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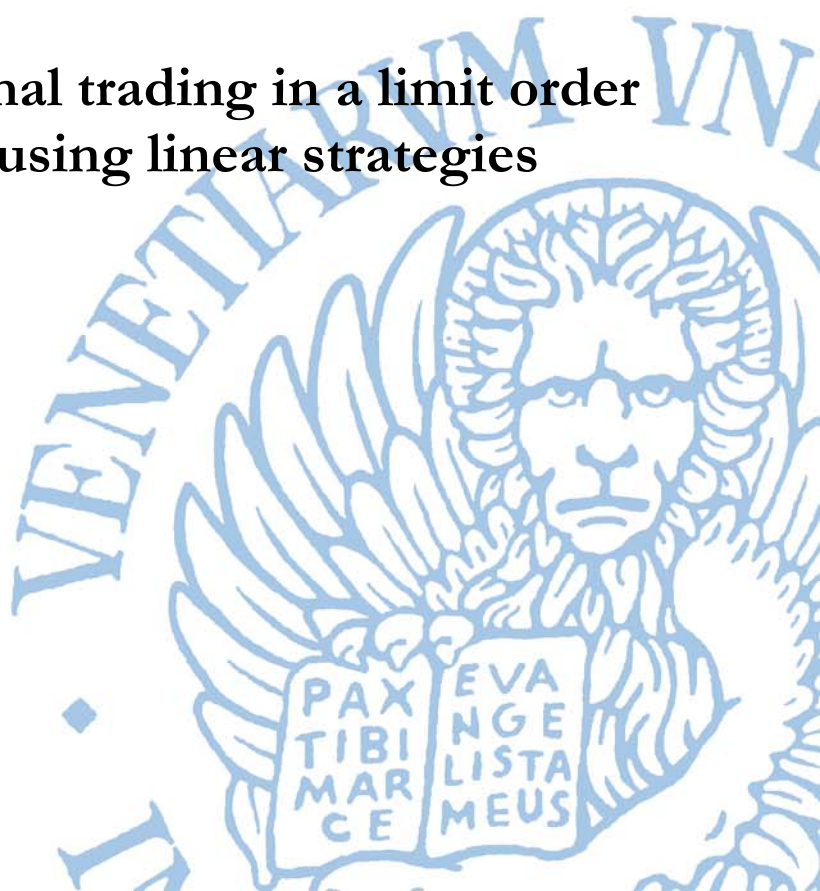
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book using linear strategies**

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Keywords: continuous double auction, dynamic equilibrium, optimal trading strategies, evolution strategies.

JEL Codes: D44, D82, C63, C72.

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Optimal trading in a limit order book using linear strategies

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1 Introduction

The Continuous Double Auction (CDA) is a market protocol that allows the submission at any time of binding proposals to purchase or sell a specified quantity of an asset. This kind of auction is very common and most stock exchanges are nowadays run using computerized CDAs that promote liquidity provision, efficient exchange and rapid incorporation of new information into prices, [?]. Even disregarding more or less relevant protocolary details and implementation minutiae, the formal analysis of CDAs is difficult because of the gargantuan number of options available to traders. Real traders react, just to mention a few things, to exogenous information, cancel and resubmit orders, sometimes in fractions of a second, respond to endogenous variations in the state of the book and devise various strategies that make use of both quantities and limit prices.

Not surprisingly, trading optimally in such an environment is difficult and not easily amenable to analysis in full generality. Some analytical models, hence, simplify the setup in order to get closed-form solutions, see [Foucault, 1999, Foucault et al., 2005, Rosu, 2009]. These papers consider only few different types of traders, whose intrinsic motivation to trade is “impatience”. Agents, usually exchanging only one unit, are arranged in an infinite flow, pay a cost because of delayed execution and can choose between market and limit orders. The main lesson we learn from such cleverly crafted and analytical models is that impatient traders always take liquidity, submitting market orders, while patient agents always offer liquidity and place less aggressive limit orders. This result is obtained, at times, with heroic assumptions, like the obligation to improve the extant bid/ask or considering truncated books.

[Parlour, 1998] is an equilibrium model of a two-tick book with agents having, instead, a continuum of values and costs. Under time constraints, traders strategically use market orders or queue their limit orders depending on their valuations and on the time left before the market closes down. This analytical model was then generalized in more realistic settings where numerical methods are used to determine the dynamic Markov-perfect equilibrium, [Goettler et al., 2005]. The strategic acquisition of information is numerically analyzed in [Goettler et al., 2009].

Agents in our model have heterogenous private values or costs and trade to maximize their profit in sessions where few agents are involved. Hence, they explicitly feel some form of time pressure, in that excessive caution may result in no trading due to the closure of the market when the last trader as come. This time-related feature adds to the standard immediacy *vs* efficacy tradeoff faced in a CDA: agents may immediately transact by submitting a market(able) order, but they have to accept the most disadvantageous (to

them) current price in the book; alternatively, they can submit a more effective limit order that is preferred conditional on (uncertain) future execution.

We allow traders to issue orders whose limit prices are based on an extremely simple strategy that takes into account only the quotes at the time of submission. The strategy is linear in the best bid and in best ask and depends on the type of the trader (i.e, on his value or her cost). Strictly speaking, our agents can submit only limit orders. However, the submission of a marketable order will result in immediate execution at the best quote and, therefore, it is profit-equivalent to a market order.

Our model can be interpreted as a stochastic dynamic game with many heterogenous agents. On the first hand, we simultaneously solve several stochastic optimization problems, one for each type. The resulting equilibrium is then the outcome of “unrelated” selfish maximizations by the groups of different agents. On the other hand, one can think to a search for a Nash equilibrium in the absence of a normal-form payoff matrix. Hence, expected gains from trade must be iteratively estimated together with the optimal strategies.

The paper reaches some novel results. First, in equilibrium the most “impatient” traders, with high values or low costs, do not *always* submit market orders. In a related fashion, it is sometimes optimal, for some patient type, to issue market orders. Second, agents take mostly into account the state of the opposite side of the market but, to a lesser extent, modify their behavior based also on their own side. Third, we characterize the states of the book in equilibrium and describe the resulting transaction prices. In particular, we found that the equilibrium states form a rather small set, confirming the general flavor stemming from other works that, despite the “curse of dimensionality”, similar problems may be tackled.

The paper is organized as follows. Section 2 develops the model, defining the rules of the CDA, and describes the agents and their restricted linear strategies. The third Section is devoted to the presentation of our numerical results. We first describe the optimal strategies and then present the main statistical features of the price and the book. We also visually depict the aggressiveness of the orders in equilibrium, using a classification reminiscent of [Biais et al., 1995]. The details about the Evolutionary Strategies (ES) algorithm used to compute the equilibrium are postponed to the Appendix. Finally, Section 4 summarizes and concludes.

2 The model

This section describes the rules of our stylized CDA and then provide details on the agents and on their bidding functions. Agents are heterogeneous and face uncertainty in the time they enter the market and do not know the types of the other traders. Moreover, they use a restricted set of linear strategies based on a simple information set.

2.1 The market

We assume agents trade in a CDA in repeated sessions. In each session, they enter in the market in a random order (sampled by nature) that is unknown to them. When it is their turn to act they have to anonymously submit one limit order for one unit of an asset, specifying a reservation price (to buy or to sell). They will have no other chance to trade or modify or cancel the order during the session. It will be handy to think to the turn in which the trader enters as a *position* in the queue or as a *time* t . Assume the agent is a buyer submitting the bid B_t . His order is matched against the selling proposals already in the book, where limit prices $l_1 \leq l_2 \leq \dots$ are stored. If the bid crosses the best ask, i.e., $B_t \geq l_1$, a transaction occurs and the selling book is updated deleting the just filled order with price l_1 ; otherwise, B_t is inserted in the book of yet unfilled buying orders. Similarly, if a seller submits an ask A_t at time t , the order is matched against the buyers proposals with limit prices $l_{-1} \geq l_{-2} \geq \dots$ and a transaction immediately happens when $A_t \leq l_{-1}$. In this case, the (buying) book is updated by deleting the order whose price is l_{-1} ; otherwise, the unfilled ask is placed in the book of sellers. As customary, orders are inserted in the books preserving price priority and ties are broken using time priority. Finally, all unfilled orders are deleted at the end of the trading session.

This description captures most of the essential features of a CDA, in which any (positive) limit price can be submitted at any time to the market for immediate execution, if it finds a counterpart, or stored for future use till the end of the session.

2.2 The agents

We assume there are n buyers and n seller, drawn randomly from a larger pool of N buyers and N sellers. Each agent has a single unit to buy or a single unit to sell in any trading session. All agents are endowed with privately known values or costs. In detail, the i -th buyer can redeem one unit of the asset for the sum $v_i \in V$. His profit, conditional on trading, is then $v_i - p$,

where p is the price paid to acquire the unit of the asset. Symmetrically, the j -th seller has some cost $c_j \in C$. Her profit is $p - c_j$, if she sells her unit. Traders get no profit if they fail to trade in the session. For simplicity, we assume that values and costs belong to the same discrete set $V = C$.

Nature samples n buyers and n sellers from the pool. Assume the selected buyers (sellers) have index in $\mathcal{B} = \{i_1, i_2, \dots, i_n\} \subset \{1, \dots, N\}$ ($\mathcal{S} = \{j_1, j_2, \dots, j_n\} \subset \{1, \dots, N\}$). As seen before, agents participate to the auction sequentially, according to a random reordering of the indexes in $\mathcal{A} = \mathcal{B} \cup \mathcal{S}$. Let $\sigma \in \mathcal{P}(\mathcal{A})$ be a permutation of the selected traders. Every agent is risk-neutral and maximizes his expected profit from trading, conditional on the information H_t available when it is his/her turn to act. The i -th buyer, say, has to solve the following problem:

$$\max_{B_t \leq v_i} E[\pi_i(B_t | v_i, H_t)], \quad (1)$$

where π_i is the profit and the expectation is taken over all possible permutations $\sigma \in \mathcal{P}(\mathcal{A})$ and over all $\binom{n}{N}^2$ possible choices of \mathcal{A} .

The reader should be aware that “information” in this paper is not related to fundamental news that affect the fair value of the asset but, instead, pertains only to the state and history of the book, and to present and past transaction prices.

Agents face two sources of uncertainty: first, they do not know \mathcal{A} and, hence, they are unaware of the types (values or costs) of the other traders; second, they do not know their position in the queue σ and, say, they do not know whether they are the first or the last to issue limit orders. Both sources of uncertainty are likely to be relevant and impact their revenues. We feel that both features are typical in real markets where agents do not easily know values or costs of the other traders and do not know whether many orders will come after any submission. In particular, due to the asynchronous clearing mechanism of the continuous double auction, the successful execution depends on the position in the queue. We incorporate individual rationality in the choice of the bid B_t in (1) and, hence, an agent never considers bids larger than his value. Sellers solve an identical problem picking the best ask A_t to maximize profits, subject to the constraint $A_t \geq c_j$.

The information H_t can provide some guidance on the state of the book. [Fano et al., 2010] studies the extreme case where $H_t = \emptyset$ and only the values/costs are known to agents. More generally, we will assume that H_t contains the best ask and the best bid at the time of submission: $H_t = (a_t, b_t)$, $t = 1, \dots, 2n$. As every agent has one unique chance to submit an order, he/she will use the information to take a decision on the (limit price of) the order to be submitted. The knowledge of (a_t, b_t) gives perfect informa-

tion on the price to be paid or received if a marketable order is used. In this case, a buyer will pay a_t and a seller will get b_t for a unit transaction. However, and perhaps more importantly, H_t provides (partial) information on the two sources of uncertainty mentioned above. For example, high limit prices (a_t, b_t) may reveal that \mathcal{A} contains more buyers with high values than sellers with low costs and, consequently, prices have drifted up in that particular session. Moreover, a small (large) spread $a_t - b_t$ provides tentative evidence on the accumulation of past orders and suggests, say, that we are close to the end (beginning) of the session or that further narrowing of the spread is unlikely (probable). Other definitions of H_t may include the second-best quotes or information related to the midprice and one or more of the past transaction prices.

We do not attempt in this paper to model the beliefs of one agent on others' values or actions and, in this sense, there is no bayesian flavor in the approximate equilibrium we look for. This work differs from [Goettler et al., 2005], where agents hold beliefs on the probabilities of execution, and resembles [Parlour, 1998] with the addition of a richer book modelization.¹

We restrict the set of strategies available to traders in such a way that they are linear in the outstanding best bid and best ask. Accordingly, the i -th buyer will bid

$$B_t(v_i, H_t) = v_i - (\alpha_i a_t + \beta_i b_t + \gamma_i), \quad (2)$$

facing the state H_t in position t in the queue. Symmetrically, the j -th seller will ask

$$A_t(c_j, H_t) = c_j + (\alpha_j a_t + \beta_j b_t + \gamma_j), \quad (3)$$

dealing with state H_t at time t . Observe that the strategy is a deterministic function of information contained in H_t at the time of submission. In particular, any specific agent will always bid or ask the same limit price in the same position of the queue with the same state of the book. However, the fact that the same action is taken does not allow other traders to exploit this conduct as orders are anonymous and the ordering is randomized. The linear shape of the bidding functions in (2) and (3) is restrictive but it is able to simply capture some important strategic features and hugely eases the interpretation of the results. In particular, it allows to investigate the significance of the two best offers on both sides of the book. A naive approach would suggest that a buyer, say, would be interested only in the asks and, in particular, in the best outstanding ask. However, some reflections show that competition with other buyers may have a role, taking the form

¹In [Parlour, 1998] agents also perfectly know their position in the queue.

of a conflict to gain priority that would increase the probability of delayed execution, whenever it is not possible or convenient to hit the opposite side.

The bid and ask in (2) and (3) can be intuitively interpreted as a state-contingent adjustment of the value or cost of the agent. Consider, for example, a seller whose cost is c . She would naturally submit asks above her cost to gain positive profits and, hence, A_t should exceed c by a markup. Using (3), the markup can be directly be quantified as $\alpha a_t + \beta b_t + \gamma$, and the same can be said in (2) for a “markdown” relative to the value of a buyer.

3 Results

We solve the joint profit maximization problem outlined in the previous section using an Evolution Strategies (ES) algorithm, see [Beyer and Schwefel, 2002]. The choice to use numerical methods is due to the analytical intractability of the setup we have described but, still, there are technical difficulties in tackling such a high-dimensional and stochastic problem and we had to recourse to non-trivial optimization techniques. For each type (i.e., value or cost), we cumulate gains from trade in τ trading sessions and compute the average profit, keeping fixed the strategies of all other agents in the pool. Then, the strategies played by each type are ranked, selected, married and recombined as customary in ES. Another period of τ trading sessions is then started in the next iteration. This optimizing process ceases when an endogenous mutation rate drops to very low values or approaches zero. A detailed account of the algorithm and the code we used is provided in Appendix.

In the following, after the specification of the parameters of the model, we first describe the optimal strategies that are evolved and then illustrate the equilibrium book that is created by agents acting optimally, together with the statistical properties of the transaction price they give raise to. Despite the lack of an analytical description, both the strategies and the states of the book can be analyzed in detail and reveal interesting structures.

3.1 The parameters

Without loss of generality, we restrict the set of values and costs to be included in the interval $[0, 1]$. We assume that the set of values and costs is

$$V = \left\{ \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m} \right\},$$

where $m = 20$. As a trivial consequence, no transaction price will be smaller than 0.05 or bigger than 0.95 as no agents is asking that low or asking that

much. Given that V has 19 elements, we have 19 types for buyer and 19 for sellers, totalizing a total of 38 types of agents. The parameter m can be used to modulate the granularity of the heterogeneity in values and costs.

We assume that $n = 10$ buyers and the same number of sellers are sampled to participate to each session. The traders are randomly selected from a bigger pool of $N = 760$ traders (380 buyers and 380 sellers, there are exactly 20 instances for each of the types). Other parameters must be initialized to start the ES numerical algorithm and are described in the Appendix. For future reference, observe that our setup is symmetric in the number and values/costs of buyers/sellers and, therefore, *ex ante* expected equilibrium price is 0.5.

3.2 The strategies

The equilibrium parameters that define the strategies of the intramarginal traders are reported in Table 1. Extramarginal types, as expected, are very rarely involved in transactions and collect gains very close to zero. As a consequence, the ES algorithm is pointlessly trying to maximize an almost null constant function. Hence, the mutation rates of extramarginal agents stay typically bounded away from zero, indicating that there has been no convergence to a meaningful strategy.² For this reason, the Table does not report the parameters of extramarginal agents.

The top panel of Table 1 shows that α is increasing in the value of the buyers. These agents are sensitive to the outstanding ask in the book and are ready to *increase* their bid when the ask *decreases*. This behavior often results in the submission of marketable orders, when the ask is moderate, and is much more pronounced for very impatient traders with high values. At the same time, the buyers take into account (to a smaller extent) also the best bid on their own side. In this case, the effect is weaker and lead to a lower (higher) bid if the book already contains a high (low) best bid: in the presence of an aggressive best bid, buyers tend to submit less aggressive orders; conversely, when the best bid is not aggressive (i.e., it is relatively low), buyers tend to gain priority by submitting a higher bid.

Rather symmetrically, the sellers behave according to the same principles, as seen in the bottom panel of Table 1. Figure 1 depicts graphically the α and β coefficients for intramarginal buyers (right) and sellers (left). The pictures confirm the greater sensitivity of buyers (sellers) to the best ask (bid). A

²Deeply extramarginal traders never manage to transact in equilibrium. Hence, ES has the impossible task to maximize a constantly null profit function. It is not surprising that mutation is kept at high levels in such a ill-conditioned situation.

Values of intramarginal buyers										
	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
α	0.02	0.08	0.07	0.19	0.18	0.25	0.22	0.32	0.33	0.40
β	-0.04	-0.06	-0.08	-0.05	-0.08	-0.06	-0.12	-0.04	-0.06	-0.08
20γ	1.31	1.15	1.78	0.31	1.06	0.52	2.04	0.42	1.32	0.99
Costs of intramarginal sellers										
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
α	0.19	0.18	0.17	0.11	0.16	0.17	0.08	0.10	0.10	0.06
β	-0.34	-0.28	-0.24	-0.22	-0.17	-0.12	-0.14	-0.06	-0.04	-0.04
20γ	5.97	5.31	4.48	4.47	2.72	2.03	2.65	1.52	0.93	1.00

Table 1: Equilibrium strategies of intramarginal traders whose value is in $[0.50, 0.95]$ (top panel) and intramarginal sellers whose cost is in $[0.05, 0.50]$ (bottom panel). Each column of the table reports the parameters α , β and γ of (2) and (3).

first interesting conclusion is that, even in our simple strategic setup, agents in equilibrium act based on *both* sides of the book.

3.3 The price and the book in equilibrium

Figure 2 shows the density of the transaction price. The smallest (largest) observed price is 0.276 (0.690), the average price over 1683 transactions is 0.506, with a median of 0.507. The density is not far from being gaussian and, for comparison, a normal pdf with same mean and variance is plotted. A formal Shapiro-Wilk normality test cannot reject the null of normally distributed prices (p -value 0.40).

The strategies described previously also shape the set of states that support the equilibrium. Recall that, in principle, the set of feasible states, i.e., the couples (a_t, b_t) of best ask and best bid, is given by the whole half-triangle whose vertices are $(0,0)$, $(1,0)$ and $(1,1)$. However, other papers have shown that often the set of recurrent states in equilibrium can be a “small” subset of the feasible ones. [Foucault, 1999] and [Foucault et al., 2005] present equilibrium models where only a few states of the book are “played”. More generally, [Pakes and McGuire, 2001] argue that the small measure of the recurrent set is useful to compute an equilibrium by iterative methods and describe an algorithm that may break the “curse of dimensionality” usually faced in similar models.

In our different setup, we confirm that the equilibrium states are proba-

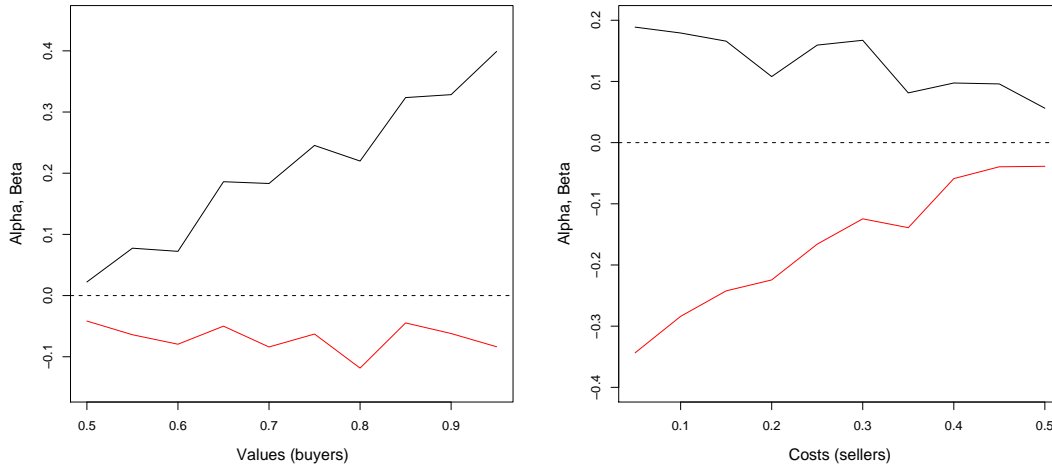


Figure 1: Coefficients α (black) and β (red) for intramarginal buyers (left) and sellers(right).

bilistically a “small” set. Figure 3 shows the 2-dimensional joint density of (a_t, b_t) in two different ways. On the left, the peaks show that, with high probability, the book is close to $(0.6, 0.4)$, $(1.0, 0.0)$, $(0.6, 0.0)$ and $(1.0, 0.4)$. The first, inner peak correspond to the normal mode of the market, when several bids and asks have been submitted. Traders in this case must compete in a full book that contains competitive offers. The corner point $(1.0, 0.0)$ is the initial state (empty book) that is relatively frequent given the small number of agents in the market. The other two peaks on the boundary are the states reached after the first bid or ask are deposited and start filling a previously empty book.

The right part of Figure 3 is a color-coded representation of the book density, with high (low) values in yellow (red), giving substance to the previous claim that some states are encountered much more frequently while others are virtually never faced by traders. Table 2 shows the frequency in which the best bid and ask are inside some tight intervals (around the *a priori* equilibrium price that is 0.5). The best bid and ask are both in $[0.35, 0.65]$ 37% of times and, often, the spread is smaller. Given that 23.6% of equilibrium states are such that either $b_t = 0.00$ or $a_t = 1.00$, it is clear that traders experience some states with relatively high frequency. Profitable strategies should provide good gains in such relevant states. Correct behavior in rare states is much less important, as far as cumulated profits are concerned.

The stationary nature of the book in equilibrium can be appreciated in

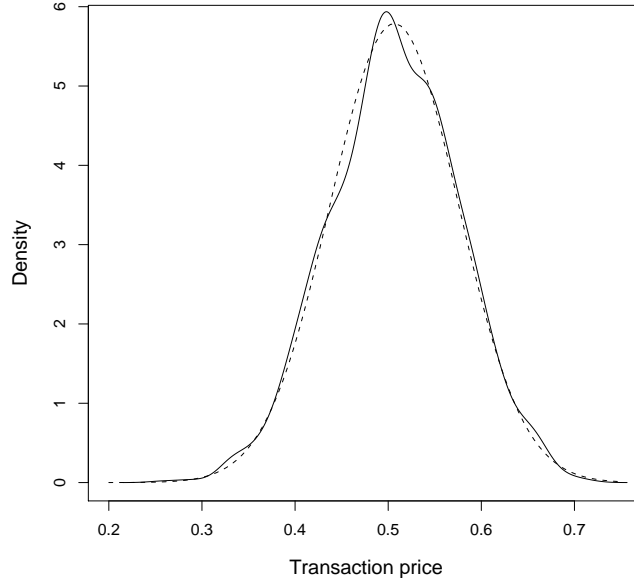


Figure 2: Density of the transaction prices (solid). A normal density with same mean and standard deviation is shown as a dashed line for comparison.

Figure 4, where we show the average³ best bid and ask (thick lines) as a function of the position in the queue, that proxies the time in which an agent enters the market in a session. The average best bid and ask are, on average, almost perfectly symmetric and monotonic: the best ask (bid) is steadily decreasing (increasing), as long as more agents take part to the auction. Therefore, the initial spread of 1 smoothly narrows down to about 0.4 after the 5-th trader and reach 0.2 at the end of the session. The traders typically find a rather “full” book if they enter their submission, say, after the 5-th agent. Having the burden to open the market, on the contrary, implies that the book is “sparser” and the spread much larger. The bid and the ask of the most impatient agents, with $v = 0.95$ and $c = 0.05$, are also shown in Figure 4, using thin dashed lines. They are obtained assuming that the agents face exactly the average book depicted with the thick lines. The stronger buyer, say, increase his bid with his position in the queue: if he is the first, he bids (approximately) 0.5; if his turn comes later (confronted with the *average book*), the bid becomes more aggressive and he typically issues a marketable order when he is sixth or later (when the thin line of

³The averages are obtained using 7600 states of the book (couples (a_t, b_t)) generated in 10 different runs.

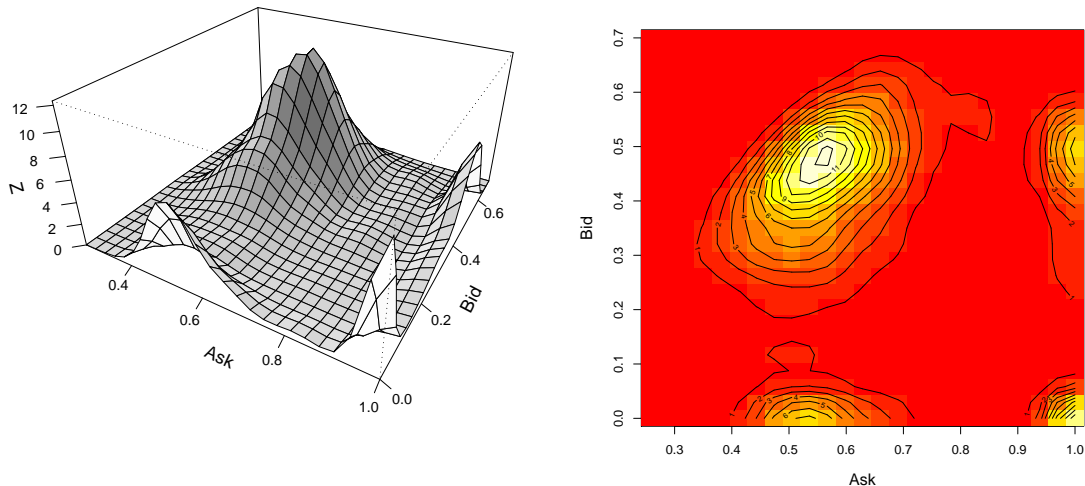


Figure 3: Joint density of the state of the book (a_t, b_t) .

Frequency of states	Bid b_t		
	0.45-0.55	0.40-0.60	0.35-0.65
Ask	0.06	0.12	0.15
a_t	0.12	0.23	0.28
	0.16	0.29	0.37

Table 2: Fraction of times in which the best bid and ask are enclosed in some intervals

the individual bid intersects the thick line of the best ask). The numerical results show, in this respect, an important novelty if compared to some of the works mentioned previously, where impatient traders always go market: even the most aggressive buyer can submit limit orders and this occurs most probably in the first part of the session. In these cases, it is optimal to “close” the spread, placing aggressive bids that exceed 0.5 instead of accepting the expensive asks on the other side.

A very similar analysis could be conducted for the the lowest-cost seller’s ask that is decreasing in the position, thus being more aggressive in the proximity of the end of the session. We skip, for brevity, similar comments that could be made for other intramarginal buyers or sellers. We stress that Figure 4, though insightful, does not represent the *deterministic* state of the

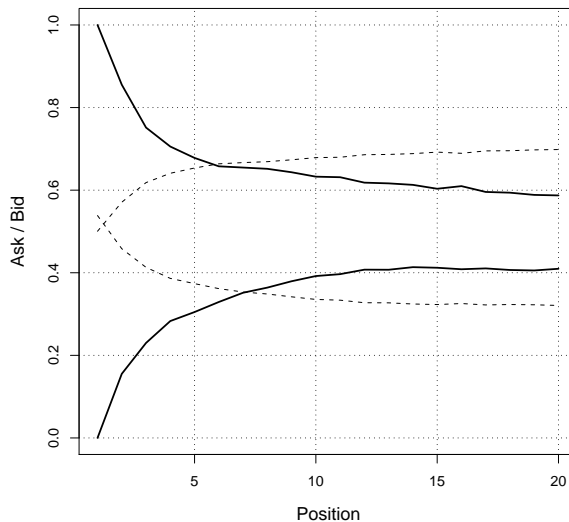


Figure 4: State of the book: the average best bid and best ask are shown as a function of the position in the queue (thick lines). The bids and asks of the buyer with $v = 0.95$ and the seller with $c = 0.05$ as functions of the position (thin lines).

book, nor the behavior of the agents, but only averages. Given the stochastic nature of the auction, it may well be that the state of the book is different from the one depicted and, hence, the bid or the ask can differ substantially from the thin lines in many specific instances.

Additional insight can be gleaned by looking at the kind of order that is produced, for different agents, by the strategies of Table 1. In particular, it is interesting to determine the level of aggressiveness of the order, given a certain state of the book before submission. Inspired by the classification of [Biais et al., 1995], we consider market(able), improving and *weak* orders, respectively. The first and most aggressive kind of order has a bid exceeding the best outstanding ask or an ask below the outstanding bid. Consequently, the order hit the other side of the book ending in immediate execution. The second kind of submission improves the quote (raising the best bid or lowering the best ask) without execution. Such type of orders narrow the spread and gain priority on one of the two sides of the book. Finally, the third variety is an order whose bid or ask hides behind the best outstanding quotes. Such an offer is called *weak* because it does not even gain priority and the order will, perhaps, be filled only after execution of more aggressive proposals that are already in the book.

Figure 5 shows, for all states of the book, the outcomes of equilibrium strategies for some buyers and sellers. The top row represents the effects of the submissions of three buyers with values $v = 0.95, 0.75, 0.55$. The first one is the most impatient buyer while the third one is close to be marginal. Each point in the graphs is colored according to the effect of the order: yellow if the bid/ask is market(able), orange if it is improving and red if it is weak. Take, say, the middle chart of the top row, relative to the buyer with $v = 0.75$, and assume that the book is in state $(a_t, b_t) = (0.70, 0.40)$. As the point $(0.70, 0.40)$ is orange-colored, the agent will submit an improving bid that is going to decrease the spread⁴ and gain priority on the buy side. The examination of the upper right panel of Figure 5 shows that the buyer whose value is 0.95 very often goes market. This always occurs when the ask is lower than 0.65, a very frequent situation (see the contour levels). However, when the ask is 0.70 or larger, the agent finds more convenient to leave an improving order on the book, with some exceptions given by the red states situated in the right upper corner.

The other two charts relative to different buyers show that, as expected, the use of market orders shrinks as the value decreases. The more “marginal” agent with $v = 0.55$, more often than not, issues weak or improving orders and rarely resorts to market(able) bids. A very similar pattern can be observed in the bottom row of the figure, that is relative to sellers. While the most impatient one is very often going market, the progressive reduction of the yellow area at the expense of the orange and red colors, shows that weaker agents mainly use improving submissions or hide behind the quotes.

4 Conclusion

We present an equilibrium trading model in a continuous double auction. Heterogeneous agents, with private values and costs, enter the market sequentially and use (minimal) information about the book to bid or ask to maximize gains. The allowed strategies are linear functions of the best quotes at the time of submission. The simplicity of the bidding functions allows a straightforward and intuitive description of the way agents compute their optimal offers. This may have the advantage to be more behaviorally plausible than other equilibrium results where, in essence, traders memorize huge look-up tables.

In equilibrium, impatient traders often issue market orders but there are

⁴In more detail, a look at equation (2) and at the coefficients of Table 1 reveals that the agent places a bid of $0.75 - (0.25 \cdot 0.70 - 0.06 \cdot 0.40 + 0.52/20) = 0.573$. As the reader can see, it is a clearly improving bid that reduces the spread from 0.3 to 0.127.

states in which they “close” the book and do not accept bloodsucking conditions. At the same time, patient traders that usually provide liquidity with limit orders, may “steal the deal” in the few cases in which it is possible. We feel both features are realistic and not easily found in other analytical works, see for instance [Foucault, 1999, Foucault et al., 2005] and [Rosu, 2009]. In such models, however, the extreme behavior of the most intramarginal (i.e., impatient) traders is needed for tractability. The unavoidable assumption that the cost of waiting is either very high or very low, effectively resulting in only two types of traders, is a device to close the model. In this respect, our work generalizes the setup, allowing more degrees of impatience through different values and costs. As a consequence, more flexible results can be obtained but, on the other hand, this painfully makes an analytical solution difficult (or impossible) and we have to rely only on numerical methods.

In our equilibrium, offers depend on both sides of the book, with traders generally more sensitive to the opposite side of the market. This feature is present in many other equilibrium models and implies that it is optimal to improve the existing quotes if the spread is large.

The set of equilibrium states is relatively smaller and while some conditions are frequent, other states of the book are virtually never encountered. This fact has practical and technical importance, as it stresses that optimal strategies are basically fit to a small set of recurrent states. Moreover, this is encouraging as the computational complexity of finding a solution may be significantly reduced if the support of the equilibrium is small. In our framework, similar to a thin market because of the small number of traders, this means that basically there are only two families of states: a set of initial configurations, where the book is empty or sparse and a set of more mature states, where agents face a full book with small spread and competitive offers.

The equilibrium is found using an evolution strategies optimization algorithm, where groups of different agents separately maximize the gains from trade. The method uses an endogenous (meta)parameter, related to the mutation rate, that enables to assess the progress of its convergence to a steady state.

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Appendix

The numerical results in this work have been obtained using a version of Evolution Strategies. Recall that we have to solve a Nash-like problem where each type of agent maximizes his profits in a game in which the set of trading partners and the position in the queue are sampled by the nature. In detail, we consider a set of subpopulations made of the agents having same value v

(buyers) or cost (sellers). Each population independently maximizes the gain from trade over τ of sessions, given the behavior of the other populations:

$$\max_{\alpha, \beta, \gamma} \sum_{i=0}^{\tau} \pi_i(\alpha, \beta, \gamma | \text{Other types}),$$

where π_i is the profit in one session and α, β, γ are the parameters appearing in the bidding function (2) or (3). In the model, we have 19 types of buyers and sellers for a total of 38 problems that are solved in parallel.

ES is a computational evolutionary algorithm that iteratively marries, recombines, mutates and selects solutions in a population of candidates, see [Beyer and Schwefel, 2002]. Fix a type (v or c) and assume the initial guesses for the parameters in (2, 3) are the vectors $(\alpha_m^{(0)}, \beta_m^{(0)}, \gamma_m^{(0)})$, $i = 1, \dots, \lambda$, that define the strategies of the agents. Assume also that $A_m^{(0)}, B_m^{(0)}, C_m^{(0)}$, $m = 1, \dots, \lambda$, are endogenous parameters used in the mutation stage.

We solve the problem using the following steps.

1. Set $g = 0$ and initialize the population $\mathcal{P}^{(0)}$ with $y_m^{(0)} = (\alpha_m^{(0)}, \beta_m^{(0)}, \gamma_m^{(0)}, A_m^{(0)}, B_m^{(0)}, C_m^{(0)})$, $m = 1, \dots, \lambda$;
2. Repeat
 - (a) sample without replacement $n + n$ agents and let them bid/ask according to their value/cost and parameters (α, β, γ) . In this phase, a CDA is simulated and transactions may occur in any given session;
 - (b) cumulate profit for τ sessions. Assume each agent has gained $F_m^{(g)}$, $m = 1, \dots, \lambda$ during the τ sessions;
 - (c) select the best μ agents out of λ according to $F_m^{(g)}$. Let the selected agents form the population $\mathcal{Q}^{(g)}$;
 - (d) for $l = 1, \dots, \lambda$ do
 - i. sample with replacement one agent $(\alpha_k, \beta_k, \gamma_k, A_k, B_k, C_k) \in \mathcal{Q}^{(g)}$, $k \in \{1, \dots, \mu\}$
 - ii. let

$$\begin{aligned} A_l^{(g+1)} &= \exp(v\tilde{z})A_k^{(g)} \\ B_l^{(g+1)} &= \exp(v\tilde{z})B_k^{(g)} \\ C_l^{(g+1)} &= \exp(v\tilde{z})C_k^{(g)} \end{aligned}$$

where \tilde{z} is a standard normal random variable (which is freshly sampled each time it is mentioned);

iii. let

$$\begin{aligned}\alpha_l^{(g+1)} &= \alpha_k^{(g)} + \tilde{z}A_l^{(g+1)} \\ \beta_l^{(g+1)} &= \beta_k^{(g)} + \tilde{z}B_l^{(g+1)} \\ \gamma_l^{(g+1)} &= \gamma_k^{(g)} + \tilde{z}C_l^{(g+1)}\end{aligned}$$

where \tilde{z} is a standard normal random variable (which is freshly sampled each time it is mentioned);

- (e) let the new individuals $(\alpha_l^{(g+1)}, \beta_l^{(g+1)}, \gamma_l^{(g+1)}, A_l^{(g+1)}, B_l^{(g+1)}, C_l^{(g+1)})$, $l = 1, \dots, \lambda$ form the population $\mathcal{P}^{(g+1)}$;
- (f) set $g = g + 1$
- (g) repeat until termination, i.e. goto a) if needed.

Points i), ii) and iii) above are called “marriage”, “s-mutation” and “y-mutation” in ES parlance. Basically, for each type, we move from one generation to another, after profits have been collected for τ session, by selecting the best μ strategies out of λ , shocking their endogenous parameters and creating λ new mutated strategies. There is no guarantee that any of the past μ strategies belonging to generation g will be sampled to breed a new strategy and all new strategies at $g + 1$ are noisy versions of the best past ones. Our method differs from the standard ES described, say, in Figure 1 of [Beyer and Schwefel, 2002] only because we need to compute the fitnesses F_l by simulation. Hence, line 11 in Beyer and Schwefel’s flowchart is replaced by a subroutine that materially manages τ trading sessions in a CDA.

Our algorithm is a $(\mu/1, \lambda)$ -ES as only 1 parent is used for each offspring (cloning). We used the “comma” version of ES, where all parents from generation g are left behind after reproduction. Observe that some initial experiments with a “plus” ES produced worse results. We used $\mu = 10$, $\lambda = 20$, $\tau = 200$, $v = \sqrt{1/6}$ and we stopped after the evolution of $g = 250$ generations. Our code is available at virgo.unive.it/paolop/notyet.R.

One of the more interesting features of ES is that the mutation parameters A, B, C can be gauged to ascertain convergence. Figure 6 shows the final level of A and B for buyers (right) and sellers (left), in a standard run of the algorithm. Taking into account that all A and B were initialized at 0.1, we can see that while all intramarginal traders reached small levels, there are several extramarginal agents for which A and B have grown up almost ten-fold. As discussed in the text, this fact raises the attention on the lack of convergence for some extramarginal types that never trade in equilibrium.

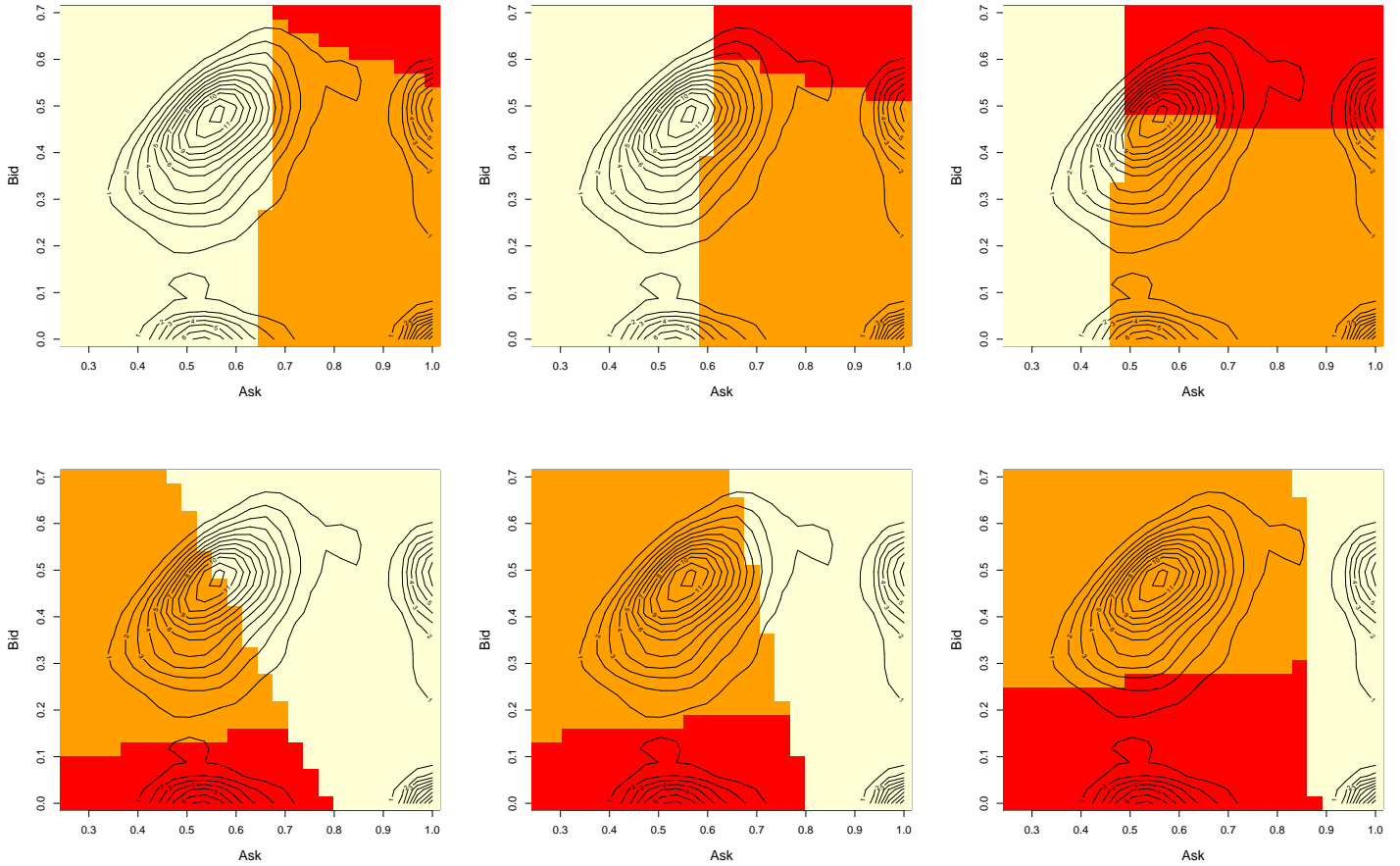


Figure 5: In the top row, the strategies used by buyers whose values are $v = 0.95, 0.75, 0.55$ (left, center, right). The bottom row depicts the strategies of sellers with costs $c = 0.05, 0.25, 0.45$ (left, center, right). States of the book are colored in yellow, orange and red if the bid of the agent results in a market(able), improving or weak order, respectively.

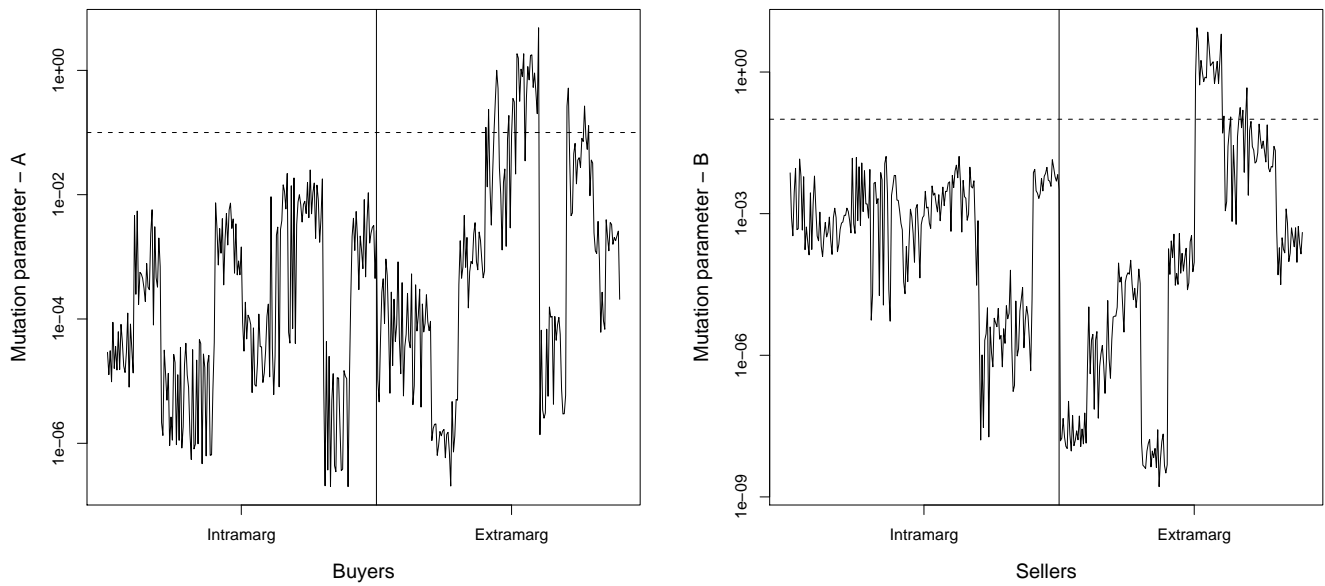


Figure 6: Final endogenous mutation parameters of the ES. On the left side, A is shown for buyers; on the right, B is plotted for sellers. The initialization level 0.1 is shown with a dashed line and a vertical line divides intramarginal traders from extramarginal ones. Notice the log-scale on the vertical axis.