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Heterogeneity, Bounded Rationality and Market Dynsfunctionality

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HETEROGENEITY, BOUNDED RATIONALITY, AND MARKET DYSFUNCTIONALITY

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ABSTRACT. As the main building blocks of the modern finance theory, homogeneity and rational expectation have faced difficulty in explaining many market anomalies, stylized factors, and market inefficiency in empirical studies. As a result, heterogeneity and bounded rationality have been used as an alterative paradigm of asset price dynamics and this paradigm has been widely recognized recently in both academic and financial market practitioners. Within the framework of Chiarella, Dieci and He (2006*a*, 2006*b*) on mean-variance analysis under heterogeneous beliefs in terms of either the payoffs or returns of the risky assets, this paper examines the effect of the heterogeneity. We first demonstrate that, in market equilibrium, the standard one fund theorem under homogeneous belief does not held under heterogeneous belief in general, however, the optimal portfolios of investors are very close to the market efficient frontier. By imposing certain distribution assumption on the heterogeneous beliefs, we then use Monte Carlo simulations to show that certain heterogeneity among investors can improve the Sharpe and Treynor ratios of the portfolios and investors can benefit from the diversity in investors' beliefs. We also show that non-normality of market equilibrium return distributions is an outcome of the market aggregation of individual investors who make rational decisions based on their beliefs. Our results explain the empirical funding that that managed funds under-perform the market index on average and show that heterogeneity can improve the market efficiency.

Key Words: Heterogeneity, bounded rationality, heterogeneous CAPM, mean-variance efficiency, Sharpe and Treynor ratios.

JEL Classification: G12, D84.

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1. INTRODUCTION

The Capital Asset Pricing Model (CAPM) developed simultaneously and independently by Sharpe (1964), Lintner (1965) and Mossin (1966) is perhaps the most influential object in modern finance. It provides the theoretical foundation for relating risks linearly with expected return of a asset. It is found on the main building blocks of the modern finance theory, i.e. homogeneity and rational expectation. However, from a theoretical perspective this paradigm has been criticized on a number of grounds, in particular concerning its extreme assumptions about homogeneous beliefs, and information about the economic environment and computational ability on the part of the rational representative economic agent. Also, this paradigm has faced difficulty in explaining many market anomalies, stylized factors, and market inefficiency in empirical studies. As the result, heterogeneity and bounded rationality have been used as an alterative paradigm of asset price dynamics and this paradigm has been widely recognized in both academic and financial market practitioners.

Literatures have made a significant contribution to the understanding of the impact of heterogeneous beliefs amongst investors on market equilibrium. Some have considered the problem in discrete time (for example, see Lintnter (1969), Rubinstein (1974) and Sharpe (2007)) and others in continuous time (for example, see Williams (1977), Detemple and Murthy (1994) and Zapatero (1998)). Equilibrium models have been developed to consider the impact of heterogeneity either in the mean-variance framework (see, Lintner (1969) and William (1977)) or in the Arrow-Debreu contingent claims economy (see, Rubinstein (1976), Abel (1989, 2002)). Given the bounded rationality of investors, heterogeneity may be caused by difference in information or difference in opinion. In the first case, investors may update their beliefs as new information become available, Bayesian updating rule is often used (see, for example, Williams (1977) and Zapatero (1998)). In the second case, investor may revise their portfolio strategies as their views of the market change over time (see, for example Lintner (1969) and Rubinstein (1975)). However, in most of the literature, the impact of heterogeneous beliefs is studied for the case of a portfolio of one risky asset and one risk-free asset (e.g. Abel (1989), Basak (2000), Zapatero (1998) and Johnson (2004)). In those papers that consider a portfolio of many risky assets and one risk-free asset, agents are assumed to be heterogeneous in the risk preferences and expected payoffs or returns of the risky assets (e.g. Williams (1977) and Varian (1985)), but not in the variances and covariances, except the early contribution of Lintner (1969) in which heterogeneity in both means and variances/covariances is investigated in a mean-variance portfolio context. However, there is no reason to assume that investors are homogeneous in variance and covariance matrix of the risky assets. As suggested by the empirical study in Chan *et al.* (1999), while future variances and covariances are more easily predictable than expected future returns, the difficulties in doing so should not be understated. These authors argue that *"While optimization (based on historical estimates of variances and covariances) leads to a reduction in volatility, the problem of forecasting covariance poses a challenge"*.

The main obstacle in dealing with heterogeneity is the complexity and heavy notation involved when the number of assets and the dimension of the heterogeneity increase. With the mean-variance framework with one risk-free asset and many risky assets, amongst all the literatures mentioned above, Lintner (1969) is the only one considering market equilibrium and asset prices by allowing for heterogeneity not only in the risky preferences and means of the risky assets but also in the variances/covariances of the risky assets across agents. Surprisingly, this significant contribution from Lintner has not been paid much attention until recent years. This might be due to the obstacle mentioned early, which makes the paper not easy to follow and to analyze. Recently Chiarella, Dieci and He (2006*a*, 2006*b*) introduce the concept of *consensus belief* and show that the consensus belief can be constructed explicitly as a weighted average of the heterogeneous beliefs. They prove that the analysis of the heterogeneous beliefs model is equivalent to the analysis of a classical homogeneous model with the consensus belief. In particular, they show that the market aggregate expected payoffs/returns of the risky assets can be measured by a weighted average of the heterogeneous expected payoffs/returns of the risky assets across the agents, while the market equilibrium price is a weighted average of the equilibrium prices under the separate beliefs of each agent. Consequently, they establish an equilibrium relation between the market aggregate expected payoff/returns of the risky assets and the market portfolio's expected payoff/returns, leading to a CAPM-like relation under heterogeneous beliefs. As a special case, their result provides a simple explanation for the observed empirical relation between cross-sectional volatility and expected returns, which is studied in Miller (1977), Bart and Masse (1981), Diether *et al.* (2002), Johnson (2004) and Ang *et al.* (2006).

Market exists because different investors trade each other for different purposes. At the same time, investors make their optimal decisions based on their information and beliefs. Among many questions on heterogeneity and bounded rationality, the question why heterogeneity matters is one of the most important ones. In particular, what is the impact of different heterogeneity on the market equilibrium and the optimal portfolios of heterogeneous agents and what do we benefit from market with many heterogeneous investors? It is the explicit construction of the consensus belief provided in Chiarella, Dieci and He (2006*a*, 2006*b*) that makes the examination of those issues possible. Within the framework of Chiarella, Dieci and He (2006*a*, 2006*b*) on mean-variance analysis under heterogeneous beliefs in both payoff and return setups, this paper examines the above issues and questions, in particular the effect of the heterogeneity on the market. The heterogeneity is measured in terms of the risk preferences (the absolute risk aversion coefficients), the expected payoffs/returns and the variance/covariance matrices of the payoffs/returns of risky assets. We first demonstrate that, under market aggregation, the standard one fund theorem under homogeneous belief in the mean-variance framework does not held in general. This offers a possible explanation on the empirical funding that managed funds under-perform the market index on average. However, in market equilibrium, the optimal portfolios of investors can be very close to the market frontier, in particular when the beliefs are formed in asset returns. We call this property the *quasi one fund theorem* under the heterogeneity and bounded rationality. To examine the market impact of the heterogeneity, we introduce certain distribution assumptions on the different aspects of the heterogeneous beliefs. We then use Monte Carlo simulations to show that certain heterogeneous beliefs, in particular in asset returns, among investors can improve the Sharpe and Treynor ratios of the optimal portfolios of investors and the market portfolio, implying that both investors and market can benefit from the diversity in investors' beliefs. We call this property the *heterogeneity diversification effect*. We also show that non-normality of market equilibrium return distributions is an outcome of the market aggregation of individual investors who make rational decisions based on their beliefs.

The paper is structured as follows. In Section 2, we review the main results developed in Chiarella et al. (2006*a*, 2006*b*) when the heterogeneity is formed in payoffs and returns of the risky assets. In Section 3, through numerical examples, we examine the impact of the heterogeneity on the one fund theorem and the mean-variance efficiency of the optimal portfolios of heterogeneity under both setups. Section 4 presents a statistic analysis on the market impact of the heterogeneity among two groups of investors whose beliefs are characterized by certain distribution. Section 5 concludes the paper.

2. Mean-Variance Asset Pricing with Heterogeneous Beliefs

In this section, we briefly review the main results in Chiarella, Dieci and He (2006*a*, 2006*b*) on the asset pricing theory under the mean-variance framework with heterogeneous beliefs. The heterogeneous beliefs are formed in either the payoffs or returns of the risky assets. In this paper, we consider the market impact of heterogeneity in both setups in the beliefs. These results provide a foundation for our analysis in the following sections.

2.1. **Heterogeneous Beliefs in Asset Payoffs.**

We first review the main result in Chiarella et al. (2006*a*) in which the heterogeneous beliefs are formed in terms of asset payoffs. The set up follows from the static mean-variance analysis. Consider a market with one risk-free asset and $K(\geq 1)$ risky assets. Let the current price of the risk-free asset be 1 and its payoff be $R_f = 1+r_f$. Let $\tilde{\mathbf{x}} = (\tilde{x}_1, \cdots, \tilde{x}_K)^T$ be the payoff vector of the risky assets, where $\tilde{x}_k = \tilde{p}_k + \tilde{d}_k (k = 1, \cdots, K)$ correspond to the cum-prices. Assume that there are I investors in the market indexed by $i = 1, 2, \cdots, I$. The heterogeneous (subjective) belief $\mathcal{B}_i = (\mathbb{E}_i(\tilde{\mathbf{x}}), \Omega_i)$ of investor i is defined with respect to the means, variances and covariances of the payoffs of the risky assets

$$\mathbf{y}_i = \mathbb{E}_i(\tilde{\mathbf{x}}) = (y_{i,1}, y_{i,2}, \cdots, y_{i,K})^T, \qquad \Omega_i = (s_{i,kl})_{K\times K},$$

where

$$y_{i,k} = \mathbb{E}_i[\tilde{x}_k], \quad s_{i,kl} = Cov_i(\tilde{x}_k, \tilde{x}_l) \tag{2.1}$$

for $i = 1, 2, \cdots I$ and $k, l = 1, 2, \cdots, K$.

Let $z_{i,o}$ and $\bar{z}_{i,o}$ be the absolute amount and the endowment of investor i in the risk-free asset, respectively, and

$$\mathbf{z}_i = (z_{i,1}, z_{i,2}, \cdots, z_{i,K})^T \qquad \text{and} \qquad \bar{\mathbf{z}}_i = (\bar{z}_{i,1}, \bar{z}_{i,2}, \cdots, \bar{z}_{i,K})^T$$

be the risky portfolio and the endowment, respectively, of investor i in absolute amount of the risky assets. Then the end-of-period wealth of the portfolio for investor i is

$$\tilde{W}_i = R_f z_{io} + \tilde{\mathbf{x}}^T \mathbf{z}_i.$$

Then, under the belief $\mathcal{B}_i$, the expected value and variance of portfolio wealth $\tilde{W}_i$ are given, respectively, by

$$\mathbb{E}_i(\tilde{W}_i) = R_f z_{i,o} + \mathbf{y}_i^T \mathbf{z}_i, \qquad \sigma_i^2(\tilde{W}_i) = \mathbf{z}_i^T \Omega_i \mathbf{z}_i. \tag{2.2}$$

Essentially, Chiarella, Dieci and He (2006a) assume that investor-i's optimal investment portfolio is obtained by maximizing the certainty-equivalent of his/her future wealth, $C_i(\tilde{W}_i) = \mathbb{E}_i(\tilde{W}_i) - \frac{\theta_i}{2} Var_i(\tilde{W}_i)$, where θ_i is the absolute risk aversion coefficient of investor i.

A market **equilibrium** is a vector of asset prices $\mathbf{p}_o$ determined by the individual optimal demands z_i^* together with the market aggregation condition

$$\sum_{i=1}^{I} \mathbf{z}_i^* = \sum_{i=1}^{I} \bar{\mathbf{z}}_i = \mathbf{z}_m, \tag{2.3}$$

which defines a **market portfolio** of risky assets [1]. To characterize the market equilibrium, a **consensus belief** is introduced.

Definition 2.1. *A belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$, defined by the expected payoff of the risky assets $\mathbb{E}_a(\tilde{\mathbf{x}})$ and the variance and covariance matrix of the risky asset payoffs Ω_a, is called a* **consensus belief** *if the market equilibrium price under the heterogeneous beliefs is also the market equilibrium price under the homogeneous belief $\mathcal{B}_a$.*

Chiarella, Dieci and He (2006*a*) show how such a consensus belief can be explicitly constructed and how the market equilibrium price can be characterized by the consensus belief in the following Proposition 2.2. Consequently, they obtain a *CAPM-like relation under heterogeneous beliefs*, which is called the *Heterogeneous CAPM* (**HCAPM**).

Proposition 2.2. *Let $\Theta = [\frac{1}{I} \sum_{i=1}^{I} (1/\theta_i)]^{-1}$. Then*

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[1] If the risk-free asset is not in net zero supply at equilibrium, then the market portfolio also contains the risk-free asset, hence the market portfolio is characterized by $\mathbf{z}_m$ and $z_{i,0}$.

(i) *the consensus belief* $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$ *is given by*

$$\Omega_a = \Theta^{-1}\Big(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}\Big)^{-1}, \tag{2.4}$$

$$\mathbf{y}_a = \mathbb{E}_a(\tilde{\mathbf{x}}) = \Theta\Omega_a\Big(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}\mathbb{E}_i(\tilde{\mathbf{x}})\Big); \tag{2.5}$$

(ii) *the market equilibrium price* $\mathbf{p}_o$ *is determined by*

$$\mathbf{p}_o = \frac{1}{R_f}\Big[\mathbb{E}_a(\tilde{\mathbf{x}}) - \frac{1}{I}\Theta\Omega_a\mathbf{z}_m\Big]; \tag{2.6}$$

(iii) *the equilibrium optimal portfolio of agent* i *is given by*

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}\Big[(\mathbf{y}_i - \mathbf{y}_a) + \frac{1}{I}\Theta\Omega_a\mathbf{z}_m\Big]; \tag{2.7}$$

(iv) *in equilibrium,*

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - r_f\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - r_f], \tag{2.8}$$

where

$$\tilde{r}_j = \frac{\tilde{x}_j}{p_{j,o}} - 1, \qquad \tilde{r}_m = \frac{\tilde{W}_m}{W_{m,o}} - 1,$$

$$\mathbb{E}_a(\tilde{r}_j) = \frac{\mathbb{E}_a(\tilde{x}_j)}{p_{j,o}} - 1, \qquad \mathbb{E}_a(\tilde{r}_m) = \frac{\mathbb{E}_a(\tilde{W}_m)}{W_{m,o}} - 1,$$

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_K)^T, \qquad \beta_k = \frac{cov_a(\tilde{r}_m, \tilde{r}_k)}{\sigma_a^2(\tilde{r}_m)}, \qquad k = 1, \cdots, K,$$

and the mean and variance/covariance of returns under the consensus belief $\mathcal{B}_a$ *are defined similarly.*

The equilibrium relation (2.8) is the standard CAPM except that the mean and variance/covariance are calculated based on the consensus belief $\mathcal{B}_a$. The β coefficients of risky assets depend upon not only the covariance between the market returns and asset returns, but also the aggregation of the heterogeneous beliefs.

2.2. **Heterogeneous Beliefs in Asset Returns.** We now review the main result in Chiarella et al. (2006*b*) in which the heterogeneous beliefs are formed in terms of asset returns. The set up follows the static mean-variance analysis in Huang-Litzenberger (1988). Consider a market with one risk-free asset and $K(\geq 1)$ risky assets. Let r_f be the return of the riskless asset and $\tilde{r}_j$ $(j = 1, 2, ..., K)$ be the return of the risky asset j. Assume asset returns of the risky assets are multivariate normally distributed. Assume that there are I investors in the market indexed by $i = 1, 2, \cdots, I$ with heterogeneous (subjective) belief $\mathcal{B}_i = (\mathbb{E}_i(\tilde{\mathbf{r}}), V_i)$ that is defined with respect to the means, variances and covariances of the returns of the risky assets

$$\mu_i = \mathbb{E}_i(\tilde{\mathbf{r}}) = (\mu_{i,1}, \mu_{i,2}, \cdots, \mu_{i,K})^T, \qquad V_i = (\sigma_{i,kl})_{K\times K},$$

where

$$\mu_{i,k} = \mathbb{E}_i[\tilde{r}_k], \quad \sigma_{i,kl} = Cov_i(\tilde{r}_k, \tilde{r}_l) \tag{2.9}$$

for $i = 1, 2, \cdots I$ and $k, l = 1, 2, \cdots, K$.

Assume investor i has concave and strictly increasing utility of wealth function $u_i(w)$ satisfying that $\theta_i := -E_i[u_i''(\tilde{W}_i)]/E_i[u_i'(\tilde{W}_i)]$ is a constant defining investor i's *global absolute risk aversion*, where

$$\tilde{W}_i = W_{i,o}\left(1 + r_f + \sum_{j=1}^{K} w_{ij}(\tilde{r}_j - r_f)\right)$$

is the portfolio wealth of agent i, $W_{i,o}$ is the initial wealth of agent i, and w_{ij} is the fraction of wealth that agent i invests in the risky asset j.

The consensus belief can be defined similarly. The following Proposition 2.3 obtained in Chiarella, Dieci and He (2006*b*) shows that the market equilibrium returns of the risky assets can be characterized similarly by a CAPM-like relation under a consensus belief

Proposition 2.3. *Let* $\Theta = [\frac{1}{I}\sum_{i=1}^{I}(1/\theta_i)]^{-1}$ *and* $\tilde{r}_m := r_f + \mathbf{w}_m^T(\tilde{\mathbf{r}} - r_f\mathbf{1})$ *be the return of the market portfolio* $\mathbf{w}_m$ *of the risky assets. Define a consensus belief* $\mathcal{B}_a$ *as follows:*

$$V_a = \Theta^{-1}\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}V_i^{-1}\right)^{-1}, \tag{2.10}$$

$$\mu_a = \mathbb{E}_a(\tilde{\mathbf{r}}) = \Theta V_a\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}V_i^{-1}\mathbb{E}_i(\tilde{\mathbf{r}})\right). \tag{2.11}$$

Then, in equilibrium, the asset return satisfies

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - r_f\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - r_f] \tag{2.12}$$

and the market risk premium is given by

$$E_a(\widetilde{r}_m) - r_f = \frac{\Theta}{I}W_{m0}\sigma_{a,m}^2, \tag{2.13}$$

where

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_K)^T, \qquad \beta_k = \sigma_{a,jm}/\sigma_{a,m}^2$$

and $\sigma_{a,m}^2 = \mathbf{w}_m^T\mathbf{V}_a\mathbf{w}_m$ *and* $\mathbf{V}_a\mathbf{w}_m = [\sigma_{a,jm}]$.

Through a different setup, the same equilibrium relation (2.8) and (2.12) are obtained. In both setups of the heterogeneous beliefs, the construction of the consensus beliefs are very similar, but the derivation of the HCAPM are different. In the payoff setup, the HCAPM is derived through the market equilibrium price, while in return setup the equilibrium price plays no role in deriving the HCAPM. From the following discussion, we see that the different setup in heterogeneity has different impact on the market equilibrium, the efficiency of the optimal portfolios, and the market performance.

3. ONE FUND THEOREM AND MEAN-VARIANCE EFFICIENCY

In the standard mean-variance framework with homogeneous beliefs, it is well known that, in the presence of a risk-less asset, one fund theorem holds. This implies that the mean-variance efficient frontier is the half line connecting the risk-free asset and the market portfolio and the optimal portfolio of investor, which is a linear combination of the risky market portfolio and the risk-less asset, is always located on the frontier. When beliefs are heterogeneous, we show that this one fund theorem does not hold anymore. This is demonstrated by considering some numerical examples of market with one risk free asset, three risky assets and two investors who have different degrees of heterogeneity for both setups.

3.1. **The Case of Payoff Setup.** We first examine the case where the heterogeneity is formed in terms of the asset payoffs. The analysis is based on Proposition 2.2. The impact of single source of heterogeneity is examined first, followed by the impact of multiple sources of heterogeneity.

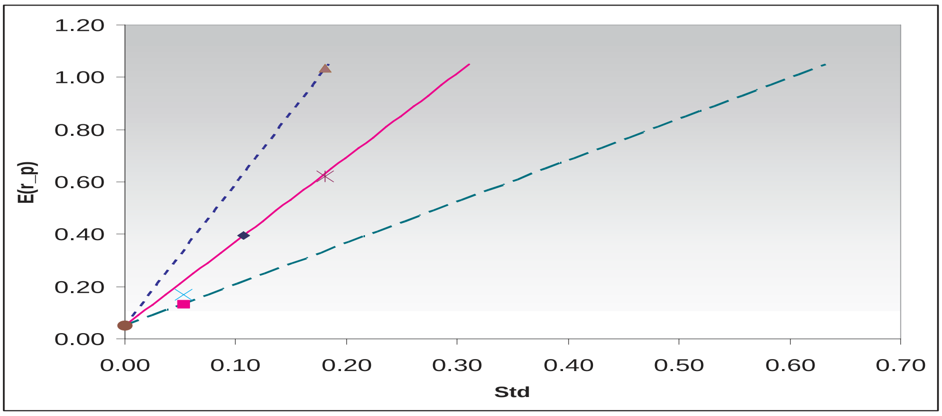
3.1.1. *Effect of heterogeneous expected payoffs.* To see the impact of the heterogeneous expected payoff, we consider the following example in which two investors have different beliefs but homogeneous beliefs in covariance matrix of the payoffs of three risky assets.

Example 3.1. *Assume that the two investors have the same covariance matrix* $\Omega_2 = \Omega_1 = \Omega_o$ *and different expected payoffs* $\mathbf{y}_1 = \mathbf{y}_o, \mathbf{y}_2 = \mathbf{y}_o + 3 \times \mathbf{1}$ *and absolute risk aversion (ARA) coefficients* $(\theta_1, \theta_2) = (3, 3), (4, 2)$ *and* $(2, 4)$ *with individual initial wealth*[2] *of* $W_{0,1} = W_{0,2} = \$10$*, market endowment of risky assets* $\mathbf{z}_m = (1, 1, 1)^T$*, risk-free rate* $r_f = 5\%$ *and*

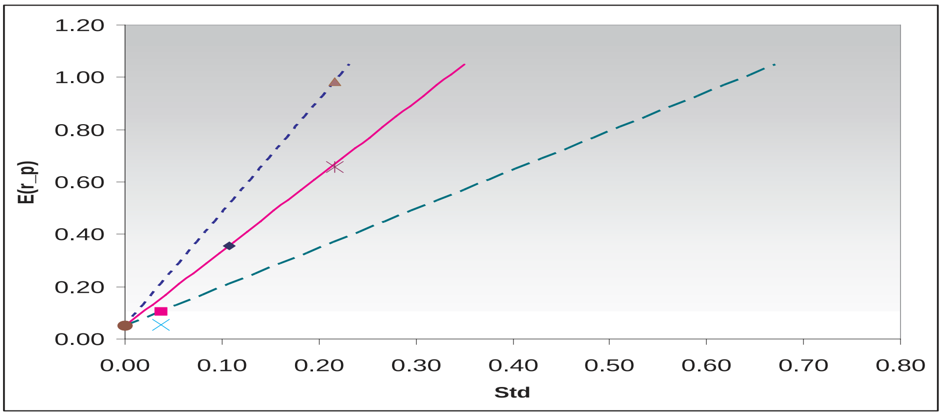
$$\mathbf{y}_o = \begin{pmatrix} 6.5974 \\ 9.3484 \\ 9.7801 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \Omega_o = \begin{pmatrix} 0.6292 & 0.1553 & 0.2262 \\ 0.1553 & 0.7692 & 0.1492 \\ 0.2262 & 0.1492 & 2.1381 \end{pmatrix}.$$

For each given ARAs (θ_1, θ_2) in Example 3.1, we can use Proposition 2.2 to construct the market consensus belief, calculate the market equilibrium price, and plot the mean-variance efficient frontiers under two heterogeneous beliefs and the consensus belief. We can also locate the optimal portfolios of the two investors under their beliefs and the consensus beliefs. For convenience, let $\mathbf{z}_i^*$ $(i = 1, 2)$ be the optimal portfolios of investor i under his/her subject belief, $\mathbf{z}_m$ be the market portfolio, $(\sigma_{\mathbf{z}_i^*}, \mu_{\mathbf{z}_i^*})$ and $(\sigma^a_{\mathbf{z}_i^*}, \mu^a_{\mathbf{z}_i^*})$ be the standard deviation and the expected return of $\mathbf{z}_i^*$ under subjective and consensus beliefs, respectively, and $(\sigma^a_{\mathbf{z}_m}, \mu^a_{\mathbf{z}_m})$ the standard deviation and the expected return of the market portfolio $\mathbf{z}_m$ under the consensus belief. Because of $\mathbf{y}_1 < \mathbf{y_2}$, investor 2 is more optimistic than investor 1. Fig. 3.1 provides the efficient frontiers

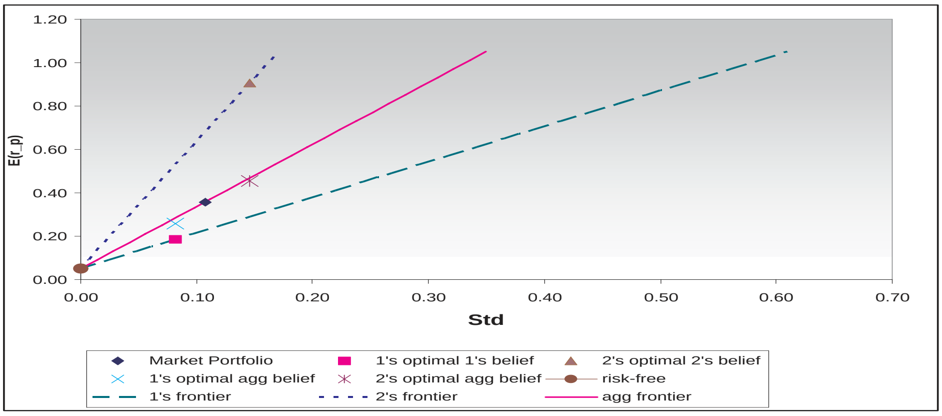
[2]The initial wealth can be easily converted to initial endowment of risky assets and risk-free asset once the equilibrium prices are recovered



(a1) $(\theta_1, \theta_2) = (3, 3)$



(a2) $(\theta_1, \theta_2) = (4, 2)$



(a3) $(\theta_1, \theta_2) = (2, 4)$

FIGURE 3.1. Mean-Variance efficient frontiers, optimal portfolios and market portfolio under heterogeneous and consensus beliefs when $\mathbf{y}_1 < \mathbf{y}_2$ and $\Omega_1 = \Omega_2$.

and the optimal portfolios of the two investors under their beliefs and consensus belief, and their relative position to the market portfolio for three combinations of ARA coefficients $(\theta_1, \theta_2) = (3, 3), (4, 2)$ and $(2, 4)$, respectively. Based on these figures, we observe the followings.

First, the optimal portfolios of the two investors are located on the frontiers under their beliefs, respectively, but not on the market efficient frontier under the consensus belie. This implies that the standard one fund theorem under homogeneous belief does not hold under heterogeneous belief. In fact, in this case,

$$\frac{1}{\Theta_a} = \frac{1}{2}[\frac{1}{\theta_1} + \frac{1}{\theta_2}], \qquad \mathbf{y}_a = \frac{\Theta_a}{2}[\frac{1}{\theta_1}\mathbf{y}_1 + \frac{1}{\theta_2}\mathbf{y}_2], \qquad \Omega_a = \Omega_o \tag{3.1}$$

and

$$\mathbf{z}_i^* = \frac{1}{\theta_i}\Omega_o(\mathbf{y}_i - \mathbf{y}_a) + \frac{1}{2}\frac{\Theta_a}{\theta_i}\mathbf{z}_m, \qquad i = 1, 2. \tag{3.2}$$

Hence, it follows from (3.2) that the optimal risky portfolios $\mathbf{z}_1^*, \mathbf{z}_2^*$ of the two investors are not proportional to the market portfolio $\mathbf{z}_m$, unless the investor belief coincides with the market belief (i.e. $\mathbf{y}_i = \mathbf{y}_a$). Therefore, the optimal portfolios of the two investors are no longer located on the frontier under the consensus belief. This is clearly illustrated in Fig 3.1 for all three combinations of the ARAs. Also, we observed that, in terms of Sharpe ratios,

$$\frac{\mu^a_{\mathbf{z}_1^*} - r_f}{\sigma^a_{\mathbf{z}_1^*}} < \frac{\mu^a_{\mathbf{z}_2^*} - r_f}{\sigma^a_{\mathbf{z}_2^*}} < \frac{\mu^a_{\mathbf{z}_m} - r_f}{\sigma^a_{\mathbf{z}_m}}$$

for all three cases. The relative higher Sharpe ratio for the optimal portfolio of investors 2 relative to investor 1 is due to his/her optimistic view on the expected payoffs, but the ratios for both investors is below the Sharpe ratio of the market portfolio. It is in this sense that we say that the optimal portfolios of two investors are inefficient. By comparing the expected returns of the optimal portfolios under subjective and consensus belief, we observe that

$$\mu^a_{\mathbf{z}_2^*} < \mu_{\mathbf{z}_2^*}, \quad \mu^a_{\mathbf{z}_1^*} > \mu_{\mathbf{z}_1^*}, \qquad \sigma^a_{\mathbf{z}_2^*} = \sigma_{\mathbf{z}_2^*}, \quad \sigma^a_{\mathbf{z}_1^*} = \sigma_{\mathbf{z}_1^*},$$

which implies that in market equilibrium the individual's optimal portfolio becomes more mean-variance inefficient (efficient) for investor who is optimistic (pessimistic). This suggests that optimistic investor with respect to the expected payoffs is worsen off under the market equilibrium.

Secondly, the market frontier under the consensus belief is located in between the frontiers under the heterogeneous beliefs, with the optimistic investor's frontier having the highest slope. This intuitive result follows from equation (3.1). When investor 1's ARA is lower, it leans more towards investor 1's frontier, see Fig. 3.1 (a2). Similarly, when investor 2's ARA is lower, it leans more towards 2's frontier, see Fig. 3.1 (a2) (although not really significant).

Thirdly, when the market is in equilibrium, the optimal portfolios of the two investors are very close to the market frontier, in particular for the investor who dominates the market. The dominance is jointly determined by optimism and risk tolerance of investor, as indicated by equations (3.1) and (3.2). Since $\mathbf{y}_2 > \mathbf{y}_1$, the belief bias $\mathbf{y}_i - \mathbf{y}_a$ is smaller for investors 2 and therefore the optimal portfolio is closer to the market frontier for investor 2 than for investor 1.

In terms of the relative position of the individual frontiers to the market frontier, the optimal portfolios to the market portfolio, one can see from Fig. 3.1 that the expected asset payoffs determine the main structure of the diagram while ARAs play a secondary role in the placement of individual optimal portfolios and change of slope of the market's frontier. On the one hand, the optimal portfolios are close to the market portfolio when the optimistic (pessimistic) investor is more (less) risk averse. On the other hand, the optimal portfolios are further away from the market portfolio when the optimistic (pessimistic) investor is less (more) risk averse.

3.1.2. *Effect of heterogeneous variance/covariance matrices.* To see the impact of the heterogeneous covariance matrices, we consider the following example in which two investors have heterogeneous beliefs in the covariance matrices but homogeneous belief in the expected payoffs of the three risky assets.

Example 3.2. *Let $\mathbf{y}_1 = \mathbf{y}_2 = \mathbf{y}_o$ and $\Omega_1 = \Omega_o$ and $\mathbf{y}_o$ and Ω_o are given in Example 3.1. Let $\Omega_2 = \Omega_1 - 0.3 \times \mathbf{1}$, where $\mathbf{1}$ is a 3×3 matrix with all elements are equal to 1. Then $\Omega_1 - \Omega_2 = 0.3 \times \mathbf{1}$ is semi-positive definite. For convenience, we denote $\Omega_1 \geq \Omega_2$ if $\Omega_1 - \Omega_2$ is semi-positive definite.*

As in the previous case, first the standard one fund theorem does not held when investors are heterogeneous and the optimal portfolios $\mathbf{z}_i^*$ of the investors are not on the market frontier under the consensus belief. This is illustrated in Fig 3.2 for three combinations of the risk aversion coefficients. In fact, in this case,

$$\frac{1}{\Theta_a} = \frac{1}{2}[\frac{1}{\theta_1} + \frac{1}{\theta_2}], \qquad \mathbf{y}_a = \mathbf{y}_o, \qquad \Omega_a^{-1} = \frac{\Theta_a}{2}[\frac{1}{\theta_1}\Omega_1^{-1} + \frac{1}{\theta_2}\Omega_2^{-1}] \tag{3.3}$$

and

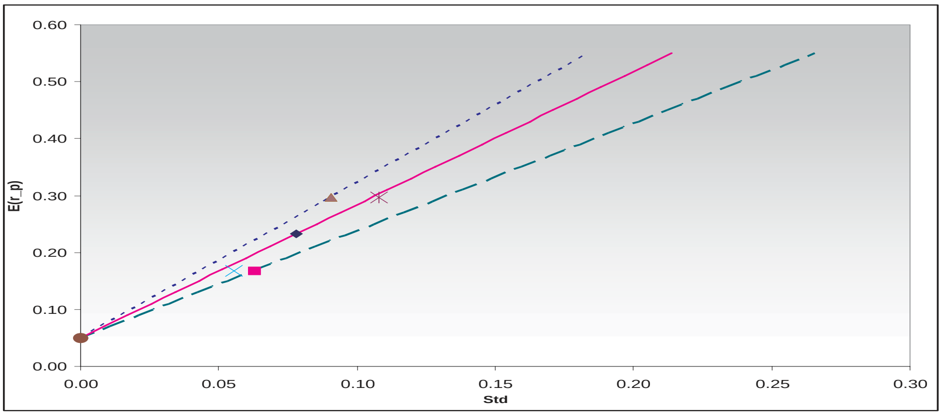
$$\mathbf{z}_i^* = \frac{\Theta_a}{2\theta_i}\Omega_i^{-1}\Omega_a\mathbf{z}_m, \qquad i = 1, 2. \tag{3.4}$$

Hence the optimal risky portfolios $\mathbf{z}_1^*, \mathbf{z}_2^*$ of the two investors are not proportional to the market portfolio $\mathbf{z}_m$, unless the investor belief in the covariance matrix is same as the covariance matrix under the consensus belief (i.e. $\Omega_i = \Omega_a$).

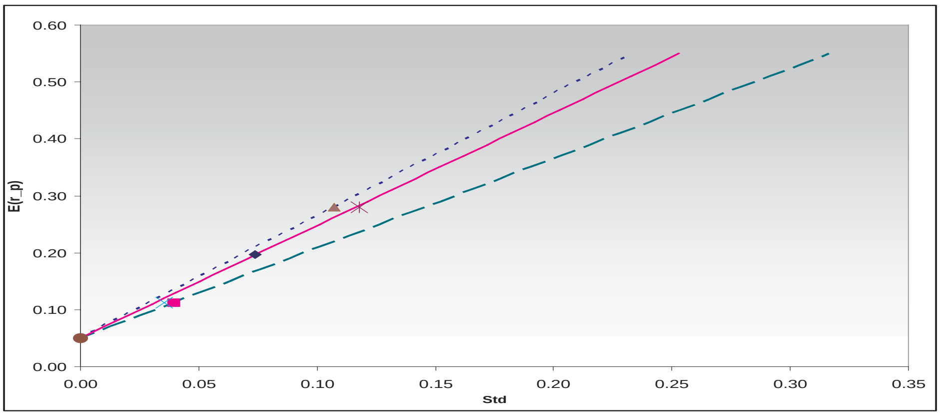
In addition, note that

$$\mu^a_{\mathbf{z}_2^*} = \mu_{\mathbf{z}_2^*}, \quad \mu^a_{\mathbf{z}_1^*} = \mu_{\mathbf{z}_1^*}, \qquad \sigma^a_{\mathbf{z}_2^*} > \sigma_{\mathbf{z}_2^*}, \quad \sigma^a_{\mathbf{z}_1^*} < \sigma_{\mathbf{z}_1^*}.$$

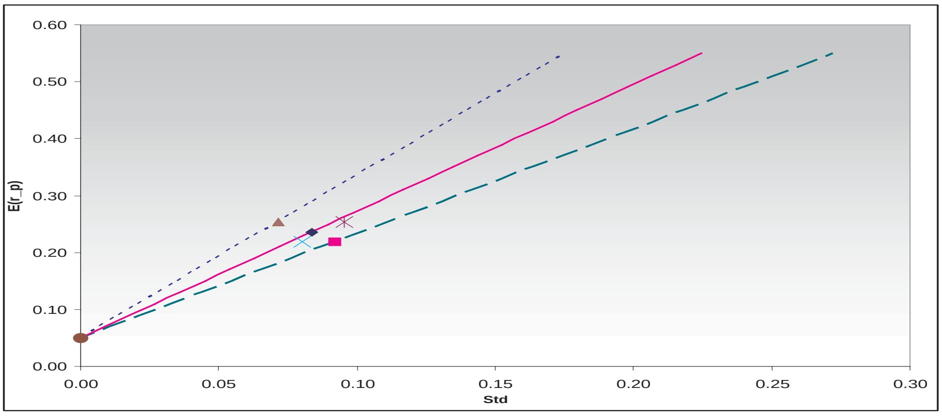
Note that, under the condition $\Omega_1 - \Omega_2$ is semi-positive definite, the variances $\sigma^2_{p,i}$ of any portfolio p corresponding to Ω_i satisfy $\sigma^2_{p,1} \geq \sigma^2_{p,2}$. If we interpret this portfolio variance as a measure of confidence of investor about the expected payoffs, then investor 2 is more confident and the most features related to optimistic investor 2 in the previous case still hold. In other words, when the market is in equilibrium, the Sharpe ratio is higher for investor 2 than investor 1 but the mean-variance efficiency is worse for investor 2. The market frontier is located in between the individual frontiers, but



(b1) $(\theta_1, \theta_2) = (3, 3)$



(b2) $(\theta_1, \theta_2) = (4, 2)$



(b3) $(\theta_1, \theta_2) = (2, 4)$

FIGURE 3.2. Mean-Variance efficient frontiers, optimal portfolios and market portfolio under heterogeneous and consensus beliefs when $\mathbf{y}_1 = \mathbf{y}_2$ and $\Omega_2 < \Omega_1$.

leaning more towards investor 2's frontier. Overall, we see that heterogeneity in covariance matrices has less effect on the structure of the portfolio frontiers comparing to the expected payoffs in the previous case in the sense that the individual frontiers are not so apart from the market's comparing to the previous section. Nevertheless, the main structure of the diagram is still determined by the covariance matrices of individuals while the ARAs determine the relative positions of individuals' optimal portfolios to the market portfolio in equilibrium

3.1.3. *Effect of heterogeneous expected payoffs and covariance matrices.* In the next example, we combine Examples 3.1 and 3.2 together and examine the joint effect of the heterogeneity in the expected payoffs and the covariance matrices.

First, we consider the situation where investor 2 is optimistic and confident in the sense that $\mathbf{y}_2 > \mathbf{y}_1$ and $\Omega_2 < \Omega_1$. Fig. A.1 in Appendix illustrates this situation for three combinations of ARAs $(\theta_1, \theta_2) = (3, 3)$, (4, 2) and (2, 4), respectively. Surprisingly, apart from those observations obtained in the previous two cases, the market's frontier is no longer located in between but below both individual frontier. Investor 2's frontier has the highest slope in Fig. A.1 (c1) and (c3) while investor 1 has the highest slope in (c2). Both investors optimal portfolios' positions under the consensus belief are very different to those under their own subjective beliefs, while investor 1's optimal portfolio is far below the market frontier. In all three combinations of ARAs, we have

$$\mu^a_{\mathbf{z}_1^*} < \mu_{\mathbf{z}_1^*}, \quad \sigma^a_{\mathbf{z}_1^*} < \sigma_{\mathbf{z}_1^*}; \qquad \mu^a_{\mathbf{z}_2^*} < \mu_{\mathbf{z}_2^*}, \quad \sigma^a_{\mathbf{z}_2^*} > \sigma_{\mathbf{z}_2^*}.$$

This suggests that the optimal portfolio for the confident investor (investor 2) becomes inefficient in market equilibrium, though it is very closer to the market frontier.

Next, we consider the situation where investor 2 is optimistic but less confident in the sense that $\mathbf{y}_2 > \mathbf{y}_1$ and $\Omega_2 > \Omega_1$. Fig. A.2 in Appendix illustrates this situation. Now amazingly, different from the previous case, market frontier regains its position between the individual frontiers, while investor 2's frontier has the highest slope, much like the case where we have only one source of heterogeneity considered in the previous cases. Also, individuals' optimal portfolios appear to almost lie on the market's frontier. However, by zooming the plot in Fig. A.2 (d3), we observe from Fig. A.2 (d4) that investors' optimal portfolios are still below the market frontier, though they are very close to it. In addition,

$$\mu^a_{\mathbf{z}_1^*} > \mu_{\mathbf{z}_1^*}, \quad \sigma^a_{\mathbf{z}_1^*} > \sigma_{\mathbf{z}_1^*}, \qquad \mu^a_{\mathbf{z}_2^*} < \mu_{\mathbf{z}_2^*}, \quad \sigma^a_{\mathbf{z}_2^*} < \sigma_{\mathbf{z}_2^*}.$$

Hence there is no mean-variance dominance for the optimal portfolios under the subjective and consensus beliefs.

In both cases, the one fund theorem does not hold and the optimal portfolios of the investors are not located on the market frontier. When an investor is optimistic and more confident, the market frontier can be below the individual frontiers, in particular when the optimistic investor is less risk averse. In this case, the market risk premium is lower and the optimal portfolio under the market belief is closer to the market frontier for the optimistic and more confident investor, but far below for the pessimistic and less confident investor. Therefore, the heterogeneity in covariance plays a very important role in determining the slope of the market frontier. This is because the aggregate return depends on not only the heterogeneous expected payoffs but also the covariance matrices.

Based on the above discussions, the effect of heterogeneity in asset payoffs can be summarized as follows. The standard one fund theorem under homogeneous belief does not hold under heterogeneous beliefs and the optimal portfolios are located below the market frontier. The heterogeneity in the covariance matrices plays the most important role in determining the relative positions of the individual frontiers to the market frontier, while the heterogeneity in expected payoffs plays the second important role. In most cases, the market frontier is located in between the individual frontiers and the optimal portfolios under the market belief are close to the market frontier. However, the market frontier can be below the individual frontiers when one investor is optimistic and more confident and the optimal portfolios of pessimistic and less confident investor can be far below the market frontier. The risk aversion coefficients determine the relative positions of the individuals' optimal portfolios to the market portfolio. The optimal portfolios are close to the market portfolio when the optimistic (pessimistic) investor is more (less) risk averse and they are further away from the market portfolio when the optimistic (pessimistic) investor is less (more) risk averse.

3.2. **The Case of Return Setup.** We now examine the case where the heterogeneity is formed in terms of returns of the risky assets. The analysis is based on Proposition 2.3. Similarly, the impact of single source of heterogeneity is examined first, followed by the impact of multiple sources of heterogeneity.

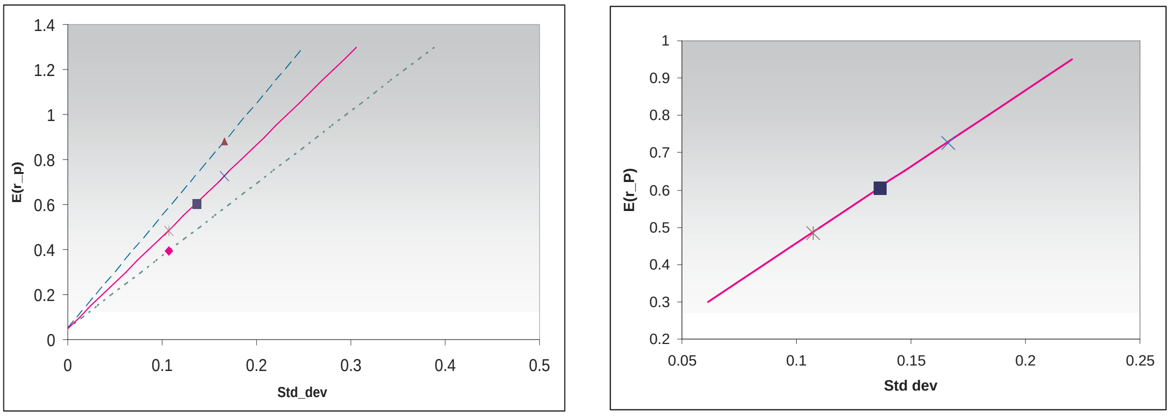
3.2.1. *Effect of heterogeneous expected returns.* To see the impact of the heterogeneous expected returns, we consider the following example in which two investors have different beliefs in the expected returns of the three risky assets.

Example 3.3. *Assume the investors have the same variance/covariance matrix $V_2 = V_1 = V_o$ and different expected returns $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_o$, $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_1 + 0.2 \times \mathbf{1}$ and absolute risk aversion (ARA) coefficients $(\theta_1, \theta_2) = (3, 3), (4, 2)$ and $(2, 4)$ with*

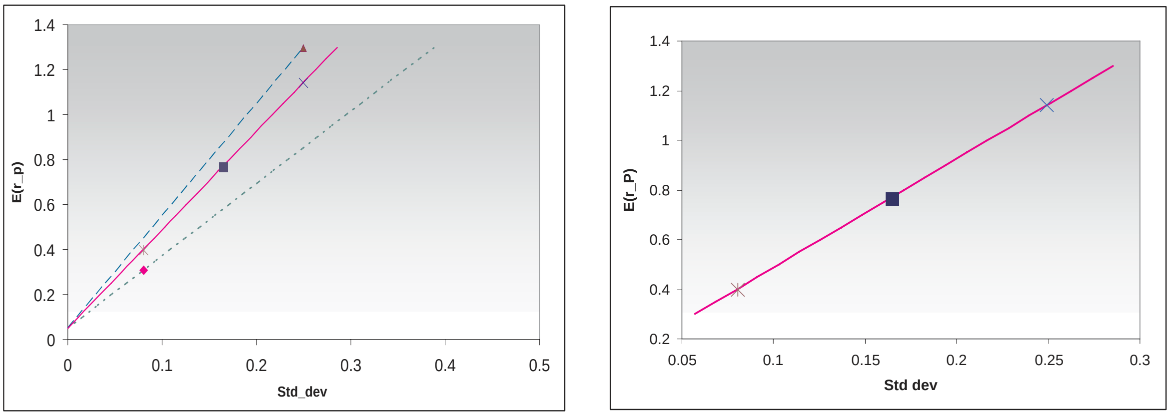
$$\boldsymbol{\mu}_o = \begin{pmatrix} 0.3633 \\ 0.2686 \\ 0.7087 \end{pmatrix}, \quad V_o = \begin{pmatrix} 0.0269 & 0.0044 & 0.0082 \\ 0.0044 & 0.0142 & 0.0035 \\ 0.0082 & 0.0035 & 0.0653 \end{pmatrix}. \tag{3.5}$$

We also assume the initial market wealth is evenly distributed between the two investors $W_{1,o} = W_{2,o} = \$10$ and there is one share available for each risky asset.

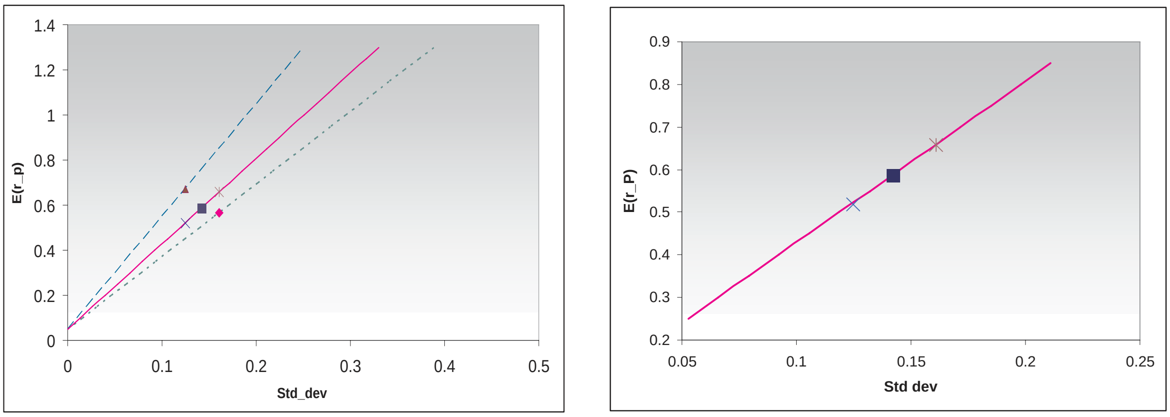
In this case, we can use Proposition 2.3 to construct the market consensus belief, to calculate the market equilibrium returns, and to plot the mean-variance efficient frontiers under two heterogeneous beliefs and the market belief, respectively. We can also locate the optimal portfolios of the two investors under their beliefs and the market



(a1) $(\theta_1, \theta_2) = (3, 3)$



(a2) $(\theta_1, \theta_2) = (4, 2)$



(a3) $(\theta_1, \theta_2) = (2, 4)$

FIGURE 3.3. Mean-Variance efficient frontiers, optimal portfolios and market portfolio under heterogeneous and consensus beliefs for $\boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$ and $V_1 = V_2$ (left panel) and their close-ups (right panel).

beliefs. For $\boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$ in Example 3.3, Fig. 3.3 provides the mean-standard deviation relationships for two heterogeneous beliefs and the market consensus belief based on all these information for three combinations of risk aversion coefficients $(\theta_1, \theta_2) = (3, 3), (4, 2)$ and $(2, 4)$, respectively. Based on these figures, we find similar effect of heterogeneity on the mean-variance frontiers to the payoff setup.

First, the standard one fund theorem does not hold in general and the optimal portfolios of the investors are below the market frontier. In equilibrium, the optimal portfolios of both investors are very close to the market efficient frontier, but not quite on it. In fact, they are too close to visualize the differences, even from the close-ups on the right panels and we call this phenomena as **quasi-one fund theorem**. However, numerically it can be verified that individual portfolios are below the market frontier. For example, in Fig 3.3 (a2), for investor 1, his portfolio has co-ordinates $(\sigma_{o_1}, \mu_{o_1}) = (0.0804, 0.398)$, the portfolio on the frontier with the same standard deviation has co-ordinates $(0.0804, 0.402)$, about 40 basis points (bp) difference in the expected return. Similarly for investor 2, the co-ordinates for his optimal portfolio and the one on the frontier with same standard deviation are $(0.2494, 1.1414)$ and $(0.2494, 1.1427)$, only 13 bp difference. Also the portfolio weights are significantly different, for example, the weights of risky assets for investor 1's optimal portfolio in Fig 3.3 (a2) are $\boldsymbol{\omega}_1^* = \begin{pmatrix} 0.1815 & 0.2764 & 0.2146 \end{pmatrix}^T$ while the efficient frontier portfolio with the same standard deviation has weights $\begin{pmatrix} 0.1935 & 0.3516 & 0.1795 \end{pmatrix}^T$. Similarly for investor 2, his/her optimal portfolio weights are $\boldsymbol{\omega}_2^* = \begin{pmatrix} 0.6094 & 1.1606 & 0.5187 \end{pmatrix}^T$ while the corresponding frontier portfolio has weights $\begin{pmatrix} 0.6002 & 1.0906 & 0.5566 \end{pmatrix}^T$. Although the differences are small, but significant enough to rule out the one fund theorem.

Secondly, the market frontier under the consensus belief is located in between the individual frontiers, with the optimistic investor's frontier having the highest slope. This results is very intuitive since the covariance matrix of the consensus belief is the same as the homogeneous covariance matrix and the expected returns under the consensus belief is just a weighted average of the subjective expected returns of the two investors. When investor 1's ARA is lower, it leans more towards investor 1's frontier, see Fig. 3.3 (a3). Similarly, when investor 2's ARA is lower, it leans more towards 2's frontier, see Fig. 3.3 (a2).

Thirdly, comparing to the optimal portfolios under the subjective beliefs, the individual's optimal portfolio under the market equilibrium becomes more mean-variance inefficient (efficient) for investor who is optimistic (pessimistic). In fact, the standard deviations of individual's optimal portfolios are unchanged in this example, but the expected return of the optimal portfolio in market equilibrium is lower for investor 2 and higher for investor 1, comparing to that under their subjective beliefs. This implies that optimistic (pessimistic) investor with respect to the expected returns is worsen (better) off under the market equilibrium. Also, the market portfolio is always located in the middle of the optimal portfolios under the consensus belief. The distance between each optimal portfolio and market portfolio is the largest (smallest) when the optimistic (pessimistic) investor in terms of expected returns is less risk averse, see Fig 3.3 (a1) and (a3).

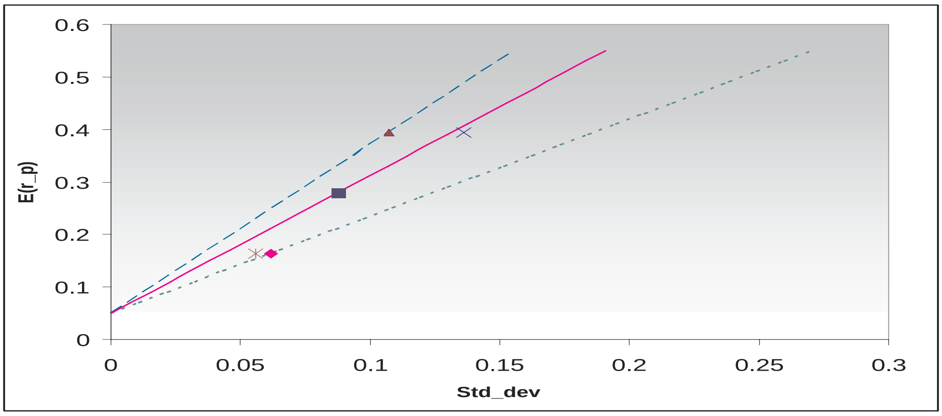
3.2.2. *Effect of heterogeneous variance/covariance matrices.* To see the impact of the heterogeneous covariance matrices, we consider in the next example that two investors have different beliefs in the covariance matrices of the returns of the three risky assets.

Example 3.4. *Let $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_1 = \boldsymbol{\mu}_o$, ARA coefficients and V are identical to their numerical value in Example 3.3, $V_2 = V_1 - 0.3\mathbf{1}_3$, where $\mathbf{1}_3$ is a 3×3 unit matrix with all elements equal to 1. Hence $V_1 - V_2$ is semi-positive definite and hence $V_2 \leq V_1$ according to the notation we introduced in the previous subsection.*

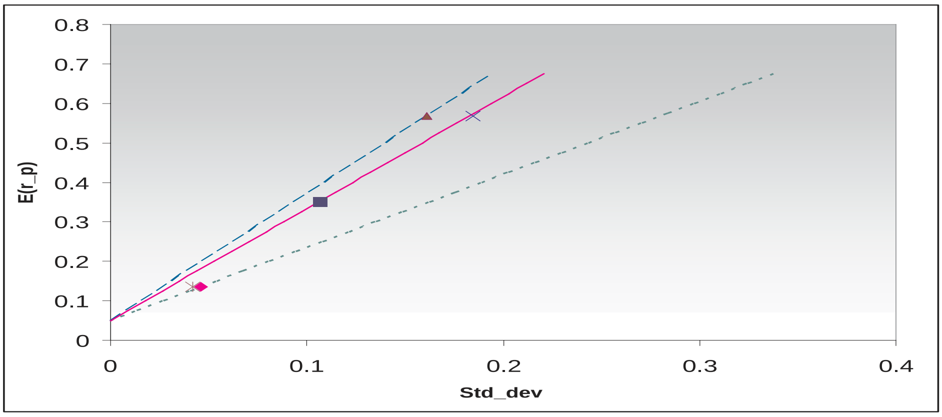
As in the previous case, the standard one fund theorem under homogeneous belief does not hold and the quasi-one fund theorem holds in this case. These features are illustrated in Fig. 3.4 for three combinations of the risk aversion coefficients. If we interpret the covariance matrix as a measure of confidence, then the observations in the previous case still hold. Namely, the market frontier under the consensus belief is located in between the individual frontiers and the market portfolio is still located in the middle of the optimal portfolios. Because of $\mu_1 = \mu_2$, the expected returns of the individual's optimal portfolio under both individual and market beliefs are the same. Correspondingly, one can see from Fig. 3.4 that the variance increases (decreases) for investor 2 (investor 1). Hence, comparing to the optimal portfolios under the subjective beliefs, the market aggregation improves (worsens) the mean-variance efficiency of the optimal portfolio for the less (more) confident investor.

Overall, we see that heterogeneity in covariance matrices has considerable effect on the structure of the portfolio frontiers. Similar to the previous example, the main structure of the diagram is determined by the covariance matrices of individuals while the ARAs determine the positions of individuals' optimal portfolios under both their own beliefs and the consensus belief.

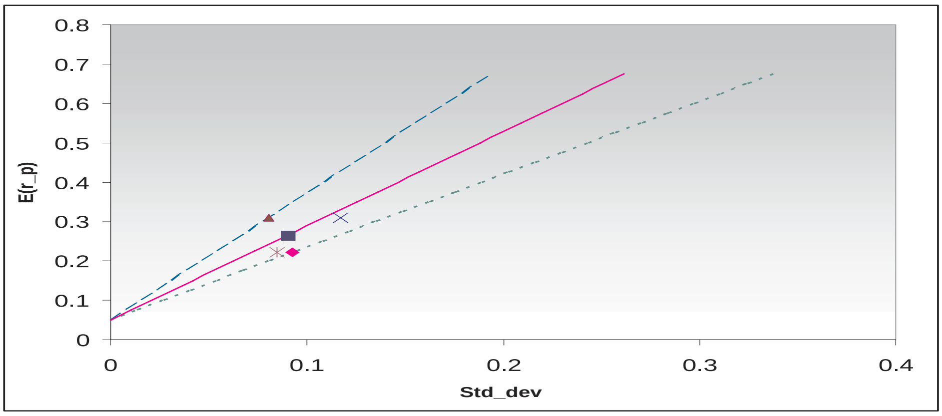
3.2.3. *Effect of heterogeneous expected returns and covariance matrices.* We now combine Examples 3.3 and 3.4 together and examine the joint effect of the heterogeneity in the expected returns and the covariance matrices. Consider two cases: (i) investor 2 is optimistic and more confident in the sense that $\mu_2 > \mu_1$ and $V_2 < V_1$, and (ii) investor 2 is optimistic but less confident in the sense that $\mu_2 > \mu_1$ and $V_2 > V_1$. Figs A.3 and A.4 in the Appendix illustrate both situations for three combinations of the risk aversion coefficients $(\theta_1, \theta_2) = (3, 3)$, (4, 2) and (2, 4). The features in the previous two cases are still present and the combined heterogeneity in both expected returns and covariance matrices have even more significant effect on the structure of the portfolio frontiers in the sense that the individual frontiers are much apart compared to the previous two cases. However, comparing Figs A.3 and A.4, one can see that there is one significant difference between these two cases. In the first case the optimal frontier of investor 2 has the highest slop, while in the second case the optimal frontier of investor 2 has the lowest slop. Also, in the market equilibrium, comparing with investor 1, investor 2's optimal portfolio is closer (further) to (from) the market's



(b1) $(\theta_1, \theta_2) = (3, 3)$



(b2) $(\theta_1, \theta_2) = (4, 2)$



(b3) $(\theta_1, \theta_2) = (2, 4)$

FIGURE 3.4. Mean-Variance efficient frontiers, optimal portfolios and market portfolio under heterogeneous and consensus beliefs for $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ and $V_2 \leq V_1$.

frontier when the frontier under his/her subjective belief has the highest (lowest) slope amongst all three frontiers, see Figs A.3 and A.4. This means that, for the investor who is more optimistic and confident, his/her optimal portfolio will be more closer to the market frontier under the consensus belief. One significant difference of return setup from the payoff setup is that the market frontier is always in the middle of the two individual frontiers.

Based on the above discussions, the effect of the heterogeneity in the asset returns is very similar to the in the asset payoff considered in the previous subsection. (i) The standard one fund theorem under homogeneous belief does not hold under heterogeneous beliefs. The optimal portfolios of investors are not located on the market frontier. However, they are very close to the market frontier and hence quasi-one fund theorem holds under heterogeneous beliefs. If we interpret bounded rationality as heterogeneous agents making optimal decisions under their subjective beliefs, then bounded rationality leads to an almost perfectly rational market in the sense of the quasi-one fund theorem. (ii) Different aspects of heterogeneity affect the market differently. The heterogeneity in covariance plays the most important role in determining the relative positions of the individual and market frontiers. This follows from comparing Figs 3.4-A.4. The heterogeneity in the expected returns plays the second important role in determining the relative positions of the frontiers. This follows from comparing Figs 3.3, A.3 and A.4. The risk aversion coefficients determine the closeness of the individuals' optimal portfolios to the market portfolio. The differences among the expected returns of the market portfolio and the individual optimal portfolios can be significant when the optimistic investor is less risk averse. In addition, within all the combinations of different aspects of heterogeneity considered, the market always generates market risk premium between the risk premia of the individual optimal portfolios since the market portfolio is always located in between the individual optimal portfolios.

4. STATISTICAL ANALYSIS OF THE AGGREGATE MARKET BEHAVIOUR

This section presents a statistical analysis on the market impact of the heterogeneity among investors of two groups within the heterogeneous CAPM framework for both payoff and return setups. Empirically, it is a challenge to measure the degree of heterogeneity and its uncertainty among the investors. To overcome this challenge, we assume that the beliefs within each group can be characterized by certain probability distributions. We assign different distributions to the beliefs of two investors from two different groups. By simulating different aspect of heterogeneity, we conduct statistical analysis through Monte Carlo simulations. The impact of the heterogeneity on the market behaviour is then examined through the statistics of market equilibrium return distributions, the beta coefficients, and two performance measures—Sharpe and Treynor ratios. Similar to the previous section, the analysis is conducted for two different setups.

4.1. **The Case of Payoff Setup.** We assume that there are two investors, one risk-free asset and three risky assets in the market. We assume that investors form their beliefs in terms of asset payoffs. We take $\theta_o = 3$, the expected payoff $\mathbf{y}_o$ and the covariance matrix Ω_o defined in Example 3.1 as the benchmark of homogeneous belief case and the heterogeneity is measured by the dispersion from this benchmark. The initial wealth of individuals are $W_{1,o} = W_{2,o} = \$10$. There is one share available for each asset and the risk-free rate is $r_f = 5\%$ p.a. Under this setup, we arrive a set of equilibrium prices $\mathbf{p}_o = (4.8323, 7.3614, 5.7237)^T$ for the benchmark case. Correspondingly, we obtain the expected returns vector $\mathbb{E}_a(\tilde{\mathbf{r}}) = (0.36375, 0.2688, 0.7087)^T$, asset betas $\boldsymbol{\beta} = (0.9098, 0.6345, 1.9101)^T$, and the market expected return $\mathbb{E}_a(r_m) = 0.3948$ for the benchmark case.

4.1.1. *Impact of heterogeneous ARA's.* First, we assume that, within each group, the investors are homogeneous except in their risk aversions, represented by their *absolute risk aversion coefficients* (ARA). Within each group, we assume investor's ARAs are normally distributed, with a mean 3 and standard deviation as a percentage of θ_o, i.e. $\theta_i \sim \mathcal{N}(\theta_o, \theta_o \times \sigma_{\theta_i})$, we call such distribution as a *mean-preserved spread (MPS)* in θ_i for group i. The diversity in the beliefs between two groups is measured by the change of σ_{θ_i}. We need to truncate the distribution so the ARAs are strictly positive. By changing σ_{θ_i}, we examine the impact on the expected returns of the risky assets, individual optimal portfolios and market portfolio. Impact on asset betas and optimal portfolios' betas are also looked at. The statistic result based on 10,000 simulations is presented in Table 1 in the Appendix, from which we have the following observations. In our tables, we denote the three risky assets, optimal portfolio for investor from group 1 and 2, and the market portfolio by A, B, C, O1, O2 and M respectively. the heterogeneity is measure by the difference between σ_{θ_i}.

First, the heterogeneity in ARAs generates non-normal distributions for the asset expected return, portfolio expected return and the beta coefficients for the risky assets and optimal portfolios, as indicated by the JB test (JB-PV)[3], negative skewness (Skew) and positive excess kurtosis (ExKurt). In addition, the standard deviation (SD) increases systematically as σ_{θ_i} increases. Secondly, the MPS in ARAs reduces expected return for all risky asset, including the market portfolio and individual optimal portfolios, and the betas, leading to a decrease in Sharpe (SR) and Treynor (TR) ratios. Both the optimal portfolios perform similarly in terms of their betas and all the four moments under all scenarios. In particular, the optimal portfolios of the two investors achieve the same Sharpe and Treynor ratios as the market portfolio under all scenarios.

We now give an explanation to the above observations. First of all, because investors are homogeneous in the mean and covariance matrix of the asset payoffs, the market belief is the same as the investors' belief $\mathbf{y}_a = \mathbf{y}_i, \Omega_a = \Omega_i$ for $i = 1, 2$. Hence the standard one fund theorem under homogeneous belief still holds. Therefore, the optimal portfolios of the investors perform equally to the market portfolio.

[3]We draw conclusions from the Jarque-Bera test of normality at 1% significance level since we run 10,000 simulations, i.e. we reject the null hypothesis only if $(JB - PV) < 0.01$

This explains the equal Sharpe and Treynor ratios of the optimal portfolios of the investors and the market portfolio under all scenarios. Secondly, Chiarella, Dieci and He (2006*a*) showed that the aggregate market equilibrium price is a weighted average of each agent's equilibrium price under his/her belief as if he/she was the only agent in the market. More precisely, if we define $\mathbf{p}_{i,o}$ as the equilibrium price vector of the risky assets for investor i as if he/she were the only investor in the market, then we would have

$$\mathbf{p}_{i,o} = \frac{1}{R_f}[\mathbb{E}_i(\tilde{\mathbf{x}}) - \theta_i \Omega_i \bar{\mathbf{z}}_i],$$

where $\bar{\mathbf{z}}_i$ is the initial endowment of investor i. Then the market equilibrium prices can be rewritten as

$$\mathbf{p}_o = \Theta\Omega_a \left[\frac{1}{I}\sum_{i=1}^{I} \theta_i^{-1}\Omega_i^{-1}\mathbf{p}_{i,o}\right] = \frac{\Theta}{IR_f}\Omega_a \left[\sum_{i=1}^{I} \theta_i^{-1}\Omega_i^{-1}\mathbb{E}_i(\tilde{\mathbf{x}}) - \bar{\mathbf{z}}_i\right]. \qquad (4.1)$$

When investors are homogeneous except ARAs, a lower θ_i leads to a higher price $\mathbf{p}_{i,o}$. It then follows from equation (4.1) that the market equilibrium price is then dominated by the investor who is less risk averse, leading to higher market equilibrium prices and hence lower equilibrium returns. When σ_{θ_i} increases, the dominance of the less risk averse investors becomes significant, reducing the averages of the expected returns of the risky assets as well as portfolios. At the meantime, because of the convexity of $1/\theta_i$ in equation (4.1), a MPS in ARAs leads to right skewed distributions for the market equilibrium prices and hence left skewed distributions for returns, leading to negative skewness for both returns and betas. In addition, the spread of the market equilibrium prices when σ_{θ_i} is large leads to more spread in distributions, resulting in high standard deviations. This analysis suggests that the heterogeneity in risk aversion coefficients is responsible for the non-normality in the market.

4.1.2. *Impact of heterogeneous beliefs in expected asset payoffs.* We now assume that investors are homogeneous except in the expected payoffs. The ARAs and covariance matrices are the same as in the benchmark homogeneous case. Within each group, we consider a MPS in agents' belief of expected asset payoffs. More precisely, let

$$\mathbf{y}_i \sim \mathbf{y}_o[1 + \sigma_{\delta_i}\mathcal{N}(0,1)], \qquad i = 1, 2.$$

This means that, within each group, investors beliefs in the expected asset payoffs are independently normally distributed with volatility expressed as a percentage of the expected payoff, which is constant across all assets. The diversity of the beliefs between two groups is measured by the change of σ_{δ_i}. We need to truncate the distribution to ensure that the expected payoff is strictly positive. The resulting statistics based on 10,000 simulations are reported in Tab. 2 in the Appendix, from which we obtain the following observations that are very different from the previous case. (i) The average of the expected returns of all the risky assets and portfolios and the beta coefficients do not change much when σ_{δ_i} changes. Also, the market expected returns has zero volatility and the volatilities associated with individual assets are very small, which means

MPS in expected payoffs does not affect the market return. In addition, the JB tests indicate that the expected returns of the risky assets and optimal portfolios and the beta coefficients are normally distributed under many scenarios. (ii) The Treynor ratios do not change at all. The Sharpe ratios are the same in all scenarios except for the optimal portfolios of investors, which decreases systematically as σ_{δ_i} increases. In addition, both the optimal portfolios have approximately the same Sharpe ratios, which are below the Sharpe ratio for the market portfolio, indicating that heterogeneity in expected payoffs are the potential causes for investors under-performing the market.

We now give an explanation to the above observations. It follows from

$$\mathbf{y}_a = \frac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2), \qquad \mathbf{p}_o = \frac{1}{R_f}[\mathbf{y}_a - \frac{\theta_o}{2}\Omega_o \mathbf{z}_m]$$

that, on average, the MPS distribution in the expected asset payoffs does not change the market aggregate expected payoffs $\mathbf{y}_a$ and hence the equilibrium price. Therefore the average of the expected returns for all the risky assets and portfolios does not change when σ_{δ_i} changes. The small standard deviations for the risky asset returns and betas lead to the same Treynor and Sharpe ratios under all scenarios. The under-performance of the optimal portfolios of the investors comparing to the market portfolio is due to their biased expected payoffs from the market, either optimistic or pessimistic to the market belief. Overall, the heterogeneity in the expected payoffs has no significant effect on the market. It does not change the systematic and unsystematic risks of the risky assets. However the unsystematic risk for the optimal portfolios of the investors increases as σ_{δ_i} increases and this is due to their bias towards the expected payoffs.

4.1.3. *Impact of heterogeneous beliefs in covariance matrices of the asset payoffs.* We now assume that investors are homogeneous except in the covariance matrices of the asset payoffs. We decompose the covariance matrix Ω_o to $\Omega_o = D_o C D_o$, where D_o is a diagonal matrix consisting of the standard deviations of the asset payoffs and C is the correlation matrix. Within each group of investors, a MPS to the covariance matrices is introduced as follows. Let

$$\Omega_i = D_i C D_i, \qquad D_i = D_o \times (\epsilon_i + 1)I, \qquad \epsilon_i \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_{\epsilon_i}), \qquad i = 1, 2,$$

where $D_o = diag(\sigma_{o1}, \sigma_{o2}, \sigma_{o3})$, σ_{oj} is the volatility of asset j's return under the homogeneous belief and $\sigma_{ij} = \sigma_{oj}(1 + \epsilon_i) \overset{\text{iid}}{\sim} \mathcal{N}(\sigma_{oj}, \sigma_{\epsilon_i}^2 \sigma_{oj}^2)$. Then $\mathbb{E}(D_i) = D_o$ and $\mathbb{E}(\Omega_i) = \Omega_o$. We need to truncate the distribution to ensure that the standard deviations are strictly positive. The resulting statistics based on 10,000 simulations are reported in Tab. 3 in the Appendix, from which we observe a similar effect to the previous case for the heterogeneous beliefs in the expected payoffs.

In this case, the market aggregate expected payoff $\mathbf{y} = \mathbf{y}_o$ and

$$\Omega_a^{-1} = \frac{1}{2}(\Omega_1^{-1} + \Omega_2^{-1}). \tag{4.2}$$

Therefore, on average Ω_a and hence the equilibrium prices are unchanged. Because of the convex relationship (4.2), the distributions of the expected returns of the risky assets and portfolios are skewed to the left. Overall, based on the statistics in Tab. 3, the heterogeneity in the covariance matrices has no significant impact on the market equilibrium returns, Sharpe and Treynor ratios, and normality of the distributions for the expected returns and beta coefficients. The optimal portfolios perform approximately equally to the market portfolio. However, when investor 2 is much more diverse in his/her belief in volatility of asset payoff compare with investor 1, the diversity can make the return distribution for the optimal portfolio of investor 2 to have extremely large kurtosis and negative skewness. This suggests that investor 2 may be subjected to potential huge losses. We notice similar effects when we consider heterogeneity in ARA in Tab. 1, in that case, the effects were even more significant.

4.1.4. *Impact of two or three sources of heterogeneity.* In the previous discussion, we consider the impact of only one source of the heterogeneity. We found that the heterogeneity in the risk aversion coefficients has significant impact on the market. It can generate non-normality of the expected returns and betas and havs significant impact on both systematic and unsystematic risks, measured by the changes in the Sharpe and Treynor ratios. Also, the heterogeneity in the expected payoffs and variances do not have much inpact on the overall market. In Tabs 4 to 6, we consider the impact of more than one source of heterogeneity. Overall it carries on from the impact of the single source of heterogeneity. When the heterogeneity in ARAs is involved, the market is dominated by the heterogeneity in ARAs. In the case of heterogeneity in both expected payoffs and the covariance matrices, there is no significant impact on the market, although the impact of the heterogeneous beliefs in the expected payoffs dominates.

4.2. **The case of return setup.** Similar to the previous subsection, we assume that there are two investors from two different group, one risk-free asset and three risky assets in the market. For the benchmark case, $\theta_1 = \theta_2 = \theta_o = 3$, the expected return $\boldsymbol{\mu}_o$ and the return covariance matrix V_o are defined in Example 3.3. For this benchmark case, we obtain identical equilibrium prices, expected returns and betas as in the payoff setup.

4.2.1. *Impact of heterogeneity in expected asset returns.* We first examine the impact of the heterogeneity in the expected returns. Assume two investors are homogeneous except in expected returns. Let $\theta_1 = \theta_2 = \theta_o = 3$ and $V_1 = V_2 = V_o$. Assume that the expected returns of the two investors are normally distributed[4],

$$\tilde{\boldsymbol{\mu}}_i = \boldsymbol{\mu}_o + \tilde{\boldsymbol{\delta}}_i, \qquad \tilde{\delta}_{i,j} \overset{\text{iid}}{\sim} \sigma_{\delta_i}\mathcal{N}(0,1), \qquad i = 1, 2, j = 1, 2, 3.$$

[4]This means that, within each group, investors' beliefs of each asset's expected rate of return are independently normally distributed with the same standard deviation, also we assume investors forms their belief independently, i.e. investor 1's belief does not affect investor 2's belief.

For each given $(\sigma_{\delta_1}, \sigma_{\delta_2})$, we run 10,000 simulations with the expected returns of the risky assets that are normal distributions with the given standard deviations. We denote the three risky assets, the market portfolio, the optimal portfolios of investor 1 and 2 by A, B, C, M, O_1 and O_2, respectively. For $(\sigma_{\delta_1}, \sigma_{\delta_2}) = (1\%, 1\%), (1\%, 2\%), (1\%, 3\%),$ $(0, 2\%)$ and $(0, 3\%)$, the summary statistics for both returns and beta coefficients are given in Tab. 7 in the Appendix. The heterogeneity of the two investors is measured by the dispersion between σ_{δ_1} and σ_{δ_2}. Based on this table, we have the following observations.

The mean of expected returns of M and O_2 increases as the dispersion in belief of the expected returns of investor 2 increases. For fixed $\sigma_{\delta_1} = 1\%$, as σ_{δ_2} increases by 1%, the mean of A, B and C's expected return stays constant, while the means for both M and O_2 increase systematically, in particular for O_2. Comparing the cases of $(\sigma_{\delta_1}, \sigma_{\delta_2}) = (1\%, 2\%)$ and $(0, 3\%)$, the later generates higher expected returns for M and O_2, about 5 basis points (bp) for M and 10bp for O_2.

The standard deviations (SD) increase systematically as the belief of the expected returns is more dispersed for all assets and portfolios, especially for O_2 where the increase in the average expected return is the largest. Clearly, high dispersion in beliefs causes one's optimal portfolio to have higher volatility, typically the investor who is more dispersed in his/her beliefs. The skewness (Skew) and excess kurtosis (ExKurt) for individual asset returns are close to zero, while they are significantly positive for the portfolios when dispersion is large. When the dispersion increases, the skewness increase systematically for M and O_2 but decrease for O_1 when investor 1 reduces his/her dispersion in expected returns to zero. The JB tests show that the returns of all three risky assets A, B and C are likely to be normally distributed, while the returns are non-normal for M and O_2, especially when dispersion in expected returns is large. Returns of O_1 is likely to be normally distributed when investor does not vary belief at all.

Both Sharpe and Treyor ratios increase systematically for all portfolios. The Sharpe ratios for M and O_2 are higher for larger difference in $(\sigma_{\delta_1}, \sigma_{\delta_2})$. Also M has the highest Sharpe ratio follow by O_1 and O_2, while asset B has the lowest Sharpe ratio. Applying the HCAPM, the Treynor ratios are the same for given $(\sigma_{\delta_1}, \sigma_{\delta_2})$ across all the assets and portfolios. However, the Treynor ratio increases for more dispersed beliefs in $(\sigma_{\delta_1}, \sigma_{\delta_2})$. The improvement of both Sharpe and Treyor ratios due to the high dispersion in beliefs is not observed for the payoff setup. This suggests that, measured by Sharpe and Treyor ratios, investors and market are benefit from investor diversity in their belief.

Noting that assets A and B have betas less than 1 while C has a beta coefficient larger 1, and portfolios $O1$ and O_2 have betas close to 1. The mean value for the beta coefficients decreases systematically, this suggests that greater dispersion beliefs causes the beta coefficients to decrease. The standard deviationfor the beta coefficients increases systematically as the belief of the expected returns is more dispersed. Skewness increases for high beta stock C ($\beta > 1$) and decreases for low beta stocks A and

B ($\beta < 1$). The J-B test shows the beta of all assets and portfolios are are likely to be normally distributed except for stock B, when dispersion in beliefs is large.

4.2.2. *Impact of heterogeneity in covariance matrices.* Different from the previous case, we now assume that the investors are homogeneous except in the covariance matrices in asset returns. Let $\theta_1 = \theta_2 = \theta_o = 3$ and $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \boldsymbol{\mu}_o$. Assume investors' beliefs of the correlation structure of the asset returns are homogenous and fixed, however the beliefs of standard deviations of asset returns are independently normally distributed for each investor, that is, $V_o = D_o C D_o$ where $D_o = diag(\sigma_{o1}, \sigma_{o2}, \sigma_{o3})$ and C is the correlation matrix and $\tilde{V}_i = \tilde{D}_i C \tilde{D}_i$ where $\tilde{D}_i = D_o + \epsilon_i \mathbf{I}$ and $\epsilon_i \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_{\epsilon_i}^2)$. Hence the standard deviation of asset j's return under investor i's belief is the random variable $\tilde{\sigma}_{ij} \sim \mathcal{N}(\sigma_{oj}, \sigma_{\epsilon_i}^2)$, which is independent for each asset j and investor i. By conducting the same Monte Carlo simulations, we summarize the resulting statistics in Tab. 8 in the Appendix.

Similar to the previous case, we measure the heterogeneoty or diversity by the dispersion in $(\sigma_{\epsilon_1}, \sigma_{\epsilon_2})$ of the covariance matrices. Large σ_{ϵ_2} or difference between σ_{ϵ_1} and σ_{ϵ_2} correspond to more dispersed or diversified beliefs in covariance matrices. Based on Tab. 8, we have the following observations. (i) The expected return of the risky assets are unchanged, but the Sharpe and Treynor ratios increase systematically when the beliefs in covariance matrices are more dispersed. (ii) The four moments, including the mean, standard deviation, skewness and kurtosis, of the expected returns for M and O_2 increase systematically when the beliefs in covariance matrices are more dispersed, while that for O_1 is unchanged. In particular, both M and O_2 have extremely high skewness and kurtosis when $(\sigma_{\epsilon_1}, \sigma_{\epsilon_2}) = (1\%, 3\%)$ and $(\sigma_{\epsilon_1}, \sigma_{\epsilon_2}) = (0\%, 3\%)$. It seems that when the dispersion of variance-covariance of returns is too large, the return distributions of the portfolios do not have finite moment except for the first moment. Also, the JB tests indicate that the expected return of M and O_2 follow non-normal distributions. (iii) For the beta coefficients of the risky assets and the optimal portfolios, the Mean and Skew decrease while the SD and ExKurt increase systematically when the beliefs in covariance matrices are more dispersed. Also, the JB test indicates that the beta coefficients follow non-normal distributions.

4.2.3. *The impact of heterogeneous ARA coefficients.* In the homogeneous case when $V_i = V_o$ and $\boldsymbol{\mu}_i = \boldsymbol{\mu}_o$ for all i, it is clear from Proposition 2.3 that any changes in the ARA coefficient will not affect the market consensus belief $\mathcal{B}_a$ and standard CAPM holds. However, this is no longer the case when either the beliefs of expected returns or variance/covariance of returns are heterogeneous. In this subsection, we consider three cases to assess the impact of heterogeneous ARA coefficients, case(i): $\boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$, $V_o = V_1 = V_2$, case (ii):$\boldsymbol{\mu}_o = \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$, $V_1 \neq V_2$, and case (iii):$\boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$, $V_1 \neq V_2$.

For case (i), consider our example in subsection 4.2.1 and let $\sigma_{\delta_i} = 1\%$ and $\theta_i \sim \mathcal{N}(3, \sigma_{\theta_i})$ for $i = 1, 2$. We run 10,000 simulations for different combinations of $(\sigma_{\theta_1}, \sigma_{\theta_2})$ and base our discussion on the summary statistics in Tab. 9 in the Appendix.

The impact on the first two moments of the expected returns under the market consensus belief is minimal, only M and O2 have an increasing trend in their mean and standard deviation of expected returns. Dispersion in ARA coefficients have positive effects on the skewness and kurtosis of the portfolios, which is strongest for O2, follow by M. This is expected since investor 2 has a larger dispersion in ARA than investor1, meaning that investor 2 has a higher chance of achieving large positive expected return for his/her portfolio. Distribution of the expected returns of the portfolios are non-normal while the distributions are likely to be normal for the risky assets under most scenarios, which is similar to the case in subsection 4.2.1. This suggests that dispersion in ARA has little or no effect on the distribution of individual asset expected returns. The Sharpe ratios appears to be constant under all scenarios, however, the Treynor ratios seem to be increasing systematically.The mean of beta coefficients is decreasing for all except O2 and standard deviation is increasing for all. This, together with the performance of the Treynor ratios, indicates that the dispersion in ARA reduces the systematic risk of the portfolios. JB tests show distributions for beta are likely to be normal for small dispersion in ARA.

For case (ii), consider our example in subsection 4.2.2 and let $\sigma_{\epsilon_i} = 1\%$ and $\theta_i \sim \mathcal{N}(3, \sigma_{\theta_i})$ for $i = 1, 2$. We run 10,000 simulations for different combinations of $(\sigma_{\theta_1}, \sigma_{\theta_2})$ and base our discussion on the summary statistics in Tab.10 in the Appendix. The results are quite similar to the previous case, except the following differences. The distribution of beta coefficients for risky assets is more likely to be normally distributed when the dispersion in ARA is large. Also, skewness of the beta coefficient for risky asset is now systematically decreasing while the kurtosis are increasing.

For case (iii), we combine the previous two cases to see the effect of dispersion in ARA coefficients when beliefs in both expected returns and varaince/covaraince matrix are heterogeneous and base our discussion on the summary statistics in Tab.11 in the Appendix. The results are somewhat similar to the previous two cases. We observe some common characteristics of impact of heterogeneous ARA coefficients. The dispersion in ARA coefficients can affect the skewness and kurtosis of the portfolios' expected returns. It improves the Treynor ratios systematically, but not Sharpe ratios.

Based on the above analysis, we obtain the following overall features on the impact of the heterogeneity when beliefs are formed in returns. (i) The heterogeneity has more impact on the distribution of portfolios' expected returns than assets' expected returns. (ii) Diversity in beliefs leads to better performance especially for the portfolios as well as the risky assets in terms of Sharpe and Treynor ratios. (iii) When beliefs in expected return and/or variance/covariance of returns are heterogeneous, investors with a large dispersion in his/her risk tolerance can create a large positive skewness and kurtosis for the expected return of his/her portfolio and the market portfolio, which is a more favorable outcome than a symmetric distribution. (iv) Market aggregation of heterogeneous beliefs can lead to non-normality in return distributions of the market portfolio as well as the individual optimal portfolios.

5. Conclusion

This paper examines the impact on the market when investors are heterogeneous and bounded rational. Within the framework of Chiarella, Dieci and He (2006*a*, 2006*b*) on mean-variance analysis under heterogeneous beliefs in terms of either the payoffs or returns of the risky assets, this paper analyzes the effect of the heterogeneity. Through some numerical examples and statistical analysis based on Monte Carlo simulations, we examine the mean-variance efficiency and the diversity effect of the heterogeneous beliefs. Our main findings can be summarized as follows. (i) For both payoff and return setups, the standard one fund theorem under homogeneous belief does not hold under heterogeneous beliefs and the optimal portfolios are located below the market frontier. This may help us to understand the empirical finding that the managed funds under-perform the index portfolio on average. However, for the return setup, the optimal portfolios of investors can be very close to the market frontier in equilibrium. This implies that investors' portfolio selections under their subjective beliefs are almost perfectly rational under the market's belief, and this phenomenon is referred as the quasi-one fund theorem. (ii) The market frontier is located in between the frontiers of investors under their beliefs in most cases, while different aspect of heterogeneity plays different role. The heterogeneity in the covariance matrices plays the most important role in determining the relative positions of the individual frontiers and the market frontier, while the heterogeneity in expected payoffs/returns plays the second important role, which is controlling how far apart are the individual frontiers from the market frontier. The risk aversion coefficients determine the relative positions of the individual optimal portfolios to the market portfolio. (iii) For the payoff setup, the diversity in heterogeneity in risk aversion coefficients has significant impact on the market, it can generate non-normality for the expected returns and betas and both systematic and unsystematic risks, measured by the changes in the Sharpe and Treynor ratios. However the belief dispersions in the expected payoffs and covariance matrices of the risky assets have no significant impact on the market. (iv) For the return setup, heterogeneity has more impact on the distribution of portfolios' expected returns than assets' expected returns. Diversity in heterogeneity leads to better performance for the portfolios as well as the risky assets in terms of Sharpe and Treynor ratios. When beliefs in expected return and/or variance/covariance of returns are heterogeneous, investors with a large dispersion in his/her risk tolerance can create a large positive skewness and kurtosis for the expected return of his/her portfolio and the market portfolio. Also, market aggregation of heterogeneous can lead to non-normality in return distributions.

References

Abel, A. (1989), Asset prices under heterogeneous beliefs: implications for the equity premium, working paper 09-89, Rodney L. White Center for Financial Research.

Abel, A. (2002), 'An exploration of the effects of pessimism and doubt on asset returns', *Journal of Economic Dynamics and Control* **26**, 1075–1092.

Ang, A., Hodrick, R., Xing, Y. and Zhang, X. (2006), 'The cross-section of volatility and expected returns', *Journal of Finance* **61**(1), 259–299.

Bart, J. and Masse, I. (1981), 'Divergence of opinion and risk', *Journal of Financial and Quantitative Analysis* **16**, 23–34.

Basak, S. (2000), 'A model of dynamic equilibrium asset pricing with heterogeneous beliefs and extraneous beliefs', *Journal of Economic Dynamics and Control* **24**, 63–95.

Chan, L., Karceski, J. and Lakonishok, J. (1999), 'On portfolio optimization: forcasting covariance and choosing the risk model', *The Review of Financial Studies* **12**, 937–974.

Chiarella, C., Dieci, R. and He, X. (2006*a*), Aggregation of heterogeneous beliefs and asset pricing theory: a mean-variance analysis, Technical Report 186, Quantitative Finance Research Center, University of Techonology, Sydney.

Chiarella, C., Dieci, R. and He, X. (2006*b*), A dynamic heterogeneous beliefs capm, working paper, University of Technology, Sydney.

Detemple, J. and Murthy, S. (1994), 'Intertemporal asset pricing with heterogeneous beliefs', *Journal of Economic Theory* **62**, 294–320.

Diether, K., Malloy, C. and Scherbina, A. (2002), 'Differences of opinion and cross section of stock returns', *Journal of Finance* **57**, 2113–2141.

Huang, C.-F. and Litzenberger, R. (1988), *Foundations for Financial Economics*, Elsevier, North-Holland.

Johnson, T. (2004), 'Forecast dispersion and the cross section of expectated returns', *Journal of Finance* **59**, 1957–1978.

Lintner, J. (1965), 'The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets', *Review of Economics and Statistics* **47**, 13–37.

Lintner, J. (1969), 'The aggregation of investor's diverse judgements and preferences in purely competitive security markets', *Journal of Financial and Quantitative Analysis* **4**, 347–400.

Miller, E. (1977), 'Risk, uncertainity, and divergence of opinion', *Journal of Finance* **32**, 1151–1168.

Mossin, J. (1966), 'Equilibrium in a capital asset market', *Econometrica* **35**, 768–783.

Rubinstein, M. (1974), 'An aggregation theorem for securities markets', *Journal of Financial Economics* **1**, 225–244.

Rubinstein, M. (1975), 'Security market efficiency in an arrow-debreu economy', *American Economic Review* **65**, 812–824.

Rubinstein, M. (1976), 'The strong case for the generalized logarithmic utility model as the premier model of financial markets', *Journal of Finance* **31**, 551–571.

Sharpe, W. (1964), 'Capital asset prices: A theory of market equilibrium under conditions of risk', *Journal of Finance* **19**, 425–442.

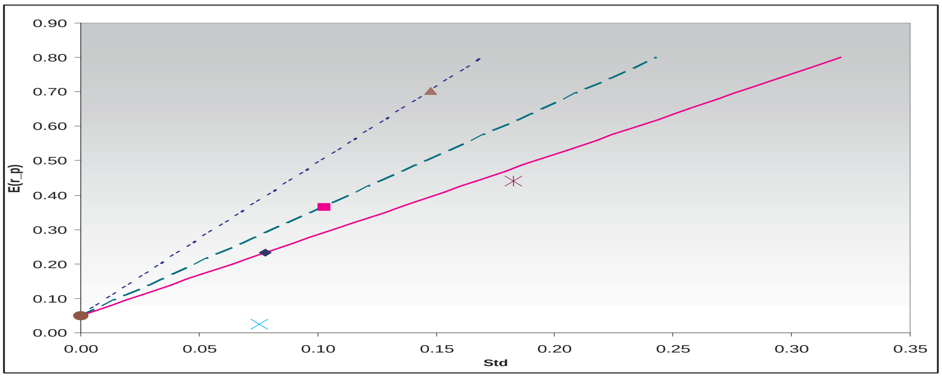
Sharpe, W. (2007), *Investors and marekts, portfolio choice, asset prices, and investment advice*, Princeton.

Varian, H. (1985), 'Divergence of opinion in complete markets', *Journal of Finance* **40**, 309–317.

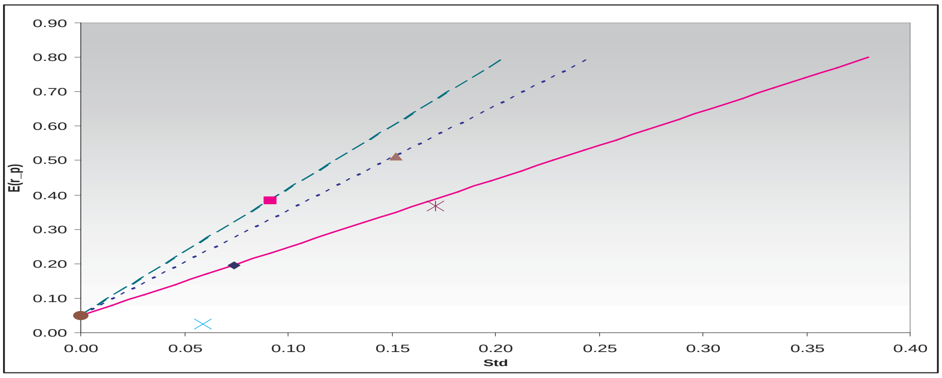
Williams, J. (1977), 'Capital asset prices with heterogeneous beliefs', *Journal of Financial Economics* **5**, 219–239.

Zapatero, F. (1998), 'Effects of financial innovations on market volatility when beliefs are heterogeneous', *Journal of Economic Dynamics and Control* **22**, 597–626.

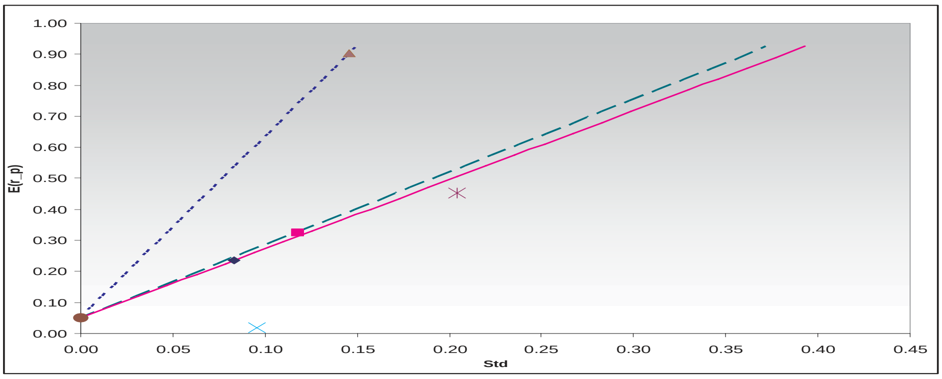
APPENDIX A. FIGURES AND TABLES



(c1) $(\theta_1, \theta_2) = (3, 3)$

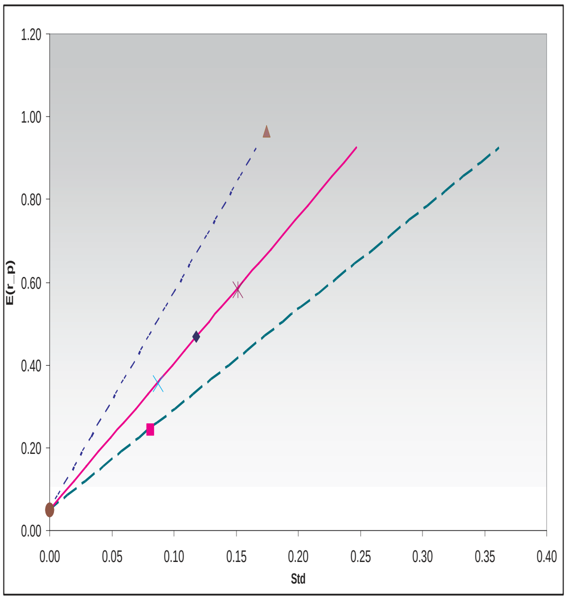


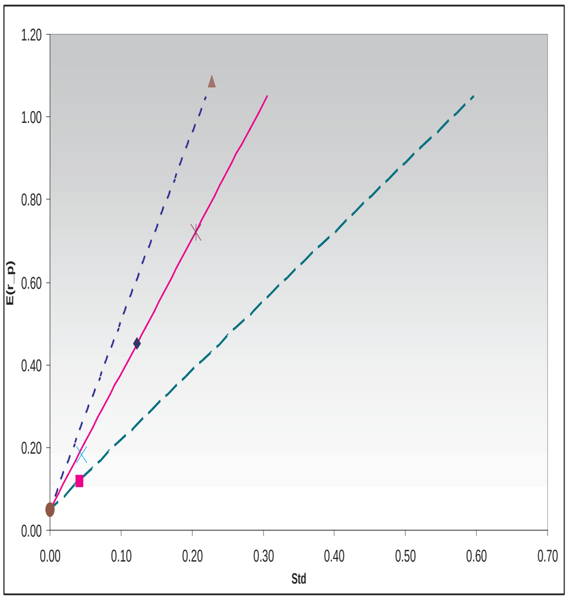
(c2) $(\theta_1, \theta_2) = (4, 2)$



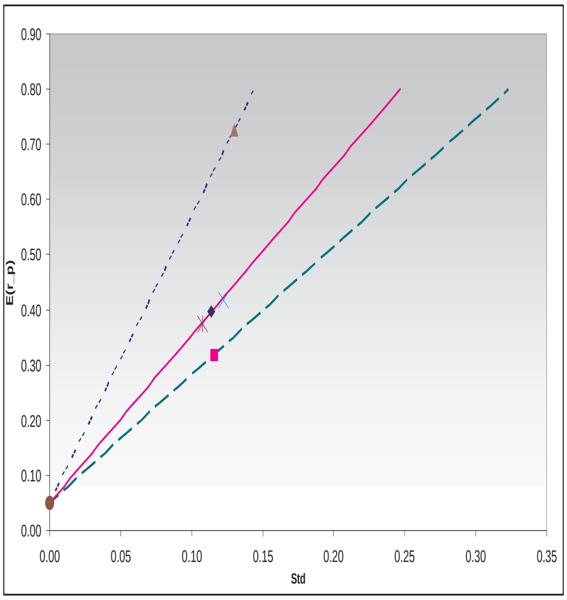
(c3) $(\theta_1, \theta_2) = (2, 4)$

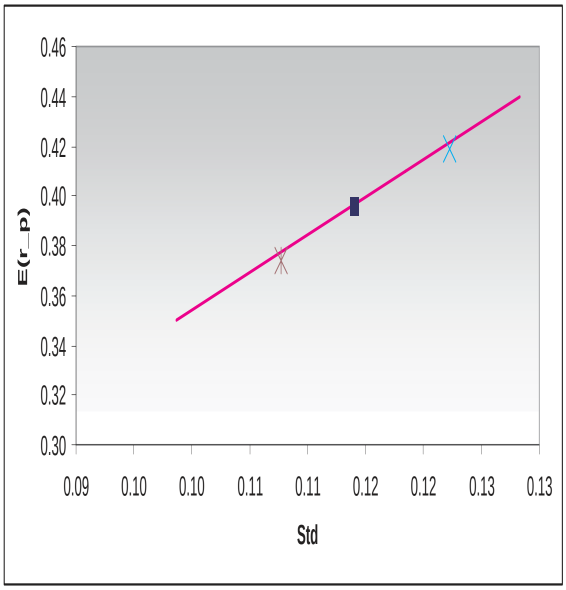
FIGURE A.1. Mean-Variance efficient frontiers, optimal portfolios and market portfolio under heterogeneous and consensus beliefs for $\mathbf{y}_1 < \mathbf{y}_2, \Omega_2 < \Omega_1$.





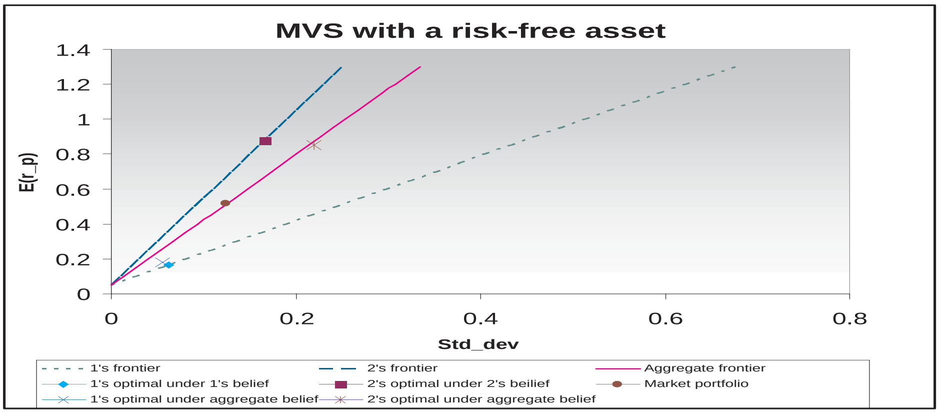
(d1) $\theta_1 = \theta_2$ (d2) $\theta_1 > \theta_2$, ,



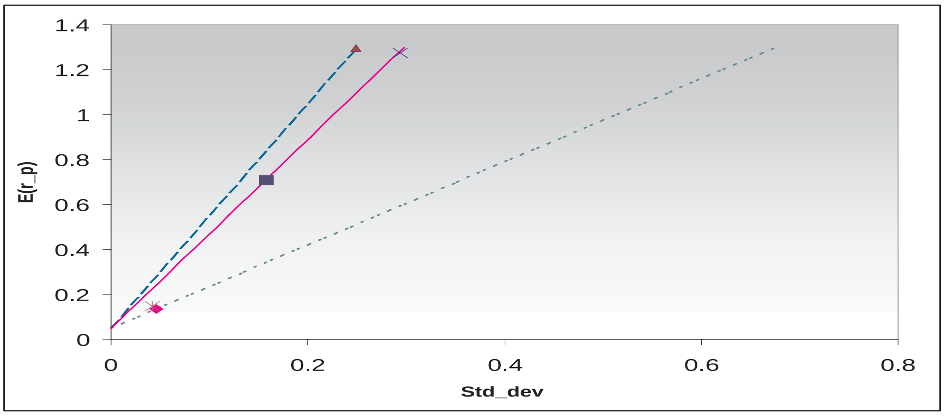


(d3) $\theta_1 < \theta_2$

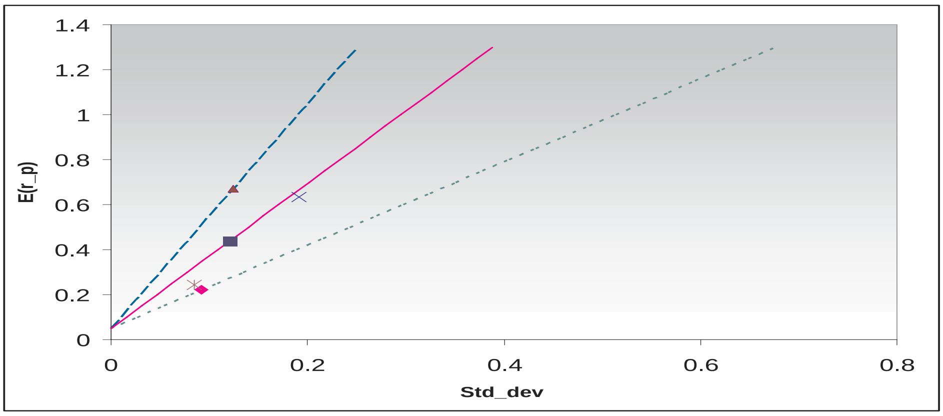
FIGURE A.2. Mean-Variance efficient frontiers, optimal portfolios and market portfolio under heterogeneous and consensus beliefs for $\mathbf{y}_1 < \mathbf{y}_2, \Omega_1 < \Omega_2$.



(c1) $(\theta_1, \theta_2) = (3, 3)$

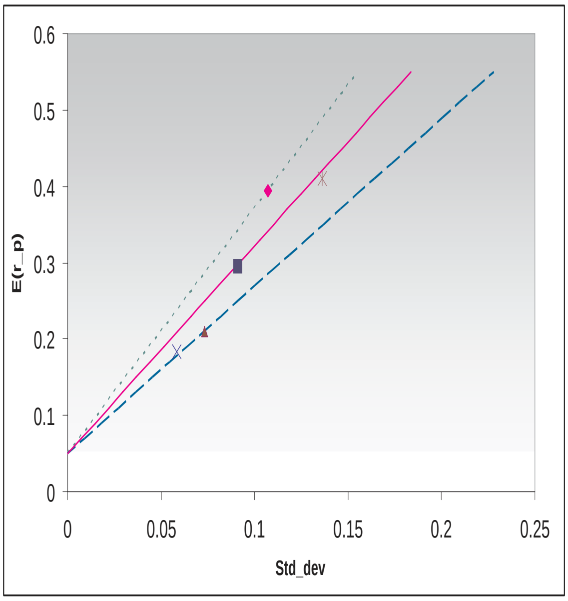


(c2) $(\theta_1, \theta_2) = (4, 2)$

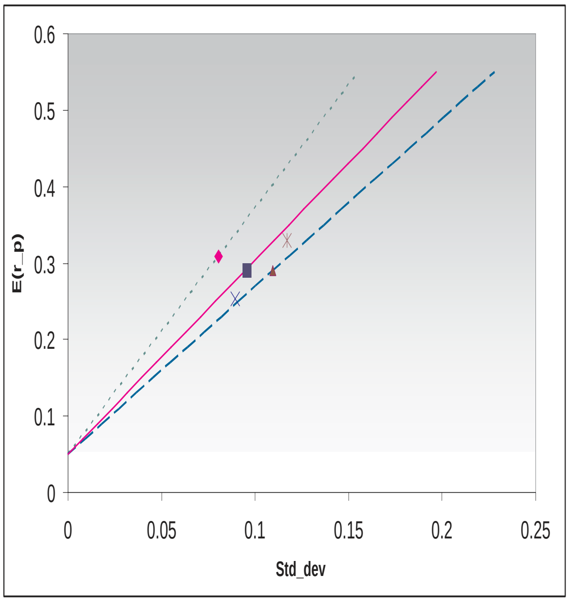


(c3) $(\theta_1, \theta_2) = (2, 4)$

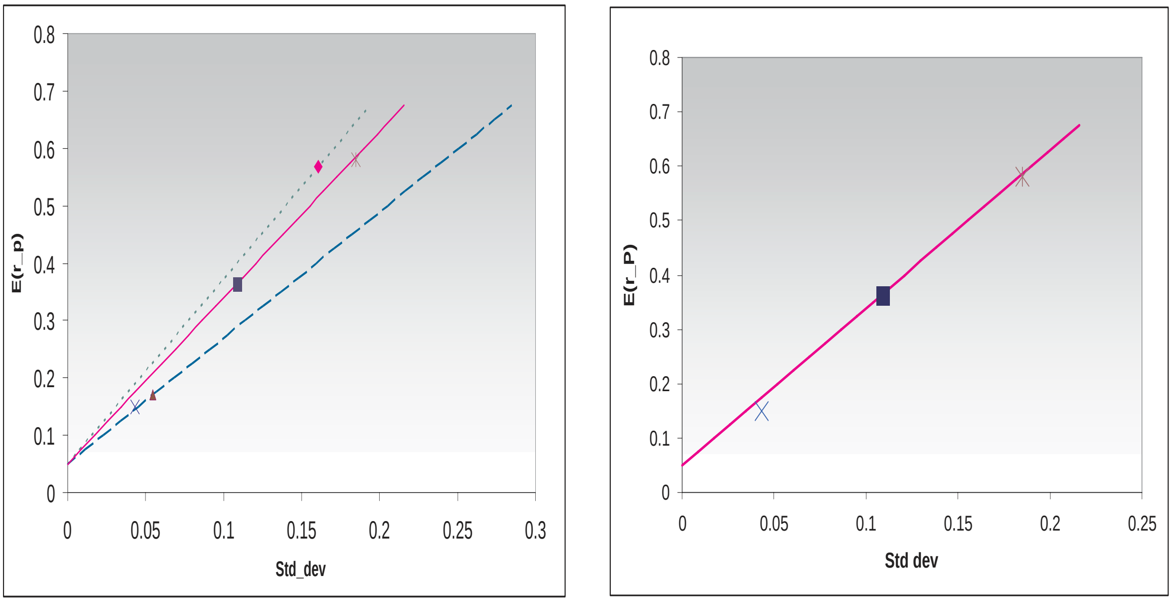
FIGURE A.3. Mean-Variance efficient frontiers, optimal portfolios and market portfolio under heterogeneous and consensus beliefs for $\boldsymbol{\mu}_1 < \boldsymbol{\mu}_2, V_2 < V_1$.



(d1) $\theta_1 = \theta_2$



(d2) $\theta_1 > \theta_2$, ,



(d3) $\theta_1 < \theta_2$

FIGURE A.4. Mean-Variance efficient frontiers, optimal portfolios and market portfolio under heterogeneous and consensus beliefs for $\boldsymbol{\mu}_1 < \boldsymbol{\mu}_2, V_1 < V_2$.

$\mathbb{E}_a(\tilde{r})$							β				
Mean							**Mean**				
$(\sigma_{\theta_1}, \sigma_{\theta_2})$	**A**	**B**	**C**	**M**	**O1**	**O2**	**A**	**B**	**C**	**O1**	**O2**
$(5\%, 5\%)$	0.3638	0.2688	0.7092	0.3947	0.3945	0.3949	0.9098	0.6344	1.9107	0.9995	1.0005
$(5\%, 10\%)$	0.3626	0.268	0.7069	0.3936	0.3936	0.3935	0.9091	0.6341	1.9081	0.9982	1.0018
$(5\%, 15\%)$	0.3617	0.2673	0.7057	0.3925	0.3923	0.3926	0.9085	0.6337	1.9067	0.9943	1.0057
$(0\%, 10\%)$	0.3629	0.2682	0.7075	0.3939	0.3938	0.394	0.9093	0.6342	1.9088	0.9973	1.0027
$(0\%, 15\%)$	0.3623	0.2677	0.7069	0.393	0.3932	0.3928	0.9089	0.6339	1.9081	0.9948	1.0052
SD							**SD**				
$(5\%, 5\%)$	0.0145	0.0094	0.0382	0.0123	0.0173	0.0172	0.0097	0.0047	0.0427	0.0353	0.0353
$(5\%, 10\%)$	0.0229	0.0149	0.0602	0.0194	0.0344	0.0175	0.0153	0.0074	0.0673	0.0566	0.0566
$(5\%, 15\%)$	0.0326	0.0212	0.0854	0.0279	0.0517	0.0174	0.0218	0.0106	0.0955	0.0805	0.0805
$(0\%, 10\%)$	0.0205	0.0133	0.0536	0.0174	0.0345	0.0013	0.0136	0.0066	0.06	0.0504	0.0504
$(0\%, 15\%)$	0.0307	0.0199	0.08	0.0262	0.0514	0.0031	0.0205	0.01	0.0895	0.0761	0.0761
Skew							**Skew**				
$(5\%, 5\%)$	0.0876	0.0685	0.1574	0.0246	-0.0488	-0.0904	0.0876	0.0685	0.1574	0.0323	-0.0323
$(5\%, 10\%)$	-0.0038	-0.0345	0.1079	-0.1059	-0.138	-0.1187	-0.0038	-0.0345	0.1079	-0.1842	0.1842
$(5\%, 15\%)$	-0.1489	-0.1953	0.018	-0.3042	-0.2025	-0.0526	-0.1489	-0.1953	0.018	-0.3648	0.3648
$(0\%, 10\%)$	-0.1717	-0.1987	-0.0748	-0.2615	-0.1095	-3.4536	-0.1717	-0.1987	-0.0748	-0.2615	0.2615
$(0\%, 15\%)$	-0.3033	-0.3448	-0.1563	-0.4434	-0.2027	-3.9927	-0.3033	-0.3448	-0.1563	-0.4434	0.4434
ExKurt							**ExKurt**				
$(5\%, 5\%)$	-0.0117	-0.0182	0.0244	-0.0274	0.0356	0.0302	-0.0117	-0.0182	0.0244	0.0249	0.0249
$(5\%, 10\%)$	0.0168	0.0214	0.0304	0.0459	-0.0486	0.1198	0.0168	0.0214	0.0304	0.064	0.064
$(5\%, 15\%)$	0.1524	0.1808	0.1235	0.2827	-0.0339	0.0887	0.1524	0.1808	0.1235	0.1737	0.1737
$(0\%, 10\%)$	0.004	0.0233	-0.0405	0.0804	-0.045	21.6489	0.004	0.0233	-0.0405	0.0804	0.0804
$(0\%, 15\%)$	0.1147	0.166	-0.0114	0.3156	-0.017	27.3091	0.1147	0.166	-0.0114	0.3156	0.3156
JB-PV							**JB-PV**				
$(5\%, 5\%)$	0.0016	0.0187	0	0.4843	0.106	0.0009	0.0016	0.0187	0	0.3677	0.3677
$(5\%, 10\%)$	0.0679	0.3366	0	0	0	0	0.0679	0.3366	0	0	0
$(5\%, 15\%)$	0	0	0.0319	0	0	0.0193	0	0	0.0319	0	0
$(0\%, 10\%)$	0	0	0.0067	0	0	0	0	0	0.0067	0	0
$(0\%, 15\%)$	0	0	0	0	0	0	0	0	0	0	0
SR											
$(5\%, 5\%)$	1.9106	1.8357	2.5774	3.2153	3.2153	3.2153					
$(5\%, 10\%)$	1.9042	1.8296	2.5688	3.2045	3.2045	3.2045					
$(5\%, 15\%)$	1.8981	1.8237	2.5606	3.1943	3.1943	3.1943					
$(0\%, 10\%)$	1.9061	1.8314	2.5713	3.2076	3.2076	3.2076					
$(0\%, 15\%)$	1.9012	1.8267	2.5648	3.1995	3.1995	3.1995					
TR											
$(5\%, 5\%)$	0.3447	0.3447	0.3447	0.3447	0.3447	0.3447					
$(5\%, 10\%)$	0.3436	0.3436	0.3436	0.3436	0.3436	0.3436					
$(5\%, 15\%)$	0.3425	0.3425	0.3425	0.3425	0.3425	0.3425					
$(0\%, 10\%)$	0.3439	0.3439	0.3439	0.3439	0.3439	0.3439					
$(0\%, 15\%)$	0.343	0.343	0.343	0.343	0.343	0.343					

TABLE 1. Impact of risk-aversion coefficients $\theta_i \sim \mathcal{N}(\theta_o, \theta_o \times \sigma_{\theta_i})$ for the payoff setup with $\mathbf{y_1} = \mathbf{y}_2 = \mathbf{y}_o$ and $\Omega_1 = \Omega_2 = \Omega_o$.

$\mathbb{E}_a(\tilde{r})$							β				
Mean							**Mean**				
$(\sigma_{\delta_1}, \sigma_{\delta_2})$	**A**	**B**	**C**	**M**	**O1**	**O2**	**A**	**B**	**C**	**O1**	**O2**
$(1\%, 1\%)$	0.3638	0.2688	0.7087	0.3948	0.3948	0.3949	0.9099	0.6345	1.9102	0.9998	1.0002
$(1\%, 2\%)$	0.3638	0.2688	0.7089	0.3948	0.3948	0.3948	0.9101	0.6346	1.9107	1	1
$(1\%, 3\%)$	0.3639	0.2689	0.7089	0.3948	0.3947	0.395	0.9101	0.6348	1.9108	0.9997	1.0003
$(0\%, 2\%)$	0.3639	0.2688	0.7087	0.3948	0.3948	0.3949	0.9102	0.6345	1.91	0.9998	1.0002
$(0\%, 3\%)$	0.3639	0.2688	0.7092	0.3948	0.3948	0.3949	0.9102	0.6345	1.9115	0.9999	1.0001
SD							**SD**				
$(1\%, 1\%)$	0.0029	0.0019	0.0077	0	0.0053	0.0053	0.0084	0.0054	0.0223	0.0154	0.0154
$(1\%, 2\%)$	0.0046	0.0029	0.0118	0	0.0084	0.0084	0.0133	0.0085	0.0343	0.0243	0.0243
$(1\%, 3\%)$	0.0064	0.0042	0.0172	0	0.0119	0.0119	0.0186	0.0121	0.0498	0.0346	0.0346
$(0\%, 2\%)$	0.0042	0.0027	0.0108	0	0.0076	0.0076	0.0121	0.0077	0.0314	0.022	0.022
$(0\%, 3\%)$	0.0062	0.004	0.0163	0	0.0113	0.0113	0.018	0.0115	0.0471	0.0329	0.0329
Skew							**Skew**				
$(1\%, 1\%)$	0.0549	0.0235	0.0692	n/a	0.008	-0.008	0.0549	0.0235	0.0692	0.008	-0.008
$(1\%, 2\%)$	0.0996	0.0857	0.0894	n/a	0.0371	-0.0371	0.0996	0.0857	0.0894	0.0371	-0.0371
$(1\%, 3\%)$	0.1148	0.0919	0.1625	n/a	-0.0042	0.0042	0.1148	0.0919	0.1625	-0.0042	0.0042
$(0\%, 2\%)$	0.064	0.0688	0.1087	n/a	-0.002	0.002	0.064	0.0688	0.1087	-0.002	0.002
$(0\%, 3\%)$	0.1058	0.1173	0.1304	n/a	-0.0014	0.0014	0.1058	0.1173	0.1304	-0.0014	0.0014
ExKurt							**ExKurt**				
$(1\%, 1\%)$	0.0553	-0.0456	0.0639	n/a	0.0156	0.0156	0.0553	-0.0456	0.0639	0.0156	0.0156
$(1\%, 2\%)$	0.1327	-0.0403	0.0566	n/a	-0.0526	-0.0526	0.1327	-0.0403	0.0566	-0.0526	-0.0526
$(1\%, 3\%)$	0.0386	0.0981	0.0152	n/a	-0.0024	-0.0024	0.0386	0.0981	0.0152	-0.0024	-0.0024
$(0\%, 2\%)$	-0.08	0.0596	0.0316	n/a	0.0104	0.0104	-0.08	0.0596	0.0316	0.0104	0.0104
$(0\%, 3\%)$	-0.0216	0.0687	-0.0443	n/a	0.0787	0.0787	-0.0216	0.0687	-0.0443	0.0787	0.0787
JB-PV							**JB-PV**				
$(1\%, 1\%)$	0.0431	0.41	0.0079	n/a	0.0989	0.0989	0.0431	0.41	0.0079	0.0989	0.0989
$(1\%, 2\%)$	0	0.0016	0.0007	n/a	0.1779	0.1779	0	0.0016	0.0007	0.1779	0.1779
$(1\%, 3\%)$	0	0.0001	0	n/a	0.0157	0.0157	0	0.0001	0	0.0157	0.0157
$(0\%, 2\%)$	0.0087	0.0092	0	n/a	0.0253	0.0253	0.0087	0.0092	0	0.0253	0.0253
$(0\%, 3\%)$	0	0	0	n/a	0.2747	0.2747	0	0	0	0.2747	0.2747
SR											
$(1\%, 1\%)$	1.9113	1.8364	2.5784	3.2164	3.2149	3.2149					
$(1\%, 2\%)$	1.9113	1.8364	2.5784	3.2164	3.2126	3.2126					
$(1\%, 3\%)$	1.9113	1.8364	2.5784	3.2164	3.2089	3.209					
$(0\%, 2\%)$	1.9113	1.8364	2.5784	3.2164	3.2133	3.2133					
$(0\%, 3\%)$	1.9113	1.8364	2.5784	3.2164	3.2096	3.2096					
TR											
$(1\%, 1\%)$	0.3448	0.3448	0.3448	0.3448	0.3448	0.3448					
$(1\%, 2\%)$	0.3448	0.3448	0.3448	0.3448	0.3448	0.3448					
$(1\%, 3\%)$	0.3448	0.3448	0.3448	0.3448	0.3448	0.3448					
$(0\%, 2\%)$	0.3448	0.3448	0.3448	0.3448	0.3448	0.3448					
$(0\%, 3\%)$	0.3448	0.3448	0.3448	0.3448	0.3448	0.3448					

TABLE 2. Impact of heterogeneous beliefs in expected asset payoffs $\mathbf{y}_i \sim \mathbf{y}_o[1 + \sigma_{\delta_i}\mathcal{N}(0,1)], \quad i = 1,2$ with $\theta_1 = \theta_2 = 3$ and $\Omega_1 = \Omega_2 = \Omega_o$.

$\mathbb{E}_a(\tilde{r})$							β				
Mean							**Mean**				
$(\sigma_{\epsilon_1}, \sigma_{\epsilon_2})$	**A**	**B**	**C**	**M**	**O1**	**O2**	**A**	**B**	**C**	**O1**	**O2**
$(1\%, 1\%)$	0.3636	0.2687	0.7088	0.3948	0.3948	0.3948	0.9097	0.6343	1.9107	1	1
$(1\%, 2\%)$	0.3636	0.2688	0.7088	0.3948	0.3948	0.3947	0.9097	0.6346	1.9104	0.9999	1.0001
$(1\%, 3\%)$	0.3635	0.2687	0.709	0.3947	0.3948	0.3945	0.9096	0.6345	1.9109	1.0001	0.9999
$(0\%, 2\%)$	0.3636	0.2687	0.709	0.3948	0.3949	0.3947	0.9096	0.6344	1.9109	1.0002	0.9998
$(0\%, 3\%)$	0.3636	0.2687	0.709	0.3947	0.3949	0.3945	0.9099	0.6344	1.9109	1.0002	0.9998
SD							**SD**				
$(1\%, 1\%)$	0.0047	0.0032	0.0142	0.0031	0.0043	0.0044	0.0119	0.0084	0.0268	0.0089	0.0089
$(1\%, 2\%)$	0.0075	0.005	0.0221	0.0048	0.0087	0.0043	0.0191	0.013	0.042	0.0142	0.0142
$(1\%, 3\%)$	0.0107	0.0071	0.0315	0.0069	0.0131	0.0044	0.0271	0.0186	0.0598	0.0201	0.0201
$(0\%, 2\%)$	0.0067	0.0046	0.0198	0.0044	0.0087	0.0001	0.017	0.012	0.0376	0.0126	0.0126
$(0\%, 3\%)$	0.0101	0.0069	0.0297	0.0065	0.013	0.0003	0.0257	0.0181	0.0568	0.0188	0.0188
Skew							**Skew**				
$(1\%, 1\%)$	0.0229	0.0806	0.0934	0.0389	0.0412	0	0.0437	0.0557	0.062	0.0101	-0.0101
$(1\%, 2\%)$	0.014	-0.0182	0.1298	0.0207	0.0274	0.0031	0.0795	-0.0456	0.1019	-0.0103	0.0103
$(1\%, 3\%)$	0.0097	-0.0024	0.0318	-0.0533	-0.0121	-0.0095	0.0472	0.0126	-0.0083	-0.0473	0.0473
$(0\%, 2\%)$	-0.0193	-0.0646	-0.0124	-0.0539	-0.01	-2.1302	-0.0113	-0.054	-0.0348	-0.0407	0.0407
$(0\%, 3\%)$	-0.0662	-0.0568	0.0632	0.0007	0.0666	-2.0914	-0.0138	-0.0538	0.0229	0.0174	-0.0174
ExKurt							**ExKurt**				
$(1\%, 1\%)$	-0.021	0.0632	0.0066	-0.0069	-0.0111	-0.0007	-0.0415	0.0529	0.0304	0.0568	0.0568
$(1\%, 2\%)$	0.0241	0.0115	-0.0095	-0.0976	-0.0559	-0.0585	0.0039	0.0464	-0.0098	-0.0314	-0.0314
$(1\%, 3\%)$	-0.0446	0.0667	0.0214	0.0484	-0.0102	0.0498	0.0075	-0.0081	-0.0331	-0.0543	-0.0543
$(0\%, 2\%)$	0.048	-0.0708	0.0092	-0.0284	-0.0425	8.1368	0.0309	-0.0279	0.0219	-0.0406	-0.0406
$(0\%, 3\%)$	0.046	-0.0302	-0.0209	0.0377	0.0426	7.9146	-0.0434	-0.0263	-0.0579	0.0345	0.0345
JB-PV							**JB-PV**				
$(1\%, 1\%)$	0.4109	0.0019	0.0007	0.2809	0.2378	0.0001	0.1421	0.0421	0.0335	0.4683	0.4683
$(1\%, 2\%)$	0.2478	0.2626	0	0.0962	0.2788	0.4863	0.0051	0.1127	0.0002	0.2542	0.2542
$(1\%, 3\%)$	0.3889	0.3938	0.3914	0.0573	0.1341	0.4466	0.154	0.1354	0.2493	0.0841	0.0841
$(0\%, 2\%)$	0.4541	0.0108	0.135	0.0751	0.3689	0	0.2631	0.0749	0.3293	0.1788	0.1788
$(0\%, 3\%)$	0.0168	0.0563	0.0328	0.2567	0.017	0	0.423	0.0776	0.3214	0.3939	0.3939
SR											
$(1\%, 1\%)$	1.911	1.836	2.5783	3.2161	3.2159	3.2159					
$(1\%, 2\%)$	1.9108	1.836	2.5779	3.2159	3.2154	3.2154					
$(1\%, 3\%)$	1.9103	1.8356	2.5776	3.2155	3.2144	3.2144					
$(0\%, 2\%)$	1.9109	1.8361	2.5783	3.2162	3.2158	3.2158					
$(0\%, 3\%)$	1.9106	1.8356	2.5778	3.2157	3.2147	3.2147					
TR											
$(1\%, 1\%)$	0.3448	0.3448	0.3448	0.3448	0.3448	0.3448					
$(1\%, 2\%)$	0.3448	0.3448	0.3448	0.3448	0.3448	0.3448					
$(1\%, 3\%)$	0.3447	0.3447	0.3447	0.3447	0.3447	0.3447					
$(0\%, 2\%)$	0.3448	0.3448	0.3448	0.3448	0.3448	0.3448					
$(0\%, 3\%)$	0.3447	0.3447	0.3447	0.3447	0.3447	0.3447					

TABLE 3. Impact of heterogeneous beliefs in variance/covariance of asset payoffs $\Omega_i = D_i C D_i, D_i = D_o \times (\epsilon_i + 1) I, \epsilon_i \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_{\epsilon_i}), D_o = diag(\sigma_{o1}, \sigma_{o2}, \sigma_{o3}), \sigma_{ij} = \sigma_{oj}(1 + \epsilon_i) \overset{\text{iid}}{\sim} \mathcal{N}(\sigma_{oj}, \sigma_{\epsilon_i}^2 \sigma_{oj}^2)$ with $\mathbf{y}_i = \mathbf{y}_o$ and $\theta_1 = \theta_2 = 3$.

$\mathbb{E}_a(\tilde{r})$							β				
Mean							**Mean**				
$(\sigma_{\theta_1}, \sigma_{\theta_2})$	**A**	**B**	**C**	**M**	**O1**	**O2**	**A**	**B**	**C**	**O1**	**O2**
$(5\%, 5\%)$	0.3632	0.2684	0.7079	0.3943	0.3943	0.3942	0.9095	0.6343	1.9095	1.0001	0.9999
$(5\%, 10\%)$	0.363	0.2682	0.7077	0.3938	0.3937	0.394	0.9095	0.6341	1.909	0.9976	1.0024
$(5\%, 15\%)$	0.362	0.2675	0.7066	0.3927	0.3927	0.3927	0.9087	0.6338	1.9081	0.9945	1.0055
$(0\%, 10\%)$	0.3632	0.2685	0.7085	0.3941	0.3943	0.394	0.9095	0.6344	1.9104	0.9978	1.0022
$(0\%, 15\%)$	0.3624	0.2678	0.7072	0.3931	0.3933	0.3929	0.909	0.634	1.9085	0.9946	1.0054
SD							**SD**				
$(5\%, 5\%)$	0.0147	0.0096	0.0389	0.0123	0.0179	0.0181	0.0127	0.0072	0.048	0.0384	0.0384
$(5\%, 10\%)$	0.0231	0.015	0.0607	0.0194	0.0352	0.0178	0.0174	0.0091	0.0708	0.059	0.059
$(5\%, 15\%)$	0.0329	0.0214	0.0863	0.028	0.0524	0.018	0.0234	0.0119	0.0987	0.0827	0.0827
$(0\%, 10\%)$	0.021	0.0136	0.055	0.0177	0.0355	0.0055	0.0162	0.0086	0.0647	0.0539	0.0539
$(0\%, 15\%)$	0.0309	0.0201	0.0807	0.0263	0.0519	0.0062	0.0223	0.0113	0.0923	0.0779	0.0779
Skew							**Skew**				
$(5\%, 5\%)$	0.0536	0.0274	0.122	-0.0326	-0.0627	-0.0383	0.1018	0.0648	0.1479	0.005	-0.005
$(5\%, 10\%)$	-0.0058	-0.0361	0.0961	-0.1187	-0.1522	-0.0234	0.0368	0.0226	0.1088	-0.2353	0.2353
$(5\%, 15\%)$	-0.1536	-0.1994	0.0087	-0.318	-0.2063	-0.0856	-0.0874	-0.0951	0.0326	-0.3904	0.3904
$(0\%, 10\%)$	-0.225	-0.2566	-0.1165	-0.3373	-0.1664	-0.0747	-0.0993	-0.0861	-0.061	-0.3231	0.3231
$(0\%, 15\%)$	-0.2727	-0.3118	-0.1287	-0.4204	-0.1799	-0.4489	-0.1963	-0.175	-0.1036	-0.4185	0.4185
ExKurt							**ExKurt**				
$(5\%, 5\%)$	-0.0003	0.0126	0.0277	0.0078	0.069	0.034	-0.0046	0.079	0.0248	0.0744	0.0744
$(5\%, 10\%)$	0.0157	0.0095	-0.0011	0.0269	-0.008	0.0427	0.0446	0.028	-0.0059	0.0952	0.0952
$(5\%, 15\%)$	0.1159	0.1689	0.0535	0.286	-0.0105	0.0342	0.072	0.1584	0.0617	0.2871	0.2871
$(0\%, 10\%)$	0.1378	0.1828	0.0787	0.272	0.095	0.0732	0.091	0.0606	0.0669	0.3105	0.3105
$(0\%, 15\%)$	0.091	0.1309	-0.0363	0.2921	-0.0397	1.2884	0.0558	0.0398	-0.0325	0.2803	0.2803
JB-PV							**JB-PV**				
$(5\%, 5\%)$	0.0912	0.4821	0	0.4064	0.014	0.2322	0.0002	0.0082	0	0.3088	0.3088
$(5\%, 10\%)$	0.0766	0.3321	0.0005	0	0	0.4338	0.2143	0.4448	0	0	0
$(5\%, 15\%)$	0	0	0.4827	0	0	0.0018	0.0006	0	0.1866	0	0
$(0\%, 10\%)$	0	0	0	0	0	0.0031	0	0.001	0.0178	0	0
$(0\%, 15\%)$	0	0	0	0	0	0	0	0	0.0001	0	0
SR											
$(5\%, 5\%)$	1.9081	1.8333	2.5741	3.2111	3.2096	3.2096					
$(5\%, 10\%)$	1.9056	1.8309	2.5708	3.2069	3.2054	3.2054					
$(5\%, 15\%)$	1.8995	1.8251	2.5625	3.1966	3.1951	3.1952					
$(0\%, 10\%)$	1.9074	1.8326	2.5732	3.2099	3.2084	3.2084					
$(0\%, 15\%)$	1.9016	1.827	2.5653	3.2001	3.1985	3.1985					
TR											
$(5\%, 5\%)$	0.3443	0.3443	0.3443	0.3443	0.3443	0.3443					
$(5\%, 10\%)$	0.3438	0.3438	0.3438	0.3438	0.3438	0.3438					
$(5\%, 15\%)$	0.3427	0.3427	0.3427	0.3427	0.3427	0.3427					
$(0\%, 10\%)$	0.3441	0.3441	0.3441	0.3441	0.3441	0.3441					
$(0\%, 15\%)$	0.3431	0.3431	0.3431	0.3431	0.3431	0.3431					

TABLE 4. Impact of risk-aversion coefficients with dispersion in expected payoffs: $(\sigma_{\delta_1}, \sigma_{\delta_2}) = (1\%, 1\%)$.

$\mathbb{E}_a(\tilde{r})$							β				
Mean							**Mean**				
$(\sigma_{\theta_1}, \sigma_{\theta_2})$	**A**	**B**	**C**	**M**	**O1**	**O2**	**A**	**B**	**C**	**O1**	**O2**
$(5\%, 5\%)$	0.3633	0.2685	0.7082	0.3943	0.3945	0.3942	0.9094	0.6343	1.9098	1.0003	0.9997
$(5\%, 10\%)$	0.363	0.2683	0.7079	0.3939	0.3939	0.3938	0.9092	0.6344	1.9093	0.9982	1.0018
$(5\%, 15\%)$	0.3619	0.2674	0.7061	0.3926	0.3924	0.3928	0.9087	0.6338	1.9073	0.9941	1.0059
$(0\%, 10\%)$	0.3629	0.2682	0.7074	0.3939	0.3938	0.3939	0.9094	0.6342	1.9087	0.9972	1.0028
$(0\%, 15\%)$	0.3618	0.2674	0.7058	0.3926	0.3924	0.3928	0.9084	0.6337	1.9072	0.9936	1.0064
SD							**SD**				
$(5\%, 5\%)$	0.0152	0.0099	0.0405	0.0126	0.0177	0.0178	0.0155	0.0097	0.0502	0.0364	0.0364
$(5\%, 10\%)$	0.0235	0.0153	0.0621	0.0197	0.0348	0.0177	0.0194	0.0113	0.0727	0.057	0.057
$(5\%, 15\%)$	0.0331	0.0216	0.0868	0.0281	0.052	0.018	0.0249	0.0137	0.0993	0.0812	0.0812
$(0\%, 10\%)$	0.0212	0.0138	0.0557	0.0178	0.035	0.0045	0.0186	0.0109	0.0658	0.0516	0.0516
$(0\%, 15\%)$	0.0311	0.0202	0.0812	0.0265	0.0516	0.0055	0.0238	0.013	0.0932	0.0769	0.0769
Skew							**Skew**				
$(5\%, 5\%)$	0.1003	0.0828	0.1988	0.0361	-0.0559	-0.0469	0.0837	0.0599	0.1895	0.0088	-0.0088
$(5\%, 10\%)$	0.0162	-0.0179	0.1247	-0.104	-0.1398	-0.0484	0.0749	0.0448	0.1336	-0.2182	0.2182
$(5\%, 15\%)$	-0.1522	-0.1896	0.012	-0.3136	-0.2039	-0.0719	-0.0532	-0.0175	0.0318	-0.3801	0.3801
$(0\%, 10\%)$	-0.1794	-0.204	-0.0731	-0.2924	-0.154	-0.0663	-0.0413	-0.0312	-0.0294	-0.3084	0.3084
$(0\%, 15\%)$	-0.3252	-0.3604	-0.164	-0.4889	-0.2394	-1.0135	-0.1775	-0.0822	-0.1269	-0.4973	0.4973
ExKurt							**ExKurt**				
$(5\%, 5\%)$	-0.0064	0.0337	0.1477	0.0449	0.0501	0.0136	-0.0309	-0.0008	0.1358	0.0093	0.0093
$(5\%, 10\%)$	-0.0017	-0.0011	0.0625	0.0335	0.013	-0.0716	-0.082	0.0103	0.0636	0.1301	0.1301
$(5\%, 15\%)$	0.0796	0.1013	0.0376	0.2251	-0.0115	0.118	0.022	0.0489	0.0469	0.2501	0.2501
$(0\%, 10\%)$	0.03	0.0732	0.0041	0.1449	-0.0105	0.1835	-0.0161	-0.0104	0.0073	0.1471	0.1471
$(0\%, 15\%)$	0.288	0.316	0.0898	0.5372	0.085	5.218	0.2859	0.0959	0.0728	0.5501	0.5501
JB-PV							**JB-PV**				
$(5\%, 5\%)$	0.0002	0.0026	0	0.2222	0.044	0.1535	0.0024	0.0504	0	0.0789	0.0789
$(5\%, 10\%)$	0.1966	0.2351	0	0	0	0.0489	0.0023	0.1831	0	0	0
$(5\%, 15\%)$	0	0	0.3389	0	0	0.0007	0.0854	0.4707	0.2729	0	0
$(0\%, 10\%)$	0	0	0.0116	0	0	0	0.228	0.4343	0.4821	0	0
$(0\%, 15\%)$	0	0	0	0	0	0	0	0.0005	0	0	0
SR											
$(5\%, 5\%)$	1.9083	1.8336	2.5747	3.2117	3.2115	3.2115					
$(5\%, 10\%)$	1.9059	1.8315	2.5713	3.2077	3.2074	3.2074					
$(5\%, 15\%)$	1.8987	1.8243	2.5615	3.1954	3.1952	3.1952					
$(0\%, 10\%)$	1.9059	1.8312	2.571	3.2073	3.2071	3.2071					
$(0\%, 15\%)$	1.8989	1.8245	2.5619	3.1959	3.1956	3.1956					
TR											
$(5\%, 5\%)$	0.3443	0.3443	0.3443	0.3443	0.3443	0.3443					
$(5\%, 10\%)$	0.3439	0.3439	0.3439	0.3439	0.3439	0.3439					
$(5\%, 15\%)$	0.3426	0.3426	0.3426	0.3426	0.3426	0.3426					
$(0\%, 10\%)$	0.3439	0.3439	0.3439	0.3439	0.3439	0.3439					
$(0\%, 15\%)$	0.3426	0.3426	0.3426	0.3426	0.3426	0.3426					

TABLE 5. Impact of risk-aversion coefficients with dispersion in covariance matrices of asset payoffs: $(\sigma_{\epsilon_1}, \sigma_{\epsilon_2}) = (1\%, 1\%)$.

$\mathbb{E}_a(\tilde{r})$							β				
Mean							**Mean**				
$(\sigma_{\theta_1}, \sigma_{\theta_2})$	**A**	**B**	**C**	**M**	**O1**	**O2**	**A**	**B**	**C**	**O1**	**O2**
$(5\%, 5\%)$	0.3636	0.2687	0.7087	0.3945	0.3946	0.3945	0.9099	0.6346	1.91	1.0001	0.9999
$(5\%, 10\%)$	0.3626	0.268	0.7069	0.3935	0.3936	0.3935	0.909	0.6343	1.908	0.9981	1.0019
$(5\%, 15\%)$	0.3622	0.2678	0.7072	0.3929	0.3931	0.3927	0.9088	0.6342	1.9086	0.9954	1.0046
$(0\%, 10\%)$	0.3632	0.2684	0.7081	0.3941	0.3942	0.3939	0.9097	0.6344	1.9096	0.9979	1.0021
$(0\%, 15\%)$	0.3624	0.2678	0.7076	0.3931	0.3936	0.3926	0.9091	0.6341	1.9095	0.9956	1.0044
SD							**SD**				
$(5\%, 5\%)$	0.0154	0.01	0.0407	0.0125	0.0184	0.0185	0.0175	0.011	0.0539	0.0395	0.0395
$(5\%, 10\%)$	0.0238	0.0155	0.0628	0.0199	0.0352	0.0185	0.0214	0.0125	0.0762	0.0589	0.0589
$(5\%, 15\%)$	0.0331	0.0216	0.0872	0.028	0.052	0.0185	0.0263	0.0147	0.1019	0.0819	0.0819
$(0\%, 10\%)$	0.0212	0.0138	0.0557	0.0176	0.0351	0.0069	0.0201	0.012	0.069	0.0537	0.0537
$(0\%, 15\%)$	0.0317	0.0206	0.0829	0.0268	0.0526	0.0076	0.0254	0.0143	0.0974	0.0793	0.0793
Skew							**Skew**				
$(5\%, 5\%)$	0.081	0.0596	0.1381	-0.0113	-0.0474	-0.0628	0.0617	0.0726	0.1316	0.0442	-0.0442
$(5\%, 10\%)$	-0.0125	-0.0441	0.1091	-0.1409	-0.1538	-0.0434	0.0797	0.053	0.154	-0.2237	0.2237
$(5\%, 15\%)$	-0.1309	-0.168	0.0562	-0.2925	-0.2127	-0.0296	-0.0118	0.0022	0.1084	-0.4102	0.4102
$(0\%, 10\%)$	-0.1549	-0.1841	-0.0554	-0.28	-0.1317	0.0194	0.015	0.0162	0.0153	-0.2676	0.2676
$(0\%, 15\%)$	-0.2899	-0.3459	-0.1216	-0.4724	-0.2188	-0.3453	-0.0787	-0.0766	-0.0417	-0.4743	0.4743
ExKurt							**ExKurt**				
$(5\%, 5\%)$	0.0263	0.0435	0.0704	0.0284	0.0305	0.023	0.0524	0.0185	0.1022	0.0761	0.0761
$(5\%, 10\%)$	0.0149	0.0572	0.0516	0.0748	0.0144	-0.0389	-0.0485	0.0443	0.0569	0.1537	0.1537
$(5\%, 15\%)$	0.0749	0.1168	0.0685	0.2257	0.0111	0.1287	0.0176	-0.0075	0.0838	0.2837	0.2837
$(0\%, 10\%)$	0.0461	0.025	-0.0512	0.1009	-0.0389	0.0621	0.1163	0.0009	-0.0578	0.1041	0.1041
$(0\%, 15\%)$	0.2614	0.3263	0.0895	0.5188	0.0829	1.3969	0.1167	0.0802	0.0512	0.5243	0.5243
JB-PV							**JB-PV**				
$(5\%, 5\%)$	0.0037	0.0351	0	0.2409	0.1271	0.0335	0.0238	0.0115	0	0.0588	0.0588
$(5\%, 10\%)$	0.1615	0.0997	0	0	0	0.1525	0.0031	0.0638	0	0	0
$(5\%, 15\%)$	0	0	0.0271	0	0	0.0153	0.1655	0.0158	0	0	0
$(0\%, 10\%)$	0	0	0.0449	0	0	0.327	0.0497	0.1975	0.4102	0	0
$(0\%, 15\%)$	0	0	0	0	0	0	0.0003	0.002	0.1357	0	0
SR											
$(5\%, 5\%)$	1.9098	1.835	2.5762	3.2139	3.2122	3.2122					
$(5\%, 10\%)$	1.9041	1.8296	2.5688	3.2045	3.2028	3.2029					
$(5\%, 15\%)$	1.9004	1.8262	2.5638	3.1984	3.1966	3.1967					
$(0\%, 10\%)$	1.907	1.8324	2.5726	3.2093	3.2076	3.2076					
$(0\%, 15\%)$	1.9016	1.8271	2.5654	3.2003	3.1985	3.1986					
TR											
$(5\%, 5\%)$	0.3445	0.3445	0.3445	0.3445	0.3445	0.3445					
$(5\%, 10\%)$	0.3435	0.3435	0.3435	0.3435	0.3435	0.3435					
$(5\%, 15\%)$	0.3429	0.3429	0.3429	0.3429	0.3429	0.3429					
$(0\%, 10\%)$	0.3441	0.3441	0.3441	0.3441	0.3441	0.3441					
$(0\%, 15\%)$	0.3431	0.3431	0.3431	0.3431	0.3431	0.3431					

TABLE 6. Impact of risk-aversion coefficients with dispersion in both expected payoffs and covariance matrices of asset payoffs: $(\sigma_{\delta_1}, \sigma_{\delta_2}) = (1\%, 1\%)$ and $(\sigma_{\epsilon_1}, \sigma_{\epsilon_2}) = (1\%, 1\%)$.

$\mathbb{E}_a(\tilde{r})$							β				
Mean							**Mean**				
$(\sigma_{\delta_1}, \sigma_{\delta_2})$	**A**	**B**	**C**	**M**	**O1**	**O2**	**A**	**B**	**C**	**O1**	**O2**
$(1\%, 1\%)$	0.3633	0.2687	0.7087	0.3952	0.3952	0.3952	0.9079	0.6336	1.9088	1	1
$(1\%, 2\%)$	0.3633	0.2687	0.7089	0.3956	0.3951	0.3961	0.9071	0.6327	1.9079	0.999	1.001
$(1\%, 3\%)$	0.3632	0.2686	0.7087	0.396	0.3951	0.3968	0.9063	0.6318	1.9073	0.9985	1.0015
$(0\%, 2\%)$	0.3632	0.2686	0.7089	0.3954	0.395	0.3959	0.9071	0.6329	1.9088	0.999	1.001
$(0\%, 3\%)$	0.3636	0.2686	0.7088	0.3961	0.395	0.3972	0.9069	0.6316	1.9066	0.9979	1.0021
SD							**SD**				
$(1\%, 1\%)$	0.0069	0.007	0.0071	0.0075	0.0083	0.0084	0.0204	0.0146	0.0346	0.0107	0.0107
$(1\%, 2\%)$	0.0112	0.0112	0.0112	0.0118	0.0096	0.0161	0.0326	0.0233	0.055	0.0171	0.0171
$(1\%, 3\%)$	0.0158	0.0159	0.0159	0.0167	0.0112	0.024	0.0467	0.0328	0.0773	0.0242	0.0242
$(0\%, 2\%)$	0.01	0.01	0.0099	0.0106	0.0053	0.0159	0.0293	0.0204	0.0492	0.0153	0.0153
$(0\%, 3\%)$	0.0152	0.0149	0.015	0.0157	0.0078	0.0236	0.044	0.0313	0.0729	0.0227	0.0227
Skew							**Skew**				
$(1\%, 1\%)$	0.003	-0.0197	0.0301	0.0414	0.0286	0.0655	-0.0174	-0.0575	0.0666	-0.0168	0.0168
$(1\%, 2\%)$	0.0295	0.0027	-0.0007	0.0466	0.0547	0.0693	-0.0028	-0.1246	0.0796	0.0046	-0.0046
$(1\%, 3\%)$	-0.0022	0.0525	-0.0477	0.1451	0.0839	0.177	-0.0385	-0.1504	0.0054	0.0388	-0.0388
$(0\%, 2\%)$	0.0165	0.0134	0.0075	0.0775	0.0016	0.1028	-0.0239	-0.0868	0.0254	0.0315	-0.0315
$(0\%, 3\%)$	0.0391	0.0288	0.0009	0.1075	-0.0014	0.1441	-0.0148	-0.1772	0.0157	0.052	-0.052
ExKurt							**ExKurt**				
$(1\%, 1\%)$	0.0923	-0.068	-0.0733	-0.0524	-0.039	-0.005	-0.0015	-0.0135	-0.0579	-0.095	-0.095
$(1\%, 2\%)$	-0.0355	-0.0033	-0.1005	0	-0.0721	0.0286	0.015	-0.0211	0.004	0.0655	0.0655
$(1\%, 3\%)$	0.0263	-0.0374	0.0024	-0.0155	-0.1174	0.0571	-0.042	-0.0918	0.0084	0.0431	0.0431
$(0\%, 2\%)$	0.1105	0.0055	0.0123	0.0348	0.0185	0.0453	-0.0535	0.0916	0.0446	0.0049	0.0049
$(0\%, 3\%)$	0.0278	0.0972	-0.034	-0.0169	-0.0628	0.0095	-0.0255	0.0684	0.0433	-0.0038	-0.0038
JB-PV							**JB-PV**				
$(1\%, 1\%)$	0.1684	0.2757	0.1533	0.1356	0.3691	0.028	0.2239	0.0615	0.0124	0.1205	0.1205
$(1\%, 2\%)$	0.3718	0.0084	0.1221	0.1643	0.028	0.0154	0.052	0	0.0051	0.4017	0.4017
$(1\%, 3\%)$	0.1382	0.0752	0.15	0	0.0002	0	0.2012	0	0.0385	0.1939	0.1939
$(0\%, 2\%)$	0.0628	0.1446	0.0751	0.0052	0.0704	0	0.3428	0.0003	0.3866	0.4361	0.4361
$(0\%, 3\%)$	0.2385	0.0702	0.2144	0	0.4387	0	0.2721	0	0.449	0.105	0.105
SR											
$(1\%, 1\%)$	1.9114	1.8377	2.5783	3.2178	3.2169	3.2169					
$(1\%, 2\%)$	1.9116	1.8376	2.5791	3.2196	3.2175	3.2175					
$(1\%, 3\%)$	1.9104	1.8369	2.5786	3.2207	3.2165	3.2165					
$(0\%, 2\%)$	1.9107	1.8371	2.5793	3.2188	3.2172	3.2172					
$(0\%, 3\%)$	1.9129	1.8367	2.579	3.2214	3.2176	3.2176					
TR											
$(1\%, 1\%)$	0.3452	0.3452	0.3452	0.3452	0.3452	0.3452					
$(1\%, 2\%)$	0.3456	0.3456	0.3456	0.3456	0.3456	0.3456					
$(1\%, 3\%)$	0.346	0.346	0.346	0.346	0.346	0.346					
$(0\%, 2\%)$	0.3454	0.3454	0.3454	0.3454	0.3454	0.3454					
$(0\%, 3\%)$	0.3461	0.3461	0.3461	0.3461	0.3461	0.3461					

TABLE 7. Impact of heterogeneous beliefs in expected returns on market equilibrium.

$\mathbb{E}_a(\tilde{r})$							β				
Mean							**Mean**				
$(\sigma_{\epsilon_1}, \sigma_{\epsilon_2})$	**A**	**B**	**C**	**M**	**O1**	**O2**	**A**	**B**	**C**	**O1**	**O2**
$(1\%, 1\%)$	0.3633	0.2686	0.7087	0.3989	0.399	0.3988	0.9	0.628	1.8923	1.0003	0.9997
$(1\%, 2\%)$	0.3633	0.2686	0.7087	0.4062	0.3989	0.4135	0.8854	0.6178	1.8615	0.9839	1.0161
$(1\%, 3\%)$	0.3633	0.2686	0.7087	0.54	0.3988	0.6812	0.8597	0.5998	1.8075	0.9553	1.0447
$(0\%, 2\%)$	0.3633	0.2686	0.7087	0.4035	0.3949	0.4121	0.8912	0.6218	1.8738	0.9811	1.0189
$(0\%, 3\%)$	0.3633	0.2686	0.7087	0.4656	0.3949	0.5362	0.8675	0.6053	1.8239	0.955	1.045
SD							**SD**				
$(1\%, 1\%)$	0	0	0	0.0164	0.0228	0.0233	0.0421	0.0294	0.0884	0.046	0.046
$(1\%, 2\%)$	0	0	0	0.03	0.023	0.0553	0.0706	0.0492	0.1484	0.0781	0.0781
$(1\%, 3\%)$	0	0	0	11.5619	0.0228	23.124	0.1149	0.0802	0.2416	0.1283	0.1283
$(0\%, 2\%)$	0	0	0	0.0275	0	0.055	0.0644	0.045	0.1355	0.0709	0.0709
$(0\%, 3\%)$	0	0	0	4.5271	0	9.0542	0.1092	0.0762	0.2295	0.1202	0.1202
Skew							**Skew**				
$(1\%, 1\%)$	n/a	n/a	n/a	0.2749	0.3419	0.4378	0.008	0.008	0.008	-0.0662	0.0662
$(1\%, 2\%)$	n/a	n/a	n/a	1.274	0.4041	1.5679	-0.273	-0.273	-0.273	-0.3393	0.3393
$(1\%, 3\%)$	n/a	n/a	n/a	99.9677	0.3844	99.9679	-0.9851	-0.9851	-0.9851	-1.0222	1.0222
$(0\%, 2\%)$	n/a	n/a	n/a	1.6325	n/a	1.6325	-0.5349	-0.5349	-0.5349	-0.5349	0.5349
$(0\%, 3\%)$	n/a	n/a	n/a	99.8536	n/a	99.8536	-1.0829	-1.0829	-1.0829	-1.0829	1.0829
ExKurt							**ExKurt**				
$(1\%, 1\%)$	n/a	n/a	n/a	0.1298	0.2681	0.5305	-0.0041	-0.0041	-0.0041	-0.0056	-0.0056
$(1\%, 2\%)$	n/a	n/a	n/a	7.2506	0.2919	9.6691	0.7263	0.7263	0.7263	0.6553	0.6553
$(1\%, 3\%)$	n/a	n/a	n/a	9992.6892	0.4107	9992.7084	3.1604	3.1604	3.1604	3.0742	3.0742
$(0\%, 2\%)$	n/a	n/a	n/a	10.4256	n/a	10.4256	1.2405	1.2405	1.2405	1.2405	1.2405
$(0\%, 3\%)$	n/a	n/a	n/a	9977.335	n/a	9977.335	3.7342	3.7342	3.7342	3.7342	3.7342
JB-PV							**JB-PV**				
$(1\%, 1\%)$	n/a	n/a	n/a	0	0	0	0.0548	0.0548	0.0548	0.0257	0.0257
$(1\%, 2\%)$	n/a	n/a	n/a	0	0	0	0	0	0	0	0
$(1\%, 3\%)$	n/a	n/a	n/a	0	0	0	0	0	0	0	0
$(0\%, 2\%)$	n/a	n/a	n/a	0	n/a	0	0	0	0	0	0
$(0\%, 3\%)$	n/a	n/a	n/a	0	n/a	0	0	0	0	0	0
SR											
$(1\%, 1\%)$	1.9191	1.8543	2.5833	3.2342	3.2262	3.2262					
$(1\%, 2\%)$	1.9343	1.8819	2.5944	3.2662	3.2433	3.2472					
$(1\%, 3\%)$	1.9653	1.9672	2.6061	3.3538	3.2695	3.3185					
$(0\%, 2\%)$	1.9305	1.8712	2.588	3.2542	3.2347	3.239					
$(0\%, 3\%)$	1.9553	1.9449	2.5998	3.329	3.2579	3.2979					
TR											
$(1\%, 1\%)$	0.3489	0.3489	0.3489	0.3489	0.3489	0.3489					
$(1\%, 2\%)$	0.3562	0.3562	0.3562	0.3562	0.3562	0.3562					
$(1\%, 3\%)$	0.49	0.49	0.49	0.49	0.49	0.49					
$(0\%, 2\%)$	0.3535	0.3535	0.3535	0.3535	0.3535	0.3535					
$(0\%, 3\%)$	0.4156	0.4156	0.4156	0.4156	0.4156	0.4156					

TABLE 8. Impact of heterogeneous beliefs in variance/covariance of asset returns on market equilibrium.

$\mathbb{E}_a(\tilde{r})$							β				
Mean							**Mean**				
$(\sigma_{\theta_1}, \sigma_{\theta_2})$	**A**	**B**	**C**	**M**	**O1**	**O2**	**A**	**B**	**C**	**O1**	**O2**
$(5\%, 5\%)$	0.3634	0.2687	0.7088	0.3961	0.3961	0.396	0.9069	0.6328	1.9068	1	1
$(5\%, 10\%)$	0.3634	0.2686	0.7087	0.3971	0.396	0.3983	0.9058	0.6318	1.9042	0.9986	1.0014
$(5\%, 15\%)$	0.3632	0.2688	0.7088	0.3999	0.3958	0.4041	0.9013	0.6294	1.896	0.9937	1.0063
$(0\%, 10\%)$	0.3633	0.2686	0.7086	0.397	0.3951	0.3989	0.9058	0.6318	1.904	0.9972	1.0028
$(0\%, 15\%)$	0.3633	0.2686	0.7088	0.3997	0.3953	0.404	0.9021	0.6294	1.8969	0.9938	1.0062
SD							**SD**				
$(5\%, 5\%)$	0.007	0.0071	0.0071	0.0143	0.0194	0.0193	0.0386	0.0268	0.0756	0.0372	0.0372
$(5\%, 10\%)$	0.0071	0.0071	0.007	0.0212	0.0192	0.0369	0.0554	0.0387	0.1127	0.0574	0.0574
$(5\%, 15\%)$	0.0071	0.0071	0.0071	0.0306	0.0191	0.0574	0.0755	0.0529	0.1565	0.0806	0.0806
$(0\%, 10\%)$	0.0071	0.0071	0.0071	0.0198	0.0083	0.0375	0.0509	0.0353	0.1038	0.0522	0.0522
$(0\%, 15\%)$	0.0071	0.0071	0.0072	0.0302	0.0083	0.0591	0.0737	0.0513	0.1519	0.0782	0.0782
Skew							**Skew**				
$(5\%, 5\%)$	0.0201	0.0053	0.0216	0.1914	0.2813	0.3084	0.0468	0.0129	0.0586	-0.0332	0.0332
$(5\%, 10\%)$	-0.0264	-0.0076	0.0631	0.3848	0.2598	0.572	-0.006	-0.0624	-0.0532	-0.1756	0.1756
$(5\%, 15\%)$	-0.0241	-0.0288	-0.0214	0.8734	0.2417	1.0491	-0.2506	-0.2354	-0.2746	-0.3785	0.3785
$(0\%, 10\%)$	-0.0003	-0.0594	0.0319	0.5916	0.0518	0.6718	-0.219	-0.2257	-0.2605	-0.3168	0.3168
$(0\%, 15\%)$	-0.0198	0.0143	0.0211	1.1664	0.0844	1.2213	-0.4233	-0.4292	-0.4391	-0.484	0.484
ExKurt							**ExKurt**				
$(5\%, 5\%)$	-0.016	0.078	0.0013	0.0621	0.1875	0.105	0.0324	0.0041	0.0351	0.0581	0.0581
$(5\%, 10\%)$	0.0722	0.0366	0.0362	0.1857	0.1064	0.4766	0.0064	-0.04	-0.0472	0.0464	0.0464
$(5\%, 15\%)$	-0.135	0.0576	-0.0014	1.672	0.1425	2.2404	0.2009	0.2822	0.253	0.3526	0.3526
$(0\%, 10\%)$	0.0029	0.0449	-0.0147	0.6724	0.0633	0.8216	0.134	0.1151	0.1215	0.1721	0.1721
$(0\%, 15\%)$	0.0339	-0.0458	0.0342	3.3556	0.0221	3.555	0.5308	0.4949	0.5706	0.6042	0.6042
JB-PV							**JB-PV**				
$(5\%, 5\%)$	0.322	0.2747	0.3227	0	0	0	0.1292	0.1323	0.0441	0.1978	0.1978
$(5\%, 10\%)$	0.189	0.2793	0.0277	0	0	0	0.0375	0.028	0.0596	0	0
$(5\%, 15\%)$	0.0138	0.2513	0.3168	0	0	0	0	0	0	0	0
$(0\%, 10\%)$	0.0018	0.0347	0.4086	0	0.0464	0	0	0	0	0	0
$(0\%, 15\%)$	0.4313	0.4553	0.4598	0	0.0024	0	0	0	0	0	0
SR											
$(5\%, 5\%)$	1.9118	1.8376	2.5788	3.2182	3.2174	3.2174					
$(5\%, 10\%)$	1.9117	1.8369	2.5784	3.2176	3.2168	3.2168					
$(5\%, 15\%)$	1.9107	1.8382	2.5787	3.218	3.2172	3.2172					
$(0\%, 10\%)$	1.9116	1.837	2.5781	3.2175	3.2166	3.2167					
$(0\%, 15\%)$	1.9112	1.8371	2.5787	3.2177	3.2169	3.2169					
TR											
$(5\%, 5\%)$	0.3461	0.3461	0.3461	0.3461	0.3461	0.3461					
$(5\%, 10\%)$	0.3471	0.3471	0.3471	0.3471	0.3471	0.3471					
$(5\%, 15\%)$	0.3499	0.3499	0.3499	0.3499	0.3499	0.3499					
$(0\%, 10\%)$	0.347	0.347	0.347	0.347	0.347	0.347					
$(0\%, 15\%)$	0.3497	0.3497	0.3497	0.3497	0.3497	0.3497					

TABLE 9. Impact of ARA coefficients with heterogeneous beliefs in expected returns.

$\mathbb{E}_a(\tilde{r})$							β				
Mean							**Mean**				
$(\sigma_{\theta_1}, \sigma_{\theta_2})$	**A**	**B**	**C**	**M**	**O1**	**O2**	**A**	**B**	**C**	**O1**	**O2**
$(5\%, 5\%)$	0.3633	0.2686	0.7087	0.3999	0.3999	0.3999	0.8985	0.6269	1.8891	1.0001	0.9999
$(5\%, 10\%)$	0.3633	0.2686	0.7087	0.4011	0.3996	0.4026	0.8972	0.626	1.8863	0.9976	1.0024
$(5\%, 15\%)$	0.3633	0.2686	0.7087	0.4029	0.3998	0.4061	0.8958	0.625	1.8834	0.9966	1.0034
$(0\%, 10\%)$	0.3633	0.2686	0.7087	0.4009	0.3987	0.4031	0.8971	0.6259	1.8861	0.9964	1.0036
$(0\%, 15\%)$	0.3633	0.2686	0.7087	0.4035	0.3989	0.4081	0.894	0.6237	1.8795	0.9933	1.0067
SD							**SD**				
$(5\%, 5\%)$	0	0	0	0.0206	0.0289	0.029	0.0526	0.0367	0.1105	0.0576	0.0576
$(5\%, 10\%)$	0	0	0	0.0259	0.0291	0.0429	0.0653	0.0456	0.1374	0.0725	0.0725
$(5\%, 15\%)$	0	0	0	0.0343	0.0291	0.0619	0.0839	0.0585	0.1763	0.0919	0.0919
$(0\%, 10\%)$	0	0	0	0.0247	0.023	0.0438	0.0621	0.0433	0.1305	0.069	0.069
$(0\%, 15\%)$	0	0	0	0.0337	0.0231	0.0632	0.0811	0.0566	0.1705	0.0897	0.0897
Skew							**Skew**				
$(5\%, 5\%)$	n/a	n/a	n/a	0.2718	0.383	0.3853	0.076	0.076	0.076	0.011	-0.011
$(5\%, 10\%)$	n/a	n/a	n/a	0.4041	0.3851	0.6125	0.0588	0.0588	0.0588	-0.1118	0.1118
$(5\%, 15\%)$	n/a	n/a	n/a	0.7492	0.3671	0.9992	-0.0818	-0.0818	-0.0818	-0.267	0.267
$(0\%, 10\%)$	n/a	n/a	n/a	0.4505	0.3388	0.595	-0.0124	-0.0124	-0.0124	-0.0986	0.0986
$(0\%, 15\%)$	n/a	n/a	n/a	0.8861	0.4042	1.0542	-0.2119	-0.2119	-0.2119	-0.3284	0.3284
ExKurt							**ExKurt**				
$(5\%, 5\%)$	n/a	n/a	n/a	0.0954	0.2738	0.3192	-0.0218	-0.0218	-0.0218	-0.014	-0.014
$(5\%, 10\%)$	n/a	n/a	n/a	0.3638	0.4319	0.7356	0.0872	0.0872	0.0872	-0.0144	-0.0144
$(5\%, 15\%)$	n/a	n/a	n/a	1.3809	0.2761	2.194	0.1195	0.1195	0.1195	0.1322	0.1322
$(0\%, 10\%)$	n/a	n/a	n/a	0.4348	0.2935	0.744	0.0315	0.0315	0.0315	0.0789	0.0789
$(0\%, 15\%)$	n/a	n/a	n/a	1.809	0.5389	2.3755	0.1425	0.1425	0.1425	0.2203	0.2203
JB-PV							**JB-PV**				
$(5\%, 5\%)$	n/a	n/a	n/a	0	0	0	0.0074	0.0074	0.0074	0.1324	0.1324
$(5\%, 10\%)$	n/a	n/a	n/a	0	0	0	0.0115	0.0115	0.0115	0	0
$(5\%, 15\%)$	n/a	n/a	n/a	0	0	0	0.0002	0.0002	0.0002	0	0
$(0\%, 10\%)$	n/a	n/a	n/a	0	0	0	0.2842	0.2842	0.2842	0	0
$(0\%, 15\%)$	n/a	n/a	n/a	0	0	0	0	0	0	0	0
SR											
$(5\%, 5\%)$	1.9207	1.8542	2.5825	3.2342	3.2262	3.2262					
$(5\%, 10\%)$	1.92	1.8525	2.5843	3.2345	3.2264	3.2265					
$(5\%, 15\%)$	1.9195	1.8538	2.583	3.234	3.2259	3.226					
$(0\%, 10\%)$	1.9216	1.8524	2.5842	3.235	3.2268	3.227					
$(0\%, 15\%)$	1.922	1.8531	2.5833	3.2347	3.2264	3.2266					
TR											
$(5\%, 5\%)$	0.3499	0.3499	0.3499	0.3499	0.3499	0.3499					
$(5\%, 10\%)$	0.3511	0.3511	0.3511	0.3511	0.3511	0.3511					
$(5\%, 15\%)$	0.3529	0.3529	0.3529	0.3529	0.3529	0.3529					
$(0\%, 10\%)$	0.3509	0.3509	0.3509	0.3509	0.3509	0.3509					
$(0\%, 15\%)$	0.3535	0.3535	0.3535	0.3535	0.3535	0.3535					

TABLE 10. Impact of ARA coefficients with heterogeneous beliefs in variance/covaraince of asset returns.

$\mathbb{E}_a(\tilde{r})$							β				
Mean							**Mean**				
$(\sigma_{\theta_1}, \sigma_{\theta_2})$	**A**	**B**	**C**	**M**	**O1**	**O2**	**A**	**B**	**C**	**O1**	**O2**
$(5\%, 5\%)$	0.3633	0.2686	0.7087	0.4001	0.4001	0.4	0.8981	0.6265	1.8888	1.0001	0.9999
$(5\%, 10\%)$	0.3633	0.2686	0.7087	0.4015	0.4002	0.4028	0.8963	0.6254	1.8848	0.9983	1.0017
$(5\%, 15\%)$	0.3633	0.2685	0.7087	0.4038	0.3998	0.4078	0.8937	0.623	1.879	0.9943	1.0057
$(0\%, 10\%)$	0.3634	0.2686	0.7087	0.4009	0.3993	0.4026	0.8975	0.626	1.8867	0.9979	1.0021
$(0\%, 15\%)$	0.3635	0.2686	0.7088	0.4028	0.3992	0.4063	0.8963	0.6248	1.8837	0.996	1.004
SD							**SD**				
$(5\%, 5\%)$	0.007	0.007	0.0071	0.0219	0.0307	0.03	0.056	0.0392	0.1153	0.0594	0.0594
$(5\%, 10\%)$	0.0071	0.0071	0.0071	0.0271	0.0302	0.0443	0.0686	0.0479	0.1417	0.0741	0.0741
$(5\%, 15\%)$	0.0071	0.0071	0.0072	0.035	0.0301	0.0625	0.0857	0.0597	0.1781	0.0926	0.0926
$(0\%, 10\%)$	0.0071	0.0072	0.0071	0.0258	0.0247	0.0444	0.0652	0.0457	0.1347	0.0697	0.0697
$(0\%, 15\%)$	0.0072	0.0071	0.0072	0.0338	0.0244	0.0624	0.0823	0.0574	0.1715	0.0889	0.0889
Skew							**Skew**				
$(5\%, 5\%)$	0.0013	-0.0071	-0.0003	0.3035	0.4145	0.3907	0.1016	0.1224	0.067	0.0205	-0.0205
$(5\%, 10\%)$	0.0159	0.0414	0.0121	0.3848	0.3397	0.584	0.1014	0.1048	0.0832	-0.0853	0.0853
$(5\%, 15\%)$	0.0103	-0.0265	-0.0048	0.707	0.3485	0.9495	-0.0265	-0.024	-0.0369	-0.1974	0.1974
$(0\%, 10\%)$	-0.0203	0.0002	0.0417	0.4271	0.3928	0.6023	0.042	0.0487	0.0328	-0.1183	0.1183
$(0\%, 15\%)$	-0.009	0.0134	0.0087	0.8419	0.3717	1.052	-0.1581	-0.1387	-0.1707	-0.3047	0.3047
ExKurt							**ExKurt**				
$(5\%, 5\%)$	-0.0374	-0.048	-0.0327	0.2458	0.4393	0.4387	0.0603	0.0505	0.069	0.0672	0.0672
$(5\%, 10\%)$	-0.0097	0.0324	0.0365	0.3801	0.0907	0.7661	0.1435	0.0554	0.0899	-0.0732	-0.0732
$(5\%, 15\%)$	0.0324	0.0199	0.0254	1.3076	0.2327	2.0648	0.1076	0.1312	0.1132	0.1496	0.1496
$(0\%, 10\%)$	0.0394	-0.0811	-0.0242	0.4587	0.5774	0.7941	0.0578	0.0467	0.092	0.074	0.074
$(0\%, 15\%)$	-0.0835	-0.0016	0.0594	1.7181	0.2675	2.3427	0.1894	0.1822	0.1915	0.2203	0.2203
JB-PV							**JB-PV**				
$(5\%, 5\%)$	0.2536	0.4063	0.1998	0	0	0	0	0	0.0088	0.2747	0.2747
$(5\%, 10\%)$	0.2064	0.1925	0.3297	0	0	0	0	0	0.0006	0.0008	0.0008
$(5\%, 15\%)$	0.2644	0.488	0.1423	0	0	0	0.0498	0.0171	0.0223	0	0
$(0\%, 10\%)$	0.4861	0.2542	0.2082	0	0	0	0.1148	0.0878	0.0702	0	0
$(0\%, 15\%)$	0.2187	0.1402	0.4507	0	0	0	0	0	0	0	0
SR											
$(5\%, 5\%)$	1.9202	1.8527	2.5836	3.2349	3.2261	3.226					
$(5\%, 10\%)$	1.923	1.8538	2.5835	3.2364	3.2275	3.2276					
$(5\%, 15\%)$	1.9199	1.8526	2.5835	3.2347	3.2258	3.226					
$(0\%, 10\%)$	1.9204	1.8526	2.5852	3.2361	3.2272	3.2273					
$(0\%, 15\%)$	1.9211	1.8523	2.5842	3.2355	3.2266	3.2267					
TR											
$(5\%, 5\%)$	0.3501	0.3501	0.3501	0.3501	0.3501	0.3501					
$(5\%, 10\%)$	0.3515	0.3515	0.3515	0.3515	0.3515	0.3515					
$(5\%, 15\%)$	0.3538	0.3538	0.3538	0.3538	0.3538	0.3538					
$(0\%, 10\%)$	0.3509	0.3509	0.3509	0.3509	0.3509	0.3509					
$(0\%, 15\%)$	0.3528	0.3528	0.3528	0.3528	0.3528	0.3528					

TABLE 11. Impact of ARA coefficients with heterogeneous beliefs in both expected returns and variance/covaraince of returns.