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A Unifying Approach to Asset Pricing

Eckhard Platen

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A Unifying Approach to Asset Pricing

Eckhard Platen[1]

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Abstract. This paper introduces a general market modeling framework under which the Law of One Price no longer holds. A contingent claim can have in this setting several self-financing, replicating portfolios. The new Law of the Minimal Price identifies the lowest replicating price process for a given contingent claim. The proposed unifying asset pricing methodology is model independent and only requires the existence of a tradable numeraire portfolio, which turns out to be the growth optimal portfolio that maximizes expected logarithmic utility. By the Law of the Minimal Price the inverse of the numeraire portfolio becomes the stochastic discount factor. This allows pricing in extremely general settings and avoids the restrictive assumptions of risk neutral pricing. In several ways the numeraire portfolio is the "best" performing portfolio and cannot be outperformed by any other nonnegative portfolio. Several classical pricing rules are recovered under this unifying approach. The paper explains that pricing by classical no-arbitrage arguments is, in general, not unique and may lead to overpricing. In an example, a surprisingly low price of a zero coupon bond with extreme maturity illustrates one of the new effects that can be captured under the proposed benchmark approach, where the numeraire portfolio represents the benchmark.

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[1]University of Technology Sydney, School of Finance & Economics and Department of Mathematical Sciences, PO Box 123, Broadway, NSW, 2007, Australia

1 Introduction

The *Law of One Price* is commonly assumed to be valid in most of the finance literature. It states that a replicable payoff can only be hedged by portfolios that follow one and the same value process. Classical asset pricing theories, as developed, for instance, in Debreu (1959), Sharpe (1964), Lintner (1965), Merton (1973a, 1973b), Ross (1976) and Harrison & Kreps (1979), are consistent with the Law of One Price. A more recent description of various asset pricing approaches can be found, for instance, in Cochrane (2001). The use of a stochastic discount factor, as described in his book, comes closest to the unifying pricing approach we will propose.

The current paper also presents a general modeling framework, which only assumes the existence of a "best" performing, strictly positive, tradable portfolio, the *numeraire portfolio*, see Long (1990). The approach is fairly self-contained and may not fit easily into the universe of existing literature on asset pricing. Its core results are model independent and, therefore, very general and robust. The presented new approach covers a much wider modeling world than captured by classical theories. It chooses as its central building block a benchmark, the numeraire portfolio, which turns out to be the growth optimal portfolio that maximizes expected logarithmic utility from terminal wealth, see Kelly (1956), Latané (1959), Breiman (1960), Hakansson (1971), Merton (1973a), Roll (1973) and Markowitz (1976).

Nonnegative portfolios, when denominated in units of the numeraire portfolio, trend downward or are at most trendless, see Becherer (2001), Bühlmann & Platen (2003), Platen & Heath (2006) and Karatzas & Kardaras (2007). Several results of fundamental importance follow from this property via a few basic arguments, as will be shown.

The current paper extends significantly the benchmark approach, which originally has been developed in Platen & Heath (2006) for the special case of jump-diffusion markets. The most striking feature of the extended modeling world is the possible co-existence of several self-financing portfolios that perfectly replicate the same contingent claim but follow different paths with their value processes. Obviously, the presence of different replicating portfolios contradicts the classical Law of One Price. Instead the new *Law of the Minimal Price* will be derived, which identifies for a given replicable contingent claim the unique minimal replicating portfolio process. As a consequence, the inverse of the numeraire portfolio becomes the stochastic discount factor in the resulting pricing formula.

By exploiting this new law, the paper will illustrate in Section 6 in an example how to replicate in two different ways a fixed cash amount, payable at a given maturity date: The first self-financing, replicating portfolio is a, so called, savings bond, which invests purely in the deterministic savings account. The second self-financing, replicating portfolio process is formed by a less expensive derivative, the, so called, fair zero coupon bond. Its price is given by the minimal replicating price process when choosing the S&P500 as numeraire portfolio under an appropriate model. In Figure 1.1, the logarithms of both resulting self-financing,

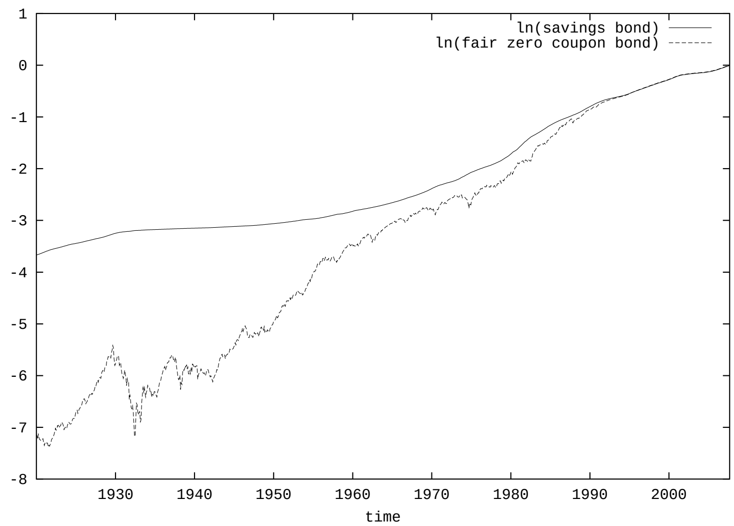

Figure 1.1: Logarithms of savings bond and fair zero coupon bond.

replicating portfolio processes are displayed over the period from 1920 until 2007 using US interest rate and S&P500 data. Remarkably, the fair zero coupon bond is priced in 1920 only at about 3% of the price of the classical savings bond. Details explaining the model and further illustrative graphs will be shown in Section 6.

The fact that the Law of One Price may not hold in the suggested richer modeling world does not generate a, so called, strong arbitrage in the sense of this paper. Some weak forms of classical arbitrage will be allowed in the suggested general framework that are typically excluded under classical approaches. The paper argues that there is no economic reason to exclude any weaker form of arbitrage than the kind of strong arbitrage that will be automatically excluded under the benchmark approach. Pricing by classical no-arbitrage arguments will be not unique in the given general setting.

Under the Law of the Minimal Price derivative pricing becomes an investment decision. In the resulting pricing formula the inverse of the numeraire portfolio is the stochastic discount factor and the expectation is taken with respect to the real world probability measure. No change of probability measure is performed when pricing, which is different to classical risk neutral pricing. In a complete market the resulting real world pricing formula generalizes the risk neutral pricing formula, as well as, the common pricing formulae involving a state price density, pricing kernel, deflator or stochastic discount factor. It also generalizes the actuarial pricing formula. These formulae represent the central pricing rules in their respective streams of literature which are all consistent with the Law of One Price. New important effects emerge under the benchmark approach which open interesting areas of research, for instance, the least expensive pricing and hedging of extreme maturity derivatives.

The paper introduces the numeraire portfolio in Section 2. It derives the Law of the Minimal Price in Section 3. Section 4 discusses strong arbitrage. Section 5 demonstrates in several ways that the numeraire portfolio is the "best" performing portfolio. Finally, Section 6 provides an example where the Law of One Price fails and the Law of the Minimal Price leads to the least expensive zero coupon bond price.

2 The Numeraire Portfolio as Benchmark

We consider a financial market in continuous time with d risky, nonnegative, primary securities, $d \in \{1, 2, \ldots\}$. These securities could be, for instance, shares, currencies or other traded assets. Denote by S_t^j the value of the corresponding jth *primary security account*, $j \in \{0, 1, \ldots, d\}$, at time $t \geq 0$. This nonnegative account holds units of the jth primary security together with all its accumulated dividends or interest payments. The 0th primary security account S_t^0 denotes the value of the locally riskless *savings account* at time $t \geq 0$, which is a roll-over short term bond account. We emphasize that the particular dynamics of the primary security accounts will be not relevant for the core statements of the paper, which makes these results very general and also robust.

The market participants can form self-financing portfolios with primary security accounts as constituents. A portfolio value S_t^δ at time t is characterized by the number δ_t^j of units held in the jth primary security account S_t^j for all $j \in \{0, 1, \ldots, d\}$, $t \geq 0$. For simplicity, assume that the units of the primary security accounts are perfectly divisible, and that for all $t \geq 0$ the quantities $\delta_t^0, \delta_t^1, \ldots, \delta_t^d$, for any given strategy $\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \ldots, \delta_t^d)^\top,\ t \geq 0\}$, depend only on information available at time t. The portfolio value at time t is given by the sum

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j \, S_t^j.$$

We only consider *self-financing* portfolios, where changes in their value only arise due to changes in the values of the primary security accounts. For simplicity, we neglect any market frictions or liquidity effects. Furthermore, let $E_t(X)$ denote the conditional expectation, under the real world probability, given the information available at time $t \geq 0$.

Definition 2.1 *For given initial value $x > 0$, a strictly positive, finite, tradable, self-financing portfolio S^{δ_*}, with $S_0^{\delta_*} = x$, is called a* numeraire portfolio *if for all observable times $t \geq 0$ and positive real numbers $h > 0$, the expected returns of all nonnegative portfolios S^δ, when denominated in units of S^{δ_*}, are* *never greater than zero as long as* $S_t^\delta > 0$*, that is,*

$$E_t\left(\frac{\frac{S_{t+h}^\delta}{S_{t+h}^{\delta_*}}}{\frac{S_t^\delta}{S_t^{\delta_*}}} - 1\right) \leq 0. \tag{2.1}$$

The notion of a numeraire portfolio was originally introduced by Long (1990) in a rather special setting. Later it was generalized in Bajeux-Besnainou & Portait (1997) and Becherer (2001). These authors worked under assumptions that imply the classical Law of One Price. More recently, Platen (2002), Bühlmann & Platen (2003), Platen & Heath (2006) and Platen (2006b) emphasized that one obtains still a viable financial market model as long as a numeraire portfolio exists. Therefore, the single main assumption of the paper is the following:

Assumption 2.2 *There exists a numeraire portfolio* S^{δ_*}*, with* $S_0^{\delta_*} = x > 0$.

This assumption is satisfied for almost all financial market models used in practice. For instance, it has been verified for jump-diffusion markets in Platen & Heath (2006). Karatzas & Kardaras (2007) and Kardaras & Platen (2008) provide extremely weak conditions for the existence of a numeraire portfolio. As soon as one specifies the dynamics of a particular financial market model it is usually straightforward to describe its numeraire portfolio, which will be later identified as the growth optimal portfolio.

From now on, let us choose the numeraire portfolio as *benchmark*. The benchmarked value of a portfolio S^δ is of particular interest and is given by the ratio

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}$$

for all $t \geq 0$. By Assumption 2.2 the Definition 2.1 of the numeraire portfolio leads directly to the following conclusion.

Corollary 2.3 *The benchmarked values of any nonnegative portfolio* S^δ *satisfy the property*

$$\hat{S}_t^\delta \geq E_t\left(\hat{S}_s^\delta\right) \tag{2.2}$$

for all $0 \leq t \leq s < \infty$.

Consequently, the current observed benchmarked value of a nonnegative portfolio is always greater than or equal to its expected future benchmarked value. This means, if there were any trend in a benchmarked nonnegative portfolio, then this trend could only point downward. As will become clear later, the property (2.2) is the central structural property of a financial market. We call it the *supermartingale property*, since $\hat{S}^\delta$ forms a, so called, supermartingale, see Shiryaev (1984). As will be shown it allows to answer, in full generality, fundamental questions in finance.

To clarify the uniqueness of a numeraire portfolio consider two strictly positive portfolios that are supposed to be numeraire portfolios. According to Corollary 2.3 the first portfolio, when expressed in units of the second one, must satisfy the supermartingale property (2.2). By the same argument, the second portfolio, when expressed in units of the first one, must also satisfy the property (2.2). Consequently, by Jensen's inequality these portfolios must be identical, and the value process of a numeraire portfolio is unique. Note that the stated uniqueness does *not* imply that the number of units invested has to be unique. Redundancies in the set of primary security accounts could allow different strategies of forming the numeraire portfolio.

3 The Law of the Minimal Price

The supermartingale property (2.2) ensures that the maximum expected return of a benchmarked nonnegative portfolio can at most equal zero. In the case when it equals zero for all time instances, the current benchmarked value of the price process is always the best forecast of its future benchmarked values. In this case one has equality in relation (2.2) and we call such a price process *fair*. A benchmarked fair price process is trendless and is a, so called, *martingale*, see Shiryaev (1984).

It may be puzzling to some readers that discounting with another discount factor than the inverse of the savings account ought to be particularly meaningful. However, our main theorem below will give a valid reason why the inverse of the numeraire portfolio is the appropriate stochastic discount factor. In general, not all primary security accounts and portfolios need to be fair in our general setting. This creates new, potentially surprising but important effects in financial market models that have not been captured by classical theories. We emphasize in the following one such new effect which can be directly linked to the classical Law of One Price. This law states for a given payoff that the price processes of all replicating portfolios must be the same. However, the introductory example illustrates in Figure 1.1 that this law may not hold in general, which will be confirmed in Section 6. The following Law of the Minimal Price emerges instead.

Theorem 3.1 (Law of the Minimal Price) *If a fair portfolio process replicates a given nonnegative payoff at a given maturity date, then it represents the minimal replicating portfolio among all nonnegative portfolios that replicate this payoff.*

Proof: Recall that a stochastic process $\hat{S}^{\delta}$ which satisfies relation (2.2) is a supermartingale. This means it trends downward or has no trend. If equality holds in (2.2), then the process has no trend, which means that it is fair and its benchmarked value forms a martingale. In the following we rely on a basic, and also very intuitive, mathematical fact: Within a family of nonnegative supermartingales, which all coincide at a given future time, it is the martingale which attains the minimal possible process, see Shiryaev (1984). In this sense the fair replicating portfolio provides the unique least expensive value process. This fundamental optimality property of the fair portfolio yields directly the proof of Theorem 3.1. $\square$

The Law of the Minimal Price in Theorem 3.1 provides the basis for a very reasonable pricing concept. For a given payoff the corresponding fair replicating portfolio characterizes according to Theorem 3.1 the least expensive hedge portfolio. In a competitive market this is also the correct price process from an economic point of view. We emphasize that the Law of the Minimal Price leads to a unique system of derivative prices. However, this does not mean that primary security accounts or portfolios need to be fair, which leads to interesting new effects that have not been captured in the classical literature. For instance, in our introductory example the savings account will turn out to be not fair. In this sense it is a "bad" investment when compared with the fair zero coupon bond. It is important to underline the fact that no wealth needs to be thrown away or received from any extra sources to observe various new effects under the benchmark approach.

Now, define a *contingent claim* H_T as a nonnegative payoff, expressed in units of the domestic currency, which is delivered at maturity $T \in (0, \infty)$ and has finite expected benchmarked value

$$E_0\left(\frac{H_T}{S_T^{\delta_*}}\right) < \infty. \tag{3.1}$$

By the above derived Law of the Minimal Price one identifies the corresponding fair price process, which follows the minimal possible price of the corresponding derivative, via the following real world pricing formula:

Corollary 3.2 *If for a contingent claim H_T, $T \in (0, \infty)$, there exists a fair portfolio S^{δ_H} that replicates this claim at maturity T such that $H_T = S_T^{\delta_H}$, then its minimal replicating price at time $t \in [0, T]$ is given by the* real world pricing formula

$$S_t^{\delta_H} = S_t^{\delta_*}\, E_t\left(\frac{H_T}{S_T^{\delta_*}}\right). \tag{3.2}$$

This general pricing formula is called the real world pricing formula because it is based on the conditional expectation E_t with respect to the real world probability measure. Note, to perform pricing via (3.2) we need only the existence of the numeraire portfolio. However, this is also necessary, otherwise the "best" performing portfolio would explode and the underlying model would become useless. In the case when (3.1) does not hold, then there does not exist any reasonable finite price for the claim H_T. One obtains in (3.2) the inverse of the numeraire portfolio as stochastic discount factor, see Cochrane (2001). Thus, one recovers from the real world pricing formula (3.2) the well-known pricing via stochastic discount factor. Similarly, also the pricing with state price density, pricing kernel or deflator emerge from (3.2) by employing the numeraire portfolio accordingly. When formulated in the literature, all these classical pricing formulae are assumed to be consistent with the Law of One Price, see for instance Ingersoll (1987), Long (1990), Constatinides (1992), Duffie (2001) and Cochrane (2001). The real world pricing formula (3.2) not only unifies these pricing formulae it still provides a unique price in such general settings where the classical approaches are no longer applicable.

Additionally, it generalizes also the following important pricing formula, which emerges from (3.2) in the case when the contingent claim H_T is independent of $S_T^{\delta_*}$. In this special case one obtains from (3.2) the *actuarial pricing formula*

$$S_t^{\delta_H} = P(t,T)\, E_t(H_T) \tag{3.3}$$

with fair zero coupon bond price

$$P(t,T) = S_t^{\delta_*}\, E_t\left((S_T^{\delta_*})^{-1}\right). \tag{3.4}$$

To obtain (3.3) one simply exploits the fact that the expectation of a product of independent random variables equals the product of their expectations. The fair zero coupon bond $P(t,T)$ with maturity T provides the discount factor in the actuarial pricing formula (3.3). This classical formula has been postulated on an intuitive basis by actuaries for centuries without derivation, which is here provided most generally via the Law of the Minimal Price.

In the real world pricing formula (3.2) no change of probability measure is employed. This avoids the rather technical and restrictive assumptions needed for the classical risk neutral approach, see Ross (1976), Harrison & Kreps (1979) and Platen & Heath (2006). Unfortunately, the verification of these important conditions is often omitted or overlooked in the literature, which sometimes causes confusion among academics and practitioners. The benchmark approach avoids the problem of measure change altogether. To highlight the link between real world pricing and classical risk neutral pricing, let us rewrite for $t = 0$ the real world pricing formula (3.2) in the form

$$S_0^{\delta_H} = E_0\left(\Lambda_T\, \frac{S_0^0}{S_T^0}\, H_T\right). \tag{3.5}$$

Here we employ the benchmarked normalized savings account $\Lambda_T = \frac{\hat{S}_T^0}{\hat{S}_0^0}$, which is the candidate for the state price density. By the supermartingale property (2.2) of the normalized benchmarked savings account process $\Lambda = \{\Lambda_t = \frac{\hat{S}^0_t}{\hat{S}^0_0}, t \geq 0\}$ we have $1 = \Lambda_0 \geq E_0(\Lambda_T)$. This relation yields together with equation (3.5) the inequality

$$S_0^{\delta_H} \leq \frac{E_0\left(\Lambda_T \frac{S_0^0}{S_T^0} H_T\right)}{E_0(\Lambda_T)}. \tag{3.6}$$

If the savings account is not fair, which means its benchmarked value is downward trending, then equality does not hold in relation (3.6). To guarantee equality in (3.6) one needs to ensure that the particular case is given where the savings account is fair, that is, its benchmarked value has no trend. Only in this special case the expression on the right hand side of (3.6) can be interpreted by Bayes' formula as the conditional expectation of the discounted contingent claim under the equivalent risk neutral probability measure Q, which is characterized by the state price density $\Lambda_T = \frac{dQ}{dP}$, see Duffie (2001). In this case the relation (3.6) yields for any contingent claim H_T the classical *risk neutral pricing formula*

$$S_0^{\delta_H} = E_0^Q\left(\frac{S_0^0}{S_T^0} H_T\right),$$

see for instance Harrison & Kreps (1979) or Duffie (2001). Here E_0^Q denotes the conditional expectation under the equivalent risk neutral probability measure Q at time $t = 0$. The risk neutral expectation is taken when the savings account is the discount factor. By inequality (3.6) it follows that the fair derivative price is never more expensive than the price obtained under formal application of the standard risk neutral pricing rule, including the case when the benchmarked savings account is not fair.

In the introductory example we observed that the initial fair zero coupon bond price is lower than the one of the corresponding savings bond. As will become clear in Section 6, the savings bond is in this case not fair. Under classical risk neutral pricing one assumes in a complete market that the benchmarked savings account has no downward trend. As will be argued later in the context of the introductory example, this assumption appears to be unrealistic not only for the US market. Therefore, the assumption that the savings account is fair should be interpreted more as a mathematical convenience in the development of the theory of asset pricing but not as any economic fact. Due to the well observed equity premium it is economically more reasonable to accept that the benchmarked savings account trends somehow downward.

It shall be noted that utility indifference pricing, in the sense of Davis (1997), for not fully replicable contingent claims, leads in the case of jump-diffusion markets, again to the real world pricing formula (3.2), see Platen & Heath (2006). The hedgable part of such claim is then replicated via the corresponding minimal self-financing, replicating portfolio. Its benchmarked unhedgable part is priced by its expected benchmarked value. When independent totally unhedgable benchmarked contingent claims are pooled then the uncertainty can be asymptotically diversified away, due to the Law of Large Numbers, see Shiryaev (1984). In principle, the real world pricing formula provides a projection of any benchmarked contingent claim into the set of current benchmarked prices. Forthcoming work will explain in detail why the real world pricing formula provides also for unhedgable claims the economically correct price.

4 Strong Arbitrage

Now, let us discuss in our general modeling world the question of arbitrage, which has been a key argument in the classical Arbitrage Pricing Theory, see Ross (1976) and Harrison & Kreps (1979). It is simply a fact that any kind of potential arbitrage opportunities can only be exploited by market participants. These investors have to use their portfolios of total tradable wealth when aiming to exploit arbitrage opportunities. Due to the established legal concept of limited liability, only portfolios of nonnegative total tradable wealth need to be considered. Consequently, for studying realistic questions on arbitrage a concept is sufficient which focusses on nonnegative self-financing portfolios.

An obvious, strong form of arbitrage arises when a market participant can generate strictly positive wealth from zero initial capital. This leads to the following definition of *strong arbitrage*, which was motivated in Platen (2002) by the supermartingale property (2.2).

Definition 4.1 *A nonnegative portfolio S^δ is a* strong arbitrage *if it starts with zero initial capital, that is $S_0^\delta = 0$, and generates strictly positive wealth with strictly positive probability at a later time $t \in (0, \infty)$, that is, $P(S_t^\delta > 0) > 0$.*

Independently, Loewenstein & Willard (2000) argued on purely economic grounds that the exclusion of the above strong arbitrage is sufficient from an equilibrium perspective. Weaker forms of arbitrage may still exist in the resulting model. This does not harm the core properties of the market model as will be described in this paper. All key financial tasks, including portfolio optimization, derivative pricing, as well as hedging, can still be performed consistently. As discussed in Platen & Heath (2006) there is no problem that, so-called, *free snacks* and *cheap thrills* may exist, in the sense of Loewenstein & Willard (2000). Also *free lunches with vanishing risk*, as excluded in Delbaen & Schachermayer (1998), can be present without causing any financial or economic complications. Also, so called, bubbles as discussed in Heston, Loewenstein & Willard (2007) can be present and explained in our setting, as will be detailed in forthcoming work. The credit constrained weak forms of arbitrage cannot really be exploited in practice because they always rely on adequate collateral, see Liu & Longstaff (2004). This, however, questions their benefit from the practical point of view. The following important statement will be derived below.

Theorem 4.2 *There does not exist any nonnegative portfolio that is a strong arbitrage.*

Proof: For a nonnegative portfolio S^δ, which starts with zero initial capital, it follows by Corollary 2.3 that

$$0 = S_0^\delta = x\,\hat{S}_0^\delta \geq x\,E_0\left(\hat{S}_t^\delta\right) = x\,E(\hat{S}_t^\delta) \geq 0,$$

for $t > 0$, where $E(\cdot) = E_0(\cdot)$ denotes expectation. By the nonnegativity of S_t^δ and the strict positivity of $S_t^{\delta_*}$, the event $S_t^\delta > 0$ can only have zero probability, that is

$$P\left(S_t^\delta > 0\right) = 0.$$

This leads to the conclusion that S_t^δ equals zero for all $t > 0$, which proves Theorem 4.2 by using Definition 4.1. □

Theorem 4.2 states that strong arbitrage is automatically excluded in the given general setting of the benchmark approach. Therefore, pricing a contingent claim by the exclusion of strong arbitrage does not make sense. According to the key no-arbitrage argument of the classical Arbitrage Pricing Theory, see Ross (1976), the existence of a replicating portfolio is sufficient to identify in a complete market with its value the price of the corresponding contingent claim. From our previous discussion it becomes clear that the candidate price process may not represent the minimal possible replicating portfolio process. The savings bond in our introductory example provides an illustration for a replicating portfolio that does not represent the minimal possible replicating portfolio process, as will be explained in detail in Section 6.

The above Theorem 4.2 can be interpreted as an extended version of the First Fundamental Theorem of Asset Pricing, addressing the problem of strong arbitrage, see Ross (1976), Harrison & Kreps (1979) and Delbaen & Schachermayer (1998). The Law of the Minimal Price formulated in Theorem 3.1 could then be interpreted as an extended version of the Second Theorem of Asset Pricing, addressing unique derivative pricing in a complete frictionless market.

5 Best Performance of the Numeraire Portfolio

To understand how it happens that the real world pricing formula identifies the least expensive derivative price, it is helpful to confirm in precise terms that the numeraire portfolio is the "best" performing, strictly positive, tradable portfolio. In this section we list four such precise manifestations. First, the definition of the numeraire portfolio itself, given by condition (2.1), expresses the fact that this portfolio performs "best" in the sense that the expected returns of benchmarked nonnegative portfolios never become strictly positive. This means that the numeraire portfolio performs so well that any benchmarked nonnegative portfolio can only trend downward or is at most trendless.

To formulate a second precise manifestation of "best" performance, define the *long term growth rate* g^δ of a strictly positive portfolio S^δ as the upper limit

$$g^\delta = \limsup_{t\to\infty} \frac{1}{t} \ln\left(\frac{S_t^\delta}{S_0^\delta}\right). \tag{5.1}$$

The long term growth rate (5.1) is defined pathwise almost surely. It does not involve any expectation. By exploiting again the supermartingale property (2.2), the following fascinating property of the numeraire portfolio emerges which makes it ideal for long term investment:

Theorem 5.1 *The numeraire portfolio S^{δ_*} achieves the maximum long term growth rate. This means, when compared with any other strictly positive portfolio S^δ, it follows*

$$g^\delta \leq g^{\delta_*}. \tag{5.2}$$

Proof: Consider a strictly positive portfolio S^δ with the same initial capital as the numeraire portfolio, that is, $S_0^\delta = S_0^{\delta_*} = x > 0$. By Corollary 2.3 we can use the following inequality, mentioned in Doob (1953), where for any $k \in \{1, 2, \ldots\}$ and $\varepsilon \in (0,1)$ one has

$$\exp\{\varepsilon\, k\}\, P\left(\sup_{k\leq t<\infty} \hat{S}_t^\delta > \exp\{\varepsilon\, k\}\right) \leq E_0\left(\hat{S}_k^\delta\right) \leq \hat{S}_0^\delta = 1.$$

One finds for fixed $\varepsilon \in (0,1)$ that

$$\sum_{k=1}^\infty P\left(\sup_{k\leq t<\infty} \ln\left(\hat{S}_t^\delta\right) > \varepsilon\, k\right) \leq \sum_{k=1}^\infty \exp\{-\varepsilon\, k\} < \infty.$$

By the Lemma of Borel and Cantelli, see Shiryaev (1984), there exists a random variable k_ε such that for all $k \geq k_\varepsilon$ and $t \geq k$ it holds that

$$\ln\left(\hat{S}_t^\delta\right) \leq \varepsilon\, k \leq \varepsilon\, t.$$

Therefore, it follows for all $k > k_\varepsilon$ the estimate

$$\sup_{t\geq k} \frac{1}{t} \ln\left(\hat{S}_t^\delta\right) \leq \varepsilon,$$

which implies that

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left(\frac{S_t^\delta}{S_0^\delta}\right) \leq \limsup_{t\to\infty} \frac{1}{t} \ln\left(\frac{S_t^{\delta_*}}{S_0^{\delta_*}}\right) + \varepsilon. \tag{5.3}$$

Since the inequality (5.3) holds for all $\varepsilon \in (0,1)$ one obtains with (5.1) the relation (5.2). $\square$

According to Theorem 5.1, the trajectory or path of the numeraire portfolio outperforms in the long run those of all other strictly positive portfolios that start with the same initial capital. Also this property is again model independent. As a consequence of Theorem 5.1, an investor who aims in the long run for the highest possible wealth, has to invest her or his total tradable wealth into the numeraire portfolio. Consider the case of investing in the long run in a portfolio S^δ with a long term growth rate g^δ strictly less than g^{δ_*}. After sufficiently long time t the portfolio value

$$S_t^\delta \approx \frac{S_0^\delta \, S_t^{\delta_*}}{S_0^{\delta_*}} \exp\{-t(g^* - g)\} \to 0$$

will become negligible compared to the value of the numeraire portfolio $S_t^{\delta_*}$ for $t \to \infty$ due to (5.1) and (5.2).

Of course, over short and medium time periods, almost any strictly positive portfolio can generate by chance larger returns than those exhibited by the numeraire portfolio. However, the following third manifestation of "best" performance will show that such outperformance cannot be achieved systematically. To formulate a corresponding statement in a precise manner we will use the following definition:

Definition 5.2 *A nonnegative portfolio S^δ* systematically outperforms *a strictly positive portfolio $S^{\tilde{\delta}}$ if*

(i) both portfolios start from the same initial capital $S_0^\delta = S_0^{\tilde{\delta}}$;

(ii) at a later time $t \geq 0$, S_t^δ is at least equal to $S_t^{\tilde{\delta}}$, that is $P(S_t^\delta \geq S_t^{\tilde{\delta}}) = 1$ and

(iii) the probability for S_t^δ being strictly greater than $S_t^{\tilde{\delta}}$ is strictly positive, that is, $P\left(S_t^\delta > S_t^{\tilde{\delta}}\right) > 0$.

For instance, the introductory example illustrates a case, where a fair zero coupon bond systematically outperforms the savings account when it starts with the same initial capital, as will become clear in Section 6. Consequently, systematic outperformance of one portfolio by another one is possible in our general setting. The above notion of systematic outperformance was introduced in Platen (2004) motivated by the supermartingale property (2.2). This definition can also be related to the notion of relative arbitrage studied in Fernholz & Karatzas (2005) and there is also a connection to the notion of a maximal element described in Delbaen & Schachermayer (1998). It follows now the third precise manifestation of "best" performance of the numeraire portfolio:

Theorem 5.3 *The numeraire portfolio cannot be systematically outperformed by any nonnegative portfolio.*

Proof: Consider a nonnegative portfolio S^δ with benchmarked value $\hat{S}^\delta_t = 1$ at a given time $t \geq 0$, where $\hat{S}^\delta_s \geq 1$ almost surely at some time $s \in [t, \infty)$. Then it follows by the supermartingale property (2.2) that

$$0 \geq E_t\left(\hat{S}^\delta_s - \hat{S}^\delta_t\right) = E_t\left(\hat{S}^\delta_t - 1\right) \geq 0.$$

Since one has $\hat{S}^\delta_s \geq 1$ and $E_t(\hat{S}^\delta_s) \leq 1$, it can only follow that $\hat{S}^\delta_s = 1$. This means that one obtains at time s the equality $S^\delta_s = S^{\delta_*}_s$. Therefore, according to Definition 5.2, the portfolio S^δ does not systematically outperform the numeraire portfolio. □

As a consequence of Theorem 5.3 one can conclude that no active fund manager can systematically outperform the numeraire portfolio. Obviously, if the current market portfolio is not the numeraire portfolio, which is most likely, and a fund manager approximates well the numeraire portfolio, then by Theorem 5.1 this fund will outperform in the long run the market portfolio.

Finally, as a fourth precise manifestation of "best" performance of the numeraire portfolio we derive its growth optimality which has been established under classical assumptions, for instance, in Long (1990) and Becherer (2001). The *expected growth* $g^\delta_{t,h}$ of a strictly positive portfolio S^δ over the time period $(t, t+h]$ is given by the conditional expectation

$$g^\delta_{t,h} = E_t\left(\ln\left(\frac{S^\delta_{t+h}}{S^\delta_t}\right)\right)$$

for $t, h \geq 0$. To identify the strictly positive portfolio that maximizes the expected growth, let us perturb at time t the investment in a given strictly positive portfolio $S^{\underline{\delta}}$ by some small fraction $\varepsilon \in (0, \frac{1}{2})$ with some nonnegative portfolio S^δ. For analyzing the changes in the expected growth of the perturbed portfolio S^{δ_ε}, we define the *derivative of expected growth* in the direction of S^δ as the limit

$$\left.\frac{\partial g^{\delta_\varepsilon}_{t,h}}{\partial \varepsilon}\right|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(g^{\delta_\varepsilon}_{t,h} - g^{\underline{\delta}}_{t,h}\right) \tag{5.4}$$

for $t, h \geq 0$. Obviously, if the portfolio that maximizes expected growth coincides in (5.4) with the portfolio $S^{\underline{\delta}}$, then the resulting derivative of expected growth will always be less than or equal to zero for all nonnegative portfolios S^δ. This leads to the following definition of growth optimality:

Definition 5.4 *A strictly positive portfolio $S^{\underline{\delta}}$ is called* growth optimal *if the corresponding derivative of expected growth is less than or equal to zero for all* *nonnegative portfolios S^δ, that is,*

$$\left.\frac{\partial g^{\delta_\varepsilon}_{t,h}}{\partial \varepsilon}\right|_{\varepsilon=0} \leq 0$$

for all t, $h \geq 0$.

Note that this definition is different to the classical characterization of a growth optimal portfolio, which is typically based on the maximization of expected logarithmic utility from terminal wealth, used in Kelly (1956) and later also employed in a stream of literature, including Latané (1959), Breiman (1960), Hakansson (1971), Merton (1973a), Roll (1973) and Markowitz (1976), among many others. The following statement offers a convenient possibility for the identification of the numeraire portfolio:

Theorem 5.5 *The numeraire portfolio is growth optimal.*

Proof: For two consecutive times t and $t+h$ with $h > 0$; $\varepsilon \in (0, \frac{1}{2})$; and a nonnegative portfolio S^δ, with $S^\delta_t > 0$, consider the perturbed portfolio S^{δ_ε} with the choice $S^{\delta}_t = S^{\delta_*}_t$ in (5.4), yielding a portfolio ratio $A^{\delta_\varepsilon}_{t,h} = \varepsilon\, A^\delta_{t,h} - (1-\varepsilon)\, A^{\delta_*}_{t,h} > 0$. One then obtains by the inequality $\ln(x) \leq x - 1$ for $x \geq 0$, the relations

$$G^{\delta_\varepsilon}_{t,h} = \frac{1}{\varepsilon}\,\ln\left(\frac{A^{\delta_\varepsilon}_{t,h}}{A^{\delta_*}_{t,h}}\right) \leq \frac{1}{\varepsilon}\left(\frac{A^{\delta_\varepsilon}_{t,h}}{A^{\delta_*}_{t,h}} - 1\right) = \frac{A^{\delta}_{t,h}}{A^{\delta_*}_{t,h}} - 1 \tag{5.5}$$

and

$$G^{\delta_\varepsilon}_{t,h} = -\frac{1}{\varepsilon}\,\ln\left(\frac{A^{\delta_*}_{t,h}}{A^{\delta_\varepsilon}_{t,h}}\right) \geq -\frac{1}{\varepsilon}\left(\frac{A^{\delta_*}_{t,h}}{A^{\delta_\varepsilon}_{t,h}} - 1\right) = \frac{A^{\delta}_{t,h} - A^{\delta_*}_{t,h}}{A^{\delta_\varepsilon}_{t,h}}. \tag{5.6}$$

Because of $A^{\delta_\varepsilon}_{t,h} > 0$ one obtains from (5.6) for $A^\delta_{t,h} - A^{\delta_*}_{t,h} \geq 0$ the inequality

$$G^{\delta_\varepsilon}_{t,h} \geq 0, \tag{5.7}$$

and for $A^\delta_{t,h} - A^{\delta_*}_{t,h} < 0$ because of $\varepsilon \in (0, \frac{1}{2})$ and $A^\delta_{t,h} \geq 0$ the relation

$$G^{\delta_\varepsilon}_{t,h} \geq -\frac{A^{\delta_*}_{t,h}}{A^{\delta_\varepsilon}_{t,h}} = -\frac{1}{1-\varepsilon+\varepsilon\,\frac{A^{\delta}_{t,h}}{A^{\delta_*}_{t,h}}} \geq -\frac{1}{1-\varepsilon} \geq -2. \tag{5.8}$$

Summarizing (5.5)–(5.8) yields the upper and lower bounds

$$-2 \leq G^{\delta_\varepsilon}_{t,h} \leq \frac{A^{\delta}_{t,h}}{A^{\delta_*}_{t,h}} - 1, \tag{5.9}$$

where by Definition 2.1

$$E_t\left(\frac{A^{\delta}_{t,h}}{A^{\delta_*}_{t,h}}\right) \leq 1. \tag{5.10}$$

By using (5.9) and (5.10) it follows by the Dominated Convergence Theorem, see Shiryaev (1984), that

$$\begin{aligned}\frac{\partial g^{\delta_\varepsilon}_{t,h}}{\partial \varepsilon}\bigg|_{\varepsilon=0} &= \lim_{\varepsilon\to 0+} E_t\left(G^{\delta_\varepsilon}_{t,h}\right) = E_t\left(\lim_{\varepsilon\to 0+} G^{\delta_\varepsilon}_{t,h}\right)\\ &= E_t\left(\frac{\partial}{\partial\varepsilon}\ln\left(\frac{A^{\delta_\varepsilon}_{t,h}}{A^{\delta_*}_{t,h}}\right)\bigg|_{\varepsilon=0}\right) = E_t\left(\frac{A^{\delta}_{t,h}}{A^{\delta_*}_{t,h}}\right) - 1.\end{aligned}$$

This proves by condition (2.1) and Definition 5.4 that the numeraire portfolio S^{δ_*} is growth optimal. $\square$

As shown in Platen & Heath (2006) and Platen (2006a) the growth optimal portfolio plays a central role in the intertemporal capital asset pricing model, see Merton (1973a), and the mean variance portfolio optimization, see Markowitz (1976). From an investor's perspective and also for the purpose of derivative pricing it is desirable to have a good proxy for the numeraire portfolio which can be identified by Theorem 5.5 as the growth optimal portfolio. For instance, for continuous markets it is well known how to construct theoretically the growth optimal portfolio, see Merton (1973a). To implement the optimal strategy one needs to estimate instantaneous expected returns, volatilities and correlations. Unfortunately, a sufficiently accurate estimation of expected returns from available market data seems to be not feasible in practice, even for the simplest security price models when exhausting all historical data. The observation periods necessary are simply too long under any reasonable model assumptions. However, in Platen & Heath (2006) a Diversification Theorem states in a jump-diffusion market under a mild regularity condition that globally diversified portfolios evolve rather similarly, which is easily confirmed by empirical evidence. Furthermore, if the number of primary security accounts tends to infinity and the fractions in a portfolio converge to zero, then this diversified portfolio approximates asymptotically the numeraire portfolio. In Le & Platen (2006) a proxy for the numeraire portfolio of the world equity market has been constructed. As permitted by Theorem 5.1, its long term growth rate turns out to be significantly larger than that of the market portfolio. This indicates that a globally diversified market index can be used as proxy for the numeraire portfolio and, thus, as benchmark for the benchmark approach. Such portfolio is of value as benchmark for fund management but its inverse is also useful as stochastic discount factor for derivative pricing. For the latter purpose one finally needs to model its dynamics to be able to calculate expectations of benchmarked contingent claims, as will be demonstrated in the following example.

6 An Extreme Maturity Zero Coupon Bond

This final section provides details on the introductory example, partly visualized in Figure 1.1. In a continuous market model we interpret a US dollar savings account as our locally riskless security S_t^0 and the S&P500 total return index as numeraire portfolio $S_t^{\delta_*}$. There may always exist better proxies for the numeraire portfolio. However, when choosing any better performing portfolio as proxy, the following illustrations, involving an extremely long time period, would become even more dramatic due to Theorem 5.1.

We use as short rate for the US dollar savings account of a generic US-investor the US 90 Day T Bill Rate plus 0.4%. For the S&P500 accumulation index we employ monthly S&P500 total return data for the period from January 1920 until September 2007, reconstructed by Global Financial Data. The logarithm of the savings account discounted S&P500 is shown in Figure 6.1, covering the

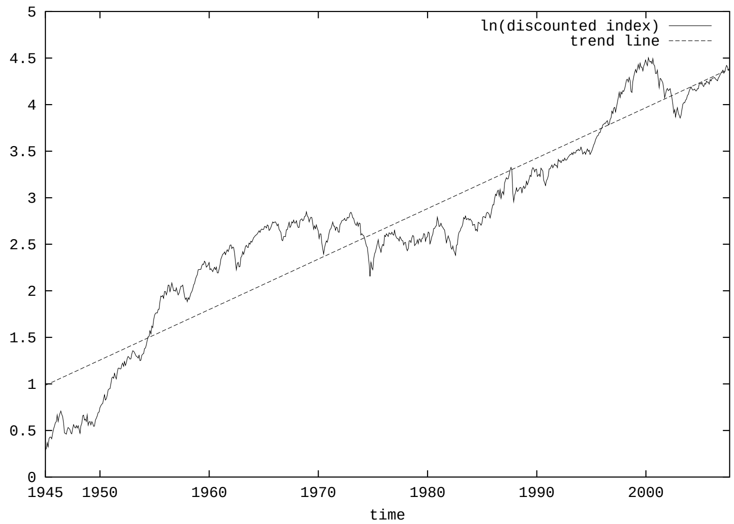

Figure 6.1: Logarithm of discounted S&P500 total return index.

period after the second world war. To simplify our calculations it is assumed that the short rate is deterministic. By making the short rate stochastic we would mainly complicate the exposition, but would still obtain very similar and eventually slightly more pronounced results, due to a stochastic effect on bond prices resulting from Jensen's inequality.

Let $P^*(t,T) = \frac{S_t^0}{S_T^0}$ denote the *savings bond* which replicates in a self-financing manner at the maturity date $T > 0$ the payoff of \$1. Its logarithm is shown in Figure 1.1, whereas its benchmarked value is displayed in Figure 6.2. The benchmarked savings bond shows a systematic downward trend. Therefore, it does not seem to form a fair price process. This systematic negative trend makes perfect economic sense. It simply reflects the presence of the well-known equity premium.

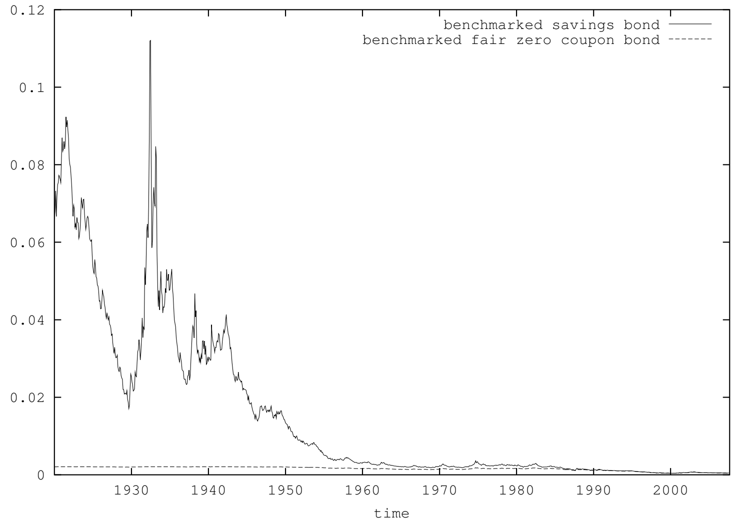

Figure 6.2: Benchmarked savings bond and benchmarked fair zero coupon bond.

On the other hand, by (3.2) the minimal value at time $t \in [0,T]$ of all self-financing portfolios that replicate \$1 at maturity T equals the price of the fair zero coupon bond

$$P(t,T) = S_t^{\delta_*}\, E_t \left(\frac{1}{S_T^{\delta_*}} \right), \tag{6.1}$$

see also (3.4).

To calculate the price of a fair zero coupon bond, one needs to compute the conditional expectation in (6.1). For this purpose one has to model the distribution of the benchmarked payoff $(S_T^{\delta_*})^{-1}$. It is clear from Figure 6.2 that due to the downward trend of the benchmarked savings bond, any realistic model should concentrate the probability density for $(S_T^{\delta_*})^{-1}$ at rather low values. One possibility for modeling $(S_T^{\delta_*})^{-1}$ reasonably well is suggested in Platen & Heath (2006):

The discounted numeraire portfolio $\bar{S}_t^{\delta_*} = \frac{S_t^{\delta_*}}{S_t^0}$ satisfies in a general continuous market the stochastic differential equation

$$d\,\bar{S}_t^{\delta_*} = \alpha_t\, dt + \sqrt{\bar{S}_t^{\delta_*}\, \alpha_t}\, dW_t. \tag{6.2}$$

Here $W = \{W_t, t \geq 0\}$ is a Wiener process, and $\alpha = \{\alpha_t, t \geq 0\}$ is a strictly positive process, modeling the trend of $\bar{S}_t^{\delta_*}$. If at time t the trend α_t of the discounted numeraire portfolio is modeled by the exponential function $\alpha_t = \alpha \exp\{\eta\, t\}$, then the stylized version of the *minimal market model* (MMM) emerges from (6.2). Under this simple two parameter model the benchmarked savings bond is not fair, see Platen & Heath (2006). This models the systematic downward trend of the benchmarked savings bond observed in Figure 6.2. Of course, other models could be constructed with savings bonds that are not fair. Such models would yield qualitatively similar results as those reported below.

To calibrate the model using our historical data we need to estimate the *net growth rate* η and the *scaling parameter* α. By standard linear regression we identify the average slope of the logarithm of the discounted S&P500 accumulation index using the data for the period from January 1945 until September 2007, which we consider to be reasonably reliable. Figure 6.1 includes the respective trendline. This gives us an estimate for the net growth rate η of about 0.0511, with an R^2 of 0.88. The estimated net growth rate is consistent with results from various studies on the equity premium in the literature, where the net growth rate for the US market during the last century was mostly estimated close to about 5%, see for instance Dimson, Marsh & Staunton (2002). However, in our calculations below it would qualitatively not matter, if we would use a two percent larger or smaller net growth rate. The remaining parameter in the MMM is the scaling parameter α, which can also be estimated by linear regression. For this purpose we exploit the fact that, under the stylized MMM, the slope of the quadratic variation $V_{t_i} = \sum_{\ell=1}^{i} (Z_{t_\ell} - Z_{t_{\ell-1}})^2$, $i \in \{1, 2, \ldots\}$, of the square root of the normalized index $Z_t = \sqrt{\frac{\bar{S}_t^{\delta_*}}{\alpha \exp\{\eta t\}}}$, equals asymptotically $\frac{1}{4}$, for vanishing h and $t_i = h\,i$, as is explained in Platen & Heath (2006). The quadratic variation V_t with the resulting trend line is shown in Figure 6.3, which yields the estimate $\alpha \approx 0.01429$ for the

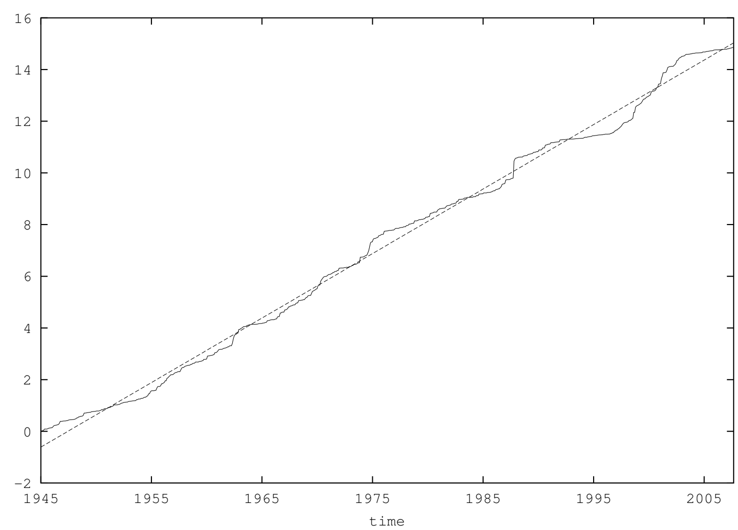

Figure 6.3: Quadratic variation V_t and trend line.

scaling parameter with an R^2 value of 0.995. We add for completeness that the initial value $\bar{S}_0^{\delta_*}$ of the discounted S&P500 was 0.3865. Under the stylized MMM the explicitly known transition density of the discounted numeraire portfolio $\bar{S}_t^{\delta_*}$ is a noncentral chi-square density. This yields by (6.1) for the fair zero coupon bond the explicit pricing formula

$$P(t,T) = P^*(t,T)\left(1 - \exp\left\{-\frac{2\,\eta\,\bar{S}_t^{\delta_*}}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}\right\}\right) \qquad (6.3)$$

for $0 \leq t \leq T < \infty$, see Platen & Heath (2006). Figure 6.2 plots the evolution of the benchmarked fair zero coupon bond price with maturity T in September

2007 as the dashed line. The price of the benchmarked fair zero coupon bond remains by (6.3) before maturity always below that of the benchmarked savings bond. This reflects the fact that the benchmarked savings bond is not fair. Only the benchmarked fair zero coupon bond provides with its current value the best forecast for the benchmarked payoff at maturity. It represents the minimal replicating price process. All other benchmarked replicating portfolios have some downward trend and constitute somehow not the best investment, since one can do better by using the fair zero coupon bond. Only, the benchmarked fair zero coupon bond is trendless and in this sense the best investment with respect to the given payoff.

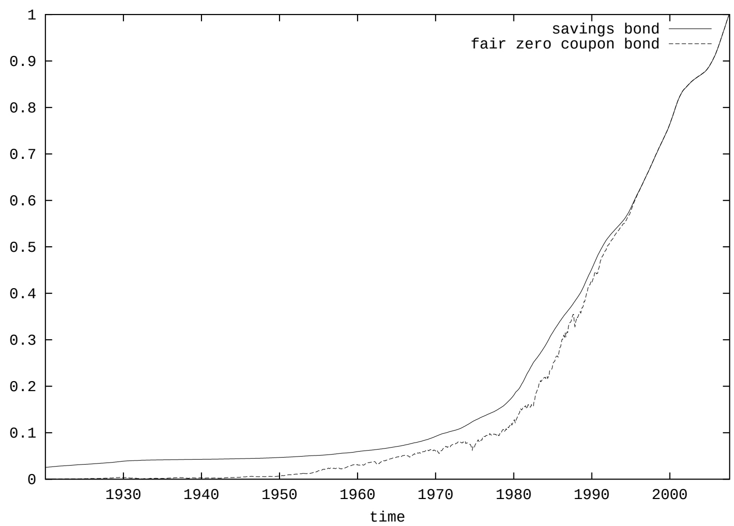

Figure 6.4: Savings bond and fair zero coupon bond.

Figure 6.4 displays in US dollar denomination the price evolution of the savings bond and the fair zero coupon bond. Both self-financing portfolios replicate the payoff of \$1 at maturity. We emphasize that no wealth is thrown away or comes from extra sources. In Figure 6.4 the two wealth processes start with significantly different initial capital but replicate the same payoff. Obviously, this contradicts the Law of One Price. The savings bond has in our example in January 1920 a price of $P^*(0,T) \approx \$0.0255$. The fair zero coupon bond is far less expensive and priced at only $P(0,T) \approx \$0.0008$. Consequently, the fair zero coupon bond costs less than 3.2% of the price of the savings bond.

The above MMM demonstrates that a market can have two self-financing replicating portfolios that are significantly different. Obviously, this is possible as soon as a primary security account is not fair, which seems to be the case for most developed economies when looking at their downward trending benchmarked savings accounts. A look at Figure 6.2 shows that any model, which captures reasonably well the likely distribution of the low benchmarked payoff $(S_T^{\delta_*})^{-1}$ at maturity, would yield a similar low initial benchmarked fair zero coupon bond price. There is sufficient robustness in what we discussed above to make similar sense under alternative models which capture downward trending savings accounts.

Finally, let us demonstrate that the obtained fair zero coupon bond price is realistic also from a hedging point of view. For this purpose we form a self-financing hedge portfolio applying delta hedging under the above described MMM. We obtain from (6.3) the number of units

$$\begin{aligned}\delta_t^* &= \frac{\partial \bar{P}(t,T)}{\partial \bar{S}_t^{\delta_*}} \\ &= P^*(0,T)\,\exp\left\{\frac{-2\,\eta\,\bar{S}_t^{\delta_*}}{\alpha\,(\exp\{\eta\,T\}-\exp\{\eta\,t\})}\right\}\frac{2\,\eta}{\alpha\,(\exp\{\eta\,T\}-\exp\{\eta\,t\})}\end{aligned}$$

of the benchmark $S_t^{\delta_*}$ to be held at time $t \in [0,T)$. The remaining wealth is invested in the savings account S_t^0. Figure 6.5 displays the fraction $\frac{\delta_t^*\,S_t^0}{P(t,T)}$

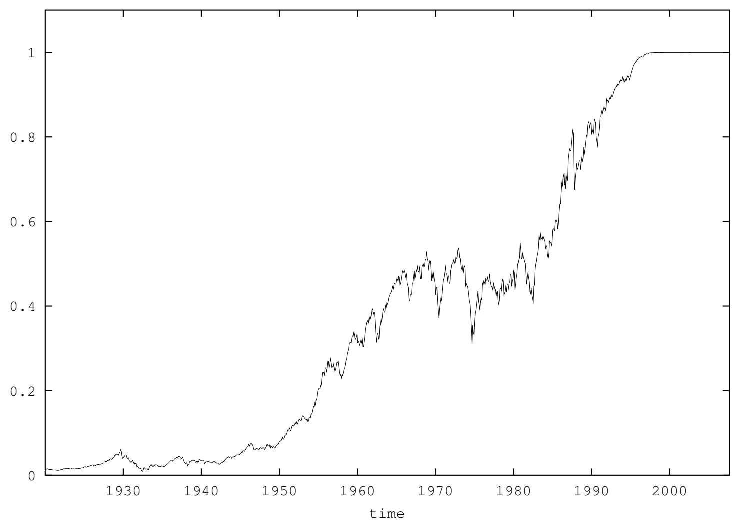

Figure 6.5: Fraction invested in the savings account.

invested in the savings account at time t, as it evolves for our data set. The self-financing hedge portfolio which we form in our hedge simulation starts in January 1920. At the end of each month the fraction of wealth invested in the S&P500 is reallocated in a self-financing manner according to the above prescription. We note from Figure 6.5 that initially almost no wealth is invested in the savings account. For a long time the fair zero coupon bond exploits almost fully the superior long term growth of the S&P500. This is also very intuitive. Closer to maturity the wealth is systematically shifted to the savings account. About ten years before maturity almost all wealth becomes fully invested in the savings account. The resulting benchmarked profit and loss, which is defined as the difference between the benchmarked fair zero coupon bond price and its initial benchmarked price minus the benchmarked gains from trade, turns out to be surprisingly small. The observed maximum absolute benchmarked profit and loss amounts in our case only to a value of about 0.00008, which is surprisingly small given monthly reallocation. The path of the hedge portfolio, when additionally plotted in Figure 6.4, would be visually identical to the path of the already displayed fair zero coupon bond.

The above example demonstrates that the classical Law of One Price can be violated under the benchmark approach. It may be a disadvantage for a market participant to rely on it when investing for the long run. The likely presence of portfolios in the real market that are not fair suggests new challenges for research. It also provides potentially new business lines, for instance, involving least expensive extreme maturity derivatives. The above presented benchmark approach provides a robust and general theoretical basis for exploring systematically a range of new effects so far not studied under the classical theories.

Conclusion

The paper presented a unifying approach to asset pricing. It requires only the existence of the numeraire portfolio. This portfolio turns out to represent in several ways the “best” performing, tradable positive portfolio. It was demonstrated that in this general setting the classical Law of One Price does no longer hold. It has been replaced by the Law of the Minimal Price, according to which the minimal replicating price process of a given contingent claim can be found as the one which is trendless when expressed in units of the numeraire portfolio. The paper indicates that by exploiting the Law of the Minimal Price certain extreme maturity derivatives will become significantly less expensive than suggested by the currently prevailing pricing paradigms. The inverse of the numeraire portfolio becomes the stochastic discount factor when determining the minimal replicating price process. Classical pricing rules are unified and generalized by the resulting real world pricing formula. Weak forms of arbitrage are allowed under the proposed benchmark approach. They do not harm the economic viability of the market model. Since strong arbitrage is automatically excluded by the existence of the numeraire portfolio, pricing by excluding strong arbitrage makes no sense. New effects can be captured under the proposed general approach which will be the topic of forthcoming work.

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