EXTRACTING THE JOINT VOLATILITY STRUCTURE OF FOREIGN EXCHANGE AND INTEREST RATES FROM OPTION PRICES

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1. Introduction

To companies operating in an ever more globalised marketplace, fluctuating exchange rates and volatile interest rates represent two significant sources of risk. Derivative financial instruments such as options are increasingly used to protect against unwanted exposures, in a way like one would buy insurance to safeguard against adversity. It has long been understood that the dynamics of exchange rates and interest rates are linked by fundamental economic relationships, but to date no attempt has been made to employ a unified framework in which information about the volatilities and correlations of these economic variables can be extracted from the prices of actively traded options.

Efficient financial markets subsume all information about an asset in its current price and it is a standing hypothesis in models of these markets that future asset prices are not predictable. The purpose of models for pricing by arbitrage in general and term structure models in particular is to process information available in liquid market prices in order to manage risk. Derivative financial instruments are the vehicle for trading risk; the corresponding hedge portfolio determines both the risk management strategy and the arbitrage–free price for bearing the particular risk embodied in the instrument. However, once derivatives are actively traded in a liquid market, their prices incorporate additional market information to which the model must be calibrated. Standard fixed income derivatives such as caps and swaptions are examples of such a development.

Term structure models directly specifying the arbitrage–free dynamics of market observable forward LIBOR[1] or swap rate processes (cf. Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997) (BGM), Jamshidian (1997), as well as Musiela and Rutkowski (1997a)) have quickly gained an eminent role in the management of interest rate risk by leading financial institutions. This is in particular due to the fact that they provide a pricing methodology in which the market practice of pricing caps or swaptions by Black/Scholes–like formulas can be applied in a manner consistent with the absence of arbitrage, hence the name *Market Models*. Musiela and Rutkowski (1997b) give a self–contained and up–to–date treatment of this development, as well as the necessary background, and Schlögl (2002b) extends this methodology to multiple currencies, incorporating exchange rate risk.

Closed form solutions which mirror market practice for standard derivatives of course facilitate the calibration to market data, but this resolves the problem only superficially.

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[1] London Interbank Offer Rate

For example, the BGM model of lognormal forward LIBOR can easily be fitted to prices for the individual "caplet" components of a cap contract. However, a term structure of forward rate volatilities must be backed out of *cap* prices for different maturities and information on correlation between various forward rates is required as well.

When these models are implemented in financial institutions, the most pressing problem remains the calibration to market data. In the case of the term structure of interest rates in a single currency, the problem of calibrating the lognormal *Market Models* to observed prices has inspired a large body of literature.[2] However, although swaption prices depend on correlation between forward LIBORs as well as variances, all of these approaches take the correlation structure as exogenously given and do not attempt to fit it to market prices. This raises the question of how much of the variance/covariance structure for interest rates is implied when cap/floor, swaption and possibly reset cap prices are considered jointly, and what modelling freedom remains when this information has been taken into account. Furthermore, the abundance of relationships between the term structure of interest rates and exchange rates means that even more information about correlation between these variables can be implied when market prices for derivative financial instruments are considered within a consistent framework across two or more currencies.

Implied volatilities from prices of interest rate caps only determine *norms* of the (vector–valued) forward LIBOR volatility function, while swaption volatilities give information about inner products. Thus the variance/covariance matrix is not uniquely determined and one of the key problems is how this gap can be filled while still ensuring a good fit to the market prices that are observable. Brace and Womersley (2000) approach this problem by applying recently developed techniques of semidefinite programming (SDP).[3] Their idea is to find, in the space of symmetric positive semidefinite matrices, the one that has minimal distance from a variance/covariance matrix estimated from time series data of the underlying variables (i.e. simply compounded forward rates), while satisfying the constraint that volatilities implied by liquid at–the–money cap and swaption prices should be fitted perfectly. Thus prices, which are forward–looking in the sense that they embody an aggregation of the expectations of the market participants, are the primary inputs, while the remaining gaps are filled in a consistent manner by statistically estimated values from historical data, arguably the best choice in the absence of further information.

In order to embed the original calibration problem in a class of SDP problems for which algorithms are readily available, Brace and Womersley (2000) reformulate it in a way which greatly increases the number of variables and the number of constraints in the optimisation. Even in the single currency case, if one wishes to solve the calibration problem without introducing needlessly restrictive assumptions on the market data, we need to formulate the algorithm to fit the original problem, as well as identify those structural features present in the data which allow us to reduce the dimension of the problem.

For the multicurrency case, foreign exchange and interest rate risk must be specified in a consistent way, imposing restrictions on the volatilities of forward interest and exchange rates when these are modelled jointly. These restrictions are made explicit in Schlögl (2002b). They provide the link between interest rate and currency risk. This opens a way to approach a problem often encountered by risk managers and options traders: While currency (FX) options are very actively traded for maturities up to two years, market

[2]cf. Rebonato (1999a, 1999b), Pedersen (1999), and Wu (2001)

[3]cf. Helmberg, Rendl, Vanderbei and Wolkowicz (1996), Vandenberghe and Boyd (1996), Overton and Wolkowicz (1997), and Wolkowicz, Saigal and Vandenberghe (2000).

information becomes thin for longer time horizons. Interest rate options, however, are liquid up to longer maturities, in particular in terms of the maturity of the underlying assets. Modelling currency and interest rate risk jointly allows long–dated foreign exchange volatilities to be extrapolated from liquid prices for short–dated currency options and the volatilities of forward interest rates in the component currencies. This is an example of how structural relationships between FX and interest rates can be exploited to extract additional information relevant to risk management. Further correlation information can be implied at the short end of the maturity spectrum, where market data is available on FX options as well as interest rate derivatives in domestic and foreign currencies.

2. Model setup in a single currency

The basic building block of the model to be calibrated is the δ–compounded simple forward interest rate, i.e. a market observable rate such as forward LIBOR.[4] Let $B(t,T)$ denote the price at time t of a zero coupon bond paying one monetary unit at maturity T. Then the forward LIBOR $L(t,T)$ is given by

$$L(t,T)=\frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)}-1\right)$$

In any arbitrage–free model, $L(t,T)$ must clearly be a martingale under the equivalent probability measure $\mathbf{P}_{T+\delta}$ associated with taking $B(\cdot,T+\delta)$ as the numeraire.[5] Furthermore, assume deterministic volatility for $L(\cdot,T)$, i.e.

$$dL(t,T)=L(t,T)\lambda(t,T)dW_{T+\delta}(t)$$

where $W_{T+\delta}$ is a standard Brownian motion of dimension d under $\mathbf{P}_{T+\delta}$ and, given a time horizon T^*, $\lambda:[0,T]\times[0,T^*]\to\mathbb{R}^d$ is a deterministic, vector–valued function of its arguments. Then $L(\cdot,T)$ is a lognormal martingale under $\mathbf{P}_{T+\delta}$ and a caplet contract paying

$$\delta\max(0,L(T,T)-\kappa)$$

at time $T+\delta$, for a fixed cap level κ, is priced according to the Black/Scholes–type formula

$$\begin{aligned}&B(0,T+\delta)E_{T+\delta}[\delta\max(0,L(T,T)-\kappa)]\\ =\;&\delta B(0,T+\delta)\Big(L(0,T)\mathcal{N}(h_1)-\kappa\mathcal{N}(h_2)\Big)\end{aligned}\tag{1}$$

with

$$h_{1,2}=\frac{\ln\frac{L(0,T)}{\kappa}\pm\frac{1}{2}\nu^2(0,T)}{\nu(0,T)}$$

and

$$\nu(0,T)^2=\int_0^T\lambda^2(t,T)dt\tag{2}$$

where $E_{T+\delta}$ denotes the expectation operator under $\mathbf{P}_{T+\delta}$ and $\mathcal{N}(\cdot)$ is the cumulative distribution function of the standard normal distribution.

[4]London Interbank Offer Rate. This is as opposed to models constructed from continuously compounded spot and/or forward rates, for example following the approach of Heath, Jarrow and Morton (1992).

[5]For a detailed discussion of numeraire assets and their associated equivalent martingale measures, see Geman, El Karoui and Rochet (1995). Note that any asset with a strictly positive price process is a valid choice of numeraire.

The construction of consistent term structure models based on lognormality assumptions of this type has been explored extensively in the literature.[6] Because they reflect market practice both in the compounding conventions of the interest rates modelled as well as in the pricing of caps and floors,[7] these models are generically known as the *Lognormal Forward Rate Market Models.* The starting point of this calibration study is the discrete tenor version of the model, as put forward by Musiela and Rutkowski (1997a).

Given a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T^*]}, \mathbf{P}_{T^*})$ satisfying the usual conditions, let $\{W_{T^*}(t)\}_{t\in[0,T^*]}$ denote a d–dimensional standard Wiener process and assume that the filtration $\{\mathcal{F}_t\}_{t\in[0,T^*]}$ is the usual $\mathbf{P}_{T^*}$–augmentation of the filtration generated by $\{W_{T^*}(t)\}_{t\in[0,T^*]}$.

The model is set up on the basis of assumptions[8]

(BP.1) For any date $T \in [0,T^*]$, the price process of a zero coupon bond $B(t,T)$, $t\in[0,T]$ is a strictly positive special martingale[9] under $\mathbf{P}_{T^*}$.

(BP.2) For any fixed $T\in[0,T^*]$, the *forward process*

$$F_B(t,T,T^*) = \frac{B(t,T)}{B(t,T^*)}, \qquad \forall\, t\in[0,T]$$

follows a martingale under $\mathbf{P}_{T^*}$.

Note that assumption **(BP.2)** means that $\mathbf{P}_{T^*}$ can be interpreted as the *time* T^* *forward measure* and implies that the bond price dynamics are arbitrage–free.

Consider now the discrete tenor structure $\mathbb{T} = \{T_0 = 0, T_1, \ldots, T_i, \ldots, T_N = T^*\}$. One may specify the dynamics of each $L(\cdot, T_i)$ as

$$dL(t,T_i) = L(t,T_i)\lambda(t,T_i)dW_{T_{i+1}}(t)$$

where $W_{T_{i+1}}$ is a standard Brownian motion under $\mathbf{P}_{T_{i+1}}$ and the probability measures are linked via the Radon/Nikodym derivatives given in terms of the Doléans exponential as[10]

$$\frac{d\mathbf{P}_{T_i}}{d\mathbf{P}_{T_{i+1}}} = \mathcal{E}_{T_i}\left(\int_0^{\cdot} \gamma(u,T_i,T_{i+1})\cdot dW_{T_{i+1}}(u)\right) \qquad \mathbf{P}_{T_{i+1}}\text{–a.s.} \tag{3}$$

with

$$\gamma(t,T_i,T_{i+1}) = \frac{\delta L(t,T_i)}{1+\delta L(t,T_i)}\lambda(t,T_i) \qquad \forall\, t\in[0,T_i] \tag{4}$$

In particular, we have

$$dW_{T_i}(t) = dW_{T_{i+1}}(t) - \gamma(u,T_i,T_{i+1})dt \tag{5}$$

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[6] See Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997), Jamshidian (1997), as well as Musiela and Rutkowski (1997a).

[7] A *cap* is a sequence of *caplets* paying $\delta\max(0, L(T_i,T_i)-\kappa)$ at $T_i+\delta$ for $T_i = i\delta$, $i = 0,1,\ldots,n-1$ and a *floor* is a sequence of *floorlets* paying $\delta\max(0,\kappa - L(T_i,T_i))$ at $T_i+\delta$ for $T_i=i\delta$, $i=0,1,\ldots,n-1$.

[8] Labelling matches that of Musiela and Rutkowski (1997a)

[9] Musiela and Rutkowski (1997a) define a *special martingale* as a process X which admits a decomposition $X = X_0 + M + A$, where $X_0\in\mathbb{R}$, M is a real–valued local martingale and A is a real–valued predictable process of finite variation.

[10] The Doléans exponential is given by

$$\mathcal{E}_t\left(\int_0^{\cdot}\alpha(u)dW(u)\right) := \exp\left\{\int_0^t \alpha(u)dW(u) - \frac{1}{2}\int_0^t \alpha^2(u)du\right\}$$

Thereby the discrete tenor lognormal forward LIBOR model is completely specified. Note that in this context, one can speak equivalently of volatilities and links between forward measures — when one is specified, the other is fixed.

Since the goal of the present study is calibration to market data, and at any one time we only observe forward LIBORs (and forward swap rates) for a discrete tenor structure, the discrete tenor version of the model is sufficient. The calibrated model can then be extended to continuous tenor in various ways, either explicitly or implicitly interpolating interest rates and volatilities.[11]

Besides caps and floors, the model also needs to be calibrated to another set of liquidly traded interest rate derivatives, swaptions. The underlying financial variable of a swaption is the forward swap rate. Consider an option (a *payer swaption*), expiring in T_m, which if exercised has option holder enter into a swap running n δ–periods, where every r δ–periods the option holder pays a fixed rate κ and receives the r–period floating rate $L(T_{m+jr}, T_{m+jr}, r)$, given in terms of the δ–compounded rates as

$$(1 + r\delta L(T_{m+jr}, T_{m+jr}, r)) = \prod_{k=0}^{r-1}(1 + \delta L(T_{m+jr}, T_{m+jr+k}))$$

Floating rates are fixed at the beginning of an r–period and paid at the end (i.e. in arrears).

This contingent claim can be approximately priced at time 0 by the Black/Scholes–type formula[12]

$$\delta \sum_{j=1}^{n} \text{roz}(j,r)B(0,T_{m+j})(\omega(0,m,n,r)\mathcal{N}(h_1) - \kappa\mathcal{N}(h_2))$$

with

$$\begin{aligned} h_{1,2} &= \frac{\ln\frac{\omega(0,m,n,r)}{\kappa} \pm \frac{1}{2}\zeta(m,n,r)}{\sqrt{\zeta(m,n,r)}} \\ \zeta(m,n,r) &= \int_0^{T_m} \sigma_0^2(t,m,r,n)dt \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} A_i^{(m,n)}(0)A_j^{(m,n)}(0)\int_0^{T_m}\lambda(t,T_{m+i-1})\lambda(t,T_{m+j-1})dt \end{aligned} \tag{6}$$

The often quoted Black implied volatility $\beta(m,n,r)$ of the swaption is given by

$$\zeta(m,n,r) = \beta^2(m,n,r)T_m$$

Note that correlation between forward LIBORs enters the swaption price via the scalar product of the volatility vectors λ in (6).

3. The calibration problem

In the single currency case, the task is to find forward LIBOR volatility functions $\lambda(\cdot,\cdot)$ such that the model prices for at–the–money[13] caps and swaptions agree with the market

[11] Brace, Gatarek and Musiela (1997) propose one possible extension; Musiela and Rutkowski (1997a) suggest another. Schlögl (2002a) makes the interpolation explicit.

[12] This formula has been derived by numerous authors in numerous papers, see in particular Brace, Dun and Barton (2001). Since this formula is central to the calibration approach, its derivation is summarised in the appendix for the reader's convenience.

[13] At–the–money means that the option strike κ is equal to the current (i.e. time 0) value of the underlying, e.g. $\kappa = L(0,T_m)$ for a caplet and $\kappa = \omega(0,m,n,r)$ for a swaption. Note that the at–the–money concept is somewhat more complicated for caps, since there is only one strike, but potentially a

prices of these instruments. The market prices provide information about integrals over scalar products of the volatility vectors $\lambda(t,T)$ (cf. (2) and (6)). Given the discrete tenor structure $\mathbb{T}$, the smallest unit of volatility information is

$$\int_{T_{i-1}}^{T_i} \lambda(t,T_j)\lambda(t,T_k)dt \qquad N > j,k \geq i \geq 0 \tag{7}$$

Thus one is taking a completely nonparametric approach by assuming $\lambda(t,T)$ to be stepwise constant in calendar time t on the intervals $[T_{i-1},T_i]$. After calibration, one may choose any functional form for $\lambda(\cdot,T_j)$ on the calendar time intervals, as long as the values of the integrals (7) are preserved.

Now let $\mathbb{S}_i$ denote the set of $i \times i$ real symmetric matrices. Brace and Womersley (2000) propose to use semidefinite programming (SDP) to calibrate the market model to observed prices of caps and swaptions and introduce the $N-1$ positive semidefinite matrices $\Gamma^{(k)} \in \mathbb{S}_{N-k}$ for $k = 1,\ldots,N-1$, defined by

$$\Gamma^{(k)} = \left(\Gamma^{(k)}_{i,j}\right)$$

with

$$\Gamma^{(k)}_{i,j} = \frac{1}{\delta}\int_{T_{k-1}}^{T_k} \lambda(t,T_{j+k-1})\lambda(t,T_{i+k-1})dt, \qquad i,j = 1,\ldots,N-k$$

Note that $\Gamma^{(1)} \in \mathbb{S}_{N-1},\ldots,\Gamma^{(N-1)} \in \mathbb{S}_1$.

The variable of the SDP optimisation is the $\frac{1}{2}N(N-1) \times \frac{1}{2}N(N-1)$ block–diagonal positive semidefinite matrix

$$\Gamma = \begin{pmatrix} \Gamma^{(1)} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma^{(k)} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \Gamma^{(N-1)} \end{pmatrix}$$

For later use, define $g := \frac{1}{2}N(N-1)$.

The goal of the approach of Brace and Womersley (2000) is to find a Γ which is as close as possible to a $\overline{\Gamma}$ estimated from historical data (under some metric), subject to the constraint that Γ matches implied at–the–money volatilities observed from market prices exactly. Rewriting (6) in terms of Γ yields

$$\begin{aligned} \zeta(m,n,r) &= \sum_{i=1}^{n}\sum_{j=1}^{n} A_i^{(m,n)}(0)A_j^{(m,n)}(0)\sum_{k=1}^{m}\delta\Gamma^{(k)}_{m+i-1,m+j-1} \\ \Leftrightarrow \quad \beta^2(m,n,r) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{1}{m}A_i^{(m,n)}(0)A_j^{(m,n)}(0)\Gamma^{(k)}_{m+i-1,m+j-1} \end{aligned} \tag{8}$$

Formulating the calibration problem in this manner addresses the key problem that the implied volatility structure and most of all the correlation structure of forward LIBORs is underdetermined by the market prices of actively traded interest rate derivatives. Out of all possible volatility structures which support the option prices observed in the market, the

different at–the–money level for each caplet. The implied volatility "smile" of different volatilities for different strikes contradicts the Black/Scholes paradigm of the present model and is thus left to further research.

optimisation selects the one closest to the historical volatilities. Thus the model is exactly fitted to the "market's view" of interest rate volatilities, and the remaining gaps in the volatility structure are filled with information gleaned from time series data. Furthermore, by simultaneously fitting the entire volatility structure, the SDP approach extracts implied correlations as well as single rate volatilities, as opposed to other methods proposed in the literature, which take the correlations between forward LIBORs as given[14].

In order to employ readily available SDP algorithms, Brace and Womersley (2000) find it necessary to reformulate the problem. As discussed in appendix B, this results in a very inefficient implementation and it is thus desirable to develop an algorithm better suited to the problem. In addition, two further useful extensions to the semidefinite programming approach suggest themselves, both also requiring adaptation of the existing algorithms. Firstly, with a view to statistical parsimony and tractability, it may be desirable to limit the rank of the matrix Γ. Secondly, by allowing for a bid/ask spread in observed market prices, one may achieve a closer fit to the target matrix $\overline{\Gamma}$ and/or a smoother evolution of Γ in calibrations carried out daily.[15] This means the constraints (8) become inequalities, a problem which is not addressed in the existing SDP literature.

4. Construction of the target matrix $\overline{\Gamma}$

Under the lognormal Market Model paradigm, the forward LIBOR volatilities $\lambda(t,T)$ may depend on calendar time t and maturity T deterministically. However, when using (backward looking) empirical data to estimate the target volatility matrix $\overline{\Gamma}$ for (forward looking) model calibration, it is simply consistent to assume time homogeneity for λ, i.e. $\lambda(t,T)=\lambda(0,T-t)$. In this case we have

$$\delta\Gamma^{(1)}_{ij}=\int_0^\delta \lambda(s,T_i)\lambda(s,T_j)ds=\int_{k\delta}^{(k+1)\delta}\lambda(s,T_{i-k})\lambda(s,T_{j-k})ds=\delta\Gamma^{(1+k)}_{i-k,j-k}\quad\forall\ i,j>k$$

It is therefore sufficient to estimate $\Gamma^{(1)}_{ij}$ for all i,j.

Typically, the variance/covariance matrix $(\text{cov}[\ln L(t,T_i),\ln(t,T_j)])_{ij}$ is estimated from time series data on *fixed time–to–maturity* forward rates. The model, on the other hand, is based on *fixed maturity* forward rates. By the non-parametric volatility specification, $\lambda(s,T_i)$ is assumed to be constant on $[0,\delta)$, i.e.

$$\delta\Gamma^{(1)}_{ij}=\int_0^\delta \lambda(s,T_i)\lambda(s,T_j)ds$$

and thus in the time homogeneous case,

$$\lambda(s,T_i)=\lambda(0,T_i-s)=\lambda(s,T_{i-1}+s)\quad\text{on } s\in(0,\delta)$$

resulting in

$$\delta\Gamma^{(1)}_{ij}=\int_0^\delta \lambda(s,T_{i-1}+s)\lambda(s,T_{j-1}+s)ds \tag{9}$$

$$\approx \text{cov}[\ln L(s,T_{i-1}+s),\ln L(s,T_{j-1}+s)] \tag{10}$$

[14]e.g. Rebonato (1999a), Pedersen (1999) and Wu (2001)

[15]Note that the calibration method discussed here — with the exception of the target matrix $\overline{\Gamma}$ — works on cross–sectional data only, i.e. the market prices for any one given day. The market practice of recalibrating the model whenever prices change is implicitly assumed. However, a method which yields a smoother evolution of the model calibration over time is desirable.

e.g. the estimator for $\delta\Gamma^{(1)}_{12}$ is approximately[16] given by the empirical covariance between the logarithms of the spot δ–LIBOR $L(s, T_0 + s)$ and the one δ–period forward δ–LIBOR $L(s, T_1 + s)$, where the covariance is normalised to one δ–period.

To calculate $\text{cov}[\ln L(s, T_{i-1} + s), \ln L(s, T_{j-1} + s)]$, consider the following example. Let there be daily data available for n trading days and let there be m trading days in a year and $m/4$ trading days in a δ–period. For this data, calculate the empirical variance of each logarithmic rate, as well as the variance of all pairwise sums of logarithmic rates. Scale these for one δ–period by multiplying by $m/4$. The desired covariances are then given by

$$\begin{aligned}\text{cov}[\ln L(s, T_{i-1} + s), \ln L(s, T_{j-1} + s)] \;=\; \frac{1}{2}\Big(&\text{var}[\ln L(s, T_{i-1} + s) + \ln L(s, T_{j-1} + s)] \\ &-\text{var}[\ln L(s, T_{i-1} + s)] - \text{var}[\ln L(s, T_{j-1} + s)]\Big)\end{aligned}$$

If the assumption of piecewise constant $\lambda(\cdot, \cdot)$ is relaxed, for example by allowing exponential decay of $\lambda(s, T)$ in $(T-s)$, the relationship between (9) and (10) must be appropriately adjusted. However, lacking further information on the form of $\lambda(s, T)$ on $s \in [0, \delta)$, one can reasonably proceed under assumption of piecewise constant volatilities.

5. Reducing the dimension of the problem

5.1. **Assumptions of time homogeneity without loss of generality for a typical data set.** In a typical data set, we have caps of length 1, 2, 3, 4, 5, 7, and 10 years. Furthermore, we have a set of options on swaps of length 1, 2, 3, 4, 5, 7, and 10 years, where the option maturities are 3 months, 6 months, or 1, 2, 3, 4, or 5 years (i.e. 7×7 different at–the–money swaptions).[17] The required time horizon is thus 15 years; this time period is spanned by 59 quarterly forward LIBORs, yielding a $(\frac{1}{2} \cdot 59 \cdot 60) \times (\frac{1}{2} \cdot 59 \cdot 60)$ matrix Γ. It is therefore necessary to reduce the size of the matrix Γ in order to make the problem more tractable.

In the present example, the longest option maturity is 10 years. Thus the blocks $\Gamma^{(k)}$ making up the block diagonal matrix Γ are only required out to $k \leq 40$, reducing the size of Γ to $N' \times N'$ with $N' = \frac{1}{2}(59 \cdot 60 - 19 \cdot 20)$.

The next step is to assume time homogeneity of volatilities where there is no information in the data. Note that beyond the first year, option maturities increase in steps of one year or more. Thus within each maturity step we can assume time homogeneity $\Gamma^{(k+1)}_{ij} = \Gamma^{(k)}_{ij}$. This is without loss of generality in the sense that any desired time inhomogeneity (for example to smooth volatilities in calendar time) consistent with the data can be introduced after the fit. In our example, the thus reduced matrix $\tilde{\Gamma}$, now consisting of only nine blocks

[16] The fact that $L(s, T)$ has a level–dependent drift under all measures except for its "native" forward measure $\mathbf{P}_{T+\delta}$ means that although quadratic variation and covariation are invariant under measure transforms, variance and covariance are not. Freezing the level–dependent coefficients at their initial values makes the drift deterministic and results in the approximation (9) $\approx$ (10).

[17] In a more extensive data set, one might have caps of length 1, 2, 3, 4, 5, 7, 10, and 15 years, and swaptions with maturities 3, 6, 9, 12, 18 months and 2, 3, 4, 5, 7, 10, 15 years, on swaps with lengths 6 months, 1, 2, 3, 4, 5, 7, 10 and 15 years.

$\tilde{\Gamma}^{(h)}$, is given by

$$\Gamma^{(k)}_{i-k+1,j-k+1} = \begin{cases} \tilde{\Gamma}^{(k)}_{i-k+1,j-k+1} & k = 1,2,3 \\ \tilde{\Gamma}^{\text{int}((k-1)/4+3)}_{i-k+1,j-k+1} & 3 < k \leq 20 \\ \tilde{\Gamma}^{(8)}_{i-k+1,j-k+1} & 20 < k \leq 28 \\ \tilde{\Gamma}^{(9)}_{i-k+1,j-k+1} & 28 < k \leq 40 \end{cases} \tag{11}$$

with int(x) denoting the integer part of x. $\tilde{\Gamma}$ is thus an $N'' \times N''$ block diagonal matrix, with

$$N'' = \sum_{k=1}^{9} n_k$$

and block sizes

k	1	2	3	4	5	6	7	8	9
n_k	59	58	57	55	51	47	43	39	31

The constraints can be written in terms of $\tilde{\Gamma}$ by substitution using (11).

A further reduction in the number of variables can be achieved by noting that the longer–dated forward LIBORs do not enter into options of shorter maturities, e.g. the 12–year forward LIBOR does not enter into any of the cap contracts in our example, nor into swaptions with a maturity less than three years. Thus we can make the additional homogeneity assumptions

$$\tilde{\Gamma}^{(k)}_{i-k+1,j-k+1} = \tilde{\Gamma}^{(1)}_{i-k+1,j-k+1} \tag{12}$$

for

$k \leq$	2	3	4	5	6	7
$i,j \geq$	41	42	44	48	52	56

Note that $i,j \geq c$ means $i \geq c$ *and* $j \geq c$, thus this is not a straightforward "pruning" of each Γ. For $i \geq c$ and $j < c$, time homogeneity can only be assumed for $\tilde{\Gamma}^{(\cdot)}_{i-k+1,i-k+1}$ and the correlation coefficient, i.e. we could set

$$\tilde{\Gamma}^{(k)}_{i-k+1,j-k+1} = \frac{\tilde{\Gamma}^{(1)}_{i-k+1,j-k+1}}{\sqrt{\tilde{\Gamma}^{(1)}_{j-k+1,j-k+1}}} \sqrt{\tilde{\Gamma}^{(k)}_{j-k+1,j-k+1}} \tag{13}$$

Using (12) and (13), we can in fact prune each $\tilde{\Gamma}^{(k)}$ by removing the terms involving longer–dated forward LIBORs, at the cost of making the constraints non-linear, due to (13). The block sizes then become

k	1	2	3	4	5	6	7	8	9
n_k	59	39	39	39	39	39	39	39	31

5.2. **Other homogeneity assumptions.** While the homogeneity assumptions introduced in the previous section are constructed so that they cannot be violated by any given data set and can be relaxed post–fit without affecting the feasibility of the solution, some additional homogeneity assumptions which do impose restrictions on the solution are worth considering.

(1) As discussed in section 5.3, the assumption of complete time–homogeneity of Γ (which would collapse Γ to a single block $\Gamma^{(1)}$) is incompatible with cap price data on some days. However, assuming homogeneity $\tilde{\Gamma}^{(k)}_{i-k+1,j-k+1} = \tilde{\Gamma}^{(2)}_{i-k+1,j-k+1}$ for $k \geq 2$ is leaves sufficient freedom to always fit observed cap prices. Since the swaption

constraints do not involve any $\tilde{\Gamma}^{(k)}$ beyond $k = 7$ (maturity 5 years), one can assume time homogeneity ($\tilde{\Gamma}^{(k+1)} = \tilde{\Gamma}^{(k)}$) for $k \geq 7$, and still be sure to remain compatible with the data. Since the underlying sets of forward LIBORs for swaptions overlap in a nontrivial manner, one does not obtain any straightforward rule to determine a "minimal" $k^*_{\text{all}} < 7$, such that one could assume time homogeneity for all $k \geq k^*_{\text{all}}$. One could, however, initially assume time homogeneity for $k \geq k^* = k^*_{\text{caps}}$ and relax this assumption by incrementing k^* until a feasible solution is found.

The potential trade–off is that assuming time homogeneity for $k \geq k^*$ for some k^* may result in extreme values for the blocks $\tilde{\Gamma}^{(k)}$ for $k < k^*$; certainly this is to be expected when setting $k^* = 2$ and fitting only cap data. From this perspective, making assumptions of this type on the volatility structure does not seem promising.

(2) Again departing from linear constraints, one could assume that time inhomogeneity is restricted to volatility levels and that the correlation structure is time–invariant. In this case, we fit a positive–semidefinite, symmetric 59×59 correlation matrix ρ and volatility levels $v_j^{(k)}$. The optimisation problem then reads

5.1. Problem. *Find*

$$\rho \in \mathbb{S}_{59}, \qquad v_j^{(k)} \text{ for } 1 \leq i-k+1 \leq 60-k,\ 1 \leq k \leq 40$$

to minimise

$$\|\rho - \bar{\rho}\|$$

subject to

$$\rho \succsim 0, \qquad v_j^{(k)} \geq 0$$

and the cap and swaption price constraints given by substituting

$$\tilde{\Gamma}^{(k)}_{i-k+1,j-k+1} = \rho_{ij} v_i^{(k)} v_j^{(k)}$$

The number of variables $v_j^{(k)}$ can be reduced by the time homogeneity assumptions discussed above.

5.3. **Checking cap data for violations of time homogeneity.** To verify whether market prices for cap contracts on a given day are compatible with the assumption of time homogeneous volatilities, we decompose the value of a cap contract into two parts. The first part is made up of those caplets which are also part of a cap contract of shorter length. We then value the remaining caplets using all those "units of volatility information" (cf. (7)) which are given by the assumption of time homogeneity in terms of volatilities implied by shorter cap contracts. All other entries of Γ are ignored (implicitly setting them to zero). If the two parts add up to less than the quoted cap price, we know that those additional entries give the freedom to fit a time homogeneous Γ to the cap price. If they add up to more than the cap price, that price contradicts the assumption of time homogeneity.

Let C_i denote the price of the I-th cap contract and ℓ_i the length of this contract in number of LIBOR accrual periods δ. Typically, in market data we have

i	1	2	3	4	5	6	7
ℓ_i	3	7	11	15	19	27	39

These are "spot" cap contracts, with the first caplet written on $L(T_1, T_1)$, maturing at the beginning and paying at the end of the accrual period $[T_1, T_2]$. Typically $[T_1, T_2] = [0.25, 0.5]$. Thus C_1 is a one–year cap with payments in 6, 9 and 12 months.

The caplets in C_i are a subset of the caplets in C_{i+1}, so we start by stripping out the remaining caplets by calculating

$$R_{i+1} = C_{i+1} - C_i$$

The forward LIBORs underlying the caplets in R_{i+1} are $L(\cdot, T_j)$, $\ell_i < j \leq \ell_{i+1}$, and the corresponding caplet volatilities are determined by

$$\nu^2(0, T_j) = \delta \sum_{k=1}^{j} \Gamma^{(k)}_{j+1-k,j+1-k}$$

and under time homogeneity Γ collapses to the first block $\Gamma^{(1)}$, so

$$= \delta \sum_{k=1}^{j} \Gamma^{(1)}_{j+1-k,j+1-k}$$

The verification algorithm iterates forward over all cap contracts, beginning with the shortest. For the shortest cap, no check is necessary, as a time homogeneous Γ can always be fit to at least one cap contract. For the second contract, the implied volatilities $\nu^2(0, T_j)$ for the caplets remaining in R_2 are set to

$$\{\nu^2(0, T_j)\}_{4\leq j\leq 7} = \left\{ \delta \sum_{k=5}^{j} \Gamma^{(1)}_{j+1-k,j+1-k} \right\}_{4\leq j\leq 7} = \left\{ \delta \sum_{k=1}^{j} \Gamma^{(1)}_{j+1-k,j+1-k} \right\}_{0\leq j\leq 3}$$

where the right hand side of the above equation is the set of implied volatilities for the caplets making up the first contract, and therefore given. For the $(i+1)$-th contract, we then have

$$\begin{aligned}
\{\nu^2(0, T_j)\}_{\ell_i+1\leq j\leq \ell_{i+1}} &= \left\{ \delta \sum_{k=\ell_{i+1}-\ell_i+1}^{j} \Gamma^{(1)}_{j+1-k,j+1-k} \right\}_{\ell_i+1\leq j\leq \ell_{i+1}} \\
&= \left\{ \delta \sum_{k=1}^{j} \Gamma^{(1)}_{j+1-k,j+1-k} \right\}_{2\ell_i-\ell_{i+1}+1\leq j\leq \ell_i} \\
&= \left\{ \delta \sum_{k=1}^{j} \Gamma^{(1)}_{j+1-k,j+1-k} \right\}_{1\leq j\leq \ell_i} - \left\{ \delta \sum_{k=1}^{j} \Gamma^{(1)}_{j+1-k,j+1-k} \right\}_{1\leq j\leq 2\ell_i-\ell_{i+1}}
\end{aligned}$$

where the right hand side is again given by a set of implied volatilities for the caplets making up preceding cap contracts.

6. The calibration problem for multiple currencies

As before, consider a tenor structure $\mathbb{T} = \{T_0 = 0, T_1, \ldots, T_i, \ldots, T_N = T^*\}$. In a multicurrency setting, the objects of interest are the forward LIBORs $L_j(t, T_i)$ for each maturity $T_i \in \mathbb{T}$ and each currency $j \in \{0, \ldots, c\}$, and the forward exchange rates

$$X_{jh}(t, T_i) = \frac{B_h(t, T_i) X_{jh}(t)}{B_j(t, T_i)} \qquad T_i \in \mathbb{T},\ j, h \in \{0, \ldots, c\} \tag{14}$$

where $B_h(t,T_i)$ denotes the currency h zero coupon bond maturing in T_i and $X_{jh}(t)$ is the spot exchange rate in terms of units of currency j per unit of currency h. We have

$$X_{jh}(t) = \frac{1}{X_{hj}(t)}$$

Let $\sigma_{jh}(t,T_i)$ denote the volatility of $X_{jh}(t,T_i)$, i.e.

$$dX_{jh}(t,T_i) = X_{jh}(t,T_i)\sigma_{jh}(t,T_i)dW^{(j)}_{T_i}(t)$$

since by (14) $X_{jh}(\cdot,T_i)$ is a martingale under $\mathbf{P}^{(j)}_{T_i}$, the equivalent measure associated with taking $B_j(\cdot,T_i)$ as the numeraire. $W^{(j)}_{T_i}$ denotes a standard Brownian motion under this measure.

Note that $\sigma_{jh}(t,T_i)$ is not necessarily a deterministic function of its arguments, i.e. $X_{jh}(t,T_i)$ is not necessarily lognormal under $\mathbf{P}^{(j)}_{T_i}$.

To extend the lognormal Market Model to multiple currencies, various combinations of lognormality assumptions are possible. In order to be consistent with no arbitrage, the forward LIBOR and forward exchange rate volatilities must satisfy the relationship[18]

$$\sigma_{jh}(t,T_{i-1}) = \gamma_h(t,T_{i-1},T_i) - \gamma_j(t,T_{i-1},T_i) + \sigma_{jh}(t,T_i) \tag{15}$$

where γ is given by (4), appropriately interpreted for each currency.

Given any three volatilities in (15), the fourth is fixed. Thus (15) imposes restrictions on which variables can simultaneously be chosen to be lognormal.For example, if the $\lambda_h(\cdot,T_i)$ and $\lambda_j(\cdot,T_i)$ are deterministic for all $T_i \in \mathbb{T}$, only one $\sigma_{jh}(\cdot,T_i)$ can be chosen to be deterministic as well, so only one $X_{jh}(\cdot,T_i)$ will be lognormal under its native forward measure.[19] More importantly, it is sufficient to calibrate three out of the four volatility functions in (15).

Typically, the most liquid price data will be available for currency options of shorter maturities, out to one or two years, For longer maturities, there is little reliable price data for these instruments. However, swaptions with longer maturities are actively traded. This suggests calibrating the following volatilities

- forward LIBOR volatilities for all maturities for a "base" currency, say currency 0, i.e. $\lambda_0(t,T_i)\ \forall\ i < N$
- forward exchange rate volatilities out to some near time horizon T_m for all currency pairs $(0,h)$, $h > 1$, i.e. $\sigma_{0h}(t,T_i)$, $i \leq m$ (all other FX volatilities $\sigma_{jh}(t,T_i)$, $j > 0$ are then fixed via cross–rate relationships)
- forward LIBOR volatilities for all currencies $h > 0$ beyond the FX time horizon T_m, i.e. $\lambda_h(t,T_i)\ \forall\ m < i < N$, $h > 1$.

It is convenient to assume these volatilities to be deterministic.

The matrix Γ must be expanded to accommodate the additional volatilities. As before, Γ is a symmetric, positive definite block diagonal matrix with $k = 1,\dots,N-1$ blocks $\Gamma^{(k)}$.

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[18]For the construction of the multicurrency Market Model and the derivation of its no arbitrage conditions, see Schlögl (2002b).

[19]This implies that strictly speaking only for one maturity will a currency option be priced by a Black/Scholes formula.

To construct $\Gamma^{(k)}$, define $\theta^{(i,k)}(t)$ by

$$\theta^{(i,k)}(t) = \begin{cases} \lambda_0(t, T_{i+k-1}) & i = 1, \dots, N-k \\ \lambda_h(t, T_{\hat{\imath}+k-1}) & \begin{cases} h = \mathrm{int}((i-(N-k))/(N-m)) + 1 \\ \hat{\imath} = (i-(N-k)) \mod (N-m) \\ i = N-k+1, \dots, N-k+c(N-m) \end{cases} \\ \sigma_{0h}(t, T_{\hat{\imath}+k-1}) & \begin{cases} h = \mathrm{int}((i-(N-k+c(N-m)))/m) + 1 \\ \hat{\imath} = (i-(N-k+c(N-m))) \mod m \\ i = N-k+c(N-m)+1, \dots, (c+1)N-k \end{cases} \end{cases}$$

Then we can write

$$\delta\Gamma^{(k)}_{i,j} = \int_{T_{k-1}}^{T_k} \theta^{(i,k)}(t)\theta^{(j,k)}(t)dt \qquad i,j = 1, \dots, (c+1)N-k$$

As in the single currency case, each market price for an actively traded derivative (caplet, swaption and currency option) leads to a constraint in the SDP. For all caplets and swaptions in the base currency, as well as for all caplets and swaptions maturing after T_m in all other currencies, the constraints can be written in the form (8). Furthermore, as a result of the deterministic volatility assumption, currency options maturing at or before T_m can be priced by the Black/Scholes formula

$$B_0(0,T_i)(X_{0h}(0,T_i)\mathcal{N}(\phi_1) - K\mathcal{N}(\phi_2))$$

with

$$\phi_{1,2} = \frac{\ln(X_{0h}(0,T_i)/K) \pm \nu^2_{0h}(0,T_i)}{\nu_{0h}(0,T_i)}$$

and

$$\nu^2_{0h}(0,T_i) = \int_0^{T_i} \sigma^2_{0h}(s,T_i)ds = \sum_{k=1}^{i} \int_{T_{k-1}}^{T_k} \sigma^2_{0h}(s,T_i)ds \tag{16}$$

i.e. there is an additional set of constraints (16).

Cross–currency options bear information about correlations between forward exchange rates via

$$\begin{aligned} \sigma_{jh}(t,T_i) &= \sigma_{0h}(t,T_i) - \sigma_{0j}(t,T_i) \\ \Rightarrow \qquad \nu^2_{jh}(0,T_i) &= \int_0^{T_i} (\sigma_{0h}(t,T_i) - \sigma_{0j}(t,T_i))^2 dt \end{aligned}$$

Further correlation information can be extracted if market prices are available for derivatives which do not have Black/Scholes–type valuation formulas under the given set of lognormality assumptions, for example caps involving forward LIBORs $L_j(t,T_i)$, $i \le m$, $j > 0$.

Appendix A. Derivation of the approximate swaption pricing formula

The cashflow of the forward swap underlying the payer swaption presented in section 2 can be priced at time $t \leq T_m$ as

$$\sum_{j=0}^{n/r-1} B(t, T_{m+j(r+1)}) E_{T_{m+j(r+1)}}[\delta r(L(T_{m+jr}, T_{m+jr}, r) - \kappa)|\mathcal{F}_t]$$
$$= \sum_{j=0}^{n/r-1} B(t, T_{m+j(r+1)}) \left(E_{T_{m+j(r+1)}} \left[\frac{B(T_{m+jr}, T_{m+jr})}{B(T_{m+jr}, T_{m+j(r+1)})} - 1|\mathcal{F}_t \right] - r\delta\kappa \right)$$
$$= \sum_{j=0}^{n/r-1} (B(t, T_{m+jr}) - B(t, T_{m+j(r+1)})(1 + r\delta\kappa)$$

which can be written as

$$= \delta \sum_{j=1}^{n} B(t, T_{m+j})(L(t, T_{m+j-1}) - \text{roz}(j, r)\kappa)$$

where $\text{roz}(\cdot, \cdot)$ is Brace's *r or zero* function[20]

$$\text{roz}(j, r) = \begin{cases} r & \text{if } j \bmod r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding forward swap rate $\omega(t, m, r, n)$ is defined as the fixed rate κ[8], which would make the value of the swap contract zero, i.e.

$$\omega(t, m, r, n) = \frac{\sum_{j=1}^{n} B(t, T_{m+j}) L(t, T_{m+j-1})}{\sum_{j=1}^{n} B(t, T_{m+j}) \text{roz}(j, r)} \tag{17}$$

A payer swaption gives the option holder the right to enter into a swap contract at time T_m, with a fixed rate of κ. The payoff of the payer swaption is thus given by

$$\max \left(0, \delta \sum_{j=1}^{n} \text{roz}(j, r) B(T_m, T_{m+j}) [\omega(T_m, m, r, n) - \kappa] \right) \tag{18}$$

and for a receiver swaption (where the party in question receives fixed rate payments and pays the floating rate) by

$$\max \left(0, \delta \sum_{j=1}^{n} \text{roz}(j, r) B(T_m, T_{m+j}) [\kappa - \omega(T_m, m, r, n)] \right) \tag{19}$$

Note from (17) that $\omega(t, m, r, n)$ is a martingale under the equivalent measure associated with taking the zero coupon bond portfolio $\sum_{j=1}^{n} B(t, T_{m+j}) \text{roz}(j, r)$ as the numeraire. Thus assuming deterministic volatility for $\omega(\cdot, m, r, n)$ would allow the swaption to be priced by a Black/Scholes–like formula, the so-called Black swaption formula. In the current model framework, this is strictly speaking not compatible with the absence of

[20]cf. Brace, Dun and Barton (2001)

arbitrage. By Ito's lemma, the volatility of the forward swap rate $\omega(\cdot, m, r, n)$ is determined by the forward LIBOR volatilities by[21]

$$\sigma(t, m, r, n) = \sum_{j=1}^{n} A_j^{(m,n)}(t)\lambda(t, T_{m+j-1})$$

with

$$A_j^{(m,n)}(t) = w_j^{(m,n)}(t) + \mu(t, T_{m+j-1}) \sum_{l=j}^{n} (\text{roz}(l, r) u_l^{(m,n)}(t) - w_l^{(m,n)}(t))$$

and

$$\begin{aligned} w_j^{(m,n)}(t) &= \frac{B(t, T_{m+j})L(t, T_{m+j-1})}{\sum_{h=1}^{n} B(t, T_{m+h})L(t, T_{m+h-1})} \\ \mu(t, T_{m+j-1}) &= \frac{\delta L(t, T_{m+j-1})}{1 + \delta L(t, T_{m+j-1})} \\ u_j^{(m,n)}(t) &= \frac{B(t, T_{m+j})}{\sum_{h=1}^{n} \text{roz}(h, r) B(t, T_{m+h})} \end{aligned}$$

Thus forward swap rate volatility is not deterministic, but depends on levels of bond prices and LIBORs. However, one can argue[22] that the coefficients $w_j^{(m,n)}(t)$, $\mu(t, T_{m+j-1})$ and $u_j^{(m,n)}(t)$ vary comparatively little in t, and one is thus justified in "freezing" these coefficients at their time 0 values.[23] The approximate forward swap rate volatility

$$\sigma_0(t, m, r, n) = \sum_{j=1}^{n} A_j^{(m,n)}(0)\lambda(t, T_{m+j-1})$$

is a linear combination of deterministic volatilities $\lambda(\cdot, \cdot)$ of forward LIBORs, and the Black swaption formula applies. For the time 0 value of the payoff (18) this yields

$$\delta \sum_{j=1}^{n} \text{roz}(j, r) B(0, T_{m+j})(\omega(0, m, n, r)\mathcal{N}(h_1) - \kappa \mathcal{N}(h_2))$$

with

$$\begin{aligned} h_{1,2} &= \frac{\ln \frac{\omega(0,m,n,r)}{\kappa} \pm \frac{1}{2}\zeta(m, n, r)}{\sqrt{\zeta(m, n, r)}} \\ \zeta(m, n, r) &= \int_0^{T_m} \sigma_0^2(t, m, r, n) dt \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} A_i^{(m,n)}(0) A_j^{(m,n)}(0) \int_0^{T_m} \lambda(t, T_{m+i-1})\lambda(t, T_{m+j-1}) dt \end{aligned}$$

[21]This volatility has been derived by numerous authors in numerous papers. The notation used here is very close to Brace and Womersley (2000). Note that $A_j^{(m,n)}(t)$ depends on r. However, since all swaptions used for calibration usually will have the same r, it is omitted in the notation.

[22]This argument was first put forward by Brace, Gatarek and Musiela (1997), developed further by Brace, Dun and Barton (2001) and formalised in Brace and Womersley (2000).

[23]Dun, Schlögl and Barton (2001) show that this approximation is applicable not only to pricing, but to hedging as well.

Appendix B. The problem formulation of Brace/Womersley

The constraints (8) can be rewritten as the Frobenius product[24]

$$\Psi^{(m,n)} \bullet \Gamma = \beta^2(m,n,r)$$

where the $g \times g$ constraint matrices $\Psi^{(m,n)}$ are formed from the $A_i^{(m,n)}$ to match (8).

As discussed by Brace and Womersley (2000), there are some limits on the choice of metric measuring the closeness of Γ and the historical volatility structure $\overline{\Gamma}$, due to the necessity of maintaining a linear objective function. Thus the Frobenius norm $\|\Gamma - \overline{\Gamma}\|_F^2 = (\Gamma - \overline{\Gamma}) \bullet (\Gamma - \overline{\Gamma})$ is not a tractable objective function. They propose instead to use the 2–norm,

$$\|\Gamma - \overline{\Gamma}\|_2 = \max_{i=1,\dots,n} |\Lambda_i(\Gamma - \overline{\Gamma})| \qquad (20)$$

where $\Lambda_i(\Gamma - \overline{\Gamma})$ is the i–th eigenvalue of the matrix $\Gamma - \overline{\Gamma}$. Then the calibration problem can be formulated to minimise (20) subject to the constraints (8) and Γ positive semidefinite (denoted by $\Gamma \succsim 0$). This is equivalent to

B.1. Problem. ***The single currency calibration problem:*** *Find the real symmetric matrix Γ and an $\eta \in \mathbb{R}$, to minimise η, subject to*

$$\begin{aligned} \eta I - (\Gamma - \overline{\Gamma}) &\succsim 0 \\ \eta I + (\Gamma - \overline{\Gamma}) &\succsim 0 \\ \eta \geq 0 \quad , \quad \Gamma &\succsim 0 \end{aligned}$$

and a constraint (8) for each swaption or caplet market price to which the model is to be calibrated. I denotes the identity matrix of appropriate size.

There are a variety of freeware implementations of SDP algorithms available.[25] However, existing algorithms for constrained SDPs[26] only implement problems of the type

B.2. Problem. ***Generic SDP problem:***

$$\begin{array}{lll} \textit{Find} & X \in \mathbb{S}_n,\ x \in \mathbb{R}^n & \\ \textit{to minimise} & C \bullet X + c^\top x & \\ \textit{subject to} & A_k \bullet X = b_k & k = 1,\dots,m_s \\ & a_j^\top x = \beta_j & j = 1,\dots,m_l \\ & X \succsim 0,\ x \geq 0 & \end{array}$$

[24]The Frobenius inner product is defined as

$$X \bullet Y = \mathrm{trace}(X^\top Y) = \sum_{i=1}^n \sum_{j=1}^n X_{ij} Y_{ij} \qquad X, Y \in \mathbb{R}^{n \times n}$$

[25]For a list of links, see ...

[26]see for example SDPA, ...

To bring (B.1) in line with this implementation, we define

$$X = \begin{pmatrix} \Gamma & 0 & 0 \\ 0 & \eta I - (\Gamma - \overline{\Gamma}) & 0 \\ 0 & 0 & \eta I + (\Gamma - \overline{\Gamma}) \end{pmatrix} \tag{21}$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

$$c = 0, \quad m_l = 0$$

i.e. no vectors x. We expand each constraint (8) to

$$A^{(k)} = \begin{pmatrix} \Psi^{(m,n)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad b_k = \beta^2(m,n,r)$$

Note that the SDP algorithm will seek any $X \in \mathbb{S}_n$ that satisfies the constraints. In order to ensure that X can in fact be represented in terms of Γ, $\overline{\Gamma}$ and η as in (21), additional constraints are required. X consists of $3 \cdot \frac{1}{2}g(g+1)$ unknowns, the entries in the three symmetric blocks on the diagonal of X — note that the entries in the off–diagonal blocks of X are irrelevant (though set to zero in (21)), since they figure neither in the objective function nor in the constraints. Of the these unknowns, $\frac{1}{2}g(g+1)$ are free (Γ), plus one (η). Thus we need $2 \cdot \frac{1}{2}g(g+1) - 1$ additional constraints. $\frac{1}{2}g(g+1)$ of these can be written as

$$B^{(i,j)} \bullet X = d_{ij}$$

where the $B^{(i,j)}$ are matrices and the d_{ij} are scalars. They are given by

$B^{(i,j)}$:zero everywhere except for 2 at the positions corresponding to the positions of Γ_{ij}, Γ_{ji} in X, 1 at the positions corresponding to $(\eta I - (\Gamma - \overline{\Gamma}))_{ij}$, $(\eta I - (\Gamma - \overline{\Gamma}))_{ji}$ and -1 at the positions corresponding to $(\eta I + (\Gamma - \overline{\Gamma}))_{ij}$, $(\eta I + (\Gamma - \overline{\Gamma}))_{ji}$.

d_{ij} $=4\overline{\Gamma}_{ij}$

Thus we are using the identity

$$4\Gamma_{ij} + 2(\eta I - (\Gamma - \overline{\Gamma}))_{ij} - 2(\eta I + (\Gamma - \overline{\Gamma}))_{ij} \stackrel{!}{=} 4\overline{\Gamma}_{ij}$$

A further $\frac{1}{2}(g-1)g$ constraints[27] are given by

$$(\eta I - (\Gamma - \overline{\Gamma}))_{ij} + (\eta I + (\Gamma - \overline{\Gamma}))_{ij} = 0 \qquad \forall\ i \neq j$$

which can be written as

$$F^{(i,j)} \bullet X = 0 \qquad \forall\ i \neq j$$

with $F^{(i,j)}$ zero everywhere except for 1 at the positions corresponding to $(\eta I - (\Gamma - \overline{\Gamma}))_{ij}$ and $(\eta I + (\Gamma - \overline{\Gamma}))_{ij}$ in X. Lastly, we have

$$(\eta I - (\Gamma - \overline{\Gamma}))_{ii} + (\eta I + (\Gamma - \overline{\Gamma}))_{ii} = 2\eta$$

which yields the $g-1$ constraints

$$D^{(i)} \bullet X = 0 \qquad \forall\ 1 < i \leq g$$

[27]As above, the number of unknowns and constraints is reduced by the fact that X is known to be symmetric.

with $D^{(i)}$ zero everywhere except for 1 at the position corresponding to $(\eta I - (\Gamma - \overline{\Gamma}))_{11}$ and -1 at the position corresponding to $(\eta I + (\Gamma - \overline{\Gamma}))_{ii}$ in X.

Thus the calibration problem (B.1) has been rewritten in terms of (B.2). Unfortunately, this results in a very inefficient implementation and it would thus be desirable to develop an algorithm which solves the SDP (B.1) directly.

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