ARBITRAGE–FREE INTERPOLATION IN MODELS OF MARKET OBSERVABLE INTEREST RATES

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ABSTRACT. Models which postulate lognormal dynamics for interest rates which are compounded according to market conventions, such as forward LIBOR or forward swap rates, can be constructed initially in a discrete tenor framework. Interpolating interest rates between maturities in the discrete tenor structure is equivalent to extending the model to continuous tenor. The present paper sets forth an alternative way of performing this extension; one which preserves the Markovian properties of the discrete tenor models and guarantees the positivity of *all* interpolated rates.

Practitioners have long priced caps, floors and other interest rate derivative contracts by Black/Scholes–like[1] formulae, a practice usually attributed to the work of Black (1976). Initially, this approach was very much focused on the pricing of individual contracts, without regard to the arbitrage relationships between different fixed income instruments. The seminal paper of Ho and Lee (1986) instigated further research into models of the entire yield curve and fitting initial data, where the yield curve was given either terms of zero coupon bond prices[2] or continuously compounded short or forward rates. The latter approach was taken by Heath, Jarrow and Morton (1992) (HJM), who developed a general framework for models of the term structure of interest rates, deriving drift conditions for instantaneous forward rates which must be satisfied in *any* arbitrage–free model. Thus the fundamental objects of these models were mainly mathematical constructs, not market observables. The first work to move away from this paradigm was Sandmann and Sondermann (1989, 1994), who construct a binomial term structure model with the short rate compounded according to market conventions (e.g. three–month LIBOR) as the fundamental model variable. Taking the modelling of market observable interest rates to continuous time and bringing theoretical developments back into line with widespread market practice, Miltersen, Sandmann and Sondermann (1997)[3] (MSS) embedded the pricing of caps and floors by Black/Scholes–like formulae in a consistent HJM framework. The critical assumption for this result is that relative volatility[4] of observable market forward rates such as LIBOR is deterministic.[5]

Brace, Gatarek and Musiela (1997) (BGM) resolved key questions in the construction of such a model, in particular concerning existence and measure relationships. Given the deterministic volatility assumption, they explicitly identified forward LIBOR as lognormal martingales under the forward measure to the end of the respective accrual periods. Their

[1]cf. Black and Scholes (1973).

[2]cf. such papers as Ho and Lee (1986) or El Karoui, Lepage, Myneni, Roseau and Viswanathan (1991).

[3]See also Sandmann and Sondermann (1994) and Sandmann, Sondermann and Miltersen (1995).

[4]The relative volatility of a diffusion process $X(t)$ is σ if its quadratic variation is given by $X(t)^2\sigma^2 dt$.

[5]For a succinct treatment of the significance of this assumption, see Rady (1997).

label *Market Models* seems to have become the generally accepted descriptor for models which postulate lognormality (under the appropriate probability measure) for market observable interest rates. The same methodology can be applied to forward swap rates to derive a model which supports the market practice of pricing swaptions by a Black/Scholes–type formula, as demonstrated by Jamshidian (1997). Miltersen, Nielsen and Sandmann (2001) use a similar approach to construct a model of lognormal futures rates (implied by the prices of interest rate futures), which nests the lognormal forward LIBOR model.

Starting point of our discussion is the work of Musiela and Rutkowski (1997), who bring important clarity into the construction of lognormal market models. Besides explicitly deriving the relationships between forward measures of different maturities, they set up the model of lognormal forward LIBORs for a discrete set of maturities (the *tenor structure*) first and then extend it to a continuum of maturities by applying the assumptions of BGM, i.e. lognormality of all forward rates with compounding period δ and zero volatility for all zero coupon bonds with time to maturity less than δ. This makes transparent the fact that extending the model to continuous tenor in effect stipulates how the model will interpolate between a discrete set of maturities. Note that in all cases considered in the present paper the stochastic dynamics of interest rates are modelled in continuous time. Interpolation occurs in the *maturity dimension* only. Furthermore, the rates modelled have a fixed maturity (as opposed to a fixed time–to–maturity). In simple terms, the question addressed is how to determine the continuous time dynamics of, say, the yield for an investment maturing next August given the continuous time dynamics of three–month forward rates for accrual periods starting in March, June, September and December of every year.

Specifying lognormal dynamics for all δ–compounded forward rates, as MSS and BGM do, has the obvious advantage that all caplets and floorlets on such rates are valued by Black/Scholes–like formulae. However, this elegance comes at a price. For one, the no–arbitrage with cash property cannot be guaranteed, i.e. although the forward LIBORs modelled as lognormal will always be positive, other rates may not be.[6] Second, the Markovian properties of the discrete tenor model are lost; this is a fact which can cause considerable problems in numerical implementations, including Monte Carlo simulation, as we will discuss below. Third, one may take the pragmatic point of view that in reality one observes only a discrete number of rates on the market, with the rest determined by some interpolation method of choice. Thus one may prefer a version of the model which implies an interpolation method for future dates that is transparent and tractable. As the approach of Musiela and Rutkowski (1997) to the construction of a continuous tenor model makes clear, once volatilities are specified for continuous tenor, one has the freedom to arbitrarily interpolate *initial* observed market rates. Thereafter, interpolation is fixed by no arbitrage conditions and must be evaluated numerically.

The alternative way of extending a lognormal market model from discrete to continuous tenor proposed here addresses each of these points. It is also worthy to note that while the practical relevance of the lognormal market models makes the deterministic volatility case a natural focus of attention, Musiela and Rutkowski's forward measure approach to term structure modelling is valid for much more general volatility specifications, and this generality carries over to a large part of the results presented here. In particular, this is true for other models constructed in discrete tenor, such as the lognormal forward swap rate model of Jamshidian (1997).

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[6]Musiela and Rutkowski (1997) give an example of this.

In the present paper, the extension to continuous tenor is initially performed by deterministically interpolating zero coupon bond prices maturing before the earliest (future) date in the tenor structure (the "short bonds"), not only for the initial term structure, but at any future time as well (section 1). At any point in time, interpolation of longer–dated bonds is determined by no arbitrage conditions (section 2). Interpolating short bond prices from the two adjacent discrete tenor LIBORs introduces volatility for the short bonds and the associated interest rates (section 3). The dynamics of all interpolated rates remain Markovian in the state vector of discrete tenor forward LIBORs (section 4). The properties of the interpolated rates are discussed in section 5. Section 6 analyses the implications for the pricing of cap/floor contacts which do not fit into the discrete tenor structure and section 7 concludes.

1. Interpolation of Short–dated Bonds

Given a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \in [0,T^*]}, \mathbb{P}_{T^*})$ satisfying the usual conditions, let $\{W_{T^*}(t)\}_{t \in [0,T^*]}$ denote a d–dimensional standard Wiener process and assume that the filtration $\{\mathcal{F}_t\}_{t \in [0,T^*]}$ is the usual $\mathbb{P}_{T^*}$–augmentation of the filtration generated by $\{W_{T^*}(t)\}_{t \in [0,T^*]}$.

The model is set up on the basis of assumptions **(BP.1)** and **(BP.2)** of Musiela and Rutkowski (1997):

(BP.1) For any date $T \in [0, T^*]$, the price process of a zero coupon bond $B(t,T)$, $t \in [0,T]$ is a strictly positive special martingale under $\mathbb{P}_{T^*}$.

(BP.2) For any fixed $T \in [0, T^*]$, the *forward process*

$$F_B(t, T, T^*) = \frac{B(t,T)}{B(t,T^*)}, \qquad \forall \, t \in [0,T]$$

follows a martingale under $\mathbb{P}_{T^*}$.

Note that assumption **(BP.2)** means that $\mathbb{P}_{T^*}$ can be interpreted as the *time T^* forward measure* and implies that the bond price dynamics are arbitrage–free.

The objects to be modelled on the fixed income markets are the δ-compounded forward rates defined by

$$(1) \qquad L(t,T) = \delta^{-1} \left(\frac{B(t,T)}{B(t,T+\delta)} - 1 \right)$$

Since the compounding matches the market convention for rates such as the London Interbank Offer Rate, $L(t,T)$ is also referred to as forward LIBOR.

Consider a discrete tenor structure $\mathbb{T} = \{T_0, \ldots, T_N\}$. For notational simplicity, let $T_0 = 0$ also be the time origin and $T_{i+1} - T_i = \delta \, \forall \, i < N$. The dynamics of the rates $L(\cdot, T_i)$, $1 \le i < N$, are given as in the discrete tenor version of the lognormal forward rate model put forth by Musiela and Rutkowski (1997):[7]

$$(2) \qquad dL(t, T_i) = L(t, T_i)\lambda(t, T_i)dW_{T_{i+1}}(t)$$

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[7]Note that the focus here is on lognormal forward LIBOR models for expositional purposes only, because this represents the mainstream of the literature. The results presented here apply equally well to the discrete tenor versions of other models of market observable rates, such as the lognormal forward swap rate model of Jamshidian (1997) or extensions such models to more general volatility functions, such as the model of level–dependent LIBOR volatilities proposed by Andersen and Andreasen (1998).

where $W_{T_{i+1}}$ is a d-dimensional standard Brownian motion under the time T_{i+1} forward measure $\mathbb{P}_{T_{i+1}}$ and $\lambda : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^d$ is a deterministic function of its arguments, i.e. each $L(t, T_i)$ is a lognormal martingale under $\mathbb{P}_{T_{i+1}}$.[8]

Define the function

$$\eta(t) : [T_0, T_N[\to \{0, \ldots, N\}$$

as

$$\eta(t) = \max\{i \in \{1, \ldots, N\}|T_{i-1} < t\}$$

Similarly to the extensions to continuous tenor proposed in Brace, Gatarek and Musiela (1997) and Musiela and Rutkowski (1997), we initially make the following

1.1. ASSUMPTION. *Let volatility be zero for all zero coupon bonds $B(t_1, t_2)$ maturing by the next (future) date in the tenor structure $\mathbb{T}$, i.e. $t_1 \leq t_2 \leq T_{\eta(t_1)}$.*

Consequently, at time T_i the price of the zero coupon bond maturing in T_{i+1} is given by

$$B(T_i, T_{i+1}) = (1 + \delta L(T_i, T_i))^{-1}$$

and its dynamics are deterministic thereafter. Since they are deterministic, interpolation of interest rates with time to maturity less than δ is the same as specifying the evolution of the bond price, and arbitrary deterministic dynamics can be specified for $B(t, T_{i+1})$, $T_i < t < T_{i+1}$, with $B(T_{i+1}, T_{i+1}) = 1$. For obvious reasons, these dynamics should be monotonically increasing and continuous, which means defining the continuously compounded short rate as some positive function[9] $r(s, L(T_{\eta(s)-1}, T_{\eta(s)-1}))$, such that[10]

$$(3) \qquad \exp\left\{\int_{T_i}^{T_{i+1}} r(s, L(T_{\eta(s)-1}, T_{\eta(s)-1}))ds\right\} = 1 + \delta L(T_i, T_i) \qquad \forall \ i < N$$

Thus $r(t)$ is an $\mathcal{F}_{\eta(t)-1}$–measurable random variable. Consequently,

$$B(t, T_{\eta(t)}) = \exp\left\{-\int_t^{T_{\eta(t)}} r(s, L(T_{\eta(s)-1}, T_{\eta(s)-1}))ds\right\}$$

Note that while the dynamics of the abstract variable r are in general discontinuous at each T_i, any rate associated with a compounding period greater than length zero will have continuous dynamics.

1.2. REMARK. *In a term structure model with zero volatility for zero coupon bonds $B(t, T_{\eta(t)})$ for all $T_0 \leq t \leq T_N$, the risk neutral and rolling spot LIBOR measures are identical and the short rate interpolation (3) is arbitrage free.*

PROOF: The risk neutral equivalent martingale measure is conventionally defined as the measure associated with taking the (continuously compounded) savings account as the numeraire asset, i.e. under this measure all assets discounted by the savings account are

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[8]For details of the model construction, see Musiela and Rutkowski (1997).

[9]Note that this immediately implies that all interest rates generated by the model will be positive.

[10]Given Assumption 1.1, (3) is simply a restatement of the identity (23) in Miltersen, Sandmann and Sondermann (1997).

martingales. The savings account $\beta(t)$ defined by the interpolation is

$$
\begin{aligned}
\beta(t) \;&=\; \exp\left\{ \int_0^t r\big(s, L(T_{\eta(s)-1}, T_{\eta(s)-1})\big) ds \right\} \\
(4) \qquad &=\; \left(\prod_{i=0}^{\eta(t)-2} \big(1 + \delta L(T_i, T_i)\big) \right) \exp\left\{ \int_{T_{\eta(t)-1}}^t r\big(s, L(T_{\eta(s)-1}, T_{\eta(s)-1})\big) ds \right\}
\end{aligned}
$$

For the rolling spot LIBOR measure as introduced by Jamshidian (1997), the numeraire is a roll–over strategy in which money is invested and subsequently reinvested at spot LIBOR. Investing one monetary unit in T_i at the spot LIBOR $L(T_i, T_i)$ is the same as buying k zero coupon bonds $B(T_i, T_{i+1})$, where $k = B(T_i, T_{i+1})$. Thus in continuous tenor the value at time t of the roll–over strategy is

$$
(5) \qquad \left(\prod_{i=0}^{\eta(t)-2} \big(1 + \delta L(T_i, T_i)\big) \right) \frac{B(t, T_{i+1})}{B(T_i, T_{i+1})}
$$

Since $r\big(s, L(T_{\eta(s)-1}, T_{\eta(s)-1})\big)$ is $\mathcal{F}_{T_i}$–measurable for $T_i \le s < T_{i+1}$, we have by construction that

$$
\frac{B(t, T_{i+1})}{B(T_i, T_{i+1})} = \exp\left\{ \int_{T_{\eta(t)-1}}^t r\big(s, L(T_{\eta(s)-1}, T_{\eta(s)-1})\big) ds \right\}
$$

for $T_i \le t \le T_{i+1}$. Thus the numeraire processes are identical, which means that the associated martingale measures must be[11].

The interpolation defined by (3) is consistent with no arbitrage if the savings account given by (4) discounted by a numeraire asset is a martingale under the martingale measure associated with this numeraire. It is sufficient to verify this for one measure and by the above argument, this condition is trivially fulfilled since the value of the savings account is identically equal to the value of the spot LIBOR roll–over strategy.

$$\square$$

In other words, since at any time t the dynamics of the shortest remaining bond to a discrete tenor date $B(t, T_{\eta(t)})$ are consistent with the deterministic interpolation for the continuously compounded short rate, all martingale properties (and thus no arbitrage) are preserved.

The observation that the roll–over strategy in spot LIBOR corresponds to a roll-over strategy in the shortest remaining bond to a discrete tenor date also leads us to the following

1.3. REMARK. Conditional *on the information at time T_i, for all $\mathcal{F}_{T_{i+1}}$–measurable events, the rolling spot LIBOR measure $I\!\!P_0$ is identical to the forward measure of maturity T_{i+1}, i.e.*

$$
I\!\!P_0\{A | \mathcal{F}_{T_i}\} = I\!\!P_{T_{i+1}}\{A | \mathcal{F}_{T_i}\} \qquad \forall\, A \in \mathcal{F}_{T_{i+1}}
$$

Thus $I\!\!P_0$ can be interpreted as "pasting together" a sequence of conditional forward measures. This is valid for any arbitrage free term structure model.

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[11] In a continuous tenor model, the market is (dynamically) complete, therefore the martingale measure for a given numeraire is unique. For an alternative proof, cf. Lemma 3.3 in Schlögl (2002).

PROOF: This is an immediate consequence of the fact that the time t value of the spot LIBOR roll–over strategy (5) can be written, independently of how the discrete tenor model is extended to continuous tenor, as a $T_{\eta(t)-1}$–measurable factor times $B(t, T_{\eta(t)})$, the numeraire of the time $T_{\eta(t)}$ forward measure.[12]

□

2. INTERPOLATION OF LONGER–DATED BONDS

2.1. REMARK. *Given a discrete tenor model and the interpolation of the short–dated zero coupon bonds $B(t, T_{\eta(t)})$, the continuous tenor model is completely specified.*

PROOF: Consider a completely arbitrary pair of time points t_1, t_2, $0 \leq t_1 < t_2 \leq T_N$. The time t_1 price of a zero coupon bond maturing in t_2 is given by

$$(6) \qquad B(t_1, t_2) = B(t_1, T_{\eta(t_1)}) \left(\prod_{i=\eta(t_1)}^{\eta(t_2)-1} (1 + \delta L(t_1, T_i))^{-1} \right) \frac{B(t_1, t_2)}{B(t_1, T_{\eta(t_2)})}$$

where the short–dated bond $B(t_1, T_{\eta(t_1)})$ is given by the interpolation and the bond price quotient on the far right by the no–arbitrage requirement that

$$\frac{B(t_1, t_2)}{B(t_1, T_{\eta(t_2)})} = E_{T_{\eta(t_2)}}[B(t_2, T_{\eta(t_2)})^{-1} | \mathcal{F}_{t_1}]$$

Given a choice of interpolation (3), we can determine a function g such that

$$(7) \qquad \frac{B(t_1, t_2)}{B(t_1, T_{\eta(t_2)})} = E_{T_{\eta(t_2)}}[g(L(T_{\eta(t_2)-1}, T_{\eta(t_2)-1})) | \mathcal{F}_{t_1}]$$

In the lognormal case, the right hand side can easily be evaluated numerically and depends only on $L(t_1, T_{\eta(t_2)-1})$ and deterministic volatilities.

□

When intermediate rates with LIBOR–type compounding are linearly interpolated by day–count fractions, (7) becomes very tractable for any model satisfying the assumptions (BP.1) and (BP.2). Define $r(s, L(T_{\eta(s)-1}, T_{\eta(s)-1}))$ implicitly by setting

$$(8) \quad \exp \left\{ \int_t^{T_{i+1}} r(s, L(T_{\eta(s)-1}, T_{\eta(s)-1})) ds \right\} = 1 + (T_{i+1} - t) L(T_i, T_i) \qquad \forall\ T_i \leq t < T_{i+1}$$

i.e.

$$r(s, L(T_i, T_i)) = \frac{L(T_i, T_i)}{1 + (T_{i+1} - s) L(T_i, T_i)}$$

Then

$$B(t_1, T_{\eta(t_1)}) = (1 + (T_{\eta(t_1)} - t_1) L(T_{\eta(t_1)-1}, T_{\eta(t_1)-1}))^{-1}$$

and

$$\begin{aligned} \frac{B(t_1, t_2)}{B(t_1, T_{\eta(t_2)})} &= E_{T_{\eta(t_2)}}[B(t_2, T_{\eta(t_2)})^{-1} | \mathcal{F}_{t_1}] \\ &= 1 + (T_{\eta(t_2)} - t_2) E_{T_{\eta(t_2)}}[L(T_{\eta(t_2)-1}, T_{\eta(t_2)-1}) | \mathcal{F}_{t_1}] \\ &= 1 + (T_{\eta(t_2)} - t_2) L(t_1, T_{\eta(t_2)-1}) \end{aligned}$$

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[12]For a more formal statement of this proof, see Schlögl (2002).

Note that by remark 2.1, if interpolation of short–dated bonds is specified for all times, there is no modelling freedom to interpolate longer–dated bonds, including initial model inputs. In addition to being intuitively appealing, linear interpolation by day–count fractions has the added attraction of providing one consistent interpolation method for all times and all maturities. Other popular interpolation methods, such as loglinear interpolation of discount factors or linear interpolation of continuously compounded yields, do not have this property.

3. Introducing Volatility for the Short Bonds

In some applications, setting short bond volatilities to zero may be unsatisfactory. Assumption 1.1 can certainly be relaxed in any way which satisfies the arbitrage constraints. Remark 2.1 remains valid, i.e. to extend the model to continuous tenor it is sufficient to specify the dynamics of the short bonds. However, when the short bond dynamics are stochastic, the constraint given by equation (3) is replaced by the more general formulation given in terms of instantaneous forward rates in equation (23) of Miltersen, Sandmann and Sondermann (1997):

$$(9) \qquad \exp\left\{\int_{T_i}^{T_{i+1}} f(T_i, s)ds\right\} = 1 + \delta L(T_i, T_i) \qquad \forall\, i < N$$

where $f(T_i, s)$ is the instantaneous forward rate at time T_i for maturity s.

One tractable and intuitively appealing way to introduce volatility for the interpolated short bonds is to make the interpolated rates dependent on the closest remaining forward LIBOR $L(t_1, T_{\eta(t_1)})$. Set

$$B(t_1, T_{\eta(t_1)})^{-1} = 1 + (T_{\eta(t_1)} - t_1)(\alpha(t_1)L(T_{\eta(t_1)-1}, T_{\eta(t_1)-1}) + (1 - \alpha(t_1))L(t_1, T_{\eta(t_1)}))$$

where

$$\lim_{\Delta \searrow 0} \alpha(T_i + \Delta) = 1$$
$$\lim_{\Delta \nearrow T_{i+1}-T_i} \alpha(T_i + \Delta) = 0 \qquad \forall \quad i = 0, \ldots, N-1$$

e.g.

$$\alpha(t) = \frac{T_{\eta(t)} - t}{T_{\eta(t)} - T_{\eta(t)-1}}$$

As in section 2, the prices of bonds with longer maturities must satisfy

$$\frac{B(t_1, t_2)}{B(t_1, T_{\eta(t_2)})} = E_{T_{\eta(t_2)}}\left[\left.\frac{B(t_2, t_2)}{B(t_2, T_{\eta(t_2)})}\right| \mathcal{F}_{t_1}\right]$$

$$= 1 + (T_{\eta(t_2)} - t_2)\left(\alpha(t_2)E_{T_{\eta(t_2)}}\left[L(T_{\eta(t_2)-1}, T_{\eta(t_2)-1})|\mathcal{F}_{t_1}\right]\right.$$
$$\left. + (1 - \alpha(t_2))E_{T_{\eta(t_2)}}\left[L(t_2, T_{\eta(t_2)})|\mathcal{F}_{t_1}\right]\right)$$

Using the expected value of $L(t, T_j)$ under $\mathbb{P}_{T_j}$ given in Rutkowski (1997):

$$= 1 + (T_{\eta(t_2)} - t_2)\left(\alpha(t_2)L(t_1, T_{\eta(t_2)-1}) + (1 - \alpha(t_2))L(t_1, T_{\eta(t_2)})\right.$$

$$(10) \qquad \cdot\underbrace{\left(1 + \frac{(T_{\eta(t_2)+1} - T_{\eta(t_2)})L(t_1, T_{\eta(t_2)})\left(\exp\left\{\int_{t_1}^{t_2} \lambda^2(s, T_{\eta(t_2)})ds\right\} - 1\right)}{1 + (T_{\eta(t_2)+1} - T_{\eta(t_2)})L(t_1, T_{\eta(t_2)})}\right)}_{\text{"correction factor"}}\Bigg)$$

The "correction factor" will usually be quite close to one, unless volatility λ is very high or time to maturity $t_2 - t_1$ is very long. Thus interpolation remains essentially linear.

Note that since the short bonds are no longer deterministic, we also need to evaluate the appropriate expectation to derive $B(t, T)$ for $t < T < T_{\eta(t)}$:

$$
\begin{aligned}
\frac{B(t,T)}{B(t, T_{\eta(t)})} &= E_{T_{\eta(t)}} \left[\left. \frac{B(T,T)}{B(T, T_{\eta(t)})} \right| \mathcal{F}_t \right] \\[2mm]
&= 1 + (T_{\eta(t)} - T) \Bigg(\alpha(T) L(t, T_{\eta(t)-1}) + (1 - \alpha(T)) L(t, T_{\eta(t)}) \\[2mm]
&\quad \cdot \left(1 + \frac{(T_{\eta(t)+1} - T_{\eta(t)}) L(t, T_{\eta(t)}) \left(\exp\left\{ \int_t^T \lambda^2(s, T_{\eta(t)}) ds \right\} - 1 \right)}{1 + (T_{\eta(t)+1} - T_{\eta(t)}) L(t, T_{\eta(t)})} \right) \Bigg)
\end{aligned}
$$

4. Forward Measures and the Markov Property

Both Miltersen, Sandmann and Sondermann (1997) and Brace, Gatarek and Musiela (1997) set up the model of lognormal forward LIBORs in continuous tenor, i.e. equation (2) applies to all maturities $T \in (0, T^* - \delta]$:

$$
dL(t, T) = L(t, T) \lambda(t, T) dW_{T+\delta}(t)
$$

with $\lambda : \mathbb{R} \times (0, T^*] \to \mathbb{R}^d$ a deterministic function of its arguments, i.e. each $L(t, T)$ is a lognormal martingale under $\mathbb{P}_{T+\delta}$ for a continuum of maturities up to the time horizon T^*. This has the advantage that all caps and floors on δ–compounded rates will be priced by the Black/Scholes–type formulae favoured by market practitioners.

The main disadvantage of this approach is revealed when one attempts to price instruments for which closed form solutions are unavailable. This is already the case if the cashflow underlying a derivative does not fit neatly into a tenor structure of δ–compounded rates and even more so for many popular interest rate exotics. The problem is that the continuous tenor model is infinite–dimensional even if the driving Brownian motion is one–dimensional. That is, the Markovian state variable for such a model is the entire yield curve.[13] This causes considerable difficulties for all types of numerical methods. Even the method of last resort for high–dimensional problems, Monte Carlo simulation, cannot handle infinite–dimensional state variables.

From the start, it has been part of the "folk wisdom" on these models that they do not permit a finite dimensional representation. Brace, Gatarek and Musiela (1997) derive the dynamics of the continuously compounded short rate under a given set of assumptions on the volatility structure and show that they are highly path–dependent and thus not Markovian in any finite set of state variables. Recently, Corr (2000) formalised this conjecture and showed that when the model is driven by a one–dimensional Brownian motion, a finite–dimensional representation can only exist in the trivial case of zero volatility. He also derives similarly restrictive conditions on the finite–dimensional representability of continuous tenor LIBOR models driven by a multidimensional Brownian motion. These

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[13]The term "yield curve" generally denotes the term structure interest rates for a continuum of maturities up to the time horizon. It can be represented in several equivalent ways, for example by the curve of all continuously compounded yields at time t, $y_t : (t, T^*] \to \mathbb{R}$ with $y_t(T)$ defined by

$$
B(t, T) = \exp\{-y_t(T) \cdot T\}
$$

conditions, though stopping short of precluding finite–dimensional cases, indicate that such cases are not likely to be practically useful if they do exist.

The extensions of the discrete tenor model to continuous tenor proposed in the previous sections provide a way around this problem. Since the interpolated rates are specified as functions of the discrete tenor rates, the Markovian structure of the discrete tenor model is preserved. This structure can be characterised as follows:

4.1. REMARK. *In the discrete tenor lognormal forward LIBOR model, consider the dynamics of a rate $L(\cdot, T_i)$ under some forward measure[14] $\mathbb{P}_{T_j}$. These dynamics are Markovian in a state variable vector consisting of $n = |j - 1 - i| + 1$ rates $L(\cdot, T_k)$ with $\min(i, j - 1) \leq k \leq \max(i, j - 1)$.*

PROOF: Musiela and Rutkowski (1997) show that the relationship between forward measures $\mathbb{P}_{T_k}$ and $\mathbb{P}_{T_{k+1}}$ is given by the Radon/Nikodym derivative

$$\left. \frac{d\mathbb{P}_{T_k}}{d\mathbb{P}_{T_{k+1}}} \right|_{\mathcal{F}_t} = \exp\left\{ \int_0^t \gamma(u, T_k, T_{k+1}) dW_{T_{k+1}}(u) - \frac{1}{2} \int_0^t \gamma^2(u, T_k, T_{k+1}) du \right\}$$

where $W_{T_{k+1}}$ is a Brownian motion under $\mathbb{P}_{T_{k+1}}$ and

$$\gamma(t, T_k, T_{k+1}) = \frac{\delta L(t, T_k)}{1 + \delta L(t, T_k)} \lambda(t, T_k)$$

with $\lambda(t, T_k)$ the (deterministic) volatility of $L(t, T_k)$. Thus

$$dW_{T_k}(t) = dW_{T_{k+1}}(t) - \frac{\delta L(t, T_k)}{1 + \delta L(t, T_k)} \lambda(t, T_k) dt$$

Since the dynamics of $L(\cdot, T_k)$ are given by

$$dL(t, T_k) = L(t, T_k) \lambda(t, T_k) dW_{T_{k+1}}(t)$$

the joint dynamics of all rates $L(\cdot, T_k)$ with $\min(i, j - 1) \leq k \leq \max(i, j - 1)$ under $\mathbb{P}_{T_{\max(i+1,j)}}$ are

(11) $$d\mathbb{L}(t) = \Lambda(t, \mathbb{L}(t)) dW_{T_{\max(i+1,j)}}(t) - \Psi(\mathbb{L}(t)) \ell(\mathbb{L}(t)) dt$$

where

$$\mathbb{L}(t) = \begin{pmatrix} L(t, T_{\min(i,j-1)}) \\ L(t, T_{\min(i,j-1)+1}) \\ \vdots \\ L(t, T_{\max(i,j-1)}) \end{pmatrix}$$

If d is the dimension of the driving Brownian motion, $\Lambda(t, \mathbb{L}(t))$ is an $n \times d$ matrix with

$$\Lambda_{hk}(t, \mathbb{L}(t)) = L(t, T_{\min(i,j-1)-1+h}) \lambda_k(t, T_{\min(i,j-1)-1+h})$$

$\Psi(t, \mathbb{L}(t))$ is an $n \times n$ matrix with

$$\Psi_{hk}(t, \mathbb{L}(t)) = \begin{cases} L(t, T_{\min(i,j-1)-1+h}) \lambda(t, T_{\min(i,j-1)-1+h}) \lambda(t, T_{\min(i,j-1)-1+k}) & \text{if } k > h \\ 0 & \text{otherwise} \end{cases}$$

Finally, $\ell(\mathbb{L}(t))$ is an n–dimensional vector with

$$\ell_k(\mathbb{L}(t)) = \frac{\delta L(t, T_{\min(i,j-1)-1+k})}{1 + \delta L(t, T_{\min(i,j-1)-1+k})}$$

---

[14]Note that the Markov property depends on the probability measure under consideration.

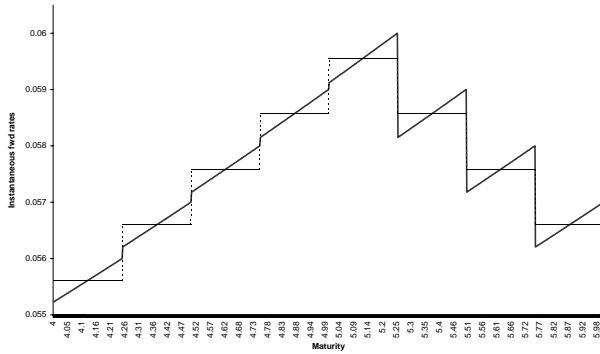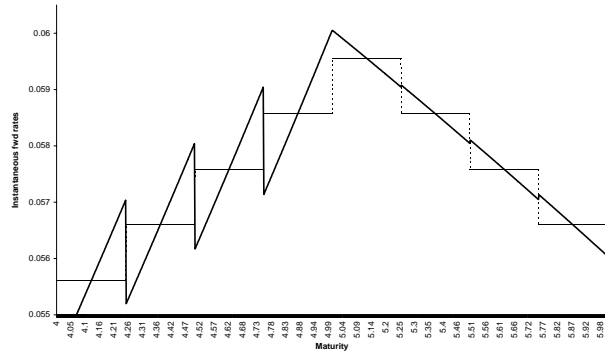<table>
<tr><td>FIGURE 1</td><td>FIGURE 2</td></tr>
</table>

Interpolation by daycount fractions vs. interpolation with short bond volatility

Conversely, under $\mathbb{P}_{T_{\min(i+1,j)}}$ we have

$$(12) \qquad d\mathbb{L}(t) = \Lambda(t, \mathbb{L}(t))dW_{T_{\min(i+1,j)}}(t) + \Psi'(\mathbb{L}(t))\ell(\mathbb{L}(t))dt$$

with

$$\Psi'_{hk}(t, \mathbb{L}(t)) = \left\{ \begin{array}{ll} L(t, T_{\min(i,j-1)-1+h})\lambda(t, T_{\min(i,j-1)-1+h})\lambda(t, T_{\min(i,j-1)-1+k}) & \text{if } 1 < k \leq h \\ 0 & \text{otherwise} \end{array} \right.$$

By the Markov property of Ito diffusions, the dynamics given by (11) resp. (12) are Markov in $\mathbb{L}(t)$.

$\square$

This result does not rely on the assumption that the LIBOR volatilities $\lambda(t, T_k)$ are deterministic functions of their arguments. Rather, if the $\lambda(t, T_k)$ are level dependent on some or all rates in $\mathbb{L}(t)$, the Markov property still holds. Also, by remark 1.3, the above result implies that under the rolling spot LIBOR measure, any rate $L(\cdot, T_i)$ is Markov in the state variable vector $\{L(t, T_k)|0 \leq k \leq i\}$.

5. A Look at the Interpolated Interest Rates

This section illustrates some of the properties of the interpolated interest rates and contrasts the two interpolation methods proposed in the previous sections. The method described in section 2 will be referred to as method 1 or *interpolation by daycount fractions*; the one introduced in section 3 will be labelled method 2 or *interpolation with short bond volatility*. For the inputs used to produce each of the plots, please refer to table 1 in the appendix. The algorithm from Brace, Musiela and Schlögl (1998) was used to generate sample paths for the evolution of the term structure.

5.1. **Term Structures.** Figures 1 and 2 show how the two methods interpolate instantaneous forward rates. For reference, the stepwise constant rates resulting from loglinear interpolation of zero coupon bond prices are included in each plot. The instantaneous forward rates jump when one moves from one δ–compounding period to the next. From equations (8) and (9), this is to be expected, as the maturities of the LIBORs underlying the interpolation are different on each interval. Instantaneous forward rates on $f(0, T)$ are determined on each interval $T \in (T_i, T_{i+1})$ by calculating

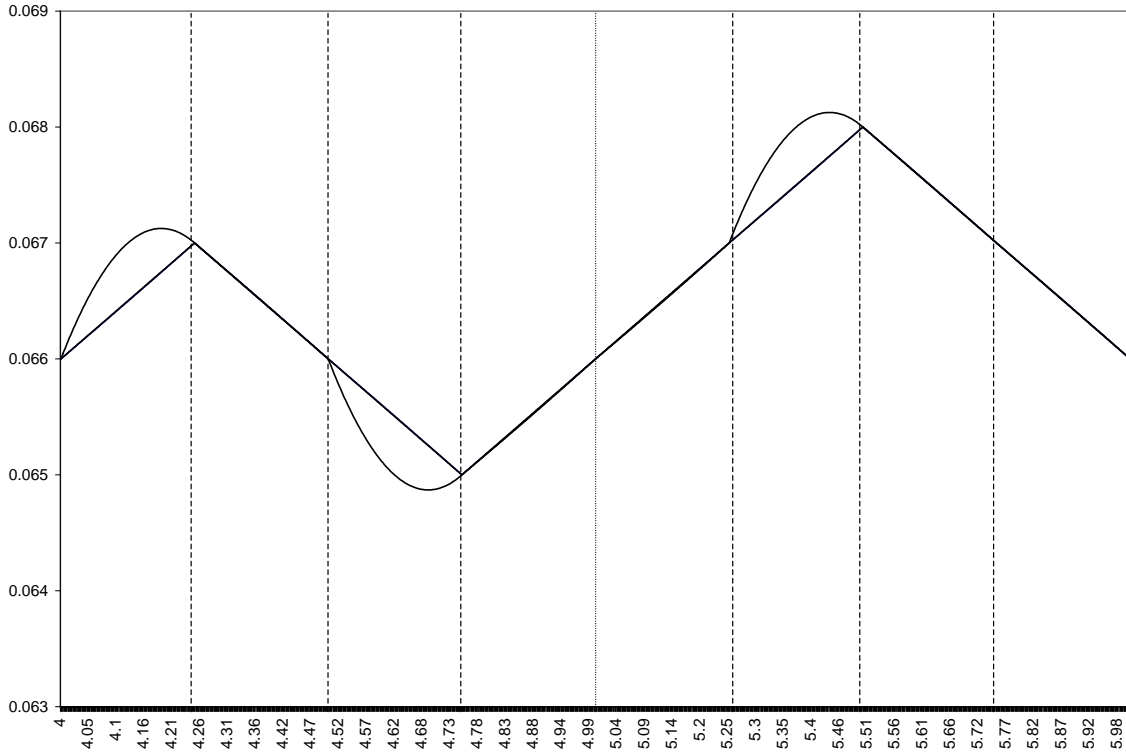$$-\partial_T \ln \frac{B(0, T)}{B(0, T_{\eta(T)})}$$

FIGURE 3. Interpolated forward LIBORs

using (9) for method 1 and (10) for method 2.

Instantaneous forward rates give the most disaggregate representation of the term structure and therefore effectively demonstrate the implications of the choice of interpolation method. However, one should not judge the usefulness of a particular method from this perspective alone. At first glance the "sawtooth" patterns of the interpolated rates for falling term structures in method 1 and rising term structures in method 2 seem to represent serious drawbacks to using either method, but one should keep in mind that the instantaneous forward rates are also purely mathematical artifacts, while the objects being modelled are market observable rates such as LIBOR. Forward LIBORs are interpolated by

$$
\begin{aligned}
L(t,T) &= \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)}-1\right) \\
&= \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T_{\eta(T)})}\frac{B(t,T_{\eta(T)})}{B(t,T_{\eta(T)+1})}\frac{B(t,T_{\eta(T)+1})}{B(t,T+\delta)}-1\right) \\
&= \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T_{\eta(T)})}(1+\delta L(t,T_{\eta(T)}))\frac{B(t,T_{\eta(T)+1})}{B(t,T+\delta)}-1\right)
\end{aligned}
$$

(13)

For method 1 this becomes

$$
\begin{aligned}
L(t,T) &= \frac{1}{\delta}\big((1+(T_{\eta(T)}-T)L(t,T_{\eta(T)-1}))(1+\delta L(t,T_{\eta(T)})) \\
&\qquad \cdot(1+(T_{\eta(T)+1}-(T+\delta))L(t,T_{\eta(T)}))^{-1}-1\big)
\end{aligned}
$$

Figure 3 shows interpolated forward three–month LIBORs for a continuum of start dates between four and six years forward. The piecewise linear graph is actually two plots so close
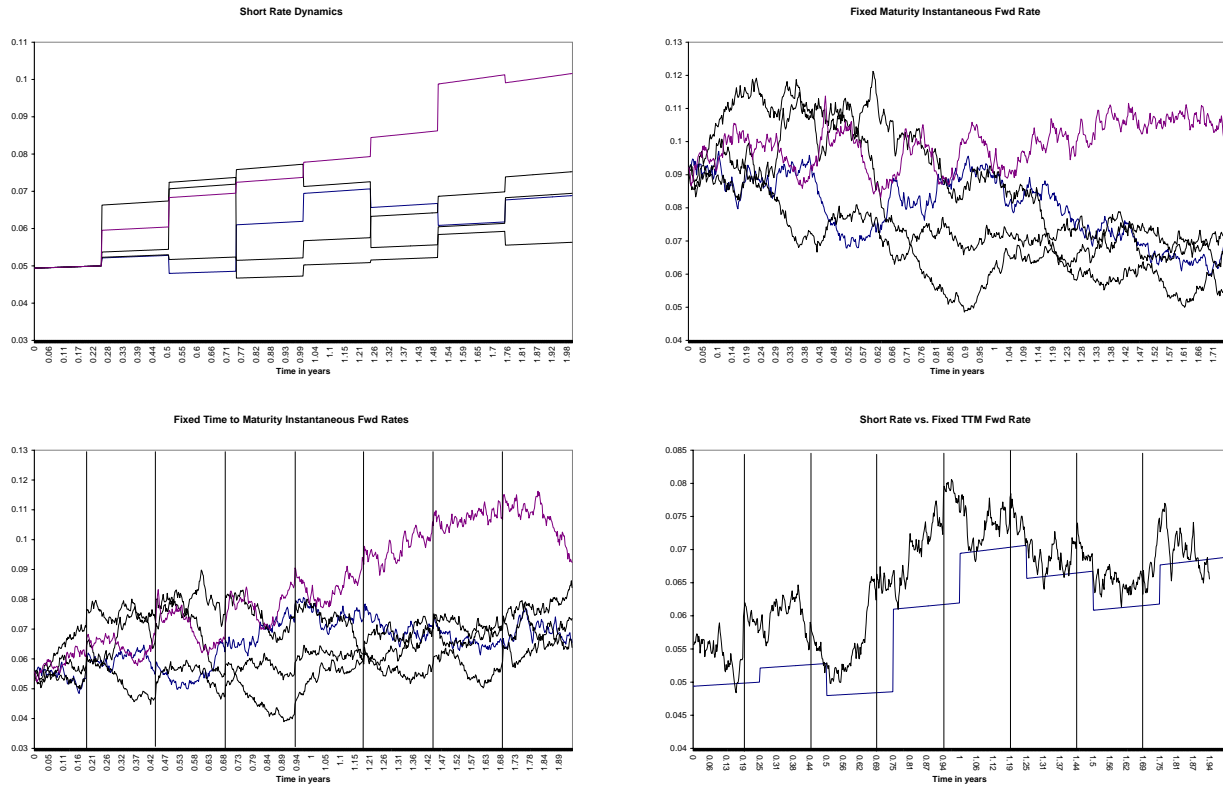
FIGURE 4. Rate dynamics under interpolation by daycount fractions

as to be indistinguishable: Interpolation by daycount fractions and loglinear interpolation of zero coupon bond prices results in nearly identical forward three–month LIBORs, even though the resulting instantaneous forward rates are very different. This also implies that if one is only concerned with rates such as forward three–month LIBORs, using loglinear interpolation of zero coupon bond prices is for all intents and purposes arbitrage free.

Introducing short bond volatility as per method 2 makes a difference for the interpolated LIBORs only when the interest rate term structure changes slope. The vertical gridlines in figure 3 represent boundaries between the accrual periods in the original discrete tenor. Interpolated forward LIBORs with start date in an accrual period immediately preceding a change in the slope of the term structure depart from the linearly interpolated plots and curve "outward". This is due to the fact that a forward LIBOR for a given start date T is interpolated using the forward LIBOR for the preceding and the two following discrete tenor start dates $T_{\eta(T)-1}$, $T_{\eta(T)}$ and $T_{\eta(T)+1}$, as can easily be seen by appropriately inserting (10) into (13).

5.2. **Rate Dynamics.** The dynamics which the two methods imply for interpolated rates are illustrated in Figures 4 and 5. Not only, as in the previous section, do equations (8) and (9) imply discontinuities in the instantaneous rates in the maturity dimension, but also in the time dimension. This becomes particularly clear in the dynamics of the continuously compounded short rate under interpolation by daycount fractions: This rate evolves deterministically within each accrual period of the discrete tenor structure and jumps at the accrual period boundaries. For instantaneous forward rates $f(t, T)$ the dynamics are stochastic as long as the time to maturity $T - t$ is greater than the accrual period length δ. The maturity T of the forward rate determines which forward LIBORs are used in the
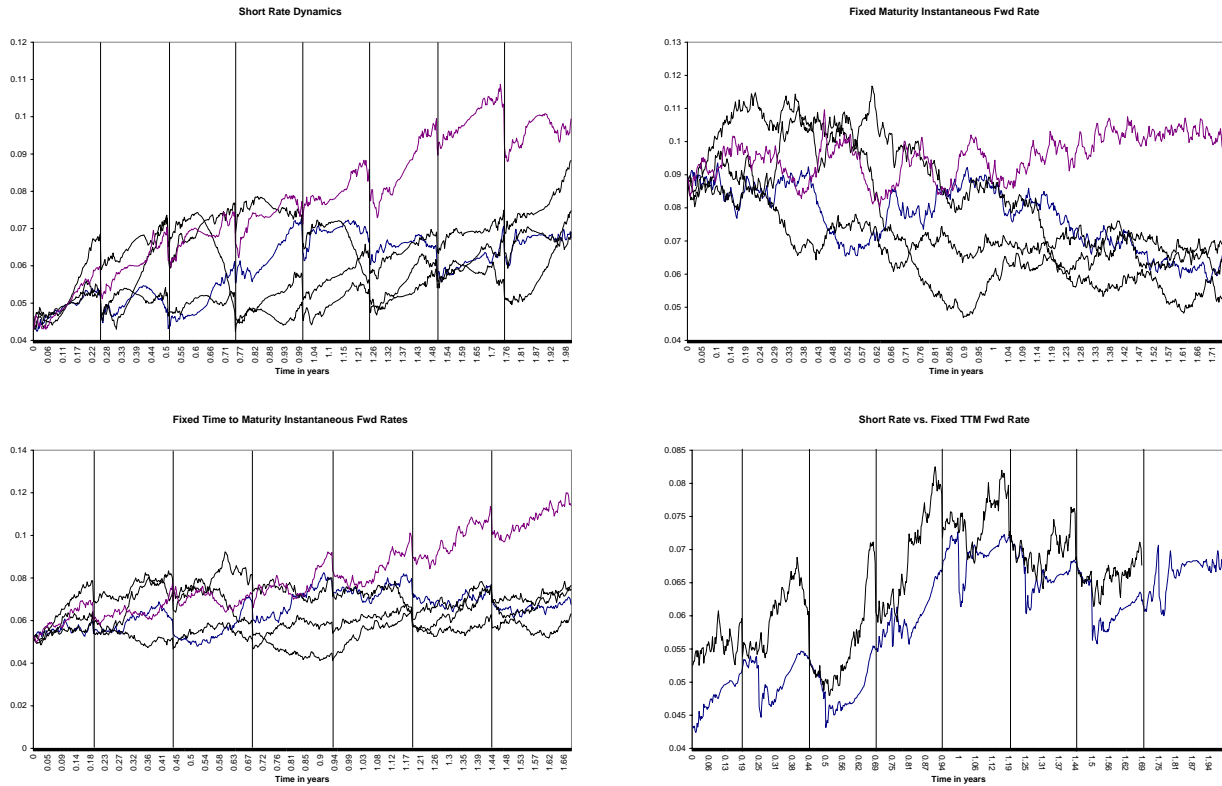
FIGURE 5. Rate dynamics under interpolation with short bond volatility

interpolation, so if one considers an instantaneous forward rate for a fixed maturity, this rate will be interpolated from the same forward LIBORs at all times. Since the LIBOR dynamics are modelled to be continuous, the interpolated rate will not jump (cf. plot in upper right of Figure 4). On the other hand, holding time to maturity (TTM) fixed means that there will be jumps when the maturity T moves across a boundary between accrual periods in the original discrete tenor structure. The bottom two plots in Figure 4) illustrate this. The time to maturity of the instantaneous forward rate shown here is 1.25δ, resulting in a jump at three–quarters of every δ–interval of the tenor structure; these points are marked by vertical gridlines in the plot. Thus different points on the forward rate curve jump at different times.

The same observations can be made for the dynamics of instantaneous rates under inter-polation with short bond volatility, as shown in Figure 5. The only qualitative difference is that now rates with times to maturity less than δ, including the continuously compounded short rate, also have stochastic dynamics (cf. the upper left plot of Figure 5).

As in the previous section, the observed discontinuities apply only to instantaneous rates. Of more practical relevance is the fact that interpolated rates for accrual periods of any arbitrary length, for example a δ–compounded forward LIBOR $L(t,T)$ for a "broken date" maturity $T \neq T_{\eta(T)}$, will have continuous dynamics: The instantaneous rates are continuous almost everywhere under the Lebesgue measure on the time line, thus any jumps are integrated out once one considers accrual periods of greater than infinitesimal length.
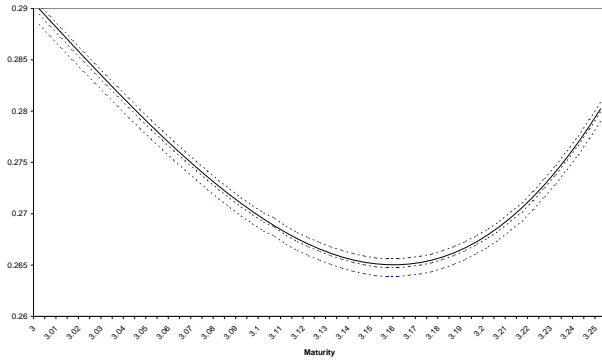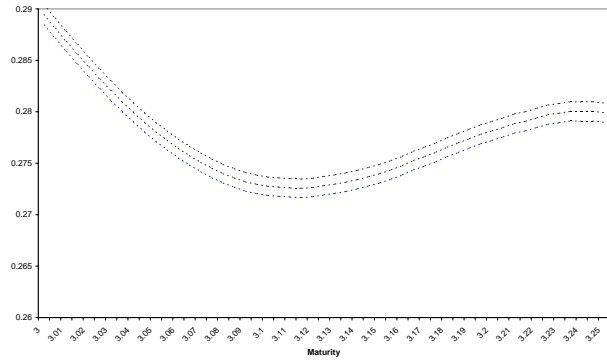
FIGURE 6 · · · · · · · · · · · · · · · · FIGURE 7

Implied volatilities resulting from
interpolation by daycount fractions vs. interpolation with short bond volatility

6. BROKEN DATE CAPLETS: A COMPARISON

Another point of interest is how the choice of interpolation method affects the pricing of derivatives. In this context, the volatility of interpolated rates is of key importance. When extending the discrete tenor model to continuous tenor, one is faced with a tradeoff of completing the model via volatilities or interest rate interpolation. By fixing the volatilities of rates for all maturities, say all δ–compounded forward LIBORs $L(\cdot, T)$, $T \in [0, T^* - \delta]$ as in MSS and BGM, the model is completely specified and there is no need to interpolate interest rates. On the other hand, when the interpolated rates are given as functions of the discrete tenor forward LIBORs, their volatilities can be calculated by an application of Ito's Lemma. The former approach has the disadvantage of rather intractable dynamics for rates other the δ–compounded forward LIBORs, while latter implies that the "broken date" forward LIBORs $L(t, T)$, $T \neq T_{\eta(T)}$ are not lognormal. Applying Ito's Lemma to (13), the relative volatility of $L(t, T)$ is

$$
\text{vol}[L(t,T)] = L(t,T)^{-1}
$$
$$
\cdot \sum_{i=0}^{N-1} \partial_{L(t,T_i)} \left(\frac{1}{\delta} \left(\frac{B(t,T)}{B(t,T_{\eta(T)})} (1 + \delta L(t, T_{\eta(T)})) \frac{B(t,T_{\eta(T)+1})}{B(t,T+\delta)} - 1 \right) \right) L(t,T_i) \lambda(t,T_i)
$$

For method 1 this becomes

$$
\text{vol}[L(t,T)] = \frac{1}{\delta} L(t,T)^{-1} \left((T_{\eta(T)} - T) \frac{1 + \delta L(t, T_{\eta(T)})}{1 + (T_{\eta(T)} - T)L(t, T_{\eta(T)})} \lambda(t, T_{\eta(T)-1}) L(t, T_{\eta(T)-1}) \right.
$$
$$
+ (1 + (T_{\eta(T)} - T)L(t, T_{\eta(T)-1}))
$$
$$
\left. \cdot \frac{\delta(1 + (T_{\eta(T)} - T)L(t, T_{\eta(T)})) - (T_{\eta(T)} - T)(1 + \delta L(t, T_{\eta(T)}))}{(1 + (T_{\eta(T)} - T)L(t, T_{\eta(T)}))^2} \lambda(t, T_{\eta(T)}) L(t, T_{\eta(T)}) \right)
$$

Figures 6 and 7 illustrate how the two interpolation methods presented in the previous sections affect the prices for "broken date" caplets. In each plot, we vary the maturity T of the forward LIBOR $L(t, T)$ underlying the caplet within one δ–accrual period of the discrete tenor structure (i.e. the caplets on the two endpoints of each plot are based on

non-interpolated rates). The strike is chosen at 1.25 times the at–the–money level and the volatility function is two–dimensional exponentially decaying.[15] This choice should provide a representative example well away from the trivial at–the–money, one–dimensional constant volatility case. The caplet prices were generated by Monte Carlo simulation; the middle line in each plot give the Monte Carlo estimate, while the outside lines are the confidence interval boundaries two standard deviations to either side of the Monte Carlo estimate. One million MC runs make these confidence bounds reasonably tight. We represent the caplet prices in terms of their Black/Scholes implied volatility, i.e. in terms of the one–dimensional constant relative volatility of the underlying rate which would result in the same caplet price. The dip in the implied volatilities is due to the fact that the interpolation of $L(T, T)$ at maturity of the caplet is in part based on $L(T_{\eta(T)-1}, T_{\eta(T)-1})$, i.e. a rate which evolves stochastically only until $T_{\eta(T)-1} < T$. This effect is less pronounced under interpolation with short bond volatility, as in this case $L(T_{\eta(T)-1}, T_{\eta(T)-1})$ plays a lesser role.

The disadvantage that the "broken date" forward LIBORs are not lognormal is greatly alleviated by the fact that their distribution is actually very close to lognormal. One can apply the argument which was first used to derive an approximate swaption formula in a lognormal forward LIBOR model.[16] Noting that the level dependence of the volatility of the interpolated forward LIBORs varies slowly compared to the rates themselves, one can derive an approximate closed–form solution for the "broken date" caplet by calculating the level dependence with respect to the initial rates, e.g. for method 1 by setting

$$
\begin{aligned}
\overline{\text{vol}}[L(t,T)] &= L(0,T)^{-1} \\
&\quad \cdot \left(\frac{\partial L(0,T)}{\partial L(0,T_{\eta(T)-1})} L(0, T_{\eta(T)-1}) \lambda(t, T_{\eta(T)-1}) + \frac{\partial L(0,T)}{\partial L(0,T_{\eta(T)})} L(0, T_{\eta(T)}) \lambda(t, T_{\eta(T)}) \right) \\
&= \frac{1}{\delta} L(0,T)^{-1} \left((T_{\eta(T)} - T) \frac{1 + \delta L(0, T_{\eta(T)})}{1 + (T_{\eta(T)} - T) L(0, T_{\eta(T)})} \lambda(t, T_{\eta(T)-1}) L(0, T_{\eta(T)-1}) \right. \\
&\quad + (1 + (T_{\eta(T)} - T) L(0, T_{\eta(T)-1})) \\
&\quad \cdot \frac{\delta(1 + (T_{\eta(T)} - T) L(0, T_{\eta(T)})) - (T_{\eta(T)} - T)(1 + \delta L(0, T_{\eta(T)}))}{(1 + (T_{\eta(T)} - T) L(0, T_{\eta(T)}))^2} \\
&\quad \left. \cdot \lambda(t, T_{\eta(T)}) L(0, T_{\eta(T)}) \right)
\end{aligned}
$$

The Black/Scholes implied volatility is then given by

$$
\frac{1}{\sqrt{T}} \sqrt{\int_0^T (\overline{\text{vol}}[L(s,T)])^2 ds}
$$

---

[15]For details see appendix.

[16]This argument first appears in Brace, Gatarek and Musiela (1997). It was developed further in Brace, Dun and Barton (1998) and formalized by Brace and Womersley (2000). This approximation works well not only for pricing, but also for hedging, as demonstrated in Dun, Schlögl and Barton (2001).

$$= \frac{1}{\sqrt{T}} \left(2 \frac{\partial L(0,T)}{\partial L(0,T_{\eta(T)-1})} L(0,T_{\eta(T)-1}) \frac{\partial L(0,T)}{\partial L(0,T_{\eta(T)})} L(0,T_{\eta(T)}) \mathrm{cov}(T_{\eta(T)-1},T_{\eta(T)}) \right.$$

$$+ \left(\frac{\partial L(0,T)}{\partial L(0,T_{\eta(T)-1})} L(0,T_{\eta(T)-1}) \overline{\lambda}(0,T_{\eta(T)-1}) \right)^2$$

$$\left. + \left(\frac{\partial L(0,T)}{\partial L(0,T_{\eta(T)})} L(0,T_{\eta(T)}) \overline{\lambda}(0,T_{\eta(T)}) \right)^2 \right)^{\frac{1}{2}}$$

with

$$\mathrm{cov}(T_{\eta(T)-1},T_{\eta(T)}) = \int_0^{T_{\eta(T)-1}} \lambda(s,T_{\eta(T)-1})\lambda(s,T_{\eta(T)})ds$$

$$\overline{\lambda}(0,T_{\eta(T)-1}) = \sqrt{\int_0^{T_{\eta(T)-1}} \lambda^2(s,T_{\eta(T)-1})ds}$$

$$\overline{\lambda}(0,T_{\eta(T)}) = \sqrt{\int_0^{T_{\eta(T)}} \lambda^2(s,T_{\eta(T)})ds}$$

As illustrated by Figure 6, the resulting approximation of Black/Scholes implied volatilities is very accurate.

7. Conclusion

Extending the models of market observable interest rates from discrete to continuous tenor by interpolation is particularly useful in implementations where there are some financial products which must be priced numerically, as the Markovian structure of the discrete tenor model is preserved. It is important to note that the interpolation method cannot be chosen arbitrarily for all maturities. Rather, it must take into account the relevant no–arbitrage conditions.

Shifting the focus from instantaneous forward rates to market observables such as forward LIBOR, the rate dynamics implied by the proposed interpolation methods are reasonable. These methods supply alternatives to the continuous tenor versions of the lognormal forward rate Market Models proposed in the literature and in those cases where moving to continuous tenor by interpolation entails a loss of tractability, for example for "broken date" caplets, very accurate approximations exist.

Appendix: Model Parameters

To generate the plots, two volatility specifications λ_1 and λ_2 were used:

$$\lambda_1(t,T) = 0.3 \qquad \forall \, t < T < T^*$$

$$\lambda_2(t,T) = \begin{pmatrix} 0.6e^{-0.8(T-t)} \\ 0.1e^{-0.01(T-t)} \end{pmatrix} \qquad \forall \, t < T < T^*$$

The length δ of the discrete tenor accrual period is in all cases 0.25.

Three initial term structures of discrete tenor forward LIBORs are considered:

(1) Linearly increasing from 4% at 0 years up to 6% at 5 years and then linearly decreasing down to 4% at 10 years.

TABLE 1

Figure	Initial Term Structure	Volatility	T^*	Fixed Maturity	Fixed Time to Maturity
1	1	λ_1	10	NA	NA
2	1	λ_1	10	NA	NA
3	2	λ_1	10	NA	NA
4	3	λ_1	2.25	1.8125	0.3125
5	3	λ_1	2.25	1.8125	0.3125
6	3	λ_2	4.25	NA	NA
7	3	λ_2	4.25	NA	NA

(2) Linearly increasing from 5% at 0 years up to 6.7% at 4.25 years, then linearly decreasing to 6.5% at 4.75 years, then linearly increasing to 6.8% at 5.5 years, and then linearly decreasing down to 5% at 10 years.

(3) Linearly increasing from 5% at 0 years up to 10% at $T^* - \delta$.

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