist also would do it.

Proof
The last statement follows from setting $p = 0$ in (4.2) and noticing that it is only a solution if $\varepsilon(0) = -1$, which is the case where a monopolist is indifferent between selling and exiting the market.■

Thus in this model firms are able to exploit the bounded rationality of consumers. The intuition of the result, in opposition to the zero profit solution, is that announcing a price lower than that of the opponent is not enough to attract all consumers, because some of them will still recall the price as higher. This means that charging a price marginally below, equal to or marginally above the opponent is irrelevant. The pressure of price competition is eroded by the inability of consumers to reward firms with lower prices, that is failing to give the right incentives for firms competing on the same good.

Here there are two trade-offs playing a role in the price-setting decision. One is a Hotelling type of trade-off, higher revenue per consumer vs higher market share (which is a function of the other firm's price), the other one is that of a monopolist, higher revenue per unit sold vs more goods sold per consumer (which has no strategic consequence). The equilibrium price depends on the strength of both.

Recall mistake and Hotelling trade-off

Each firm opts between the increase of market share achieved through lower price, and the revenue per consumer achieved through higher price. This trade-off shows up in equation (4.3) in the term $g'(0)$, which stands for the marginal decrease in the market share due to a price increase. In other words, it is the marginal change of the indifferent consumer as in Hotelling models.

The term $g'(0)$ is lower for higher variance. The intuition is straightforward, the more difficulties the consumers have in remembering and therefore comparing the prices, the smaller the marginal change in the market share due to a price variation. Suppose the change of the variance is achieved through the "spreading" of the possible random values, that is changing x to σx with $\sigma > 0$. The new density function f satisfies $\sigma f(\sigma x) = f_0(x)$, with $f_0(x)$ standing for $f(x)$ with $\sigma = 1$, so that $\int f(y)dy = \int \sigma^{-1} f_0(\sigma^{-1}y)dy = \int f_0(x)dx = 1$, where $x = \sigma^{-1}y$ and $dy/dx = \sigma$ was used.

The new variance is given by $var(y) = \sigma^2 var(x)$. The term $g'(0)$ appearing in equation (4.3) changes according to

$$\int_{-\infty}^{\infty} f^2(y)dy = \int_{-\infty}^{\infty} \sigma^{-2} f_0^2(\sigma^{-1}y)dy = \frac{1}{\sigma}\int_{-\infty}^{\infty} f_0^2(x)dx, \tag{4.4}$$

or simply $g'(0) = \sigma^{-1} g_0'(0)$, where $g_0'(0)$ is the $g'(0)$ for $\sigma = 1$. The variation in the new equilibrium price is seen in the new version of equation (4.3)

$$\begin{aligned} 2p^* g'(0) &= 1 + \varepsilon(p^*) \\ \Leftrightarrow 2p^* g_0'(0) &= \sigma\left(1 + \varepsilon(p^*)\right). \end{aligned} \tag{4.5}$$

Implicit differentiation of (4.5) yields

$$\frac{dp^*}{d\sigma} = \frac{1 + \varepsilon(p^*)}{2g_0'(0) + \sigma\varepsilon'(p^*)} > 0.$$

As expected, higher price uncertainty σ means higher price markup. Compared to the fixed demand case (take $\varepsilon(p) = \varepsilon'(p) = 0$) this influence is however smaller, because the denominator is now bigger and the numerator smaller (remember that $0 \leq 1 + \varepsilon(p^*) < 1$). The intuition is that firms must also take the diminishing demand into account. It is not true that the marginal increase of the price due to σ is always diminishing, for it depends on the value of $\varepsilon'(p^*)$. Two extreme results can however be established.

Corollary 4.1 *For a decreasing price recall error variance the equilibrium price goes to the Bertrand price.*

Corollary 4.2 *If the demand allows for a monopoly price, i.e. there is a $\hat{p}$ such that $\varepsilon(\hat{p}) = -1$, then for an increasing price recall error variance the equilibrium price goes to the monopoly price.*

The first result follows easily from (4.5) with $\sigma \to 0$ which forces the first term to be zero, and because the derivative of $g_0(\cdot)$ is a strictly positive constant, p^* must be zero. The second result follows from the fact that the first term in (4.5) is bounded (because p^* is bounded), so as $\sigma \to \infty$ it must be that $1 + \varepsilon(p^*) \to 0$.

Monopolist trade-off

Because each consumer's demanded quantity also depends on the stated (in opposition to recalled) price, firms face a monopolist-type of price setting trade-off. High price means higher per unit revenue but also less units being sold. Because of non-increasing elasticity of demand this monopolist type of decision also decreases the equilibrium price (compared to the fixed demand case). This can be seen in (4.1'), where both non-increasing elasticities are added up and set equal to -1. Notice that the left-hand side of (4.3) is positive at p^*, so in equilibrium it must be that $\epsilon(p^*) > -1$, that is the equilibrium price has the monopoly price (if it exists) as an upper bound.

Explicit equilibrium solutions

To gain some more insight several specific functional forms are considered. Suppose demand $D(p) = e^{-a(\ln p + bp)}$ so that it has a linear elasticity over all $p \geq 0$, with $\varepsilon(p) = -(a + bp)$ for some $a \in [0, 1]$ and $b \geq 0$. Equation (4.3) simplifies to

$$2pg'(0) = 1 - (a + bp),$$

so that in equilibrium

$$p^* = \frac{1-a}{b + 2g'(0)}. \tag{4.6}$$

As a comparison, for the same price elasticity a monopolist would charge price $p_M = \frac{1-a}{b}$. If the recall errors ϵ_A and ϵ_B follow a normal distribution with mean 0 and variance σ^2, which satisfies Assumption 4.1[4.6], then $g'(0) = (2\sqrt{\pi}\sigma)^{-1}$. If they follow a Gumbel distribution[4.7] with cumulative distribution function

[4.6] The sum of two independent random variables following $N(0, \sigma^2)$ is a normal random variable with $N(0, 2\sigma^2)$. So $g(p_j - p_i) = \Phi\left(\frac{1}{\sqrt{2}\sigma}(p_j - p_i)\right)$ where $\Phi(\cdot)$ is the standard normal cumulative distribution function. Therefore $\frac{d}{dx} \ln g(x) = \frac{1}{\sqrt{2}\sigma}\left(\Phi'(\frac{x}{\sqrt{2}\sigma})\right)\left(\Phi(\frac{x}{\sqrt{2}\sigma})\right)^{-1}$. The ratio $\frac{\Phi'(x)}{\Phi(x)}$ is similar to the so-called hazard function $\frac{\Phi'(x)}{\Phi(-x)}$, which is strictly positive and strictly increasing for the normal distribution. Because $\Phi'(x) = \Phi'(-x)$, $\frac{\Phi'(-x)}{\Phi(-x)}$ must be strictly increasing, therefore $\frac{\Phi'(x)}{\Phi(x)}$ is strictly positive and strictly decreasing. Hence Assumption 4.1 applies to the normal distribution.

[4.7] Also known as Fisher-Tippet or log-Weibull, this distribution is important in Order and Extreme-value Statistics. It is also widely used in the literature on random utility models and on quantal response equilibria because of its mathematical tractability.

$F(x) = e^{-e^{-\tau(x-\mu)}}$ for some $\mu \in \mathbb{R}$, $\tau > 0$ and variance $\frac{\pi^2}{6\tau^2}$, then $g(x) = \frac{1}{1+e^{-\tau x}}$[4.8]. Now it holds $g'(0) = \frac{\tau}{4}$.

Some comments can be made for this closed form solutions. When firms face consumers with a wrong price recall, they charge a price which is related to the monopolist price. Formally it is equivalent to an increase in the slope of the demand elasticity, the worse the price recall is the flatter the slope. While this slope is not an intuitive concept, it may help to recall that the monopolist price does not depend on some point elasticity but on the price at which this elasticity crosses some threshold, namely -1. The higher (in absolute terms) the slope, the smaller the price the monopolist firm, as well as the competitive firms here, will choose.

The term related to the price recall, $g'(0)$, depends on the standard deviation of the shock. For $\sigma \to 0$, the price goes to the usual Bertrand price equilibrium, that is $p^* = 0$.

Taking the linear term of the Taylor series expansion in respect to σ, at $\sigma = 0$, gives the first order impact of introducing a price shock compared to usual Bertrand, namely $\frac{1-a}{2g'(0)}$.

For small b the monopolist is able to charge a high price, because demand is very inelastic in the low prices range. Here competition among the firms compensates for the inelasticity of demand and sets a lower price.

Other comments

Equation (4.6) summarizes the main prediction of the model, i.e. the equilibrium price dependence on the demand and the recall parameters, it provides therefore a testable prediction of the model, provided that the price recall variance and the slope of the elasticity of demand is known. If valid, it can be used to estimate one from the other.

While having different motivations and intuitions the present model is formally equivalent for the basic case to one of horizontal differentiation with unitary demand and bounded support of the recall errors. It is easy to see that for each choice of parameters in this model, there is a distribution of consumers on the horizontal line and a distance (transportation or preference) cost that yield the same problem and therefore the same result. The position on the line represents the price recall bias[4.9] towards one of the firms, the distance cost represents the dispersion of recall biases, the firms being located at the extremes of the line[4.10]. The intuition is straightforward, in both cases there are consumers that are inherently inclined to one of the firms, and this hinders perfect competition.

[4.8] Again it is easy to check that Assumption 4.1 is satisfied, for $\frac{d}{dx} \ln g(x) = \frac{\tau}{1+e^{\tau x}}$, which is decreasing in x.

[4.9] It is not strictly speaking a bias, because it is drawn from the difference of two random variables, but visualizing it as a bias may help the intuition here.

[4.10] In the bounded support case firms must be placed in the extremes, otherwise the consumers that would be placed outside the section between the two firms, would have exactly the same price recall bias. One of the firms could then have a discontinuous market share increase if its (negative) price difference in comparison to the other firm would be bigger than the distance cost between the two firms. In the case of full support the firms should also be located at the "extremes", otherwise there would not be consumers with all possible values of recall biases.

This similarity while giving helpful insights to the intuition in the present model, only holds for the basic setup. See Section 4.6.2 for further discussion.

4.3.2 Firms with symmetric costs

In this section firms face a positive constant cost c for the production of the good. It is assumed that the firms only bare the costs for the goods that are purchased, that is there are no costs prior to the purchase.Because of the non-linearity of the model, the results will be more complex but still with simple and intuitive limit cases. The objective function of firm i can be written as

$$\Pi_i(p_i, p_j) = (p_i - c)g(p_j - p_i)D(p_i),$$

for $i = A, B$. To maximize profit firm i solves the first order condition

$$\frac{1}{p_i - c} + \frac{D'(p_i)}{D(p_i)} - \frac{g'(p_j - p_i)}{g(p_j - p_i)} \quad = \quad 0.$$

Given the assumptions the earlier lemmas will also be applicable here and these conditions will be sufficient.

Proposition 4.2 *If the firms face the same strictly positive unitary cost c, then there is a unique symmetric equilibrium, where equilibrium price p^* is implicitly defined by*

$$2p^* g'(0) = \frac{p^*}{p^* - c} + \varepsilon(p^*).$$

Proof
Rewriting the maximization condition as

$$\begin{aligned} \frac{p_i}{p_i - c} &= -p_i \frac{D'(p_i)}{D(p_i)} + p_i \frac{g'(p_j - p_i)}{g(p_j - p_i)} \\ \Leftrightarrow \frac{p_i}{p_i - c} &= -\varepsilon(p_i) - p_i \frac{\partial}{\partial p_i} \ln g(p_j - p_i), \end{aligned} \tag{4.7}$$

enables a similar proof to the one of Proposition 4.1. The LHS in (4.7) decreases monotonically from ∞ at $p_i = c$ to 1 when $p_i \to \infty$, and the RHS in (4.7) increases monotonically from $-\varepsilon(c) + c\frac{d}{dx} \ln g(p_j - c)$ at $p_i = c$ to ∞ as $p_i \to \infty$, so the equation is satisfied for only one $p_i \geq c$ for every p_j. This single crossing implies quasi-concavity of Π_i in p_i for any given $p_j \geq c$, and therefore sufficiency of the first order conditions above. The focus is again on symmetric equilibria, so that $p_A = p_B = p$ and therefore $g(0) = \frac{1}{2}$, the equilibrium condition being

$$\frac{p}{p - c} = -\varepsilon(p) + 2g'(0)p. \tag{4.8}$$

Again it is easy to see that there is such a p and it is unique. The reasoning in the proof of Lemma 4.3 can also be applied here, noting that a hypothetical increase in p_i of the same amount as a hypothetical increase in p_j would not only increase the RHS but also decrease the LHS. This implies that $p_i(p_j)$ increases less than p_j, in other words, the reaction curve slope is below 1. The equilibrium price p^* is therefore unique. ■

The unique symmetric equilibrium p^* can also be defined by

$$\begin{aligned} \frac{p^*-c}{p^*} &= \frac{1}{-\varepsilon(p^*)+2g'(0)p^*} \\ \Leftrightarrow \frac{p^*-c}{p^*} &= -\frac{1}{\bar{\varepsilon}(p^*)}, \end{aligned} \tag{4.9}$$

where $\bar{\varepsilon}(p) \equiv \varepsilon(p) - 2g'(0)p$. Equation (4.9) is comparable to the usual first order condition $\frac{p-c}{p} = -\frac{1}{\varepsilon(p)}$ for the monopolist. The introduction of the consumer errors leads to the situation where firms have a monopolist like behavior but their Lerner index is reduced by a term that depends on the variance of the consumers' errors. The intuition is that once the consumer is at the firm, that is when the demanded quantity is set, the firm can act as a monopolist. But because of the a priori price competition, prices cannot be set too high.

To see that the equilibrium price and Lerner index are indeed smaller than those of the monopoly, notice that introducing $2g'(0)p$ on the right side of (4.8) yields a lower right hand side. Because the left (right) hand side of (4.8) decreases (increases) with p, the equilibrium price must be strictly smaller. Notice also that the additional term $2g'(0)p^*$ depends on the price, indicating that there is a variance-price level interaction in the equilibrium Lerner index[4.11].

The variation of the equilibrium price, the Lerner index, or the markup, as a function of the cost c depends on the two terms in the denominator. For low c and consequently low p, the first term may prevail so that the Lerner index is close to constant and the markup is increasing in c. However for high costs the second term becomes predominant if $-\varepsilon(p)$ does not increase indefinitely (or at least it increases less than proportional to p), meaning that the Lerner index will be proportional to p^{-1} and the markup will be constant. In the first case the recall errors are so high compared to the costs that consumers are shopping randomly. While for high c the $-\varepsilon(p)$ can be neglected so that the markup will be proportional to the standard deviation of the errors, namely given by

$$p^* - c \approx \frac{1}{2g'(0)}. \tag{4.10}$$

Considering higher prices in the denominator is equivalent to lower variances (recall that $g'(0)$ is inversely proportional to the standard deviation), the intuition being that the price recall error is relatively smaller when compared to the absolute value of the price. Once again the price and therefore the markup increases with the price recall errors magnitude by consumers. This can be seen

[4.11] The same analysis concerning the equilibrium price is not straightforward, because it is not possible to distinguish the changes due to the cost from those due to the diminishing demand. If the demand function is normalized, that is if the new demand function $\tilde{D}(\cdot)$ is chosen such that $\tilde{D}(p+c) = D(p)$ for $p \geq c \geq 0$, there will be no change in the equilibrium markup. The FOC is formally the same:

$$\begin{aligned} \frac{1}{p_i - c} + \frac{\tilde{D}'(p_i)}{\tilde{D}(p_i)} - \frac{\int_{-\infty}^{\infty} f(u) f(u + p_j - p_i) du}{\int_{-\infty}^{\infty} f(u) F(u + p_j - p_i) du} &= 0 \\ \Leftrightarrow \frac{1}{r_i} + \frac{D'(r_i)}{D(r_i)} - \frac{\int_{-\infty}^{\infty} f(u) f(u + r_j - r_i) du}{\int_{-\infty}^{\infty} f(u) F(u + r_j - r_i) du} &= 0, \end{aligned}$$

so that the optimal markup r^* equals the no-cost equilibrium price and $p^* = r^* + c$. The markup is therefore the same.

from equation (4.8) (by implicit differentiation) that shows that p^* decreases with $g'(0)$ and therefore increases with the error variance,

$$\frac{dp^*}{dg'(0)} = -\frac{2p^*}{\frac{c}{(p^*-c)^2} + 2g'(0)} < 0.$$

4.3.3 Firms with different costs and price dispersion

Introducing asymmetric costs for the firms leads to two interesting results. Firms with higher costs for the same homogeneous good do participate in the market, with higher equilibrium prices, for there will be always consumers recalling its price as smaller. The second feature is already implicit in the previous statement: the model predicts a price dispersion situation in a homogeneous good market.

Assumptions 4.1 and 4.2 are once again taken. Firm A faces unitary cost $c_A > 0$ and firm B faces $c_B > 0$. The profit of firm $i = A, B$ is given by $\Pi_i(p_i, p_j) = (p_i - c_i)g(p_j - p_i)D(p_i)$. In order to maximize it, firm i solves $\frac{\partial \Pi_i}{\partial p_i} = 0$, that is

$$\frac{p_i}{p_i - c_i} = -\varepsilon(p_i) - p_i \frac{\partial}{\partial p_i} \ln g(p_j - p_i), \tag{4.11}$$

with $i, j = A, B$ and $i \neq j$. Proposition 4.2 can be applied here with two changes. First the issue of existence of one equilibrium, i.e. an intersection of the reaction curves implicitly defined by (4.11), must be put differently because the system of equations is now asymmetric. Take $c_B > c_A$ without loss of generality. The best response $p_B(c_A)$ of firm B to the minimum price that firm A may offer, c_A, satisfies $p_B(c_A) \geq c_B > c_A$. So $(c_A, p_B(c_A))$ lies 'above' (taking p_B as the vertical axis) or on the line $(p_A, p_B) = (c_A + t, c_B + t)$, $t \in \mathbb{R}$. On the other hand, $p_A(c_B)$ satisfies $p_A(c_B) \geq c_A$, so $(p_A(c_B), c_B)$ lies below or on that diagonal. From Proposition 4.2 it is known that the best response curves always cross the main diagonal $p_A = p_B$[4.12], and they do that only once because $\frac{\partial p_i(p_j)}{\partial p_j} < 1$ for all $p_j \geq 0$ with $i, j = A, B$, $i \neq j$. Either $p_A(c_B)$ lies on the same side of the main diagonal as $p_B(c_A)$ (the crosspoint with the main diagonal lies then below c_B, representing a price combination that is never considered by the firms) so that $p_B(p_A)$ must cross $p_A(p_B)$ only once to get to the main diagonal, or $p_A(c_B)$ lies on the other side which means that $p_A(p_B)$ will also cross the main diagonal, crossing therefore $p_B(p_A)$ as well.

The second difference is that the equilibrium will not be symmetric, being thus implicitly defined by a system of equations with equation (4.11) and its counterpart, instead of being simply defined by (4.8). The FOC do not have here a closed solution because $p_A^* = p_B^*$ does not hold. It is however possible to determine how the equilibrium prices change depending on the costs through implicit differentiation of the FOC, when costs depart from the symmetric case $c = c_A = c_B$. In Section 4.8.1 in the Appendix it is derived how both firms "react to changes" in one of the costs, which is shown in formulas (4.24) and (4.25). Some comments on those formulas are as follows.

[4.12] Take p_j^* to be the equilibrium price in the symmetric cost scenario with costs being $c = c_j$. By definition the best reply of firm i in this case is to set $p_i^* = p_j^*$, that is $p_i(p_j^*) = p_j^*$.

Cost price relation

In the symmetric case it was clear that a higher cost implies a higher equilibrium price. While the trade-off here is quite more intricate it is the case that a cost increase of one firm leads to a price increase by both firms.

Proposition 4.3 *An increase in cost c_i of firm i leads in equilibrium to an increase of p_i^* and p_j^*,*

$$\frac{\partial p_i^*}{\partial c_i} > \frac{\partial p_j^*}{\partial c_i} > 0.$$

Proof
Inspecting equation (4.11), which defines the best strategy p_i given p_j and c_i, it can be seen that the left-hand side of the equation (strictly) increases in c_i and the right-hand-side (strictly) decreases with p_j. Now for a given p_j an increase in c_i leads to an increase in the response p_i defined by (4.11). The proof of Lemma 4.3 shows that the choices of the two players are strategic complements. So the best-response p_j of firm j shall also (strictly) increase. This decreases the right-hand side of equation (4.11), which further increases the implied best response p_i. Both changes point in the same direction and that implies the result $\frac{\partial p_i^*}{\partial c_i} > 0$. From strategic complementarity it is obtained that $\frac{\partial p_j^*}{\partial c_i} > 0$. At last, from $\frac{\partial p_i(p_j)}{\partial p_j} < 1$, which is shown in the proof of Lemma 4.3 (notice that the FOC of firm j are not altered with the change in c_i), it follows that $\frac{\partial p_i^*}{\partial c_i} > \frac{\partial p_j^*}{\partial c_i}$.
■

One might have foreseen that a smaller cost gap would mean a more competitive market, meaning lower prices of both. Consider the case where the low cost firm suffers a cost increase. The high cost firm could choose to take benefit of the competitor handicap by trying to attract more consumers, that is by lowering the price. But it appears that the revenue increase that is attainable by a higher price of the high cost firm, dominates the benefits from trying to attract more consumers.

Aggregate behavior

On the aggregate level, the firms act (locally) as a monopolist. It is easy to see that the monopolist reaction to a cost change at cost c equals

$$\frac{\partial p_M}{\partial c} = \frac{1}{1 - \left(\frac{p_M - c}{p_M}\right)^2 (p_M \varepsilon'(p_M) - \varepsilon(p_M))}. \tag{4.12}$$

Recall that the monopolist price p_M and the symmetric cost equilibrium price p^* are defined in the same way given $\varepsilon(\cdot)$ and $\bar{\varepsilon}(\cdot)$, respectively. It was already shown in equation (4.9) that the competitive price follows the cost in the same way as the monopolist price, when the cost of both firms change. To see that this is also the case for a change in just one of the firms' costs, say firm i, the following term must be examined $2 \times \frac{1}{2}\left(\frac{\partial p_i^*}{\partial c_i} + \frac{\partial p_j^*}{\partial c_i}\right)$, the factor 2 appearing because an increase in c_i yields just a half increase in the average cost and the factor $\frac{1}{2}$ is needed to have the average price change (and not their sum). From formulas (4.24) and (4.25) it must be that at $c = c_A = c_B$,

$$\begin{aligned}
\frac{\partial p_i^*}{\partial c_i} + \frac{\partial p_j^*}{\partial c_i} &= \frac{1}{1 - \left(\frac{p^*-c}{p^*}\right)^2 (p^* \varepsilon'(p^*) - \varepsilon(p^*))} \\
&= \frac{1}{1 - \left(\frac{p^*-c}{p^*}\right)^2 [p^*(\varepsilon'(p^*) + 2g'(0)) - (\varepsilon(p^*) + 2p^* g'(0))]} \\
&= \frac{1}{1 - \left(\frac{p^*-c}{p^*}\right)^2 (p^* \bar{\varepsilon}'(p^*) - \bar{\varepsilon}(p^*))}. \qquad (4.13)
\end{aligned}$$

Notice that $\frac{\partial p_i^*}{\partial c_i} + \frac{\partial p_j^*}{\partial c_i}$ stands for the change in the average price of the two firms due to a unitary increase in the average cost (locally around $c_A = c_B$). Equation (4.13) tells us that the average price of the firms follows the corresponding average cost as a monopolist price follows its cost, just performing the custom substitution from $\varepsilon(\cdot)$ to $\bar{\varepsilon}(\cdot)$. In other words, the costs of the individual firms are irrelevant for the average market price determination once the average cost is known.

Recall error amplitude and price dispersion

Price dispersion, defined here as the price difference, is driven in this model by cost dispersion. Thus to analyze it, it should be checked how the price difference depends on one of the costs. This is given by

$$\frac{\partial p_i^*}{\partial c_i} - \frac{\partial p_j^*}{\partial c_i} = \frac{1}{1 - \left(\frac{p^*-c}{p^*}\right)^2 (p^* \varepsilon'(p^*) - \varepsilon(p^*)) + 8(p^* - c)^2 g'^2(0)}$$

as shown in Section 4.8.2.

The demand and the recall error effects must now be disentangled. If demand falls sharply at some given price, firms will not choose to price their goods above (or largely above) that level, no matter what the cost structure and recall error level are. The best way to study the recall error effect is to fix elasticity of demand at p^*, and this is so for a strong reason. The equilibrium price in the symmetric case is a function of elasticity (not of demand or the derivative of demand alone). Therefore fixing elasticity does not change the symmetric equilibrium outcome as it would happen if $D(p)$ or $D'(p)$ were fixed, even if just locally.

Proposition 4.4 *Price dispersion, defined here as the price difference, is ceteris paribus an increasing function of the recall error standard deviation.*

Proof
Price dispersion with $\varepsilon(p) = \varepsilon(p^*) \equiv \varepsilon^*$ for any p is

$$\frac{\partial p_i^*}{\partial c_i} - \frac{\partial p_j^*}{\partial c_i} = \frac{1}{1 + \left(\frac{p^*-c}{p^*}\right)^2 \varepsilon^* + 8g'^2(0)(p^* - c)^2}. \qquad (4.14)$$

In Section 4.8.3 it is shown that the whole denominator is decreasing in σ meaning that $\frac{\partial p_i^*}{\partial c_i} - \frac{\partial p_j^*}{\partial c_i}$ at $c = c_A = c_B$ (and therefore the price dispersion when the firms' costs are similar) is locally an increasing function of the recall error.■

Note that as consumers become more inattentive, it could be thought that the diminishing competitive pressure on the firms would lead to lower price dispersion, as it leads to higher markups, because consumers are less responsive to price gaps. It turns out that it is optimal for the low cost firm to increase this gap.

Welfare analysis

Price recall errors can be regarded as a weakening of competition, given that the incentives of a fierce price competition are smaller. There are however some counterintuitive results with important welfare and policy implications. To have some insights a comparison between the extreme cases, perfect price recall and random shopping, is made. In the first case the basic textbook equilibrium arises. The low cost firm will charge the lowest of the following two, the price it would charge as a monopolist and the cost of the other firm. The high cost firm simply takes price equal to cost. At the other extreme, the two firms are monopolists on half of the market, so they simply both charge their monopolist prices.

Profits of the high cost firm go up (not necessarily monotonically) from zero to half of the monopolist profits, from one extreme to the other. If the low cost firm charges his monopolist price in the competitive situation, i.e. with perfect price recall, then its profits decrease (not necessarily monotonically).

On the aggregate level a surprising result occurs when the cost gap is big enough.

Lemma 4.4 *Firms are worse off, on the aggregate level, when facing higher variances of the recall errors for sufficiently large cost gaps and low initial error variances.*

Proof
First it is shown that a degenerate distribution can be taken without loss of generality for the recall errors ϵ_i, $i = A, B$, so that the setting is the basic Bertrand model. From equation (4.11) it is concluded that the full support case can be arbitrarily close to this solution, because the last term goes to zero as the recall error standard deviation goes to zero, given that $g'(\cdot) \to 0$, while the other terms are bounded. Furthermore, if demand is bounded then profits also converge to the degenerate case. Considering the degenerate case is thus a simplification.

Take any c_A, $D(p)$ and corresponding $\varepsilon(p)$ such that the monopolist problem of firm A has a solution, call it p_A^M, yielding strictly positive profits. Assume that $c_B > p_A^M$. In equilibrium the market will be completely covered by firm A setting p_A^M, getting profits $\Pi_A^M = p_A^M D(p_A^M)$ and $\Pi_B^M = 0$.

Now consider a new recall error distribution with full support. It follows directly from the definition of monopolist price that the new equilibrium price of firm A, p_A^*, satisfies $p_A^* D(p_A^*) \leq p_A^M D(p_A^M)$. Moreover, because $c_B > c_A$ it must also hold for firm B that $p_B^* D(p_B^*) < p_A^M D(p_A^M)$. The aggregate profits are now given by $\Pi^{\text{agg}} \equiv p_A^* D(p_A^*) g(p_B^* - p_A^*) + p_B^* D(p_B^*) \left(1 - g(p_B^* - p_A^*)\right)$ satisfying

$$\Pi^{\text{agg}} < p_A^M D(p_A^M) \left[g(p_B^* - p_A^*) + 1 - g(p_B^* - p_A^*) \right] = \Pi_A^M.$$

This example is sufficient to prove the lemma.■

While the proof mentions an extreme case, it is clear that worse price recall can decrease the total profits in a broad set of parameters. Larger recall errors push more consumers to the firm with higher price, that is the one with higher cost, whose profits are typically lower.

Lemma 4.5 *Worse price recall decreases the welfare of both the firms and the consumers, for sufficiently large cost gaps and low initial error variances..*

Proof
Notice that the welfare decrease for the consumers alone, follows directly from the symmetric cost case. To prove the lemma it must be shown that it happens in cases where the aggregate profits also decrease, which is not the case for symmetric costs. Recall the cases from the previous proof. As the standard deviation increases, the share of consumers buying at firm B can be arbitrarily close to one half, given that an increasing number of consumers will recall the price of the high cost firm as lower. This implies further that p_A^* can be arbitrarily close to p_A^M. Taking a linear demand function for instance, it becomes clear that the extra consumer surplus that the consumers still at firm A get due to existence of two firms in the market, i.e. firm A is not a monopolist, is smaller than the welfare cost of the other half, that switched from monopolist A to duopolist B. ∎

This result is striking given the previous qualification of price recall errors as a cause of weaker competition. In fact, if consumers shop (close to) randomly then goods are being bought at high price which does not imply higher profits since consumption is being shifted from a low to a high cost firm. Whilst this observation is obvious, the counterintuitive nature of the above result is simply a consequence of it. In further extensions of this model, where the standard deviation of the price recall is somehow manipulated by the firms, it may happen that firms choose strategies that make them worse off. Figure

Furthermore, for fixed price recall error amplitude, one may wonder if competition is itself welfare decreasing. In other words, may a duopoly be something that should be avoided in comparison to a monopoly? This is indeed the case, again because consumers do not make optimal choices. If the possibility of buying at higher prices is somehow not there, in some cases the welfare is higher with a monopoly in comparison to the duopoly.

Proposition 4.5 *In a market where consumers do not perfectly recall the prices, protecting a monopoly from entrant firms is optimal from the consumer welfare point of view, as well as from the social welfare point of view, for sufficiently large cost gaps and low initial error variances.*

Proof
Straightforward conclusion from the proof of Lemma 4.5, just considering the degenerate distribution in the proof simply as the monopoly, which is then compared with duopoly. ∎

Notice that the proofs of the above two lemmas use the extreme case of recall errors with degenerate distribution, for simplicity. Given the continuity of the equilibrium prices (therefore profits and surpluses) as a function of the error standard deviation, the results apply for other non-extreme parameter choices. The statement in Proposition 4.5 is not related to parameter choices but to different market structure under the same conditions, and should not be confused with the lemmas. The proposition compares monopoly with duopoly, where the monopolist case happens to be equivalent to the duopoly with degenerate errors.

Some examples

Take again a linear elasticity, $\varepsilon(p) = -(a + bp)$, and the Gumbel distribution for the recall error. In the end of Section 4.8.1 in the Appendix, the following formulas are worked out (again locally at $c_A = c_B$):

$$\frac{\partial p_i^*}{\partial c_i} = \frac{1}{1 - a(\frac{p-c}{p})^2} \frac{1 - a(\frac{p-c}{p})^2 + \left[\frac{\tau}{2}(p-c)\right]^2}{1 - a(\frac{p-c}{p})^2 + 2\left[\frac{\tau}{2}(p-c)\right]^2}$$

and

$$\frac{\partial p_j^*}{\partial c_i} = \frac{1}{\left[1 - a(\frac{p-c}{p})^2\right]^2 (\frac{\tau}{2}(p-c))^{-2} + 2\left[1 - a(\frac{p-c}{p})^2\right]}.$$

Only the two limiting cases, $c_A = c_B = 0$ and $c_A = c_B \to \infty$ will be discussed because other cases yield intermediate results.

For $c_A = c_B = 0$ the price is driven by the recall error of consumers, and is therefore of the same magnitude. Departing from $c_A = c_B = 0$ the equilibrium prices react in the following way:

$$\frac{\partial p_i^*}{\partial c_i} = \frac{1}{1-a} - \frac{1}{\left[2b/\tau + 1\right]^2 + 2\left[1-a\right]}$$

and

$$\frac{\partial p_i^*}{\partial c_j} = \frac{1}{\left[2b/\tau + 1\right]^2 + 2\left[1-a\right]},$$

which are positive because $0 < 1 - a$. For $c_A = c_B \to \infty$ the other extreme case occurs:

$$\frac{\partial p_i^*}{\partial c_i} = 1 - \frac{1}{\left[2b/\tau + 1\right]^2 + 2}$$

and

$$\frac{\partial p_j^*}{\partial c_i} = \frac{1}{\left[2b/\tau + 1\right]^2 + 2},$$

which are also positive. Notice that as in Section 4.3.1, as costs (and therefore equilibrium prices) increase, the level of the demand elasticity, which is defined by the parameter a, becomes irrelevant (it enters the first order conditions as a/p_i). Only the slope of the elasticity b is maintained. It is thus easy to see that the derivatives at $c \to \infty$ are smaller than at $c = 0$.

In equilibrium the prices will rise with an increase of any of the two costs. Firm's pricing behavior lies between the perfect competition lower bound and the monopolist upper bound, depending on the size of the recall error (which

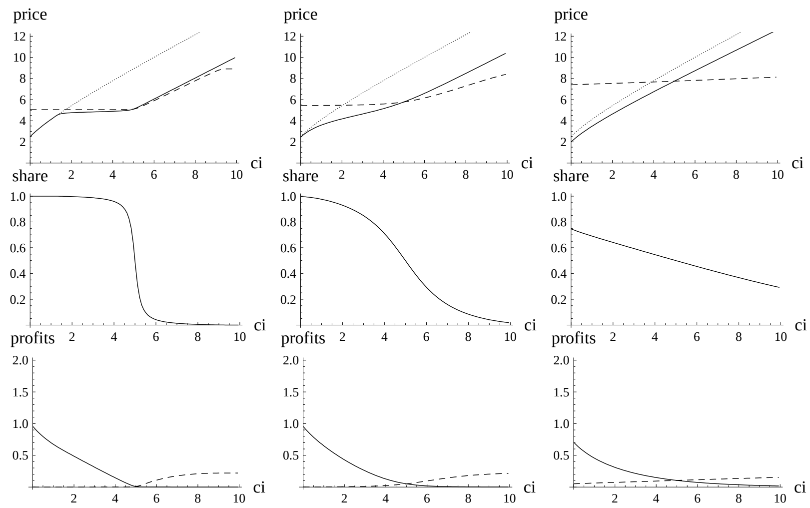

Figure 4.1: Equilibrium prices, market share of firm i and profits depending on the cost of firm i, c_i. The straight lines denote firm i, the dashed lines firm j and the dotted the monopolist with c_i. Chosen parameters $c_j = 5$, $D(p) = e^{-\frac{1}{2}\ln p - \frac{p}{5}}$ so that $\varepsilon(p) = -\frac{1}{2} - \frac{p}{5}$, Gumbel distribution. Left column: $\tau = 20$, $\text{var}(\epsilon) = \frac{\pi^2}{2400}$, center column: $\tau = 2$, $\text{var}(\epsilon) = \frac{\pi^2}{24}$, right column: $\tau = 0.2$, $\text{var}(\epsilon) = \frac{25\pi^2}{6}$.

fosters towards the upper bound) and the relative cost advantage (also towards the upper bound).

Figure 4.1 shows some examples how incorrect price recall affects the market, in the case of different costs of firms. The variance of the price error increases from left to right. In the first panel consumers are able to recall the price with very good precision, so that the outcome is close to the classic setting. A firm sets the monopolist price for low production cost, switches to a price slightly below the other firm's price for low intermediate cost, and follows his own cost for high costs. Its market share is almost 100% for low own cost and almost 0% for costs higher than the other firm's cost. In the last column the consumers are almost completely unaware of the chosen prices so that they almost shop randomly, as can be seen in the slowly decreasing market share of firm i on the second row. Because consumers almost shop randomly, firms set a price close to the monopolist price given their cost.

Figure 4.2 illustrates Lemma 4.4, where total profits decrease as the recall errors increase.

4.4 Three or more firms

In the Bertrand pricing model with no product differentiation, the number of firms is irrelevant as long as it is more than one. In this extension with more than two firms the equilibrium will present however an intuitive outcome: the price will decrease as the number of firms increases, though only slightly. Moreover this result does not depend on the number of consumers in the market, so it is

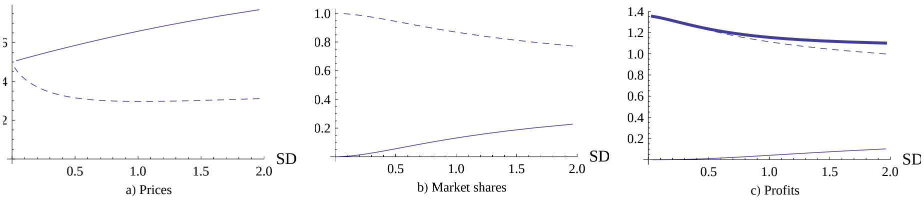

Figure 4.2: Aggregate profit decrease as a function of the standard deviation (SD) of the recall error, with $D(p) = e^{-\frac{1}{2}\ln p - \frac{p}{10}}$ so that $\varepsilon(p) = -\frac{1}{2} - \frac{p}{10}$, and Gumbel distribution. Firm i with $c_i = 0$ and straight line, firm j with $c_j = 5$ and dashed line. Total profits represented by the thick line.

entirely driven by the competition among the firms. To see this notice that the number of consumers μ just appears as a multiplicative constant in the profit function. Thus its maximization is independent of the value of μ.

Let there be n firms, $n \geq 3$. Firm i has unitary production cost c_i and charges price p_i. $D(p)$ is the demand from each consumer at price p, where the mass of consumers is taken to be 1. The number of consumers heading to firm $i = 1, \cdots, n$ is given by

$$\begin{aligned} P(p_i' < p_j', \forall j \neq i) &= P(p_i + \epsilon_i < p_j + \epsilon_j, \forall j \neq i) \\ &= \int_{-\infty}^{\infty} f(y) \prod_{j \neq i} F(y + p_j - p_i) dy \\ &= G(p_1 - p_i, \ldots, p_{i-1} - p_i, p_{i+1} - p_i \ldots, p_n - p_i), \end{aligned}$$

where for $x \in \mathbb{R}^{n-1}$

$$G(x) \equiv \int_{-\infty}^{\infty} f(y) \prod_k F(y + x_k) dy.$$

The profit of firm $i = 1, \ldots, n$ is

$$\Pi_i(p_1, \cdots, p_n) = (p_i - c_i) G(x_{-i}) D(p_i), \tag{4.15}$$

where $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = p_{-i} - \mathbf{e}^{n-1} p_i$ with $\mathbf{e}^{n-1}$ being the $(n-1)$-vector of ones, and the first order condition for its maximization is

$$\begin{aligned} \frac{\partial \Pi_i(p_1, \cdots, p_n)}{\partial p_i} &= 0 \\ \Leftrightarrow \frac{1}{p_i - c_i} + \frac{D'(p_i)}{D(p_i)} + \frac{\partial}{\partial p_i} \ln G(x_{-i}) &= 0. \end{aligned} \tag{4.16}$$

Assumption 4.1 is now generalized as follows.

Assumption 4.3 *The random shocks $\epsilon_1, \ldots, \epsilon_n$ are iid with mean 0 and full support on the real line with probability density function $f(\cdot)$ and cumulative* *distribution function $F(\cdot)$. Moreover the distribution of $\epsilon_1, \ldots, \epsilon_n$ is such that, for each $i = 1, \ldots, n$ and p, $\frac{\partial}{\partial p_i} \ln G(p_{-i} - \mathbf{e}^{n-1} p_i)$ is non-decreasing in p_i, i.e. $\sum_{j \neq i}^{n} \frac{\partial^2}{\partial x_j^2} \ln G(x_{-i}) \leq 0$.*

Assumption 4.3 assures the existence of the equilibrium, whose proof is easily obtained from the $n = 2$ case. For uniqueness it must be assured that the best response hyper-surfaces only cross once. Again this is guaranteed if $\frac{\partial p_i^*(p_{-i})}{\partial p_j} < 1$ for any $p_{-i} \geq c_{-i}$, i and $j \neq i$.

Consider a Δp_j increase in p_j. Notice that an equal increase in p_i offsets an increase in x_j, and furthermore it decreases all other x_k with $k \neq i, j$, meaning that $\frac{\partial}{\partial p_i} \ln G(x_{-i})$ is increased since $\sum_{j \neq i}^{n} \frac{\partial^2}{\partial x_j^2} \ln G(x_{-i}) \leq 0$. Following the same reasoning steps as in the $n = 2$ case, it is concluded that the change in the best response to Δp_j is some Δp_i satisfying $\Delta p_i < \Delta p_j$ which proves the uniqueness of equilibrium.

Lemma 4.6 *The imperfect price recall model for multiple firms, has a unique equilibrium if Assumptions 4.2 and 4.3 are satisfied. This equilibrium is implicitly defined by equation (4.16).*

4.4.1 Symmetric costs

For $c_i = c$ for all $i = 1, \cdots, n$ the equilibrium is straightforward.

Lemma 4.7 *The symmetric price competition model with n firms has an equilibrium with the price p^* defined by*

$$\frac{p^* - c}{p^*} = \frac{1}{-\varepsilon(p^*) + p^* \sum_{j \neq i} \frac{\partial}{\partial x_j} \ln G(0)}.$$

The above result follows directly from Lemma 4.6 and equation (4.16), where $\frac{\partial}{\partial p_i} G(x_{-i}) = -\sum_{j \neq i} \frac{\partial}{\partial x_j} G(x_{-i})$ was used, with $x_j = p_j - p_i$, $j \neq i$.

For the Gumbel distribution it is known that

$$G(x_{-i}) = \frac{1}{1 + \sum_{j \neq i} e^{-\tau x_j}},$$

so that

$$\begin{aligned}
\frac{\partial}{\partial p_i} \ln G(x_{-i}) &= -\sum_{j \neq i} \frac{\partial}{\partial x_j} \ln G(x_{-i}) \\
&= \sum_{j \neq i} \frac{\partial}{\partial x_j} \ln \left(1 + \sum_{k \neq i} e^{-\tau x_k} \right) \\
&= \sum_{j \neq i} \frac{-\tau e^{-\tau x_j}}{1 + \sum_{k \neq i} e^{-\tau x_k}} \\
&= -\tau \left[1 - \frac{1}{1 + \sum_{j \neq i} e^{-\tau x_j}} \right],
\end{aligned}$$

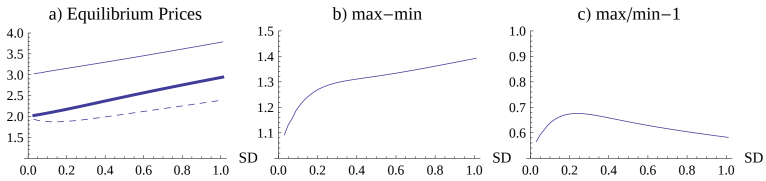

Figure 4.3: Equilibrium prices and two measures of price dispersion for $(c_1, c_2, c_3) = (1, 2, 3)$ depending on the standard deviation of the Gumbel distributed recall error. Firm 1, 2 and 3 are represented by the dashed, the thick and the thin line. Demand is $D(p) = 10p^{-\frac{1}{2}}$ so that its price elasticity is $\varepsilon(p) = -\frac{1}{2}$.

which is increasing in p_i. In the symmetric equilibrium

$$\sum_{j \neq i} \frac{\partial}{\partial x_j} \ln G(x_{-i})|_{x_{-i}=0} = \tau \frac{n-1}{n}$$

and the equilibrium price is defined by

$$\frac{p^* - c}{p^*} = \frac{1}{-\varepsilon(p^*) + \tau \frac{n-1}{n} p^*}.$$

The competition pressure due to an increase in the number of firms has a small impact. Observe that an increase in n from 2 to ∞ is equivalent, in terms of markup setting, to just halving the standard deviation of the recall error (which is proportional to $\frac{1}{\tau}$). As a corollary, note that the Lerner index does not go to zero as $n \to \infty$. Gabaix and Laibson (2004) also conclude that the number of firms has a low impact in decreasing the charged markup, while in Stahl (1989) it even brings the equilibrium price closer to the monopoly price.

4.4.2 Asymmetric costs

Due to analytical complexity, the results in the $n = 2$ case cannot be extended. It is however possible to obtain numerically the equilibrium prices for a given cost structure. The figures in the Appendix show the equilibrium prices and different price dispersion measures for cases with $n = 3$, Gumbel distributed shocks and $\varepsilon(p) = -\frac{1}{2}$ for any $p \geq 0$. Two price dispersion measures are presented, namely the difference between maximum and minimum price as well as its ratio. The standard deviation of prices seems to follow very closely the pattern of Max-Min in all cases that were checked, whereas the standard deviation/mean ratio follows Max/Min.

In the first case, depicted in Figure 4.3, the firms' costs are given by the cost vector $(c_1, c_2, c_3) = (1, 2, 3)$. Except for a minor exception (see discussion below in the $(c_1, c_2, c_3) = (1, 2, 2)$ case), all prices grow with the price recall error. The behavior of price dispersion depends on the chosen measure.

More interesting is the comparison between $(c_1, c_2, c_3) = (1, 1, 2)$ and $(1, 2, 2)$ presented in Figures 4.4 and 4.5. They differ significantly, while one might have thought the opposite given that both cases have the same cost values, the only difference being the number of firms having the different costs. If there are

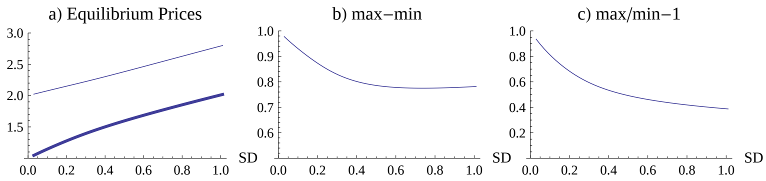

Figure 4.4: Equilibrium prices and two measures of price dispersion for $(c_1, c_2, c_3) = (1, 1, 2)$ depending on the standard deviation of the Gumbel distributed recall error. Firm 1 and 2 represented by the straight line, firm 3 by the thick line. Demand is $D(p) = 10p^{-\frac{1}{2}}$ so that its price elasticity is $\varepsilon(p) = -\frac{1}{2}$.

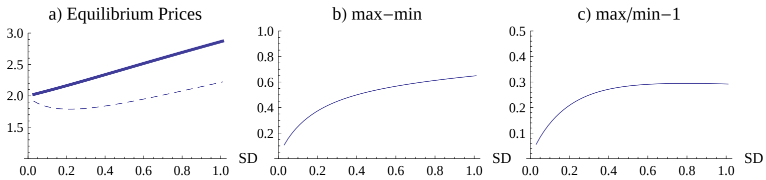

Figure 4.5: Equilibrium prices and two measures of price dispersion for $(c_1, c_2, c_3) = (1, 2, 2)$ depending on the standard deviation of the Gumbel distributed recall error. Firm 1 represented by the dashed line, firms 2 and 3 by the thick line. Demand is $D(p) = 10p^{-\frac{1}{2}}$ so that its price elasticity is $\varepsilon(p) = -\frac{1}{2}$.

two low-cost firms they compete among them setting the competitive price, i.e. price marginally above cost, for small recall errors. As errors grow larger, their market share becomes less dependent on the price and they opt for a price close to that of the high-cost firm. The price dispersion is therefore decreasing in the errors standard deviation.

If there is only one low-cost firm, it is sufficient for it to charge a price slightly below the high cost. As higher errors are considered there are two effects that become clear. Initially the need to differentiate its price from that of the high-cost firms is predominant, so that it actually chooses a lower price. But then the effect of having a market share which is less dependent on the price starts to act and the low-cost firm raises the price again. This explains the steepest price dispersion increase among the three considered cases. While a deeper analytical investigation is beyond the scope of this chapter, it becomes clear from the above examples that the cost structure of the firms in the market have a strong influence on the equilibrium prices and price dispersion.

4.5 Price dependent error variance

A shortcoming of the previous sections is the arbitrariness of the shock variance. In this section it is assumed that the recall error is in some way related to the value of the good. As a simple example one may think of the uncertain price of a cup of coffee in a bar compared to the uncertain price of an expensive computer. While in the first case the consumers may have a ±20 cents 'confidence interval', in the second case this might be around ±100 euro[4.13].

4.5.1 Exogenous variance

Setting the shock proportional to the equilibrium price would lead to an implicit definition problem. The cost of the good is however a good proxy for it. In this section it is assumed that firms face cost c to produce the good and the price observation of the consumers suffers a shock whose standard deviation is an increasing function of c.

Derivation is very similar to above and Lemmas 4.1, 4.2 and 4.3 can be applied here. Profits are still given by $\Pi_i(p_i, p_j) = p_i g(p_j - p_i)D(p_i)$ for firm i, but the standard deviation is now multiplied by c. In other words $g'(0)$ becomes $\frac{g_0'(0)}{c}$ where g_0 is just the benchmark $g(\cdot)$ function for $c = 1$. Equation (4.8) becomes now

$$\frac{p}{p-c} = -\varepsilon(p) + \frac{2}{c}g_0'(0)p.$$

Introducing this cost dependence modifies the competition pressure term in the equilibrium Lerner index. Higher costs mean lower attention (in absolute terms) paid by consumers. If constant elasticity is assumed, to abstract from demand driven changes, the above formula can be written as

$$\frac{1}{1-c/p} = -\varepsilon + \frac{2p}{c}g_0'(0),$$

which can be regarded as an equation on $\frac{p}{c}$. This means that in equilibrium the ratio $\frac{p}{c}$ is the same for any c, meaning that the markup is proportional to cost.

4.5.2 Endogenous shock variance

While the previous section has an intuitive outcome regarding the dependence of markup on the cost of a good, it would be more realistic to allow consumers to choose an effort according to the amount of money involved. For instance Sorensen (2001), while related to search costs, states that consumers put a higher search effort for pharmaceutical products that they buy more often. Here the standard deviation of the recall error will be associated to the effort they will be putting in remembering the exact prices, which is denoted by Σ. It is assumed that for a given effort level Σ the recall error will be proportional to c, in other words the effort is not related to recalling the last digits of the price but the first digits. Thus the standard deviation of a given distribution $g_0(\cdot)$ is here multiplied by Σc (in the above section $\Sigma = 1$). The consumer faces here a trade-off between mental effort and overspending.

As proxy for the costs of overspending the following is used, $\beta\Sigma c$, that is the standard deviation of the recall error chosen by the consumer, Σc, times a constant, β, related to the error distribution.[4.14] The idea is that higher recall

[4.13] See Drèze, Vanhuele, and Laurent (2006) for empirical evidence on the harder memorability of lengthier prices.

[4.14] This choice can be motivated by a more complex modeling as follows.

Let the consumer be concerned about the worst expected loss that may happen for a given standard deviation. In other words, for a given price dispersion $x \equiv \max p_i - \min p_i$ there is an associated expected loss (when the consumer mistakenly chooses the firm with the highest price) and the consumer will take into account the highest value of the expected losses across errors imply higher expected losses from choosing the firm with the highest price.

As stated before consumers will face a mental effort cost of reducing Σ, $h(\Sigma)$. It is assumed that decreasing the recall error - lower Σ - is increasingly costly, $h'(\cdot) < 0, h''(\cdot) > 0$, so that $\frac{\partial h(\Sigma)}{\partial(-\Sigma)} > 0$ and $\frac{\partial^2 h(\Sigma)}{\partial(-\Sigma)^2} > 0$. Consumers will thus compare the benefit $\beta\Sigma c$ and the cost $h(\Sigma)$ of making a memory effort Σ. Equalizing marginal benefit βc and marginal cost $h'(\Sigma)$ yields the optimal recall error of consumers.

Take for instance $h(\Sigma) = \frac{1}{\Sigma}$ as the effort cost. Then consumers minimize the losses from the recall errors given by $\beta\Sigma c + h(\Sigma)$. The minimum is attained at

$$\Sigma = \sqrt{\frac{1}{\beta c}}$$

or $\Sigma c = \sqrt{\frac{c}{\beta}}$. Recall that while the consumers choose Σ the shock standard deviation will be Σc. The idea behind this result is that consumers do make bigger mistakes when recalling a price of a more expensive good, but this mistake is only proportional to the root of it. In other words, when buying the outfit consumers exert a bigger effort in price comparison because the stakes are higher.

Once again Lemmas 4.1, 4.2 and 4.3 regarding the equilibrium in the firms' price game are applicable here, because firms take the consumers' effort as given. The equilibrium price will be characterized by

$$\frac{1}{p^* - c} = -\frac{\varepsilon(p^*)}{p^*} + 2\sqrt{\frac{c}{\beta}} g_0'(0).$$

For costs and thus prices close to zero the first term on the right hand side overweights the second, so that the equilibrium price is close to the monopoly price. But for higher costs (and assuming that $-\varepsilon(p)$ increases less than proportionally to p for high prices) the markup will be close to $\sqrt{\frac{\beta}{c}} \frac{1}{2g_0'(0)}$. This case lies between the basic case where markup is constant as $c \to \infty$ and the exogenous recall variance case where it is proportional to c.

all x. While this choice is not very intuitive, it must be noticed that a priori the consumer is not aware of the price dispersion distribution x, so she cannot calculate the expected loss.

Consider the case with $p_A < p_B$. The probability of shopping at B by mistake is $1 - g(x)$. Hence the expected loss due to false recall is $x(1 - g(x))$. The consumer is not aware of the real prices and therefore not aware of x, so that she can only evaluate what the maximum expected loss is for a given Σ. Call it $\Delta(\Sigma)$:

$$\Delta(\Sigma) \equiv \max_{x \geq 0} x \left(1 - g_\Sigma(x)\right),$$

where $g_\Sigma(x)$ is the probability of choosing the lowest price given effort Σ. While Assumption 4.1 does not guarantee the uniqueness of this maximum, it exists and is unique for many distributions (like Normal and Gumbel). Now, as seen in the discussion of equation (4.4) the standard deviation enters $g(\cdot)$ as a number by which x is divided. So for a given recall error distribution, the above maximization has the same solution if solved for $\frac{x}{\Sigma c}$, a solution which is independent of Σ. In other words the maximum expected loss $\Delta(\Sigma)$ is just (linearly) proportional to Σc, which motivates the choice for the overspending proxy. The proportionality constant is denoted by β.

4.6 Related models

In this section the price recall model is compared with the price competition model with utility uncertainty and the classic horizontal differentiation model.

4.6.1 Comparison to utility uncertainty

The literature related to this chapter takes an approach close to the quantal response equilibrium concept, as in Gabaix and Laibson (2004) and Gabaix, Laibson, and Li (2005). That is, consumers[4.15] are not able to compare two utility levels, corresponding to two alternatives, perfectly. Mathematically, instead of comparing U_1 and U_2, they compare the utility plus a shock, $U_1 + \epsilon_1$ and $U_2 + \epsilon_2$.

For the simple case of fixed demand, where consumers are willing to buy one and only one unit of the good, this is equivalent to the present model where the random component is added to the price. In a utility shock setting, consumers compare $(u - p_1) + \epsilon_1$ with $(u - p_2) + \epsilon_2$, where u is the monetary utility of having the good. This is equivalent to the comparison of $u - (p_1 + \epsilon_1')$ with $u - (p_2 + \epsilon_2')$ where $\epsilon_i' = -\epsilon_i$, $i = 1, 2$.

But once the assumption of fixed demand is relaxed, the models yield different predictions.

Let $V(p)$ be the total indirect utility function of a consumer from buying the good at price p, that is the indirect utility minus the cost in utility units. The consumers when facing p_A and p_B compare $V(p_A) + \epsilon_A$ with $V(p_B) + \epsilon_B$, where ϵ_A and ϵ_B are i.i.d. random errors. Define the probability of a given consumer to consider firm i as having the best option as $q(V(p_j) - V(p_i))$ with $j \neq i$, where $q(\cdot) = 1 - g(\cdot)$ as now consumers opt for the highest $V(\cdot)$ instead of the lowest p. Now the procedure to find the equilibrium price follows closely that of previous sections[4.16]. Equation (4.8) becomes now

$$\frac{p}{p - c} = -\varepsilon(p) + 2q'(0)V'(p)p,$$

where $g'(0)$ became $q'(0)V'(p)$ since this term follows from $-\frac{\partial}{\partial p_i} q(V(p_j) - V(p_i))$ at $p_i = p_j = p$.

Consider the simple case of constant elasticity of demand $-1 < \varepsilon(p) < 0$ for a comparison of the implications of the two approaches. Suppose demand is given by $D(p) = D_0 p^{-a}$ so that $\varepsilon(p) = -a$ for any $p \geq 0$, where $D_0 > 0$ is some constant. Assuming that the expenditure on the product on focus is small compared to the total income of the consumer, one can take the marginal utility of income λ as fixed when setting the consumer utility function for this purchasing behavior. Thus the (separable) utility of the good that yields the desired demand function is simply $U(x) = -U_0 x^{-\frac{1-a}{a}}$ with $U_0 > 0$ and x being the amount of the consumed good. This leads to the above mentioned demand

[4.15] This bounded rationality argument can also be applied to the firms, as Baye and Morgan (2004) do. The rationale for this option is however not so clear. The possible gains and losses at stake for the firms are clearly higher, as is their availability to compute the problem lengthily through.

[4.16] The assumption needed to guarantee existence and uniqueness of equilibrium differs. Here it is sufficient that $\frac{\partial}{\partial p_i} \ln q(V(p_j) - V(p_i))$ is non-increasing in p_i for any $p_j \geq 0$.

function with $D_0 = \left(\frac{1-a}{a}\frac{U_0}{\lambda}\right)^a$. The total indirect utility function as a function of price is

$$\begin{aligned} V(p) &= -U_0 D(p)^{-\frac{1-a}{a}} - \lambda D(p)p \\ &= -\left(\frac{U_0}{a}\right)^a \left(\frac{\lambda}{1-a}\right)^{1-a} p^{1-a}. \end{aligned}$$

If ϵ_i are Gumbel i.i.d. then a given consumer chooses firm A with probability $q(V(p_B) - V(p_A))$ defined by

$$q(V(p_B) - V(p_A)) = \frac{1}{1 + e^{\tau(V(p_B) - V(p_A))}} = \frac{1}{1 + e^{-\tau\left(\frac{U_0}{a}\right)^a \left(\frac{\lambda}{1-a}\right)^{1-a} (p_B^{1-a} - p_A^{1-a})}}.$$

The symmetric equilibrium price p^* is given by

$$\frac{p^* - c}{p^*} = \frac{1}{a + \frac{\tau}{2}\left(\frac{1-a}{a}U_0\right)^a \lambda^{1-a} {p^*}^{1-a}},$$

where $q'(0) = -\frac{\tau}{4}$ and $V'(p) = \left(\frac{1-a}{a}U_0\right)^a \lambda^{1-a} p^{*-a}$ are used. It is hard to compare the two models, because the utility uncertainty has more degrees of freedom. But two observations can be made that distinguish them, both related to the change from $g'(0)$ to $q'(0)V'(p)$. First, the equilibrium price equation now contains U_0 (or D_0) meaning that products with the same elasticity of demand may have different equilibrium prices. In the price recall model, both demand and market share depend solely on price so the firm incentives only depend on price. Now the market share, related to the probability of correctly choosing the good according to its utility, depends on U_0 (or D_0). Consumers make less mistakes for products with higher U_0 (that is with a higher demand parameter D_0) so the competition pressure will be stronger and the prices lower.

Second, the term contains now ${p^*}^{1-a}$ instead of p^* implying a non-linear response to price. Recall that the $g'(0)$ or $q'(0)V'(p)$ term can be interpreted as an increase in the (absolute value) of the elasticity of demand when compared to the monopolistic price setting. If a is close to 1, this increase is almost independent of p.

Therefore the bounded rationality concept not only offers a more intuitive and tractable model, but its predictions depart from the common model in the literature. Assuming a shock in the utility, instead of in the price, changes the new term showing up in the equilibrium Lerner index of the firms. The price in the utility shock model depends on many parameters, namely on the demand level D_0, the functional form of the indirect utility function, the price elasticity of demand and the error distribution, whereas only the last two parameters are to be found on the price shock model.

4.6.2 Price recall as a horizontal differentiation model

As argued in Subsection 4.3.1 the price recall model also resembles the horizontal differentiation model in the basic cases with fixed unitary demand. This equivalence is however not true for distributions with full support. If the firms are placed in the "extremes" of the infinite horizontal line, then for any strictly positive distance cost, both total prices (good plus distance cost) will be infinite for all consumers. The full support cases can however be approximated by distributions with bounded support, assuring that the latter yields an interior solution. The problem with the approximation is that in the latter case a firm can have the whole market if it chooses a sufficiently low price. Again this can be avoided by assuming high distance costs and equivalently high utility of the good to guarantee that all the market is covered. Assigning a high utility may however distort further applications of the model, for instance in welfare analysis.

In the price recall model it is natural to assume that the difference between the price recall errors follows some symmetric probability distribution, but it is intuitively not so clear why the consumers in the horizontal differentiation framework should be densely concentrated in the center of the horizontal line, a feature which is necessary for the formal equivalence. But the models diverge more once one moves away from the basic case and considers elastic demand. The reason is that in the horizontal model, the distance decreases the willingness to buy the goods, either because the total price is higher (transportation costs interpretation) or the value of the good is lower (preferred variety interpretation). Put simply, in the horizontal differentiation model the distance represents a worse alternative. In the price recall model the distance only reflects a lower probability of buying the good. In the former the consumers choose to buy less, in the latter demand remains constant. Mathematically, the demand in the horizontal differentiation model is given by the integral of different demands along the line, here it is simply demand as a function of price times the market share. In equation (4.1') the market share term remains unchanged in horizontal differentiation models, but the demand term becomes an analytically complex term[4.17]. This also makes the bounded support approximation problem more salient. Assuming a value of the good which is sufficiently high to guarantee a covered market and an interior solution, is not compatible with a low demand. Put differently, in the limit case where recall error variance goes to infinity, the market in the price recall model is split between the firms irrespective of their prices, acting both as a monopolist in their half. The demand they get is not affected. That is not so for horizontal differentiation. A robust market split only occurs for distance costs increasing towards infinite, but that would have an impact on actual demand.

The distinctness is also more evident when extensions are considered. In Section 4.5 the recall error variance is endogenized, being it a choice of the individual consumers (could be a choice of firms as well). In the horizontal differentiation model this would be equivalent to having the individual consumer choosing the distribution (or the distance cost) of the whole market. Risk aversion of firms' managers would also damage the analogy, because the focus here has been expected (therefore random) market share. Moreover, in horizontal differentiation the positioning of firms can conceptually be a decision of the agents, but here that cannot be the case.

4.7 Conclusions

In this chapter a model of bounded rationality of the demand side in price competition settings is presented. Assuming imprecise recall of the price by

[4.17]The Hotelling model with elastic demand is only mathematically tractable for very specific parameter functions. See for instance Puu (2002).

the consumers for the classic Bertrand model, some of its paradoxes are solved. Competitive firms do charge above the competitive equilibrium price, having therefore a positive profit. There is no benefit (in equilibrium) in lowering the price, for some consumers will still recall the other firm's price as lower. This has obvious welfare damaging effects, which can be obtained given the proposed characterization of the equilibrium which is equivalent to that of a monopoly price model. Moreover it was shown that in some cases with asymmetric costs both the firms and the consumers incur welfare costs compared to the classic case.

Firms with different costs do coexist in the market. Price dispersion does persist in competitive frameworks. In the simple setting, price dispersion increases with the recall errors. If the effort choice is endogenized, consumers put more effort when purchasing a valuable product.

Even though the presented model is static, it can be interpreted as a stage game of an infinitely repeated game where a Nash Equilibrium is played in every stage. The intuition is that consumers do not actually seek information before every purchase, but have a vague idea of the price they faced in previous purchases.

While these results are quite interesting by itself, this extension may turn many models more realistic on the demand side. For instance in multi-product retail pricing, the capacity of consumers of comparing price vectors of baskets of goods seems to be a central issue. It also leads to continuous demands and continuous best response functions, which are more realistic and sometimes analytically more tractable.

Here it was assumed that all consumers had the same type and degree of bounded rationality. That is actually a weak assumption because all consumers act rationally once at the store. If one were to consider consumers with heterogeneous recall errors, the trade-off of the firms would still be consisted of the two trade-offs, the same monopoly-like trade-off and a similar marginal market share dispute. The former is independent of the type of consumers and the later can be mimicked with homogeneous consumers with a different recall error distribution.

4.8 Appendix

4.8.1 Price cost partial derivatives

Equation (4.11) and its counterpart can be rewritten as

$$H_i(p_i, p_j, c_i) \equiv (p_i - c_i)(-\varepsilon(p_i) - p_i \frac{\partial}{\partial p_i} \ln g(p_j - p_i)) - p_i = 0, \qquad (4.17)$$

for $i, j = A, B$ and $i \neq j$.

The partial derivatives of the equilibrium prices with respect to both costs are obtained by implicit differentiation of the first order conditions of the two firms,

$$\begin{pmatrix} \frac{\partial p_A^*}{\partial c_A} & \frac{\partial p_A^*}{\partial c_B} \\ \frac{\partial p_B^*}{\partial c_A} & \frac{\partial p_B^*}{\partial c_B} \end{pmatrix} = - \begin{pmatrix} \frac{\partial H_A}{\partial p_A} & \frac{\partial H_A}{\partial p_B} \\ \frac{\partial H_B}{\partial p_A} & \frac{\partial H_B}{\partial p_B} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial H_A}{\partial c_A} & \frac{\partial H_A}{\partial c_B} \\ \frac{\partial H_B}{\partial c_A} & \frac{\partial H_B}{\partial c_B} \end{pmatrix}. \qquad (4.18)$$

From (4.17) it follows that

$$
\begin{aligned}
\frac{\partial H_i}{\partial p_i} &= (p_i - c_i)\left(-\varepsilon'(p_i) - \frac{\partial}{\partial p_i}\ln g(p_j - p_i) - p_i \frac{\partial^2}{\partial p_i^2}\ln g(p_j - p_i)\right) \\
&\quad -\varepsilon(p_i) - p_i \frac{\partial}{\partial p_i}\ln g(p_j - p_i) - 1, \\
\frac{\partial H_i}{\partial p_j} &= -(p_i - c_i)p_i \frac{\partial^2}{\partial p_i \partial p_j}\ln g(p_j - p_i), \\
\frac{\partial H_i}{\partial c_i} &= p_i \frac{\partial}{\partial p_i}\ln g(p_j - p_i) + \varepsilon(p_i), \\
\frac{\partial H_i}{\partial c_j} &= 0
\end{aligned}
$$

for $i, j = A, B$, $i \neq j$. Because the partial derivatives will be taken at $c_A = c_B = c$, and therefore at $p_A = p_B = p$, one can perform the change $\frac{\partial}{\partial p_j}\ln g(p_j - p_i) = -\frac{\partial}{\partial p_i}\ln g(p_j - p_i)$ so that $\frac{\partial H_i}{\partial p_j}$ can be rewritten as

$$\frac{\partial H_i}{\partial p_j} = (p_i - c_i)p_i \frac{\partial^2}{\partial p_i^2}\ln g(p_j - p_i).$$

Pricing behavior depending on own cost at $\mathbf{c_A = c_B}$

From equation (4.18),

$$\frac{\partial p_i^*}{\partial c_i} = -\frac{\frac{\partial H_j}{\partial p_j}\frac{\partial H_i}{\partial c_i} - \frac{\partial H_i}{\partial p_j}\frac{\partial H_j}{\partial c_i}}{\frac{\partial H_i}{\partial p_i}\frac{\partial H_j}{\partial p_j} - \frac{\partial H_i}{\partial p_j}\frac{\partial H_j}{\partial p_i}}.$$

Noting that $\frac{\partial H_i}{\partial p_j} = \frac{\partial H_j}{\partial p_i}$ and $\frac{\partial H_i}{\partial p_i} = \frac{\partial H_j}{\partial p_j}$, this can be written as

$$\frac{\partial p_i^*}{\partial c_i} = -\frac{\frac{\partial H_i}{\partial c_i}\frac{\partial H_i}{\partial p_i}}{\left(\frac{\partial H_i}{\partial p_i}\right)^2 - \left(\frac{\partial H_i}{\partial p_j}\right)^2} = -\frac{\frac{\partial H_i}{\partial c_i}}{\frac{\partial H_i}{\partial p_i} + \frac{\partial H_i}{\partial p_j}}\frac{\frac{\partial H_i}{\partial p_i}}{\frac{\partial H_i}{\partial p_i} - \frac{\partial H_i}{\partial p_j}}.$$

Simplifying the first fraction leads to

$$
\begin{aligned}
-\frac{\frac{\partial H_i}{\partial c_i}}{\frac{\partial H_i}{\partial p_i} + \frac{\partial H_i}{\partial p_j}} &= -\frac{-2pg'(0) + \varepsilon(p)}{(p-c)(-\varepsilon'(p) + 2g'(0)) - \varepsilon(p) + 2pg'(0) - 1} && (4.19) \\
&= -\frac{-p\left(\frac{1}{p-c} + \frac{\varepsilon(p)}{p}\right) + \varepsilon(p)}{-(p-c)\varepsilon'(p) + 2(2p-c)g'(0) - \varepsilon(p) - 1} && (4.20) \\
&= \frac{\frac{p}{p-c}}{-(p-c)\varepsilon'(p) + (2p-c)(\frac{1}{p-c} + \frac{\varepsilon(p)}{p}) - \varepsilon(p) - 1} && (4.21) \\
&= \frac{1}{1 + \left(\frac{p-c}{p}\right)^2(\varepsilon(p) - p\varepsilon'(p))}, && (4.22)
\end{aligned}
$$

where $2g'(0) = \frac{1}{p-c} + \frac{\varepsilon(p)}{p}$ from (4.9) was used. To see that the denominator is positive, substitute $\frac{p-c}{p}\varepsilon(p)$ to $2(p-c)g'(0) - 1$ following (4.9). It then becomes

$$1 - \left(\frac{p-c}{p}\right)^2 p\varepsilon'(p) - \frac{p-c}{p} + 2\frac{p-c}{p}(p-c)g'(0) =$$

$$= \left[1 - \frac{p-c}{p}\right] + \left[-\left(\frac{p-c}{p}\right)^2 p\varepsilon'(p)\right] + \left[2\frac{p-c}{p}(p-c)g'(0)\right],$$

where all square brackets are strictly positive given that the markup $p - c$ is strictly positive, due to the distribution full support. For the simplification of the other fraction the following will be needed

$$\begin{aligned} -\frac{d^2}{dx^2}\ln g(0) &= -\frac{g(0)g''(0) - g'^2(0)}{g^2(0)} \\ &= -\frac{\frac{1}{2}g''(0) - g'^2(0)}{\left(\frac{1}{2}\right)^2} \\ &= 4g'^2(0). \end{aligned} \tag{4.23}$$

The second derivative of $g(\cdot)$ at 0 equals zero because ϵ_A and ϵ_B have the same distribution, so that $g(x) = 1 - g(-x)$, therefore $g''(x) = -g''(-x)$, which implies $g''(0) = 0$. Following similar steps, the second fraction of the partial derivative becomes

$$\frac{\frac{\partial H_i}{\partial p_i}}{\frac{\partial H_i}{\partial p_i} - \frac{\partial H_i}{\partial p_j}} =$$

$$= \frac{\frac{p}{p-c} + \varepsilon(p)\frac{p-c}{p} - \varepsilon'(p)(p-c) + 4(p-c)pg'^2(0)}{\frac{p}{p-c} + \varepsilon(p)\frac{p-c}{p} - \varepsilon'(p)(p-c) + 4(p-c)pg'^2(0) + 4(p-c)pg'^2(0)}$$

$$= \frac{1 + \left(\frac{p-c}{p}\right)^2(\varepsilon(p) - p\varepsilon'(p)) + 4(p-c)^2g'^2(0)}{1 + \left(\frac{p-c}{p}\right)^2(\varepsilon(p) - p\varepsilon'(p)) + 8(p-c)^2g'^2(0)}.$$

Notice that both the denominator and the numerator are strictly positive, so that this fraction belongs to the interval $\left(\frac{1}{2}, 1\right)$. Concluding,

$$\begin{aligned} \frac{\partial p_i^*}{\partial c_i} &= \frac{1}{1 - \left(\frac{p-c}{p}\right)^2(p\varepsilon'(p) - \varepsilon(p))} \times \\ &\quad \times \frac{1 + \left(\frac{p-c}{p}\right)^2(\varepsilon(p) - p\varepsilon'(p)) + 4(p-c)^2g'^2(0)}{1 + \left(\frac{p-c}{p}\right)^2(\varepsilon(p) - p\varepsilon'(p)) + 8(p-c)^2g'^2(0)}. \end{aligned} \tag{4.24}$$

Example

Using the Gumbel distribution, so that $g'(0) = \frac{\tau}{4}$, and linear elasticity the above implicit derivative turns to

$$\frac{\partial p_i^*}{\partial c_i} = \frac{1}{1 - a(\frac{p-c}{p})^2} \frac{1 - a(\frac{p-c}{p})^2 + \left[\frac{\tau}{2}(p-c)\right]^2}{1 - a(\frac{p-c}{p})^2 + 2\left[\frac{\tau}{2}(p-c)\right]^2}.$$

At $c = 0$ the equilibrium prices will be $p^* = \frac{1-a}{b+\tau/2}$ and so

$$\begin{aligned}\frac{\partial p_i^*}{\partial c_i} &= \frac{1}{1-a}\left[1 - \frac{1}{(1-a)\left(\frac{2}{\tau}\frac{b+\tau/2}{1-a}\right)^2 + 2}\right] \\ &= \frac{1}{1-a} - \frac{1}{\left[\frac{2b}{\tau}+1\right]^2 + 2\left[1-a\right]}.\end{aligned}$$

At $c_A = c_B = c \to \infty$ the equilibrium prices will be $p^* - c = \frac{1}{b+\tau/2}$, the term $\frac{p^*-c}{p^*}$ goes thus to zero, so that

$$\begin{aligned}\frac{\partial p_i^*}{\partial c_i} &= 1 - \frac{1}{\left[\frac{\tau}{2(b+\tau/2)}\right]^{-2} + 2} \\ &= 1 - \frac{1}{\left[\frac{2b}{\tau}+1\right]^2 + 2}.\end{aligned}$$

Pricing behavior depending on other firm's cost at $\mathbf{c_A = c_B}$

The other firm responds in the following manner according to equation (4.18),

$$\begin{aligned}\frac{\partial p_j^*}{\partial c_i} &= -\frac{-\frac{\partial H_j}{\partial p_i}\frac{\partial H_i}{\partial c_i} + \frac{\partial H_i}{\partial p_i}\frac{\partial H_j}{\partial c_i}}{\frac{\partial H_j}{\partial p_j}\frac{\partial H_i}{\partial p_i} - \frac{\partial H_j}{\partial p_i}\frac{\partial H_i}{\partial p_j}} = \frac{\frac{\partial H_i}{\partial c_i}}{\frac{\partial H_i}{\partial p_i} + \frac{\partial H_i}{\partial p_j}} \frac{\frac{\partial H_j}{\partial p_i}}{\frac{\partial H_i}{\partial p_i} - \frac{\partial H_i}{\partial p_j}} \\ &= \frac{1}{1 + \left(\frac{p-c}{p}\right)^2 (\varepsilon(p) - p\varepsilon'(p))} \times \\ &\quad \frac{4(p-c)^2 g'^2(0)}{1 + \left(\frac{p-c}{p}\right)^2 (\varepsilon(p) - p\varepsilon'(p)) + 8(p-c)^2 g'^2(0)}. \qquad (4.25)\end{aligned}$$

While the first fraction is the same as above (therefore positive), the second one is different. It is however easy to see that it is also positive, because the denominator is the same as in equation (4.24) and the numerator is positive. Moreover this partial derivative is smaller than $\frac{\partial p_i^*}{\partial c_i}$, for the disappearing terms in the numerator are positive.

Example

With linear ε and Gumbel distributed recall error it turns to

$$\begin{aligned}
\frac{\partial p_j^*}{\partial c_i} &= \frac{\frac{\tau^2}{4}\frac{1}{(p-c)^2}}{\left(\frac{a}{p^2}-\frac{1}{(p-c)^2}-\frac{\tau^2}{4}\right)^2-\left(\frac{\tau^2}{4}\right)^2} \\
&= \frac{\frac{\tau^2}{4}(p-c)^2}{\left(1-a(\frac{p-c}{p})^2+(\frac{\tau}{2}(p-c))^2\right)^2-\left(\frac{\tau}{2}(p-c)\right)^4} \\
&= \frac{\frac{\tau^2}{4}(p-c)^2}{\left[1-a(\frac{p-c}{p})^2\right]^2+\left[1-a(\frac{p-c}{p})^2\right]\frac{\tau^2}{2}(p-c)^2} \\
&= \frac{1}{\left[1-a(\frac{p-c}{p})^2\right]^2(\frac{\tau}{2}(p-c))^{-2}+2\left[1-a(\frac{p-c}{p})^2\right]}.
\end{aligned}$$

For prices and price errors of comparable magnitude, that is $c=0$ and $p^*=\frac{1-a}{b+\tau/2}$, this response simplifies to

$$\begin{aligned}
\frac{\partial p_j^*}{\partial c_i} &= \frac{1}{[1-a]^2(\frac{\tau}{2}\frac{1-a}{b+\tau/2})^{-2}+2[1-a]} \\
&= \frac{1}{[1-a]^2(\frac{\tau}{2}\frac{1-a}{b+\tau/2})^{-2}+2[1-a]} \\
&= \frac{1}{[2b/\tau+1]^2+2[1-a]}.
\end{aligned}$$

The limit for $c_A=c_B=c\to\infty$, which implies $\frac{p^*-c}{p^*}\to 0$ and $p^*-c=\frac{1}{b+\tau/2}$, will be

$$\frac{\partial p_j^*}{\partial c_i} = \frac{1}{[2b/\tau+1]^2+2}.$$

4.8.2 Price dispersion with asymmetric costs

As it follows from Section 4.8.1 the price difference as a function of one of the costs can be approximated by

$$\frac{\partial p_i^*}{\partial c_i}-\frac{\partial p_j^*}{\partial c_i} = \frac{-\frac{\partial H_j}{\partial p_j}\frac{\partial H_i}{\partial c_i}-\frac{\partial H_j}{\partial p_i}\frac{\partial H_i}{\partial c_i}}{\frac{\partial H_j}{\partial p_j}\frac{\partial H_i}{\partial p_i}-\frac{\partial H_j}{\partial p_i}\frac{\partial H_i}{\partial p_j}}. \tag{4.26}$$

Taken at $c_A=c_B=c$, so that $p_i=p_j=p^*$, $\frac{\partial H_i}{\partial p_i}=\frac{\partial H_j}{\partial p_j}$, $\frac{\partial H_i}{\partial p_j}=\frac{\partial H_j}{\partial p_i}$, $\frac{\partial H_i}{\partial c_i}=\frac{\partial H_j}{\partial c_j}$ and $\frac{\partial H_i}{\partial c_j}=\frac{\partial H_j}{\partial c_i}$, equation (4.26) becomes

$$
\begin{aligned}
&\frac{\partial p_i^*}{\partial c_i} - \frac{\partial p_j^*}{\partial c_i} = \\
= \quad & -\frac{\frac{\partial H_i}{\partial c_i}\left(\frac{\partial H_j}{\partial p_j} + \frac{\partial H_j}{\partial p_i}\right)}{\left(\frac{\partial H_i}{\partial p_i}\right)^2 - \left(\frac{\partial H_i}{\partial p_j}\right)^2} \\
= \quad & \frac{\frac{\partial H_i}{\partial c_i}}{\frac{\partial H_i}{\partial p_j} - \frac{\partial H_i}{\partial p_i}} \\
= \quad & \frac{-2p^* g'(0) + \varepsilon(p^*)}{1 + \varepsilon(p^*) + (p^* - c)\varepsilon'(p^*) - 2g'(0)(2p^* - c) - 8(p^* - c)p^* g'^2(0)} \\
= \quad & \frac{1}{1 + \left(\frac{p^*-c}{p^*}\right)^2 (\varepsilon(p^*) - p^*\varepsilon'(p^*)) + 8(p^* - c)^2 g'^2(0)},
\end{aligned}
$$

where $2g'(0) = \frac{\varepsilon(p^*)}{p^*} + \frac{1}{p^*-c}$ and equation (4.23) were used in the simplification.

4.8.3 Proof of proposition 4.4

Since p^* increases with the standard deviation σ of the recall error, $\left(\frac{p^*-c}{p^*}\right)^2 \varepsilon^*$ in equation (4.14) is decreasing in σ. Write now $g_0'(0)\sigma^{-1}$ for $g'(0)$, following equation (4.4). The last term in the denominator of equation (4.14) depends on σ according to

$$
\frac{d}{d\sigma}(p^* - c)^2 g_0'^2(0)\sigma^{-2} \quad = \quad 2(p^* - c)g_0'^2(0)\sigma^{-3}\left[\sigma\frac{dp^*}{d\sigma} - (p^* - c)\right]. \tag{4.27}
$$

The sign of the derivative equals the sign of the term between brackets, because all other terms are strictly positive. Because a constant elasticity is being considered, equation (4.8) is a quadratic equation with a closed form solution, namely

$$
p^* = \frac{c}{2} + \frac{\sigma}{4g_0'(0)}\left[1 + \varepsilon^* + \sqrt{\left(1 + \varepsilon^* + \frac{2}{\sigma}cg_0'(0)\right)^2 - \frac{8}{\sigma}cg_0'(0)\varepsilon^*}\right].
$$

The term $\sigma\frac{dp^*}{d\sigma} - (p^* - c)$ in (4.27) is therefore equal to

$$
\sigma\frac{dp^*}{d\sigma} - (p^* - c) = \frac{c}{2}\left[1 - \frac{\frac{2}{\sigma}cg_0'(0) + 1 - \varepsilon^*}{\sqrt{\left(1 + \varepsilon^* + \frac{2}{\sigma}cg_0'(0)\right)^2 - \frac{8}{\sigma}cg_0'(0)\varepsilon^*}}\right].
$$

Now notice that

$$
\begin{aligned}
\left(1 + \varepsilon^* + \frac{2}{\sigma}cg_0'(0)\right)^2 - \frac{8}{\sigma}cg_0'(0)\varepsilon^* \quad &= \quad \left(\frac{2}{\sigma}cg_0'(0) + 1 - \varepsilon^*\right)^2 + 4\varepsilon^* \\
&\leq \quad \left(\frac{2}{\sigma}cg_0'(0) + 1 - \varepsilon^*\right)^2,
\end{aligned}
$$

implying that the square root term is smaller than the numerator which shows that the above expression is negative. Therefore the term $\sigma \frac{dp^*}{d\sigma} - (p^* - c)$ is negative and the last term in the denominator of equation (4.14) is decreasing in σ. Hence the whole denominator is decreasing in σ.

Bibliography

AINSLIE, G. (1991): "Derivation of "rational" economic behaviour from hyperbolic discount curves," *American Economic Review*, 81(2), 334–340.

——— (2010): "Pure hyperbolic discount curves predict "eyes-open" self control," Mimeo.

AL-NAJJAR, N. I., AND J. WEINSTEIN (2009): "The ambiguity aversion literature: A critical assessment," *Economics and Philosophy*, 25, 249–284.

ALOS-FERRER, C., A. B. ANIA, AND K. R. SCHENK-HOPPE (2000): "An evolutionary model of bertrand oligopoly," *Games and Economic Behavior*, 33(1), 1–19.

ARIELY, D., AND K. WERTENBROCH (2002): "Procrastination, deadlines, and performance: Self-control by precommitment," *Psychological Science*, 13(3), 219–224.

ARMANTIER, O., AND N. TREICH (2009): "Subjective probabilities in games: an application to the overbidding puzzle," *International Economic Review*, 50(4), 1079–1102.

BAYE, M. R., AND J. MORGAN (2004): "Price dispersion in the lab and on the internet: Theory and evidence," *RAND Journal of Economics*, 35(3), 449–466.

BEWLEY, T. F. (2002): "Knightian decision theory: part 1," *Decisions in Economics and Finance*, 25(2), 79–110.

BOSE, S., AND A. DARIPA (2009): "A dynamic mechanism and surplus extraction under ambiguity," *Journal of Economic Theory*, 144(5), 2084–2114.

BOSE, S., E. OZDENOREN, AND A. PAPE (2006): "Optimal auctions with ambiguity," *Theoretical Economics*, 1(4), 411–438.

CALVO, G. A. (1983): "Staggered prices in a utility-maximizing framework," *Journal of Monetary Economics*, 12(3), 383–398.

CAMERER, C., AND M. WEBER (1992): "Recent developments in modeling preferences: Uncertainty and ambiguity," *Journal of Risk and Uncertainty*, 5, 325–370.

CARVALHO, M. (2009): "Price Recall, Bertrand Paradox and Price Dispersion With Elastic Demand," Discussion Paper 2009-69, Tilburg University, Center for Economic Research.

Chatterjee, K., and R. V. Krishna (2009): "A dual-self representation for stochastic temptation," *American Economic Journal: Microeconomics*, 1(2), 148–67.

Chen, Y., G. Iyer, and A. Pazgal (2005): "Limited memory and market competition," available at `http://groups.haas.berkeley.edu/marketing/PAPERS/IYER/Memory_july05_final.pdf`.

Chen, Y., P. Katuscak, and E. Ozdenoren (2007): "Sealed bid auctions with ambiguity: Theory and experiments," *Journal of Economic Theory*, 136(1), 513–535.

DellaVigna, S., and U. Malmendier (2006): "Paying Not to Go to the Gym," *American Economic Review*, 96(3), 694–719.

Diamond, P. A. (1971): "A model of price adjustment," *Journal of Economic Theory*, 3(2), 156–168.

Dominiak, A., P. Dürsch, and J.-P. Lefort (2009): "A dynamic Ellsberg urn experiment," Working Papers 487, University of Heidelberg, Department of Economics.

Dow, J. (1991): "Search decisions with limited memory," *Review of Economic Studies*, 58(1), 1–14.

Dow, J., and S. R. d. C. Werlang (1992): "Uncertainty aversion, risk aversion, and the optimal choice of portfolio," *Econometrica*, 60(1), 197–204.

Drèze, X., M. Vanhuele, and G. Laurent (2006): "Consumers' immediate memory for prices," *Journal of Consumer Research*, 33(2), 163–172.

Ellison, G. (2005): "A model of add-on pricing," *Quarterly Journal of Economics*, 120(2), 585–637.

Ellsberg, D. (1961): "Risk, ambiguity, and the Savage axioms," *Quarterly Journal of Economics*, 75(4), 643–669.

Epstein, L. G. (1999): "A definition of uncertainty aversion," *Review of Economic Studies*, 66(3), 579–608.

Epstein, L. G. (2010): "A Paradox for the Smooth Ambiguity Model of Preference," *Econometrica*, 78(6), 2085–2099.

Epstein, L. G., and M. Schneider (2003): "Recursive multiple-priors," *Journal of Economic Theory*, 113(1), 1–31.

Etner, J., M. Jeleva, and J.-M. Tallon (2009): "Decision theory under uncertainty," Documents de travail du Centre d'Economie de la Sorbonne 09064, Université Panthéon-Sorbonne (Paris 1), Centre d'Economie de la Sorbonne.

Frederick, S., G. Loewenstein, and T. O'Donoghue (2002): "Time discounting and time preference: A critical review," *Journal of Economic Literature*, XL, 351–401.

Fudenberg, D., and D. K. Levine (2006): "A dual-self model of impulse control," *American Economic Review*, 96(5), 1449–1476.

Gabaix, X., and D. Laibson (2004): “Competition and consumer confusion,” Econometric Society 2004 North American Summer Meetings 663, Econometric Society.

——— (2006): “Shrouded attributes, consumer myopia, and information suppression in competitive markets,” *Quarterly Journal of Economics*, 121(2), 505–540.

Gabaix, X., D. Laibson, and H. Li (2005): “Extreme value theory and the effects of competition on profits,” mimeo.

Ghirardato, P., F. Maccheroni, and M. Marinacci (2004): “Differentiating ambiguity and ambiguity attitude,” *Journal of Economic Theory*, 118(2), 133 – 173.

Gilboa, I. (2009): *Theory of Decision under Uncertainty*. Cambridge University Press, Cambridge.

Gilboa, I., and D. Schmeidler (1989): “Maxmin expected utility with non-unique prior,” *Journal of Mathematical Economics*, 18(2), 141–153.

——— (1993): “Updating ambiguous beliefs,” *Journal of Economic Theory*, 59(1), 33–49.

Gollier, C. (2009): “Portfolio choices and asset prices: The comparative statics of ambiguity aversion,” IDEI Working Papers 357, Institut d’Économie Industrielle (IDEI), Toulouse.

Gul, F., and W. Pesendorfer (2001): “Temptation and self-control,” *Econometrica*, 69(6), 1403–1435.

——— (2004): “Self-control and the theory of consumption,” *Econometrica*, 72(1), 119–158.

Halevy, Y. (2007): “Ellsberg revisited: An experimental study,” *Econometrica*, 75(2), 503–536.

Hanany, E., and P. Klibanoff (2009): “Updating ambiguity averse preferences,” *The B.E. Journal of Theoretical Economics*, 9(1).

Hansen, L. P., and T. J. Sargent (2001): “Robust control and model uncertainty,” *American Economic Review*, 91(2), 60–66.

Hehenkamp, B. (2002): “Sluggish consumers: An evolutionary solution to the Bertrand paradox,” *Games and Economic Behavior*, 40, 44–76.

Herings, P., G. van der Laan, and D. Talman (2007): “Socially structured games,” *Theory and Decision*, 62(1), 1–29.

Hey, J. D., G. Lotito, and A. Maffioletti (2007): “Choquet OK?,” Discussion Papers 07/12, Department of Economics, University of York.

Ju, N., and J. Miao (2009): “Ambiguity, learning, and asset returns,” MPRA Paper 14737, University Library of Munich, Germany.

Kagel, J. H., and A. E. Roth (eds.) (1995): *The Handbook of Experimental Economics*. Princeton University Press, Princeton.

Kahneman, D., and A. Tversky (1979): "Prospect Theory: An Analysis of Decision under Risk," *Econometrica*, 47(2), 263–91.

Kalayci, K., and J. Potters (2009): "Buyer confusion and market prices," available at `http://center.uvt.nl/phd_stud/kalayci/buyerconfusion2009.pdf`.

Karni, E. (1988): "On the equivalence between descending bid auctions and first price sealed bid auctions," *Theory and Decision*, 25, 211–217.

Klibanoff, P., and E. Hanany (2007): "Updating preferences with multiple priors," *Theoretical Economics*, 2(3), 261–298.

Klibanoff, P., M. Marinacci, and S. Mukerji (2005): "A smooth model of decision making under ambiguity," *Econometrica*, 73(6), 1849–1892.

——— (2009): "Recursive smooth ambiguity preferences," *Journal of Economic Theory*, 144(3), 930–976.

Knight, F. H. (1921): *Risk, Uncertainty, and Profit.* Houghton Mifflin, Boston.

Kocher, M. G., and S. T. Trautmann (2011): "Selection into auctions for risky and ambiguous prospects," Forthcoming in *Economic Inquiry*, no. doi: 10.1111/j.1465-7295.2010.00347.x.

Laibson, D. (1996): "Hyperbolic discount functions, undersaving, and savings policy," Working Paper 5635, National Bureau of Economic Research.

——— (1997): "Golden eggs and hyperbolic discounting," *Quarterly Journal of Economics*, 112(2), 443–477.

Liu, H.-H., and A. M. Colman (2009): "Ambiguity aversion in the long run: Repeated decisions under risk and uncertainty," *Journal of Economic Psychology*, 30(3), 277–284.

Lo, K. C. (1998): "Sealed bid auctions with uncertainty averse bidders," *Economic Theory*, 12(1), 1–20.

Maccheroni, F., M. Marinacci, and A. Rustichini (2006): "Ambiguity aversion, robustness, and the variational representation of preferences," *Econometrica*, 74(6), 1447–1498.

McClure, S. M., D. Laibson, G. Loewenstein, and J. Cohen (2004): "Separate neural systems value immediate and delayed monetary rewards," *Science*, 306, 503–507.

Monroe, K. B., and A. Y. Lee (1999): "Remembering versus knowing: Issues in buyers's processing of price information," *Journal of the Academy of Marketing Science*, 27(2), 207–225.

Mullainathan, S. (2002): "A memory based model of bounded rationality," *Quarterly Journal of Economics*, 117(3), 735–774.

O'Donoghue, T., and M. Rabin (1999): "Doing it now or later," *American Economic Review*, 89(1), 103–124.

——— (2001): "Choice and procrastination," *Quarterly Journal of Economics*, 116(1), 121–160.

——— (2003): *Time and Decision: Economic and Psychological Perspectives on Intertemporal Choice*chap. Self awareness and self control, pp. 217–243. Russell Sage Foundation.

Ozdenoren, E., and J. Peck (2008): "Ambiguity aversion, games against nature, and dynamic consistency," *Games and Economic Behavior*, 62(1), 106–115.

Pan, X., B. T. Ratchford, and V. Shankar (2004): "Price dispersion on the internet: A review and directions for future research," *Journal of Interactive Marketing*, 18(4), 116–135.

Perloff, J. M., and S. C. Salop (1985): "Equilibrium with product differentiation," *Review of Economic Studies*, 52(1), 107–20.

Phelps, E., and R. A. Pollak (1968): "On second-best national saving and game-equilibrium growth," *Review of Economic Studies*, 35(2), 185–199.

Pires, C. P. (2002): "A rule for updating ambiguous beliefs," *Theory and Decision*, 53(2), 137–152.

Pollak, R. (1968): "Consistent planning," *Review of Economic Studies*, 35(2), 201–208.

Puu, T. (2002): "Hotelling's "Ice cream dealers" with elastic demand," *The Annals of Regional Science*, 36(1), 1–17.

Salo, A. A., and M. Weber (1995): "Ambiguity aversion in first-price sealed-bid auctions," *Journal of Risk and Uncertainty*, 11(2), 123–37.

Savage, L. J. (1954): *The Foundations of Statistics*. New York: John Wiley.

Schmeidler, D. (1986): "Integral representation without additivity," *Proceedings of the American Mathematical Society*, 97, 255–261.

——— (1989): "Subjective probability and expected utility without additivity," *Econometrica*, 57(3), 571–87.

Siniscalchi, M. (2010): "Dynamic choice under ambiguity," http://faculty.wcas.northwestern.edu/ msi661/dynchoice.pdf.

Sorensen, A. T. (2001): "An empirical model of heterogeneous consumer search for retail prescription drugs," *NBER Working Paper Series*, No. 8548.

Spiegler, R. (2006): "Competition over agents with boundedly rational expectations," *Theoretical Economics*, 1(2), 207–231.

Stahl, D. O. (1989): "Oligopolistic pricing with sequential consumer search," *American Economic Review*, 79(4), 700–712.

Strotz, R. H. (1956): "Myopia and inconsistency in dynamic utility maximization," *Review of Economic Studies*, 23(3), 165–180.

Thaler, R. H., and H. M. Shefrin (1981): "An economic theory of self-control," *Journal of Political Economy*, 89(2), 392–406.

Varian, H. R. (1980): "A model of sales," *American Economic Review*, 70(4), 651–59.