Geometric Lévy Process Pricing Model

Yoshio Miyahara, Nagoya City University,
Alexander Novikov, University of Technology, Sydney,

Abstract

We consider models for stock prices which relates to random processes with independent homogeneous increments (Levy processes). These models are arbitrage free but correspond to the incomplete financial market. There are many different approaches for pricing of financial derivatives. We consider here mainly the approach which is based on minimal relative entropy. This method is related to an utility function of exponential type and the Esscher transformation of probabilistic measures.

1 Introduction

We suppose that the financial market consists of two assets : bond B_t and stock S_t . We use the notation r for the spot interest rate assuming that it is a constant. Then, under the assumption of continuous compounding, the value of the bond at time t is

$$B_t = B_0 \exp\{rt\}. \tag{1}$$

The classical diffusion model (Merton (1973) [29], Black and Scholes (1973) [6]) for the process S_t is

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2}$$

where W_t is a standard Wiener process, μ is the expected return and σ is the stock price volatility. The solution of this equation is

$$S_t = S_0 \exp\{\sigma W_t + gt\} \tag{3}$$

with $g = \mu - \frac{1}{2}\sigma^2$.

It is well-know that contrary to a Wiener process, returns of stocks (that is $\log(S_t)$) are neither Gaussian, nor homogeneous and not having independent increments, see e.g. Amaral et al 2000, [1]. In spite of these empirical observations the classical diffusion model remains as a reference model due to its simplicity. Trying to preserve this feature of simplicity one may keep the property of having independent homogeneous increments i.e. assuming that $\log(S_t)$ is a Levy process which could be a non-Gaussian process. So, we will assume that

$$S_t = S_0 \exp\{Z_t\} \tag{4}$$

where Z_t is a Levy process, $Z_0 = 0$.

In particular, it implies the distribution function of Z_t belongs to the class of infinite divisible distributions (see, e.g. Bertoin (1996) [7] or Sato (1999) [36]). Under this assumption the process Z_t has the following representation

$$Z_t = \sigma W_t + Y_t \tag{5}$$

where W_t is a standard Wiener process, Y_t is a jump Levy process which is independent of W_t . The process Y_t has the following representation in terms of the counting Poisson measure $N(dt, dx), t \geq 0,\ x \in R\backslash\{0\}$ generated by the jumps of Z_t :

$$Y_t = bt + \int_0^t \int_{R\backslash\{0\}} xI_{\{|x|\geq 1\}} N(ds, dx) + \int_0^t \int_{R\backslash\{0\}} xI_{\{|x|<1\}} [N(ds, dx) - \nu(dx)ds]. \tag{6}$$

Here $\nu(dx)$ is the so-called Levy measure which satisfies the following condition

$$\int \min(x^2, 1)\nu(dx) < \infty \tag{7}$$

(see, e.g. [7] or [36]).

The characteristic function of Y_t is given by

$$E \exp(iuY_t) = \exp\{t\psi(iu)\} \tag{8}$$

where the cumulant function $\psi(iu)$ can be written in the form

$$\psi(iu) = iub + \int_{-\infty}^{\infty} [\exp(iux) - 1 - iuxI\{|x| < 1\}]\nu(dx) \tag{9}$$

(it is a variant of the so-called Levy-Khintchine formula).

We will call the characteristic $(\sigma^2, v(dx), b)$ in the above representation *the triplet of* Z_t. (Slightly different terminologies are used in monographs [36] and [25].)

Remark 1. Chan (1999) [13] considered a stock price model in a different form

$$dS_t = \mu S_t dt + \sigma S_{t-} dX_t \tag{10}$$

where X_t is a Levy process (satisfying some conditions). This model is actually equivalent to the model (4) under the assumption that jumps $\Delta X_t > -1$ which is necessary for positiveness of prices S_t. To our mind it is more convenient to work with characteristics of the process Z_t from (4).

Some particular cases.

1) Compound Poisson models.

$$Y_t = tc + \sum_{k=1}^{N_t} \xi_k \tag{11}$$

where N_t is a homogeneous Poisson process with intensity λ, ξ_k are iid r.v. with a distribution function $\rho(dx)$. Then (as well-known)

$$\psi(iu) = iuc + \lambda \int_{-\infty}^{\infty} [\exp(iux) - 1]\rho(dx) \tag{12}$$

and so $\nu(dx) = \lambda\rho(dx)$. (Here and below the letter c without or with index denotes a constant). This type of model was firstly considered e.g. by Merton (1976) [30] and then by many authors. We also mention only recent papers Andersen and Andreasen (1999) [4], and Zhou (1999) [41]. Note that in the last two papers it was supposed that ξ_k are Gaussian r.v.'s with mean θ and variance σ_1^2. Under this assumption it is easy to see that the process Z_t can be written in a different form:

$$Y_t = ct + \theta N_t(\lambda) + \sigma_1 B_{\alpha_t} \tag{13}$$

where B_t is a standard Wiener process independent of $N_t(\lambda)$ and W_t,

$$\alpha_t = N_t(\lambda). \tag{14}$$

An another natural choice is with ξ_k as a mixture of exponential distributions, e.g. in the form

$$\rho(dx) = (c_1 I(x < 0)\exp(-c_3 x) + c_2 I(x > 0)\exp(-c_4 x))dx \tag{15}$$

Under this choice of $\rho(dx)$ the distribution of jumps of the process S_t is a Pareto-type which has some empirical justifications ([1], [34]).

2) Logstable models. The important particular case of this class of models can be given in the following form:

$$Y_t = tc + L_t^{(\alpha)} \tag{16}$$

where $L_t^{(\alpha)}$ is a stable Levy process with only negative jumps , $1 < \alpha < 2$. The moment-generating function of this process is given by the following simple formula

$$E\exp(zL_t^{(\alpha)}) = \exp(tCz^{\alpha}),\ \ z \geq 0 \tag{17}$$

where the constant $C > 0$. From here it is easy to derive that this process has the self-similarity property that is we have the following equality by distribution

$$L_t^{(\alpha)} - L_s^{(\alpha)} = (t-s)^{1/\alpha} L_1^{(\alpha)}, t \geq s \tag{18}$$

Mandelbrot (1963) [28] was the first who suggested to model $\log(S_t)$ as a symmetric stable Levy process and he presented some statistical evidence for his approach (exploring, actually, a small sample size of about 2000 data points). But in his case the stock process S_t does not have moments and that contradicts other intensive empirical findings (see discussion in [1]). The only case of stable Levy processes for which all moments of S_t are finite is given by (17).

Between other related papers we mention only recent ones. Hurst et al (1999) [23] studied a very similar model but in terms of stable subordinators to a Wiener process. Carr and Wu (2000) [12] considered examples of stock prices for which the Logstable model (as in (17)) statistically better against several alternatives.

The Levy measure of the stable process with moment generating function (17) has a very simple form:

$$\nu(dx) = cI(x<0)|x|^{-(\alpha+1)}dx \tag{19}$$

but there are no simple formulas for density of r.v. $L_t^{(\alpha)}$. Note this density can be calculated in terms of Fast Fourier Transformation using an analytical continuation of (17). See also Carr and Madan (1999) [11].

3) Variance Gamma model and CGMY model. The Variance Gamma (VG) process introduced by Madan and Seneta (1990) [26] and then extended in Madan et al (1998) [27]. The main feature is obtained by evaluating Brownian motion with drift at random times given by a gamma process. The gamma process γ_t with mean rate μ and variance rate ν is the process of independent gamma increments with the density

$$p(x) = C_t x^{\frac{\mu^2 t}{\nu}-1}\exp(-\frac{\mu}{\nu}x),\ x>0,\ C_t = \left(\frac{\mu}{\nu}\right)^{\frac{\mu^2 t}{\nu}} / \Gamma(\frac{\mu^2 t}{\nu}) \tag{20}$$

where $\Gamma(x)$ is the gamma function. The VG process Y_t is then defined in terms of Brownian motion with drift and the gamma process with unit mean rate $\mu = 1$ as follows:

$$Y_t = \theta\gamma_t + \sigma_1 B_{\gamma_t} \tag{21}$$

where B_t is a standard Wiener process independent of γ_t.

The characteristic function of value Y_t is

$$E[\exp(iuY_t] = \left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{t/v} \tag{22}$$

with the following relations between parameters when $\sigma_1 > 0$:

$$\beta = \theta/\sigma_1^2,\quad \alpha^2 = \beta^2 + 2/\left(\nu\sigma_1^2\right). \tag{23}$$

It is easy to check that the VG process has also the another representation:

$$Y_t = \gamma_t^{pos} - \gamma_t^{neg} \tag{24}$$

where γ_t^{pos} and γ_t^{neg} are **independent** gamma processes ([27]).

The density of Y_t can be expressed in terms of the modified Bessel function of the second kind $K_\lambda(z)$ (see [27]) in the following way:

$$p(x) = C_t\, K_{t/\nu-1/2}(\alpha|x|)\,|x|^{t/\nu-1/2}\exp(\beta x),\ t>0 \tag{25}$$

where C_t denotes a normalizing constant (here and below).

There are several different representations of function $K_\lambda(z)$. The one of most convenient is the following (see e.g. Abramowitz (1972) [2]):

$$K_\lambda(z) = \int_0^\infty \exp\{-z\cosh(x)\}\cosh(\lambda x)dx \tag{26}$$

The VG model has several very atractive features. To simulate the trajectories of VG processes one need only to simulate gamma and normal distribution (or, two independent gamma distributions). Also, the Levy measure has a simple form

$$\nu(dx) = (c_1 I(x<0)\exp(-c_3 x) + c_2 I(x>0)\exp(-c_4 x))|x|^{-1}dx \qquad (27)$$

and it is of convenience for pricing models (see Albanese et al (2001) [3]).

Carr-Geman-Madan-Yor have introduced the CGMY process in [10], which is an extended process of VG process. The Lévy measure of the CGMY process is

$$\nu(dx) = c\left(I_{\{x<0\}}\exp(Gx) + I_{\{x>0\}}\exp(-Mx)\right)|x|^{-(1+Y)}dx, \qquad (28)$$

where $c > 0, G \geq 0, M \geq 0, Y < 2$. In the case that $Y \leq 0$, then $G > 0$ and $M > 0$ are assumed. We mention here that the case $Y = 0$ is the VG process case, and the case $G = M = 0$ and $0 < Y < 2$ is the symmetric stable process case.

4) The Hyperbolic model.

The Hyperbolic model is discussed by Eberlain et al (1998) [17]. The density of the terminal value of the return Z_T (up to drift term) is supposed to be the infinitely divisible distribution whose density depends on four parameters (δ, β, α) and, of course, maturity T and has a form

$$p(x) = C\exp\left\{-\alpha\sqrt{\delta^2+x^2}+\beta x\right\}, -\infty < x < \infty \qquad (29)$$

where

$$C = \frac{\sqrt{\alpha^2-\beta^2}}{2\delta\alpha K_1\left(\delta\sqrt{\alpha^2-\beta^2}\right)}; \quad \delta \geq 0, \ \ |\beta| < \alpha. \qquad (30)$$

$K_1(z)$ is the modified Bessel function of the second kind (defined above).

The characteristic function of value Z_t for all $t \geq 0$ is ([17])

$$E[\exp(iuZ_t] = (\frac{\tilde{K}_1(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{\tilde{K}_1(\delta\sqrt{\alpha^2-\beta^2})})^{t/T} \qquad (31)$$

where $\tilde{K}_\lambda(z) = K_\lambda(z)z^{-\lambda}$.

Note that the simple expression (29) is valid only for the terminal value Z_T. Alternatively, one could assume that increments $Z_{t+\delta} - Z_t$ with some $\delta > 0$ has density (29) and then check that the characteristic function of value Z_t is the same as in (31) but with δ instead of T. To calculate the density of Z_t for arbitrary t one may use the Fourier transformation of the characteristic function of the process or some other integral representations.

5) Generalized Hyperbolic (GH) Model. The class of generalized hyperbolic distributions was introduced in Barndorff-Nielsen and Halgreen (1977) [5]. Eberlein at al (1998) [17] considered it as the natural generalization of Hyperbolic model. According to the GH model the density of the random variable Z_T (up to drift term) is

$$p(x) = \tag{32}$$

$$= C_T K_{\lambda-1/2}(\alpha\sqrt{\delta^2+x^2})(\sqrt{\delta^2+x^2})^{\lambda-1/2}e^{(\beta x)}, \tag{33}$$

where

$$C_T = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^\lambda K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \tag{34}$$

and

$$\delta \geq 0, |\beta| < \alpha \quad \text{if } \lambda > 0 \tag{35}$$
$$\delta > 0, |\beta| < \alpha \quad \text{if } \lambda = 0 \tag{36}$$
$$\delta > 0, |\beta| \leq \alpha \quad \text{if } \lambda < 0. \tag{37}$$

Note that the distribution of the GH process, like that of the hyperbolic process, is given in the explicit form only at **maturity**, that is for $t = T$, and not for arbitrary $t < T$.

The characteristic function of Z_t for all $t \geq 0$ is ([17])

$$E[\exp(iu\,Z_t] = (\frac{\tilde{K}_\lambda(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{\tilde{K}_\lambda(\delta\sqrt{\alpha^2-\beta^2})})^{t/T} \tag{38}$$

6) Processes subordinated to the Wiener process. This type of model have the following form

$$Z_t = ct + \theta\alpha_t + \sigma_1 W_{\alpha_t} \tag{39}$$

where α_t is a non-negative Levy process (subordinator). The prove that the process Z_t is a Levy process can be easily done with help of conditioning of the characteristic functions of Z_t . The particular cases of this model are the Gauss-Poisson model (13) and the VG model (21).

The another interesting choice is when α_T has the Generalized Inverse Gaussian distribution with p.d.f.

$$p(x) = C_T \, x^{T/\nu - 1} \exp\{-x/\nu - \delta^2/x\}, \quad x > 0 \tag{40}$$

Then accordingly to [5] Z_t has the generalized hyperbolic distribution with the characteristic function

$$E[\exp(iuZ_t] = (\frac{\tilde{K}_{T/\nu}(\delta\sqrt{\alpha^2 - (\beta + iu)^2})}{\tilde{K}_{T/\nu}(\delta\sqrt{\alpha^2 - \beta^2})})^{t/T} \tag{41}$$

where

$$\beta = \theta/\sigma_1^2, \;\; \alpha^2 = \beta^2 + 2/(\nu\sigma_1^2) \tag{42}$$

Comparing this formula with (38) one can easily see that under of this choice of α_T we have just the another representation for the GH model when $\lambda\nu = T$.

The case of VG model can be considered as the limiting case the GH model when $\delta \to 0$. Indeed, using the asymptotics $\tilde{K}_\lambda(z) \sim z^{-2\lambda}$ as $z \to 0$ (see [2]) we get from (38) formula (21).

2 Minimal Relative Entropy Martingale Measure (MEMM) and Pricing Models

The models described in Chapter 1 are incomplete market models in general. Only the Brownian motion model and the simple Poisson process model are the exceptional cases. In fact, other geometric Lévy process models have many equivalent martingale measures if an equivalent martingale measure exists.

The pricing models for the incomplete markets are, in general, consisting of the following two parts. The first part is the part of defining the price process of underlying assets, and the second part is the part of defining the pricing rule of options.

For the second part we follows to the so-called martingale method, and we adopt the minimal entropy martingale measure (MEMM) as the suitable martingale measure among several candidate martingale measures (see [15], [19], [20], [31], etc.). Then the price of an option is given as the expectation of that option with respect to the MEMM.

2.1 Minimal Relative Entropy Martingale Measure of Geometric Lévy Processes

We will first give the definition of MEMM, and then we will see the existence problem of MEMM of the geometric Lévy processes.

1) MEMM

Let $\mathcal{P}(S)$ be the set of all equivalent martingale measures of S_t.

Definition 1 (minimal entropy martingale measure (MEMM)) *If an equivalent martingale measure P^* satisfies the following condition*

$$H(P^*|P) \leq H(Q|P) \qquad \forall Q \in \mathcal{P}(S) \tag{43}$$

where $H(Q|P)$ is the relative entropy of Q with respect to P, which is given by the following formula

$$H(Q|P) = \begin{cases} \int_\Omega \log[\frac{dQ}{dP}]dQ, & if \quad Q \ll P, \\ \infty, & otherwise, \end{cases} \tag{44}$$

then P^* is called the minimal entropy martingale measure (MEMM) of S_t.

The basic properties of MEMM are described in §2 of [31]. For example, it is known that if the MEMM exists then it is unique.

There are many reasons why we adopt the MEMM as the suitable equivalent martingale measure. One of them is the fact that the MEMM is related to the utility function of exponential type (see [20]). And another one is the relations of the MEMM with the theory of large deviation (remember Sanov's Lemma). And the third one is the fact that this measure is strongly related to the Esscher transformation.

2) Existence of MEMM

The existence problem of the MEMM of geometric Lévy processes has been discussed in [13],[33] and [21]. The most general form is given in [21].

We use the same notations as in Chapter 1. For the simplicity we assume that $B_0 = 1$, namely the value of bond is given

$$B_t = \exp\{rt\}.$$

Let a Lévy process Z_t be given in the the following general form

$$\begin{aligned} Z_t &= \sigma W_t + Y_t \\ &= \sigma W_t + bt + \int_0^t \int_{R\setminus\{0\}} x I_{\{|x|>1\}} N_p(dsdx) \\ &\quad + \int_0^t \int_{R\setminus\{0\}} x I_{\{|x|\le 1\}} [N(ds,dx) - \nu(dx)ds], \end{aligned} \tag{45}$$

and consider the following conditions.

Condition (C) There exists $\theta^* \in R$ which satisfies the following conditions :

$$(\mathbf{C})_1 \qquad \int_{\{x>1\}} e^x e^{\theta^*(e^x-1)} \nu(dx) < \infty, \tag{46}$$

$$\begin{aligned} (\mathbf{C})_2 \quad & b + (\tfrac{1}{2}+\theta^*)\sigma^2 + \int_{R\setminus\{0\}} (e^x-1)e^{\theta^*(e^x-1)} I_{\{|x|>1\}}\,\nu(dx) \\ & + \int_{R\setminus\{0\}} \left((e^x-1)e^{\theta^*(e^x-1)} - x\right) I_{\{|x|\le 1\}}\,\nu(dx) = r. \end{aligned} \tag{47}$$

Under the above assumptions the following theorem is obtained in [21].

Theorem 1 (Fujiwara-Miyahara [21, Theorem 3.1]) *Suppose that the condition **(C)** holds, and let P^* be the probability measure defined by*

$$\left.\frac{dP^*}{dP}\right|_{\mathcal{F}_t} = \exp[\theta^*\sigma W_t - \frac{1}{2}(\theta^*\sigma)^2 t + \theta^* \int_0^t \int_{R\setminus\{0\}} (e^x-1) I_{\{|x|>1\}} N_p(dsdx)$$

$$+\theta^* \int_0^t \int_{R\setminus\{0\}} (e^x-1) I_{\{|x|\le 1\}} [N(ds,dx) - \nu(dx)ds]$$

$$-t\int_{R\setminus\{0\}}\left(e^{\theta^*(e^x-1)}-1-\theta^*(e^x-1)I_{\{|x|\le 1\}}\right)\nu(dx)]. \tag{48}$$

Then the probability measure P^* is well defined and it holds that
(a)(MEMM): P^* is the MEMM of S_t.
(b)(Minimal Relative Entropy):

$$H(P^*|P)=-T[\theta^*(\frac{1+\theta^*}{2})\sigma^2)+\theta^*(b-r)+\int_{R\setminus\{0\}}\left(e^{\theta^*(e^x-1)}-1-\theta^* xI_{\{|x|\le 1\}}\right)\nu(dx)]. \tag{49}$$

(c)(Lévy process): Z_t is also a Lévy process w.r.t. P^*, and the generating triplet (A^*,ν^*,b^*) of Z_t under P^* is

$$A^* = A(=\sigma^2), \tag{50}$$
$$\nu^*(dx) = e^{\theta^*(e^x-1)}\nu(dx), \tag{51}$$
$$b^* = b+\theta^*\sigma^2+\int_{R\setminus\{0\}} xI_{\{|x|\le 1\}}d(\nu^*-\nu). \tag{52}$$

Remark 1.

If the following condition

$$\int_{R\setminus\{0\}}|x|I_{\{|x|\le 1\}}\nu(dx)<\infty \tag{53}$$

is satisfied, then the generating triplet of Z_t is $(\sigma^2,\nu,\gamma_0)_0$ (see [36, p.39]) and Z_t is expressed

$$Z_t=\sigma W_t+\gamma_0 t+\int_0^t\int_{R\setminus\{0\}} xN_p(dsdx). \tag{54}$$

In this case, the condition $(\mathrm{C})_2$ is replaced by

$$\gamma_0+(\frac{1}{2}+\theta^*)\sigma^2+\int_{R\setminus\{0\}}(e^x-1)e^{\theta^*(e^x-1)}\nu(dx)=r. \tag{55}$$

and as the corollary of Theorem 1 we obtain

Theorem 2 *Suppose that the Lévy process Z_t is given by (54) . Assume that there exists a constant θ^* which satisfies the equation (55), and let P^* be the probability measure defined by*

$$\left.\frac{dP^*}{dP}\right|_{\mathcal{F}_t}=\exp[\theta^*\sigma W(t)-\frac{1}{2}(\theta^*\sigma)^2 t+\int_0^t\int_{R\setminus\{0\}}\theta^*(e^x-1)N_p(dsdx)-$$

$$-\int_0^t \int_{R\setminus\{0\}} (e^{\theta^*(e^x-1)} - 1)\nu(dx)]. \tag{56}$$

Then the probability measure P^* is well defined and it holds that
(a)(MEMM): P^* is the MEMM of S_t.
(b)(Minimal Relative Entropy):

$$H(P^*|P) = -T[\theta^*(\frac{1+\theta^*}{2})\sigma^2) + \theta^*(\gamma_0 - r) + \int_{R\setminus\{0\}} (e^{\theta^*(e^x-1)} - 1)\nu(dx)]. \tag{57}$$

(c)(Lévy process): Z_t is also a Lévy process w.r.t. P^*, and the generating triplet $(A^*, \nu^*, \gamma_0^*)_0$ of Z_t under P^* is

$$A^* = A(=\sigma^2), \tag{58}$$
$$\nu^*(dx) = e^{\theta^*(e^x-1)}\nu(dx), \tag{59}$$
$$\gamma_0^* = \gamma_0 + \theta^*\sigma^2. \tag{60}$$

3) Comparison of the MEMM with the MMM

We see the relations of MEMM with the minimal martingale measure (MMM). The MMM was introduced by Föllmer-Schweizer in [19] and has been investigated by M. Schweizer ([35]), etc.

Suppose that the price process is a semimartingale and represented as

$$S_t = S_0 + M_t + \int_0^t \alpha d\langle M\rangle$$

then the MMM $\hat{P}$ is the (possibly signed) measure defined by

$$\frac{d\hat{P}}{dP} = \mathcal{E}\left(-\int \alpha dM\right),$$

where $\mathcal{E}(M)$ is the Doléans-Dade exponential of M.

In [21, §4] it is shown that when we apply the above result to the geometric Lévy process, then we know that under the somewhat strong assumptions (for example $\int_{R\setminus\{0\}}(e^x-1)^2\nu(dx) < \infty$) the MMM exists and is given by

$$\frac{d\hat{P}}{dP} = \mathcal{E}(\hat{M})$$

$$\hat{M}_t = \hat{\theta}\left(\sigma W_t + \int_0^t \int_{R\setminus\{0\}} (e^x - 1)[N(ds, dx) - \nu(dx)ds]\right),$$

where

$$\hat{\theta} = \frac{r - \hat{b}}{\sigma^2 + \int_{R\setminus\{0\}} (e^x - 1)^2 \nu(dx)}$$

$$\hat{b} = \frac{1}{2}\sigma^2 + b + \int_{R\setminus\{0\}} (e^x - 1 - xI_{\{|x|\leq 1\}})\nu(dx).$$

On the other hand, it is shown in [21] that the formula (6) of Theorem 1 is represented as

$$\left.\frac{dP^*}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}(M^*)_t, \qquad (61)$$

$$M_t^* = \theta^* \sigma W_t + \int_0^t \int_{R\setminus\{0\}} \left(e^{\theta^*(e^x-1)} - 1\right) [N(ds, dx) - \nu(dx)ds]. \qquad (62)$$

From the above results it follows that the MEMM is different from the MMM except the very special cases (such cases that the equivalent martingale measure is unique, namely the complete market cases).

We mentions here that the MMM has the following week points when compared with the MEMM:
1)The MMM is possibly signed measure. (MEMM is a probability measure.)
2)The class of the processes to which the MMM theory is applicable is limited compared with that of the MEMM.

4) Relations with the Esscher transformation

It is proved in [21] that

$$\left.\frac{dP^*}{dP}\right|_{\mathcal{F}_t} = \frac{e^{\theta^* \tilde{R}_t}}{E_P[e^{\theta^* \tilde{R}_t}]},$$

where $\tilde{R}_t$ is the return process of the discounted price process $\tilde{S}_t = S_t e^{-rt}$, namely

$$\tilde{R}_t = \int_0^t \frac{1}{\tilde{S}_{u-}} d\tilde{S}_u, \quad (therefore \quad \tilde{S} = \mathcal{E}(\tilde{R})).$$

This formula means that the MEMM P^* is obtained as the Esscher transformation by the return process $\tilde{R}_t$.

2.2 Pricing Models

When we start from the geometric Lévy process and adopt the MEMM as the special equivalent martingale measure, we call it the [Geometric Lévy Process & MEMM] pricing model (see [33]). We can apply the results of the previous section to the models noted in Chapter 1, and we obtain several examples of the [Geometric Lévy Process & MEMM] pricing model.

Before we see examples we investigate the Condition (C). If the condition $(C)_2$ has the sense then the condition $(C)_1$ follows. So we consider the condition $(C)_2$. Set

$$\begin{aligned} f(\theta) &= b + (\frac{1}{2}+\theta)\sigma^2 + \int_{R\setminus\{0\}} (e^x-1)e^{\theta(e^x-1)} I_{\{|x|>1\}}\,\nu(dx) \\ &\quad + \int_{R\setminus\{0\}} \left((e^x-1)e^{\theta(e^x-1)} - x\right) I_{\{|x|\le 1\}}\,\nu(dx). \end{aligned} \tag{63}$$

Then the condition $(C)_2$ is equivalent to the condition that the following equation

$$f(\theta) = r \tag{64}$$

has a solution.

It is easy to see that $f(\theta)$ is an increasing function of θ. Therefore, if $f(\theta)$ is continuous and $\lim_{\theta\to-\infty} f(\theta) < r < \lim_{\theta\to\infty} f(\theta)$ then the equation (64) has a unique solution.

Note that

$$E(S_t) = S_0 \exp(tf(0)) \tag{65}$$

If one assume that $f(0) \geq r$ (it seems that in some cases it is really natural to assume) then the solution θ^* of (64) is non-positive.

1) Brownian Motion Model

Suppose that the Lévy process Z_t consists of continuous part only. Then Z_t is in the following form

$$Z_t = bt + \sigma W_t.$$

This case is identical with Black-Scholes model, and equation $(C)_2$ has a solution $\theta^* = -\frac{1}{2} - \frac{b-r}{\sigma^2}$, and P^* is the unique risk neutral martingale measure.

2) Compound Poisson Model

Suppose that the Lévy process Z_t is compound Poisson process and that the Lévy measure $\nu(dx)$ is given in the form of

$$\nu(dx) = \lambda\rho(dx), \tag{66}$$

where $\rho(dx)$ is a probability measure on R such that $\rho(\{0\}) = 0$, and λ is a positive constant. Then the equation for θ^* is

$$\gamma_0 + \lambda \int_{R\setminus\{0\}} (e^x - 1)e^{\theta^*(e^x-1)}\rho(dx) = r. \tag{67}$$

It is easy to see that this equation has a solution θ^*, then by Theorem 2 (a) the MEMM, P^*, exists and by Theorem 2 (c) the process $Z_t = \log[S_t/S_0]$ is also a compound Poisson process with Lévy measure $\nu^*(dx) = e^{\theta^*(e^x-1)}\nu(dx)$.

3) (Brownian Motion + Compound Poisson) Model

Suppose that the Lévy process Z_t consists of continuous part and compound Poisson part. Then the equation (13) for θ^* is

$$\gamma_0 + (\frac{1}{2} + \theta^*)\sigma^2 + \lambda \int_{(-\infty,\infty)\setminus\{0\}} (e^x - 1)e^{\theta^*(e^x-1)}\rho(dx) = r. \tag{68}$$

Suppose that this equation has a solution θ^*, then the process Z_t is also a Lévy process under the MEMM P^* with the generating triplet $(\sigma^2, c^*\rho^*(dx), \gamma_0 + \theta^*\sigma^2)_0$, where

$$\lambda^* = \lambda \int e^{\theta^*(e^x-1)}\rho(dx), \quad \rho^*(dx) = \frac{e^{\theta^*(e^x-1)}\rho(dx)}{\int e^{\theta^*(e^x-1)}\rho(dx)}. \tag{69}$$

In the above example, suppose that the Lévy measure $\nu(dx)$ is discrete, namely in the following form

$$\nu(dx) = \lambda\rho(dx) = \lambda \sum_{i=1}^{\infty} p_i \delta_{a_i}(dx), \quad p_i \geq 0, i = 1, 2, \ldots, \quad \sum_{i=1}^{\infty} p_i = 1. \tag{70}$$

Then the equation (13) is in the following form.

$$\gamma_0 + (\frac{1}{2} + \theta^*)\sigma^2 + \lambda \sum_{i=1}^{\infty} p_i(e^{a_i} - 1)e^{\theta^*(e^{a_i}-1)} = r. \tag{71}$$

4) LogStable Model

The Lévy measure $\nu(dx)$ of α-stable $(0 < \alpha < 2)$ distribution is

$$\nu(dx) = c_1 I_{\{x>0\}} |x|^{-(\alpha+1)} dx + c_2 I_{\{x<0\}} |x|^{-(\alpha+1)} dx \tag{72}$$

where c_1 and c_2 are non-negative constants. In the sequel we assume that

$$c_1 > 0, \quad c_2 > 0. \tag{73}$$

Suppose that Z_t is a stable process and its generating triplet is $(0, \nu(dx), b)$, where $\nu(dx)$ is of the form (72). By Theorem 1, if the equation $f(\theta) = r$ has a solution θ^*, then it holds that Z_t is also a Lévy process w.r.t. P^*, and the generating triplet (A^*, ν^*, b^*) of Z_t under P^* is

$$\begin{aligned} A^* &= 0, && (74) \\ \nu^*(dx) &= e^{\theta^*(e^x-1)} \nu(dx) \\ &= c_1 I_{\{x>0\}} \frac{e^{\theta^*(e^x-1)}}{|x|^{(\alpha+1)}} dx + c_2 I_{\{x<0\}} \frac{e^{\theta^*(e^x-1)}}{|x|^{(\alpha+1)}} dx, && (75) \\ b^* &= b + \int_{R\setminus\{0\}} x I_{\{|x|\le 1\}} d(\nu^* - \nu). && (76) \end{aligned}$$

We will examine the conditions such that the equation $f(\theta) = r$ has a solution. When the Lévy measure $\nu(dx)$ is given by (72), the function $f(\theta)$ is

$$\begin{aligned} f(\theta) &= b + c_2 \int_{-\infty}^{-1} \frac{(e^x - 1)e^{\theta(e^x-1)}}{|x|^{(\alpha+1)}} dx \\ &\quad + c_2 \int_{-1}^{0} \frac{\left((e^x - 1)e^{\theta(e^x-1)} - x\right)}{|x|^{(\alpha+1)}} dx + c_1 \int_{0}^{1} \frac{\left((e^x - 1)e^{\theta(e^x-1)} - x\right)}{|x|^{(\alpha+1)}} dx \\ &\quad + c_1 \int_{1}^{\infty} \frac{(e^x - 1)e^{\theta(e^x-1)}}{|x|^{(\alpha+1)}} dx, && (77) \end{aligned}$$

and

$$\begin{aligned} f(0) &= b + c_2 \int_{-\infty}^{-1} \frac{(e^x - 1)}{|x|^{(\alpha+1)}} dx + c_2 \int_{-1}^{0} \frac{(e^x - 1 - x)}{|x|^{(\alpha+1)}} dx \\ &\quad + c_1 \int_{0}^{1} \frac{(e^x - 1 - x)}{|x|^{(\alpha+1)}} dx + c_1 \int_{1}^{\infty} \frac{(e^x - 1)}{|x|^{(\alpha+1)}} dx && (78) \\ &= \infty. \end{aligned}$$

It can be proved that $f(\theta)$ is a continuous increasing function on $(-\infty, 0)$ and that

$$\lim_{\theta \to -\infty} f(\theta) = -\infty, \tag{79}$$

$$\lim_{\theta \uparrow 0} f(\theta) = \infty. \tag{80}$$

Therefore the equation $f(\theta) = r$ has a negative solution θ^*. Thus we have obtained the following result.

Proposition 1 *(1) Under the assumption $c_1, c_2 > 0$, the equation $f(\theta) = r$ has a unique solution θ^*, and the solution θ^* is negative.*
(2) The MEMM P^* is determined by θ^*, and the generating triplet (A^*, ν^*, b^*) of Z_t under P^* is given by (74), (75) and (76).

As the corollary of this proposition, we obtain

Corollary 1 *Under the MEMM P^*, any moments $E_{P^*}[|S_t|^k], k = 1, 2, \dots,$ of S_t are finite.*

Remark 2 Since S_t has finite moments of any degree under theMEMM P^*, if the option O satisfies such conditions as $|O| < (S_T)^k$, etc., then the price of O is computable as the expectation $E_{P^*}[O]$.

5) Variance Gamma Model

As we have seen in Chapter 1, the variance gamma (VG) distribution has the following distribution density,

$$p(x, t) = C_t \, K_{t/v-1/2}(\alpha|x|) \, |x|^{t/v-1/2} \exp(-\beta x), \quad t > 0. \tag{81}$$

And the Lévy measure is of the following form.

$$\nu(dx) = C \left(I_{\{x<0\}} \exp(-c_1|x|) + I_{\{x>0\}} \exp(-c_2|x|) \right) |x|^{-1} dx, \tag{82}$$

where C, c_1, c_2 are positive constants. In this case, the condition (53) is satisfied, and so we can use the expressions (54) and (55). Set the left hand side of (55) as f_0, namely we set

$$f_0(\theta) = \gamma_0 + \int_{R\setminus\{0\}} (e^x - 1) e^{\theta(e^x-1)} \nu(dx). \tag{83}$$

So the equation $f_0(\theta) = r$ is

$$\begin{aligned} f_0(\theta) &= \gamma_0 + C \left(\int_{-\infty}^0 \frac{1}{|x|} e^{-c_1|x|} (e^x - 1) e^{\theta(e^x-1)} dx + \int_0^\infty \frac{1}{|x|} e^{-c_2|x|} (e^x - 1) e^{\theta(e^x-1)} dx \right) \\ &= r. \end{aligned} \tag{84}$$

It is clear that

$$f_0(\theta) = \infty \quad if \quad \theta > 0. \tag{85}$$

It can be proved that $f_0(\theta)$ is a continuous increasing function on $(-\infty, 0)$ and that

$$\lim_{\theta\to-\infty} f_0(\theta) = -\infty, \tag{86}$$
$$\lim_{\theta\uparrow 0} f_0(\theta) = f_0(0), \tag{87}$$

and

$$f_0(0) = \infty, \quad if \quad c_2 \leq 1, \tag{88}$$
$$f_0(0) < \infty, \quad if \quad c_2 > 1, \tag{89}$$

Thus we have obtained

Proposition 2 *(1) If $c_2 \leq 1$, then the equation $f_0(\theta) = r$ has a unique solution θ^*, and the solution is negative.*
(2) If $c_2 > 1$ and $f_0(0) \geq r$, then the equation $f_0(\theta) = r$ has a unique solution θ^*, and the solution is non-positive.
(3) If $c_2 > 1$ and $f_0(0) < r$, then the equation $f_0(\theta) = r$ has no solution.
(4) When the equation $f_0(\theta) = r$ has a solution θ^*, then the MEMM P^* exists and is determined by θ^*. The generating triplet $(A^*, \nu^*, \gamma_0^*)_0$ of Z_t under the MEMM P^* is given by

$$A^* = 0, \tag{90}$$
$$\nu^*(dx) = C\left(I_{\{x<0\}}\frac{e^{-c_1|x|}e^{\theta^*(e^x-1)}}{|x|}dx + I_{\{x>0\}}\frac{e^{-c_2|x|}e^{\theta^*(e^x-1)}}{|x|}dx\right), \tag{91}$$
$$\gamma_0^* = \gamma_0. \tag{92}$$

Since θ^* is non-positive, the MEMM P^* has good properties as we have seen for the stable models in the above.

6) CGMY Model

The function $f(\theta) = r$ in the equation (C_2) is

$$\begin{aligned} f(\theta) = {} & b + C\left(\int_{-\infty}^{-1} \frac{(e^x-1)e^{\theta(e^x-1)}e^{-G|x|}}{|x|^{(1+Y)}}dx \right.\\ & + \int_{-1}^{0} \frac{\left((e^x-1)e^{\theta(e^x-1)} - x\right)e^{-G|x|}}{|x|^{(1+Y)}}dx + \int_{0}^{1} \frac{\left((e^x-1)e^{\theta(e^x-1)} - x\right)e^{-M|x|}}{|x|^{(1+Y)}}dx \\ & \left. + \int_{1}^{\infty} \frac{(e^x-1)e^{\theta(e^x-1)}e^{-M|x|}}{|x|^{(1+Y)}}dx\right), \end{aligned} \tag{93}$$

and $f(0)$ is

$$\begin{aligned} f(0) &= b + C\Bigg(\int_{-\infty}^{-1} \frac{(e^x-1)e^{-G|x|}}{|x|^{(1+Y)}}dx \\ &\quad + \int_{-1}^{0} \frac{(e^x-1-x)e^{-G|x|}}{|x|^{(1+Y)}}dx + \int_{0}^{1} \frac{(e^x-1-x)e^{-M|x|}}{|x|^{(1+Y)}}dx \\ &\quad + \int_{1}^{\infty} \frac{(e^x-1)e^{-M|x|}}{|x|^{(1+Y)}}dx\Bigg). \qquad (94) \end{aligned}$$

We can carry out the same argument as we have done in the above for the stable processes and the VG processes, and we obtain the following results.

Proposition 3 *(1) If $M \le 1$, then the equation $f(\theta) = r$ has a unique solution θ^*, and the solution is negative.*
(2) If $M > 1$ and $f(0) \ge r$, then the equation $f(\theta) = r$ has a unique solution θ^*, and the solution is non-positive.
(3) If $M > 1$ and $f(0) < r$, then the equation $f(\theta) = r$ has no solution.
(4) When the equation $f(\theta) = r$ has a solution θ^*, then the MEMM P^* exists and is determined by θ^*. The generating triplet (A^*, ν^*, b^*) of Z_t under the MEMM P^* is given by

$$A^* = 0, \qquad (95)$$

$$\nu^*(dx) = C\left(I_{\{x<0\}}\frac{e^{-G|x|}e^{\theta^*(e^x-1)}}{|x|^{(1+Y)}}dx + I_{\{x>0\}}\frac{e^{-M|x|}e^{\theta^*(e^x-1)}}{|x|^{(1+Y)}}dx\right), \qquad (96)$$

$$b^* = b + \int_{R\setminus\{0\}} x I_{\{|x|\le 1\}} d(\nu^* - \nu). \qquad (97)$$

7) Hyperbolic Model and Generalized Hyperbolic Model

For the hyperbolic models and the generalized hyperbolic (GH) models, we can do the same investigation as we have done for the stable models and VG models in the above, and the results are similar to the above results. However the formulas are complicated, so we omit the details.

3 Remarks for Applications and Discussions

In order to apply the [GLP & MEMM] Pricing Models obtained in the previous subsection, we have to do the following three procedures.

1) Estimation of the price process and Lévy measure.

2) Determination of the MEMM.

3) Computation of the option prices.

We will give some comments on these problems briefly.

1) Estimation of the price process and Lévy measure. What we need to know is the generating triplet. It is not easy to solve this problem theoretically. But if we want to solve in the sense of numerical analysis, then the FFT(Fast Fourier Transform) method seems to be very useful.

2) Determination of the MEMM P^*. This part is just what we have seen in §2.2.

3) Computation of the Option Prices. An option with a payoff f_T is a functional of the price process $\{S_t, 0 \leq t \leq T\}$, and the price of it is given as $e^{-rT}E_{P^*}[f_T]$ under the [GLP & MEMM] Pricing Models. By $S_t = S_0 \exp(Z_t)$, an option is a functional of $\{Z_t, 0 \leq t \leq T\}$, and Z_t is a Lévy process under the MEMM P^*. Therefore the computation problems of option prices are reduced to the computation problems of the Lévy functionals. This is a subject of stochastic calculus of Lévy processes, and it is not easy to solve in general.

4) Discussions. As we have mentioned above (see Corollary 1), even if the price process S_t does not have the finite moments under the original probability P, it may have the finite moments under the MEMM P^*. This property is very convenient for the computation of option prices. We think that the MEMM has some relations with the idea of the exponential hedging of [14] . In fact, in [14] the exponential hedging is discussed relating with relative entropy.

References

[1] Amaral, L., Plerou, V., Gopikeishnan, P., Meyer, M., and Stanly, H. (2000) The distribution of returns of stock prices. *International Journal of Theoretical and Applied Finance,* Vol. 3, No. 3 (2000) 365-369.

[2] Abramowitz, M. (1972) "Handbook of mathematical functions with formulas, graphs and mathematical tables". Edited by Milton and Irene A. Stegun, Wiley.

[3] Albanese, C., Jaimungal, S. and Rubisov, D. (2001) Jumping in line. *Risk Magazine,* February, pp 65-68.

[4] Andersen, L. and Andreasen, J. (1999) Jump-Diffusion Processes: Volatility Smile Fitting and Numerical Methods for Pricing. *Working paper*, see www.ssrn.com.

[5] Barndoff-Nielson, O. and Halgreen, O. (1977) Infinite divisibility of the hyperbolic and general inverse Gaussian distributions. *Z. fur Wahr. und ver. Gebiete,* v.38, 309-312.

[6] Black, F., and Scholes, M. (1973) The pricing of options and corporate liabilities. *Journal of Political Economy.* Vol. 81, 637–654.

[7] Bertoin, J. (1996) "*Lévy Processes*". Cambridge Univ. Press.

[8] Borovkov, A. (1976) "*Stochastic processes in queuing theory*". Springer.

[9] Bühlmann, H., Delbaen, F., Embrechts, P., and Shiryaev, A. N. (1996) No-arbitrage, Change of Measure and Conditional Esscher Transforms. *CWI Quarterly,* 9,No. 4, 291-317.

[10] Carr, P., Geman, H., Madan, D.B., and Yor, M. (2000) The Fine Structure of Asset Returns: An Empirical Investigation. *(Preprint).*

[11] Carr, P. and Madan, D. (1999) Option valuation using the fast fourier transform. *Journal of Computational Finance,* 2, 61–73.

[12] Carr, P. and Wu, E. (2000) The Finite Moment Logstable Process And Option Pricing. *Working paper.*

[13] Chan, T. (1999) Pricing Contingent Claims on Stocks Derived by Lévy Processes. *The Annals of Applied Probability,* v. 9, No. 2, 504-528 .

[14] Delbaen, F., Grandits, P., Rheinländer, T., Samperi, D., Schweizer, M. and Stricker, C. (2000) Exponential Hedging and Entropic Penalties. *(Preprint).*

[15] Delbaen, F. and Schachermayer, W. (1996) The Variance-Optimal Martingale Measure for Continuous Processes. *Bernoulli,* v. 2, 81-106.

[16] Eberlein, E. and Keller, U. (1995) Hyperbolic distributions in finance. *Bernoulli,* 1, 281-299.

[17] Eberlein, E., Keller, U. and Prause, K. (1998) New insights into smile, mispricing and Value-at-Risk: the hyperbolic model. *Journal of Business,* v.71, 371-405.

[18] Fama, E. F. (1963) Mandelbrot and the Stable Paretian Hypothesis. *J. of Business,* 36, 420-429.

[19] Föllmer, H. and Schweizer, M. (1991) Hedging of Contingent Claims under Incomplete Information. In M.H.A. Davis and R.J. Elliot (ed.) : *Applied Stochastic Analysis*, Gordon and Breach, 389-414.

[20] Frittelli, M. (2000) The Minimal Entropy Martingale Measures and the Valuation Problem in Incomplete Markets. *Mathematical Finance,* v.10, No. 1, 39-52.

[21] Fujiwara, T. and Miyahara, Y. (2001) On the Minimal Entropy Martingale Measure for Geometric Lévy Processes. *Discussion Papers in Economics, Nagoya City University*, No. 299, pp. 1-21.

[22] Hurst, S. R., Platen, E. and Rachev, T. (1997) Subordinated Markov Index Models: A Comparison. *Financial Engineering and Japanese Markets*, v.4, 97-124.

[23] Hurst, S.R., Platen, E. and Rachev, S.T. (1999) Option pricing for a logstable asset price model. *Math. Comput. Modelling, 29* , no. 10-12, 105–119.

[24] Ikeda, N. and Watanabe, S. (1989) "*Stochastic Differential Equations and Diffusion Processes, Second Edition.*", North-Holland.

[25] Jacod, J. and Shiryaev, A.N. (1987) "*Limit Theorem for Stochastic Processes*". Springer, Berlin.

[26] Madan, D. and Seneta, E. (1990) The variance gamma (vg) model for share market returns. *Journal of Business,* v.63(4), 511-524.

[27] Madan, D., Carr, P. and Chang, E. (1998) The variance gamma process and option pricing. *European Finance Review*, v.2, 79-105.

[28] Mandelbrot, B. (1963) The variation of certain speculative prices. *J. of Business,* 36, 394-419.

[29] Merton, R.C. (1973) Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science,* v.4, 141–183.

[30] Merton, R. C. (1976), Option Pricing when Underlying Stock Returns are Discontinuous. *J. of Financial Economics*, v.3, 125-144.

[31] Miyahara, Y. (1996a) Canonical Martingale Measures of Incomplete Assets Markets. In *Probability Theory and Mathematical Statistics: Proceedings of the Seventh Japan-Russia Symposium, Tokyo 1995 (eds. S. Watanabe et al),* pp.343-352.

[32] Miyahara, Y. (1999a) Minimal Entropy Martingale Measures of Jump Type Price Processes in Incomplete Assets Markets. *Asian-Pacific Financial Markets,* Vol. 6, No. 2, pp. 97-113.

[33] Miyahara, Y. (2001) Geometric Lévy Process & MEMM Pricing Model and Related Estimation Problems. *Asia-Pacific Financial Markets,* Vol. 8, No. 1, pp. 45-60.

[34] Rachev, S. and Mittnik, S. (2000), *"Stable Paretian Models in Finance",* Wiley.

[35] Schweizer, M. (1995) On the Minimal Martingale measure and the Föllmer-Schweizer Decomposition. *Stochastic Analysis and Applications,* 13(5), 573-599.

[36] Sato, K. (1999) *"Lévy Processes and Infinitely Divisible Distributions".* Cambridge University Press.

[37] Sato, K. (2000) Density Transformation in Lévy Processes. *MaPhySto Lecture Notes,* no. 7.

[38] Shiryaev, A. N.(1999) *"Essentials of Stochastic Finance: Facts, Models, Theory"*. World Scientific.

[39] Xiao, K., Miyahara, Y., and Misawa, T. (1999) Computer Simulation of [Geometric Lévy Process & MEMM] Pricing Model. *(Preprint).*

[40] Zolotarev, V.M. (1986) *"One-dimensional Stable Distributions".* American Mathematical Society.

[41] Zhou, C. (1999) Path-dependent Option Valuation when the underlying path is discontinuous. *The J. of Financial Engineering*, v.8, N 1, pp 73-97.