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Abstract

Implementation and collapse of exchange rate pegging schemes are recurrent events. A currency crisis (pegging) is usually followed by an economic downturn (boom). This essay explains why a benevolent government should pursue fiscal and monetary policies that lead to those recurrent currency crises and subsequent periods of pegging. It is shown that the optimal policy induces a competitive equilibrium that displays a boom in periods of below average devaluation and a recession in periods of above average devaluation. A currency crisis (pegging) can be understood as an optimal policy answer to a recession (boom).

Keywords: exchange rate, business cycles.
JEL: E31, F31, F41.

1 Introduction

In June 1975 a 16% currency devaluation took place in Argentina. Up to the previous month the nominal exchange rate was fixed. The GDP, which had grown 8.5% in 1974, fell 2.7% in 1975. A reverse episode happened in 1991. The Argentinean currency devaluated 68.7% in January and only 5.9% in February. The GDP rose 8.9% in 1991, against the modest increase of 0.1% in 1990.

Instead of constituting an isolated episode, the Argentinean experience illustrates a general pattern. Frankel and Rose [5], Klein and Marion [8], and Milesi-Ferretti and Razin [13] show that implementation and collapse of exchange rate pegging schemes are recurrent events. They also show that currency crises are frequently followed by a fall below the trend in output and consumption. Kiguel and Liviatan [7] and Végh [19] provide evidence that the reverse facts plus a deterioration of the current account often accompany a pegging.

The first goal of this paper is to explain why governments optimally choose to pursue actions that lead to a recurrent implementation of pegging policies and their subsequent breakdown. The second goal is to reproduce the corresponding business cycles regularities.

The environment studied in this paper is an infinite horizon stochastic one. The model is a cash-credit two sector (tradable and non tradable) small open economy. Consumers face a cash-in-advance constraint on fraction of their purchases of non tradables. Government consumption and few other variables are stochastic processes. The government also chooses tax rates on labor income. But these tax rates are not fully state contingent. For this reason the taxation is said to be incomplete.

The essay builds on Lucas and Stokey's [11] seminal work on optimal monetary and fiscal policy. The papers addresses the problem of selecting the optimal monetary, exchange rate and incomplete labor income taxation policies when the government consumption is exogenous.

The optimal devaluation rate is a non constant function of the economy's state. As this state changes, the devaluation rate oscillates. That will lead to the implementation and collapse of exchange rate pegging policies. This policy switch can happen infinitely often. Most (if not all) of the previous research on currency crisis can explain at most one devaluation episode.

In periods of high government consumption the devaluation is higher than in times of surpluses. The intuition is simple. Whenever fiscal expenditures are relatively high, the optimal policy will prescribe a combination of higher taxation and debt issuing. Since tax rates are not fully state contingent, it is optimal to raise additional tax revenue through inflation. A higher inflation level will determine a higher rate of devaluation of the domestic currency. A positive technological shock that leads to an output rise will reduce the fiscal deficit as a fraction of GDP. The previous reasoning shows that currency devaluation and technological shocks are negatively correlated.

Consider the business cycle facts associated to a pegging. In response to shocks that decrease government consumption and increase the productivity, the optimal policy prescribes a decrease in the devaluation rate. Higher productivity leads to higher output. The combination of lower devaluation rate and higher output generates an income effect. People increase their consumption sufficiently to induce a current account deficit. In a similar fashion, shocks that increase government consumption and reduce productivity lead to a higher devaluation and induce the empirical regularities associated with currency crises.

There exists a traditional wisdom that a currency crisis triggers a recession. In this essay, a large devaluation does not cause a recession. The government optimally chooses to devaluate the currency when the economy hits a bad state. This finding has a striking policy implication. Given that the economy is facing a recession, a devaluation is an optimal answer. Any policy that prevents or just postpones the devaluation will lead to welfare losses.

Exchange rate devaluations are often viewed as a consequence of time consistency problems, as in Obstfeld [14] and Giavazzi and Pagano [6]. In this essay, devaluations are fully anticipated and are optimal choices for a government that can credibly commit to a policy.

Obstfeld and Rogoff [15] advocated the use of models with solid micro foundations to study exchange rate policy. Obstfeld [14] pointed out the relevance of understanding how the exchange rate policy is selected. Today, there exist several papers on open economy macro with micro foundations. However, few of them study the selection of the exchange rate policy. This paper attempts to answer the question raised by Obstfeld [14] using the approach advocated by Obstfeld and Rogoff [15].

Several studies on currency crises take an exchange rate pegging as given and explain why a currency crisis must happen, as in Krugman [9]. Other essays explain why a government chooses to devaluate, even if pegging is still possible, as in Obstfeld [14] and Giavazzi and Pagano [6]. Nevertheless, no essay is aimed at explaining why a pegging is ever introduced. This essay innovates by adopting a unified framework to explain both pegging episodes and currency crises.

So far, the research in the field of exchange rate based stabilization has taken the exchange rate policy as exogenous. Rebelo [16] states that it is important to understand the timing of the stabilization. This paper shows that a stabilization may be an optimal answer to a fiscal contraction. This important step is taken in the context of a model that replicates several of the stylized facts listed by Mendoza and Uribe [12], Rebelo [16], and Rebelo and Végh [17].

The paper is organized as follows. The model is described in section 2. Section 3 is devoted to characterization and examples of competitive equilibrium. The problem of selecting policies that lead to the best competitive equilibrium is studied in section 4, along with the properties of this efficient outcome. Section 5 concludes. Technical details are presented in the appendix.

2 The Economy

Consider a small country populated by a continuum of identical infinitely lived households with Lebesgue measure one and a government. A household is composed by a shopper and a worker, who is endowed with one unit of time.

That country produces two non tradable goods. The first is consumed by households (c_1^N). The second is consumed by households (c_2^N) and government (g^N). The country also produce a tradable good, which is consumed by households (c^T) and a government (g^T). This last good can also be exported (x) or imported ($-x$).

Transactions take place in this economy in a particular way. At a first stage of each date t spot markets for goods and labor services operates. At the second stage, security and currency markets operate.

A domestic currency M circulate in this economy. Two types of securities are traded: a claim B, with maturity of one period, to one unit of M and a claim B^*, with the same maturity, to one unit of some foreign currency. Foreigners do not sell or buy claims to the domestic currency. Government and residents can purchase and/or sell the claims B^* at an exogenous price, in terms of the foreign currency, q_t^*.

Workers cannot sell their services outside the country. Shoppers face a cash-in-advance constraint. The purchases of c_1^N must be paid for with the domestic currency. Except for the purchases of that good, all other transactions are liquidated during the security and currency trading session. The date t price, in terms of the foreign currency, of the tradable good is exogenous and equal to p_t^*.

Technology is described by $0 \leq y^T \leq \theta^T (l^T)^{\alpha^T}$ and $0 \leq y^N \leq \theta^N (l^N)^{\alpha^N}$, where y^T is the tradable output and l^T is the amount of labor allocated to the production of that good. A similar meaning is assigned to y^N and l^N. Both α^T and α^N lie in the set $(0, 1]$.

Let $s_t = (\theta_t^T, \theta_t^N, g_t^T, g_t^N, p_t^*, q_t^*)$. The sequence $\{s_t\}_{t=0}^{\infty}$ is a stochastic process on some probability space $(\Omega, \mathcal{F}, P)$. Each s_t has a support contained in the finite set $S = \Theta^T \times \Theta^N \times G^T \times G^N \times P^* \times Q^*$. These sets satisfy $\Theta^T \subset \mathbb{R}_{++}$, $\Theta^N \subset \mathbb{R}_{++}$, $G^T \subset \mathbb{R}_{+}$, $G^N \subset \mathbb{R}_{++}$, $P^* \subset \mathbb{R}_{++}$, and $Q^* \subset (0, 1)$. The object s^t stands for a history $(s_0, ..., s_t)$ of events and $s^\infty = (s_0, s_1, ...)$.

For a given t, S^t denotes the set of all possible histories s^t and S^∞ is the set of all possible s^∞. For a given s^t in S^t, $\mu(s^t)$ denotes the probability that this particular s^t will be realized. The realization of s_t is known at the beginning of date t. If $k \leq t$, $\mu(s^t | s^k)$ denotes the conditional probability of s^t given s^k; $S^t(s^k)$ is the set of all $s^t \in S^t$ such that the first k events in s^t are equal to s^k. In other words, $S^t(s^k)$ is the set of all possible continuations of the history s^k up to date t. Whenever there is no danger of confusion, $S^t(s^k)$ will be denoted by S_k^t. As usual, $\{[f(s^t)]_{s^t \in S^t}\}_{t=0}^{\infty}$ is a history contingent sequence and $\|f\|_\infty = \sup_t \sup_{s^t \in S^t} |f(s^t)|$.

Each good is produced by a single competitive firm. Let $l(s^t)$ denote the amount of labor supplied by each household at date t if the history s^t occurs. Other variables indexed by s^t have analogous meaning. Feasibility requires

$$l^T(s^t) + l^N(s^t) = l(s^t) \leq 1 \ , \ c_1^N(s^t) + c_2^N(s^t) + g_t^N = \theta_t^N[l^N(s^t)]^{\alpha^N} \ ,$$

$$c^T(s^t) + g_t^T + x(s^t) = \theta_t^T[l^T(s^t)]^{\alpha^T} \ . \tag{1}$$

The government finances the sequence $\{g_t^T, g_t^N\}_{t=0}^{\infty}$ by issuing and withdrawing the domestic currency; by issuing and redeeming claims B of maturity of one period to one unit of the domestic currency; by purchasing and selling B^*; and taxing labor income at a proportional tax τ. The date zero tax rate is exogenous and equal to some value τ_0. At other periods, that variable depend on the history s^{t+1} but it must satisfy the constraint

$$\tau(s^t, s_{t+1}) = \tau(s^t, \bar{s}_{t+1}), \forall s^t, s_{t+1}, \bar{s}_{t+1} \ . \tag{2}$$

The government budget constraint is

$$E(s^t)p_t^* g_t^T + p^N(s^t)g_t^N + B(s^{t-1}) + E(s^t)q_t^* B_G^*(s^t) + M(s^{t-1}) =$$

$$\tau(s^t)w(s^t)l(s^t) + q(s^t)B(s^t) + E(s^t)B_G^*(s^{t-1}) + M(s^t) \ , \tag{3}$$

where $p^N(s^t)$, $w(s^t)$ and $q(s^t)$ are the respective date t monetary prices (in terms of the domestic currency) of the non tradable good, labor services and the domestic claim; $E(s^t)$ is the nominal exchange rate; $B_G^*(s^t)$ stands for the foreign assets held by the government at the end of date t; $M(s^t)$ and $B(s^t)$ are the amount of domestic currency and public debt held by the households at the end of date t. All those variables are conditional on the history of events. A negative value for $B_G^*(s^t)$ means that the government is borrowing abroad, while a negative value for $B(s^t)$ means that the government is lending to domestic residents. At $t = 0$ the government holds an initial amounts $\bar{B}_G^*$ of foreign assets. To avoid Ponzi schemes, a standard boundedness constraint $\|B_G^*/p^*\|_\infty \leq A < \infty$ is imposed on the government foreign assets.

The function $u : \mathbb{R}_+^3 \times [0,1] \to \mathbb{R} \cup \{-\infty\}$,

$$u(c^T, c_1^N, c_2^N, l) = (c^T)^{\gamma^T}(c_1^N)^{\gamma_1}(c_2^N)^{\gamma_2}(1-l)^{\gamma^l} \ , \tag{4}$$

is the typical household period utility function. The γ's are positive and add up to 1. Intertemporal preferences are described by

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \mu(s^t) u\left(c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t)\right) \ , \tag{5}$$

where $\beta \in (0,1)$. The date t budget constraint of the typical household is

$$E(s^t)p_t^* c^T(s^t) + p^N(s^t)[c_1^N(s^t) + c_2^N(s^t)] + q(s^t)B(s^t) +$$

$$E(s^t)q_t^* B_H^*(s^t) + M(s^t) \leq [1 - \tau(s^t)]w(s^t)l(s^t) + B(s^{t-1}) +$$

$$E(s^t)B_H^*(s^{t-1}) + M(s^{t-1}) + \psi^T(s^t) + \psi^N(s^t) \ , \tag{6}$$

where $B_H^*(s^t)$ stands for the foreign assets held by the household at the end of date t if history s^t occurs and $\psi^T(s^t)$ and $\psi^N(s^t)$ are the date t profits. The constraint $\left\|B/p^N\right\|_\infty, \left\|B_H^*/p^*\right\|_\infty \leq A$ prevents Ponzi games. People face the cash-in-advance constraint

$$p^N(s^t)c_1^N(s^t) \leq M(s^{t-1}) \ . \tag{7}$$

Given initial asset holdings $(\bar{M}, \bar{B}, \bar{B}_H^*)$, a household chooses a history contingent sequence $\{[c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t), B(s^t), B_H^*(s^t)]_{s^t \in S^t}\}_{t=0}^\infty$ to maximize (5) subject to the constraints (6), (7), and $l(s^t) \leq 1$. Except for $B(s^t)$ and $B_H^*(s^t)$, all those variables are constrained to be non-negative. An additional boundedness condition $\left\|c^T\right\|_\infty, \left\|c_1^N\right\|_\infty, \left\|c_2^N\right\|_\infty, \left\|l\right\|_\infty, \left\|M/p^N\right\|_\infty < \infty$ is imposed on the consumer problem.

Adding the identities $\psi^N(s^t) + w(s^t)l^N(s^t) = p^N(s^t)[\ c_1^N(s^t) + c_2^N(s^t) + g_t^N]$ and $\psi^T(s^t) + w(s^t)l^T(s^t) = E(s^t)p_t^*[c^T(s^t) + g_t^T + x(s^t)]$ to (3) and (6) taken as equality, one obtains

$$p_t^* x(s^t) + B_G^*(s^{t-1}) + B_H^*(s^{t-1}) - q_t^* B_G^*(s^t) - q_t^* B_H^*(s^t) = 0 \ , \tag{8}$$

which is the balance-of-payments identity.

3 Competitive Equilibrium

A history contingent date t policy $(E(s^t), p^N(s^t), w(s^t), q(s^t), B_G^*(s^t), \tau(s^t))$ is denoted by $\varphi(s^t)$. A *policy* is a history contingent sequence $\varphi = \{[\varphi(s^t)]_{s^t \in S^t}\}_{t=0}^\infty$. Date t history contingent allocations $(c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t), l^N(s^t), l^T(s^t), x(s^t))$ and asset holdings $(M(s^t), B(s^t), B_H^*(s^t))$ are denoted, respectively, by $\chi(s^t)$ and $\zeta(s^t)$. Additionally, $\chi = \{[\chi(s^t)]_{s^t \in S^t}\}_{t=0}^\infty$ and $\zeta = \{[\zeta(s^t)]_{s^t \in S^t}\}_{t=0}^\infty$.

Definition 1 *A* competitive equilibrium *is an object* (φ, χ, ζ) *that satisfies the following properties: (i) given* φ*,* (χ, ζ) *provides a solution for the household problem; (ii)* $w(s^t) = p^N(s^t)\alpha^N\theta_t^N[l^N(s^t)]^{\alpha^N-1} = E(s^t)p_t^*\alpha^T\theta_t^T[l^T(s^t)]^{\alpha^T-1}$*; (iii) (1), (2), and (3) hold.*

The definition of competitive equilibrium does not place bounds on inflation (throughout this essay the term inflation will apply to the rate of increase in p^N). For future reference, it is convenient to spell out a particular boundedness requirement.

Definition 2 *A competitive equilibrium* (φ, χ, ζ) *is of* bounded inflation *if*

$$\exists \varepsilon > 0 : \varepsilon \leq \frac{p^N(s^t, s_{t+1})}{p^N(s^t)} \leq \frac{1}{\varepsilon} \ , \ \forall s^\infty \in S^\infty, \forall t \ . \tag{9}$$

The above condition prevents prices from increasing or decreasing "too much" in a single period.

The government can pursue several distinct policies. To clarify this point, consider the simple case in which the government has no source of revenue but inflation. For simplicity, assume that at date zero the government has no net debt and the public consumption is always positive. The government can balance its lifetime budget with a constant inflation rate and borrow abroad to finance temporary imbalances. It is also possible to balance the budget period by period solely with inflation tax. In this case the inflation does not need to be constant. Different policies will induce distinct competitive equilibria. Hence, the problem of selecting an efficient policy is not a trivial problem.

A set of competitive equilibrium allocations will be characterized in this section. That will reduce the problem of selecting an efficient policy to a standard constrained maximization problem.

To simplify the notation, $u(s^t)$, $u_T(s^t)$, $u_1(s^t)$, $u_2(s^t)$, and $u_l(s^t)$ will denote, respectively, the value of u and its partial derivatives $\partial u/\partial c^T$, $\partial u/\partial c_1^N$, $\partial u/\partial c_2^N$, and $\partial u/\partial l$ evaluated at the point $(c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t))$. The auxiliary variable $W(s^t)$ is defined according to

$$W(s^t) = \left\{\alpha^T c^T(s^t) - (1-\alpha^T)[x(s^t) + g_t^T]\right\} u_T(s^t) + u_1(s^t)c_1^N(s^t) +$$

$$\left\{\alpha^N c_2^N(s^t) - (1-\alpha^N)[c_1^N(s^t) + g_t^N]\right\} u_2(s^t) + u_l(s^t)l(s^t) \ .$$

There exist seven constraints with obvious economic meaning that must hold in any competitive equilibrium. A trivial condition is (1). The second is

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \mu(s^t) W(s^t) = u_1(s^0)c_1^N(s^0) +$$

$$u_2(s^0)\left[\frac{\bar{B}}{p^N(s^0)} + \frac{\bar{M}}{p^N(s^0)} - c_1^N(s^0)\right] + u_T(s^0)\frac{\bar{B}_H^*}{p_0^*} \ , \tag{10}$$

which is simply the consolidation of all date t budget constraints of the households. The third is a balance-of-payment constraint

$$-\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \mu(s^t) u_T(s^t) x(s^t) = u_T(s^0)\frac{\bar{B}_H^* + \bar{B}_G^*}{p_0^*} \ , \tag{11}$$

which requires imports to be financed by the country's initial wealth. The fourth requirement, ensuring that people's marginal rate of substitutions are consistent with the international interest rates and prices, is

$$q_t^* \frac{\mu(s^t)u_T(s^t)}{p_t^*} = \beta \sum_{s_{t+1} \in S_{t+1}} \frac{\mu(s^t, s_{t+1})u_T(s^t, s_{t+1})}{p_{t+1}^*} \ . \tag{12}$$

The fifth constraint is that households' marginal rate of substitution between tradables and non tradables must match the marginal rate of transformation between those types of goods, i.e.,

$$\frac{u_T(s^t)}{u_2(s^t)} = \frac{\alpha^N \theta_t^N [l^T(s^t)]^{1-\alpha^T}}{\alpha^T \theta_t^T [l^N(s^t)]^{1-\alpha^N}} \ . \tag{13}$$

This equation is also an implementability condition for the real exchange rate $\frac{E(s^t)p_t^*}{p^N(s^t)}$. The sixth

$$\frac{u_l(s^t, s_{t+1})}{u_2(s^t, s_{t+1})} \frac{[l^N(s^t, s_{t+1})]^{1-\alpha^N}}{\alpha^N \theta_{t+1}^N} = \frac{u_l(s^t, \bar{s}_{t+1})}{u_2(s^t, \bar{s}_{t+1})} \frac{[l^N(s^t, \bar{s}_{t+1})]^{1-\alpha^N}}{\alpha^N \bar{\theta}_{t+1}^N} \ , \tag{14}$$

is an implementability constraint for (2), while

$$(1-\tau_0)\frac{\alpha^N \theta_0^N}{[l^N(s^0)]^{1-\alpha^N}} = -\frac{u_l(s^0)}{u_2(s^0)} \ , \tag{15}$$

ensures that the allocations are consistent with τ_0.

The above constraints are not enough to characterize a competitive equilibrium. Seven other conditions have to be imposed. The inequalities

$$p^N(s^0)c_1^N(s^0) \leq \bar{M} \tag{16}$$

$$u_2(s^t) \leq u_1(s^t) \tag{17}$$

ensure that cash-in-advance constraints hold. An implementability constraint for a transversality condition is

$$\lim_{t\to\infty} \beta^t \sum_{s^t \in S_k^t} \mu(s^t) u_1(s^t) c_1^N(s^t) = 0 \ . \tag{18}$$

The boundedness of foreign debt requires

$$\sup_k \sup_{s^k \in S^k} \frac{1}{u_T(s^k)} \left| \sum_{t=k}^{\infty} \sum_{s^t \in S^t} \beta^{t-k} \mu(s^t|s^k) u_T(s^t) x(s^t) \right| < \infty \ . \tag{19}$$

A similar constraint is required to ensure that $\left\|B/p^N\right\|_\infty < \infty$. However, it is not possible to characterize that condition for all competitive equilibria. Nevertheless, it is possible to do so for all equilibria with bounded inflation. If the inflation is bounded, it is enough to require

$$\sup_k \sup_{s^k \in S^k} \frac{1}{u_2(s^k)} \left| \sum_{t=k}^{\infty} \sum_{s^t \in S^t} \beta^{t-k} \mu(s^t|s^k) W(s^t) - u_1(s^k) c_1^N(s^k) \right| < \infty \ . \tag{20}$$

Inflation is bounded if there exists a positive ε such that

$$\varepsilon \leq \frac{\beta}{\mu(s^t)u_2(s^t)} \sum_{\hat{s}_{t+1}\in S} \mu(s^t,\hat{s}_{t+1})u_1(s^t,\hat{s}_{t+1})\frac{c_1^N(s^t,\hat{s}_{t+1})}{c_1^N(s^t,s_{t+1})} \leq \frac{1}{\varepsilon}\ . \tag{21}$$

As in definition 2, ε does not depend on the histories.

Proposition 1 (a set of competitive equilibria) *Let $\bar{M} > 0$. An array χ and a price $p^N(s^0) > 0$ satisfy (1) and (10)-(21) if and only if they are components of a competitive equilibrium (φ, χ, ζ) of bounded inflation.*
Proof. *See appendix.*

4 Ramsey Equilibrium

4.1 Definition and Characterization

The concept of competitive equilibrium does not impose optimality on the government behavior. In this section, a game in which the government is a player will be considered.

At date zero, before markets open, the government announces that will follow a policy φ. That policy cannot be changed in future dates (i.e., there is some commitment device that allows the government to credible stick to φ). Then, private agents will be allowed to trade. The government is benevolent and will choose φ to maximize (5).

Private agents actions depend on the prevailing policy. To keep track of that relation, let f denote a generic function that maps a vector (s^t, φ) into the space of the pairs $(\chi(s^t), \zeta(s^t))$. As before, $f(\varphi) = \{[f(s^t,\varphi)]_{s^t\in S^t}\}_{t=0}^{\infty}$. Abusing the notation, $u(f(s^t,\varphi))$ will denote u evaluated at the corresponding $(c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t))$ coordinates of $f(s^t,\varphi)$.

Definition 3 *A* Ramsey Equilibrium *is a pair (φ, f) satisfying: (i) for all $\bar{\varphi}$, $f(\bar{\varphi})$ provides solutions for both households' and firms' problems; (ii) the policy φ solves $\max_{\bar{\varphi}} \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \beta^t u(f(s^t,\bar{\varphi}))$ subject to (1), (3) and (2). A triple (φ, χ, ζ) is a* Ramsey outcome *if there exists a f such that (φ, f) is a Ramsey equilibrium and $f(\varphi) = (\chi, \zeta)$.*

Private agents are required to behave optimally for all policies, not only for the equilibrium one. This requirement is a natural consequence of the game being studied. When the government chooses φ it knows that people and firms will behave optimally, no matter the chosen policy. So, government uses this information when choosing φ. Note that this requirement is equivalent to subgame perfection, as pointed out by Chari and Kehoe [3].

Recall that $\{g_t^T, g_t^N\}_{t=0}^{\infty}$ is a stochastic process. Thus, the government problem consists in choosing paths for money supply, domestic debt, external borrowing, and tax rates to maximize people's welfare. One can see this problem as a simplified version of the problem faced by a benevolent government that takes the expenditures as given and it is not able to design tax rates that are fully state contingent.

In a Ramsey equilibrium, the government chooses a policy that will maximize people's welfare. Therefore, it is possible to characterize Ramsey outcomes through a standard maximization problem.

Proposition 2 *Suppose that* $(p^N(s^0), \chi)$ *solve*

$$\max_{(\bar{p}^N(s^0), \bar{\chi})} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t u(\bar{c}^T(s^t), \bar{c}_1^N(s^t), \bar{c}_2^N(s^t), \bar{l}(s^t))$$

subject to (1) and (10)-(15). If $(p^N(s^0), \chi)$ *satisfies (16)-(21), then* $(p^N(s^0), \chi)$ *is a component of some Ramsey outcome* (φ, χ, ζ)*.*

Proof. Suppose that $[p^N(s^0), \chi]$ solves the problem in question. From proposition 1, there exist φ and ζ that supports $[p^N(s^0), \chi]$ as a competitive equilibrium. It will be shown that (φ, χ, ζ) is a Ramsey outcome. For a given $\bar{\varphi}$, define $f(\bar{\varphi})$ as a solution of households' and firms' for this particular $\bar{\varphi}$. Trivially, $f(\varphi) = (\chi, \zeta)$. It remains to show that (φ, f) is a Ramsey equilibrium. The constraints (1) and (10)-(14) are not enough to characterize a competitive equilibrium. Thus, $f(\varphi)$ yields the highest welfare in a set of $\bar{\varphi}$'s that is a superset of all $\bar{\varphi}$'s that can be implemented as a competitive equilibrium. But φ can be implemented as a competitive equilibrium of bounded inflation. Therefore, $f(\varphi)$ yields the highest welfare in the set of $\bar{\varphi}$'s that can be implemented as a competitive equilibrium. Hence, (φ, f) satisfies all conditions of definition 3. ■

4.2 Examples

In all incoming examples it is assumed that $\bar{B} = \bar{B}_G^* = \bar{B}_H^* = 0$, $s^0 = a$, $\tau_0 = 20\%$, $p_t^* = 1$, $q_t^* = \beta$ and $g_t^T = 0$. The sequence $\{g_t^N, \theta_t^N, \theta_t^T\}_{t=0}^{\infty}$ is a Markov process on the state space $\{a, b\}$ with transition probabilities μ_{ab} and μ_{ba}. State space and transition probabilities are example specific. A detailed explanation of how to compute the optimal allocations is provided in section 6.4.

Example 1 (benchmark economy: the optimal policies) The current example will be adopted as a benchmark to the next ones. The state space satisfies $\theta_t^N = \theta_t^T = 1$, $g_a^N = 0.05$, and $g_b^N = 0.1$. The transition probabilities are $\mu_{ab} = 0.4$ and $\mu_{ba} = 0.7$. Let $\hat{E}$ denote the rate of devaluation of the nominal exchange rate. The optimal policies are described by

$$(\hat{E}(s^t), \tau(s^t)) = \begin{cases} (13.20\%, 20.78\%) & \text{if } (s_t, s_{t+1}) = (a, a); \\ (20.41\%, 20.78\%) & \text{if } (s_t, s_{t+1}) = (a, b); \\ (13.65\%, 20.86\%) & \text{if } (s_t, s_{t+1}) = (b, a); \\ (20.88\%, 20.86\%) & \text{if } (s_t, s_{t+1}) = (b, b). \end{cases}$$

Tax rates are roughly constant. Whenever the economy hits state b (the state with higher government consumption) the domestic currency devaluates almost 21%, while in state a the devaluation is close to 13.5%. Of course, histories ending with either (a, a, b) or (b, a, b) are associated to currency crashes. Histories ending with either (b, b, a) or (a, b, a) are associated to the introduction of a currency pegging. The other possible histories are not associated to either a crash or a pegging.

In the above example the consumption is higher in state a when compared to consumption in state b. Output displays the opposite behavior. The empirical evidence mentioned in section 2 states that in periods of higher devaluation both output and consumption fall and in periods of pegging these two variables grow faster (the measurement of consumption, output, and other variables is explained in details in section 6.3). Thus, example 1 fails to reproduce some of the quantitative features found in the data. This is a general feature of the model. An economy driven only by fiscal shocks cannot account for all patterns found in the real data.

Several alternative policies could be implemented as a competitive equilibrium. To illustrate some of these possibilities, three alternative policies will be presented below.

Example 2 (benchmark economy: the Friedman Rule) The policies

$$(\hat{E}(s^{t+1}), \tau(s^{t+1})) = \begin{cases} (-2.05\%, 25.78\%) & \text{if } (s_t, s_{t+1}) = (a, a); \\ (2.35\%, 25.78\%) & \text{if } (s_t, s_{t+1}) = (a, b); \\ (-1.62\%, 25.78\%) & \text{if } (s_t, s_{t+1}) = (b, a); \\ (2.80\%, 25.78\%) & \text{if } (s_t, s_{t+1}) = (b, b); \end{cases}$$

can be implemented in the benchmark economy. The associated allocations satisfy $u_1(s^t) = u_2(s^t)$. This condition implies that the $q(s^t) = 1$, that is, the nominal interest rate is zero. This is exactly the well known Friedman Rule (for a discussion of the Friedman Rule, see [2] or [4]).

The fact that the Friedman Rule can be implemented in this economy but it is not optimal is, at first glance, surprising. The economy is a two sector cash-credit good one. In an one sector cash-credit good closed economy environment the Friedman Rule is known to be optimal.

An obvious reason for this departure from the Friedman Rule is the incomplete taxation feature. However, that is not the only cause. The Friedman Rule would be optimal if all possible distorting taxation instruments were available to the Ramsey planner. For instance, the optimality of the Friedman Rule would require, among other conditions, that the consumption of tradable and non-tradable goods could be taxed at different rates – so that the implementability constraint (13) could be dropped from the Ramsey problem.

Besides the Friedman Rule with constant tax rates, there are several other attainable policies. In example 3 the exchange rate is fixed at whenever the economy hits state a twice in a roll and devaluates 34% otherwise, while the labor income taxation is constant at 20%. In example 4 there is no taxes on labor income and the exchange rate devaluates 180% every period.

Example 3 (benchmark economy: fix at (a, a)) The policies

$$(\hat{E}(s^{t+1}), \tau(s^{t+1})) = \begin{cases} (0\%, 20\%) & \text{if } (s_t, s_{t+1}) = (a, a); \\ (33.99\%, 20\%) & \text{if } (s_t, s_{t+1}) = (a, b); \\ (33.99\%, 20\%) & \text{if } (s_t, s_{t+1}) = (b, a); \\ (33.99\%, 20\%) & \text{if } (s_t, s_{t+1}) = (b, b); \end{cases}$$

can be implemented in the benchmark economy.

Example 4 (benchmark economy: constant devaluation) The constant policies $\hat{E}(s^{t+1}) = 180.15\%$ *and* $\tau(s^{t+1}) = 0$ can be implemented in the benchmark economy.

As previously mentioned, in example 1 consumption and output are negatively correlated. However, a positive correlation between consumption and output is one of the stylized facts to be reproduced.

One of the goal of this section is to verify the ability of the Ramsey equilibrium to reproduce the stylized facts. Another goal is to obtain a better understanding of the properties of the optimal policies and the induced competitive equilibrium. To achieve these to goals several experiments were performed. Some illustrative examples will be reported below.

Example 5 (distinct transition probabilities) The economy is as in example 1, except that $\mu_{ab} = 0.2$ and $\mu_{ba} = 0.9$. The optimal policies are described by

$$(\hat{E}(s^{t+1}), \tau(s^{t+1})) = \begin{cases} (11.19\%, 17.90\%) & \text{if } (s_t, s_{t+1}) = (a, a); \\ (17.88\%, 17.90\%) & \text{if } (s_t, s_{t+1}) = (a, b); \\ (11.63\%, 17.96\%) & \text{if } (s_t, s_{t+1}) = (b, a); \\ (18.35\%, 17.96\%) & \text{if } (s_t, s_{t+1}) = (b, b). \end{cases}$$

Again, tax rates are roughly constant and the domestic currency devaluates more at state a than at state b.

A change in the transition probabilities affects the expected present value of future government expenditures. This is why optimal devaluation (as well as inflation rate) and tax rates fall when compared to example 1. Another effect is a change in the amplitude of the devaluation oscillations. The qualitative behavior of the real variables was not affected. Similar results were found with other transition probabilities.

Example 6 (higher oscilation in g^N) The economy is as in example 1, except that $g_a^N = 0$ and $g_b^N = 0.15$. The optimal policies are described by

$$(\hat{E}(s^{t+1}), \tau(s^{t+1})) = \begin{cases} (5.27\%, 17.76\%) & \text{if } (s_t, s_{t+1}) = (a, a); \\ (25.56\%, 17.76\%) & \text{if } (s_t, s_{t+1}) = (a, b); \\ (6.53\%, 17.96\%) & \text{if } (s_t, s_{t+1}) = (b, a); \\ (27.05\%, 17.96\%) & \text{if } (s_t, s_{t+1}) = (b, b). \end{cases}$$

Once more, tax rates are roughly constant and the domestic currency devaluates more at state a than at state b.

The major effect of an increase in the variation of g^N is to increase the oscillation in the devaluation rate of the domestic currency. The behavior of the real variables is the same as in example 1.

Example 7 (an economy driven only by technological shocks) The economy is identical to the one in example 1, except that $g_a^N = g_b^N = 0.075$, $\theta_a^N = 1.2$, and $\theta_b^N = 0.8$. The optimal policies are described by

$$(\hat{E}(s^{t+1}), \tau(s^{t+1})) = \begin{cases} (11.99\%, 22.32\%) & \text{if } (s_t, s_{t+1}) = (a, a); \\ (28.33\%, 22.32\%) & \text{if } (s_t, s_{t+1}) = (a, b); \\ (13.33\%, 22.34\%) & \text{if } (s_t, s_{t+1}) = (b, a); \\ (29.86\%, 22.34\%) & \text{if } (s_t, s_{t+1}) = (b, b). \end{cases}$$

Once more, tax rates are roughly constant and the domestic currency devaluates more at state a than at state b.

Devaluation rate behaves as in example 1. As in that example, the ration $\frac{g^N}{\theta^N}$ is higher in state b than in state a. So, the relative higher public expenditures will lead to higher devaluation whenever the economy hits state b.

Despite the similar behavior of the exchange rate devaluation, the real variables display different qualitative patterns when compared to example 1. In the above example, both consumption and output are higher at state a than at state b. Hence, those two variables are positively correlated.

An economy driven only by productivity shocks will display a positive correlation between consumption and output. However, this class of economy does not perform well at quantitative level. In example 7 the exchange rate oscillates at most 16 percent points and the output may increase or decrease 30%. Oscillation of this order in the output are too high.

The experiments performed so far suggest that a combination of negatively correlated fiscal and technological shocks is required to mimic all stylized facts. The behavior of an economy that is driven by this type of shocks is discussed below.

Example 8 (technological and fiscal shocks) Transition probabilities are $\mu_{ab} = 0.4$ and $\mu_{ba} = 0.7$. The state space is described by $g_a^N = 0,\ \ g_b^N = 0.15$, $\theta_a^N = \theta_a^T = 1.05$, and $\theta_b^N = \theta_b^T = 0.95$. Thus, except for the state space the economy is exactly as in example 1. The optimal policies are described by

$$(\hat{E}(s^t), \tau(s^t)) = \begin{cases} (5.48\%, 18.59\%) & \text{if } (s_t, s_{t+1}) = (a, a); \\ (28.18\%, 18.59\%) & \text{if } (s_t, s_{t+1}) = (a, b); \\ (6.87\%, 18.84\%) & \text{if } (s_t, s_{t+1}) = (b, a); \\ (29.85\%, 18.84\%) & \text{if } (s_t, s_{t+1}) = (b, b). \end{cases}$$

Tax rates are approximately constant. As before, the nominal exchange rate devaluates more at state a than at state b.

The behavior of the real variables in example 8 mimics some of the stylized facts mentioned in the introduction. Consider a history s^t in which the last three events are equal to (a, a, b). At date t the rate of devaluation jumps from 5.5% to 28.2%. Consumption falls 30%, while the GDP decreases by 2%. Similar facts are observed at histories that end with (b, a, b).

Consider a history in which the last three events are equal to (b, b, a). At date t the rate of devaluation falls from 29.9% to 6.87%. Consumption increases by 48% and GDP grows by 2%. The current account reverts from a surplus to a deficit. Similar facts are observed at histories that end with (a, b, a).

5 Conclusion

Governments often choose to pursue exchange rate policies that are later abandoned. To understand the driving forces behind the selection of these policies, this paper studied the problem of choosing optimal devaluation and taxation policies when tax rates are not fully state contigent. The main finding is that the optimal devaluation rate is correlated in a positive way to government expenditures and in a negative way to technological shocks.

The optimal devaluation policy features have a simple justification. In periods of high government expenditures, a benevolent government would like to increase tax rates. Since tax rates are not state contigent, an efficient way to raise additional tax revenue is through inflation tax. As the inflation rises, so does the devaluation rate. A negative technological shock will lead to a fall in output. Thus, the ratio between fiscal deficit and output will rise. Again, the government's willingness to raise tax rates explains why the devaluation is higher when there is a bad technology draw.

A currency crisis is often followed by a drop below the trend of consumption and output. When a country pegs the exchange rate, the opposite facts plus a current account deterioration usually take place. Ideally, a model aimed at explaining the implementation and collapse of exchange rate regimes should reproduce these stylized facts. This essay also succeeds in replicating these empirical regularities.

There is an intuitive explanation for the link between devaluation and the real side of the economy. A drop in the devaluation rate and an increase in the output will generate an income effect that leads to a consumption boom. The higher demand for tradable goods is partially offset by imports. Thus, the current account deteriorates. The opposite occurs when devaluation increases and output falls.

The notion that currency crises trigger recessions is widely accepted. In this essay neither a devaluation causes a slowdown nor a pegging causes a boom. The optimal devaluation rate reacts to technological shocks that hit the economy. If a low productivity shock leads to a recession, to prevent or postpone the devaluation is not an efficient policy.

Most (if not all) of the essays on currency devaluation take an exchange rate pegging as given. However, these papers do not try to explain why the exchange rate was ever pegged. This paper adopts a single framework to explain simultaneously both currency crises and peggings. The model can account for successive shifts between periods of low and high devaluation rate. Related papers account for only one devaluation episode.

This essay has some other contributions. Obstfeld [14] states that it is essential to consider how policies are selected to understand currency crises. Rebelo [16] makes similar statements when discussing monetary stabilization. This paper investigates how the exchange rate policy is chosen.

The essay builds a bridge between two research fields that so far have been seeing as completely apart. Today there is a large and growing body of literature on quantitative macroeconomic theory. Typical examples are the essays of Rebelo [16] and Backus, Kehoe, and Kydland [1]. On the other hand, there exist several studies that rely on reduced form models to explain exchange rate devaluations. Obstfeld [14] and Giavazzi and Pagano [6] are good examples of this investigation avenue. This paper unifies the two approaches.

As shown by Klein and Marion [8], governments often fix the nominal exchange rate. The model does not reproduce this particular fact. Obstfeld [14] and Rebelo and Végh [18] assume that there is a fixed cost of any currency devaluation. The introduction of this feature in the present model is likely to make the optimal policies to prescribe zero devaluation in some states. This is a promising research avenue.

This paper extended the research line started by Lucas and Stokey [11] to an open economy. This allowed the discussion of the optimal exchange rate policy to go beyond the usual discussion of “pegging versus floating”. Between those two policies, there are uncountable others. There is no reason to restrain the discussion only to these two extreme options.

6 Appendix

6.1 Households' First Order Conditions

If $\bar{M}$ is positive, the first order necessary and sufficient conditions for a typical household are

$$\beta^t \mu(s^t) u_T(s^t) = \lambda(s^t) E(s^t) p_t^* \ ; \tag{22}$$

$$\beta^t \mu(s^t) u_1(s^t) = [\lambda(s^t) + \xi(s^t)] p^N(s^t) \ ; \tag{23}$$

$$\beta^t \mu(s^t) u_2(s^t) = \lambda(s^t) p^N(s^t) \ ; \tag{24}$$

$$-\beta^t \mu(s^t) u_l(s^t) = \lambda(s^t)[1 - \tau(s^t)] w(s^t) \ ; \tag{25}$$

$$\lambda(s^t) = \sum_{s_{t+1} \in S} [\lambda(s^t, s_{t+1}) + \xi(s^t, s_{t+1})] \ ; \tag{26}$$

$$\lambda(s^t) q(s^t) = \sum_{s_{t+1} \in S} \lambda(s^t, s_{t+1}) \ ; \tag{27}$$

$$\lambda(s^t) E(s^t) q_t^* = \sum_{s_{t+1} \in S} \lambda(s^t, s_{t+1}) E(s^t, s_{t+1}) \ ; \tag{28}$$

$$M(s^{t-1}) \geq p^N(s^t) c_1^N(s^t) \ \& \ \xi(s^t)[M(s^{t-1}) - p^N(s^t) c_1^N(s^t)] = 0 \ ; \tag{29}$$

$$\begin{aligned} &E(s^t) p_t^* c^T(s^t) + p^N(s^t)[c_1^N(s^t) + c_2^N(s^t)] + q(s^t) B(s^t) + \\ &E(s^t) q_t^* B_H^*(s^t) + M(s^t) = [1 - \tau(s^t)] w(s^t) l(s^t) + B(s^{t-1}) + \\ &E(s^t) B_H^*(s^{t-1}) + M(s^{t-1}) + \psi^T(s^t) + \psi^N(s^t) \ ; \end{aligned} \tag{30}$$

$$\begin{aligned} &\lim_{t \to \infty} \sum_{s^t \in S_k^t} \lambda(s^t) M(s^t) = \lim_{t \to \infty} \sum_{s^t \in S_k^t} \lambda(s^t) q(s^t) B(s^t) = \\ &\lim_{t \to \infty} \sum_{s^t \in S_k^t} \lambda(s^t) E(s^t) q_t^* B_H^*(s^t) = 0 \ , \forall s^k, \forall k \ ; \end{aligned} \tag{31}$$

$$c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t), M(s^t), \lambda(s^t), \xi(s^t) \geq 0 \ , \ l(s^t) \leq 1 \ ; \tag{32}$$

$$\left\| \max \left\{ c^T, c_1^N, c_2^N, l, \frac{M}{p^N}, \left| \frac{B}{p^N} \right|, \left| \frac{B_H^*}{p^*} \right| \right\} \right\|_\infty < \infty \ ; \tag{33}$$

where $\lambda(s^t)$ and $\xi(s^t)$ are Lagrange multipliers for, respectively, budget and cash-in-advance constraints.

6.2 Proofs

Proof of proposition 1. For the "if" part, suppose that (φ, χ, ζ) is a competitive equilibrium of bounded inflation. It is needed to show that (1) and (10)-(21) hold. Constraints (1) is trivially satisfied.

It will now be shown that (10) holds. Multiplying (30) by $\lambda(s^t)$ and using (22)-(29) plus equations $\psi^N(s^t) = (1-\alpha^N)p^N(s^t)[\ c_1^N(s^t) + c_2^N(s^t) + g_t^N]$ and $\psi^T(s^t) = (1-\alpha^T)E(s^t)p_t^*[c^T(s^t) + g_t^T + x(s^t)]$ one obtains

$$\beta^t \mu(s^t) W(s^t) + \sum_{s_{t+1}} [\lambda(s^t, s_{t+1}) + \xi(s^t, s_{t+1})] M(s^t) -$$
$$[\lambda(s^t) + \xi(s^t)] M(s^{t-1}) + \lambda(s^t) q(s^t) B(s^t) - \lambda(s^t) B(s^{t-1}) +$$
$$\lambda(s^t) E(s^t) q_t^* B_H^*(s^t) - \lambda(s^t) E(s^t) B_H^*(s^{t-1}) = 0 \ . \tag{34}$$

Summing up over s^t and then from date 0 to some date k and using (27) and (28) to cancel the identical terms out one gets

$$u_T(s^0) c^T(s^0) + u_2(s^0) c_2^N(s^0) + u_l(s^0) l(s^0) + \sum_{t=1}^{k} \sum_{s^t} \beta^t \mu(s^t) W(s^t) +$$
$$[u_1(s^0) - \xi(s^0) p^N(s^0)] c_1^N(s^0) + \sum_{s^k} \sum_{s_{k+1}} [\lambda(s^k, s_{k+1}) + \xi(s^k, s_{k+1})] M(s^k) +$$
$$\sum_{s^k} \lambda(s^k) [q(s^k) B(s^k) + E(s^k) q_k^* B_H^*(s^k)] = \lambda(s^0) [\bar{M} + \bar{B} + E(s^0) \bar{B}_H^*] \ .$$

But $u_1(s^0) - \xi(s^0)p^N(s^0) = \lambda(s^0)p^N(s^0)$. So, the last equality combined to (26) yields

$$u_T(s^0) c^T(s^0) + u_2(s^0) c_2^N(s^0) + u_l(s^0) l(s^0) + \sum_{t=1}^{k} \sum_{s^t} \beta^t \mu(s^t) W(s^t) =$$
$$\lambda(s^0) [\bar{B} + \bar{M} - p^N(s^0) c_1^N(s^0) + E(s^0) \bar{B}_H^*] -$$
$$\sum_{s^k} \lambda(s^k) [M(s^k) + q(s^k) B(s^k) + E(s^k) q_k^* B_H^*(s^k)] \ . \tag{35}$$

From (22) and (24), $\lambda(s^0)E(s^0) = u_T(s^0)/p_0^T$ and $\lambda(s^0) = u_2(s^0)/p^N(s^0)$. Plugging those two expressions into (35), making $k \to \infty$, using (31) and adding $u_1(s^0)c_1^N(s^0)$ one obtains (10).

The balance-of-payments equation (8) has to hold in a competitive equilibrium. Multiplying it by $\lambda(s^t)E(s^t)$, summing up over s^t and from date 0 to some date k and applying (28) to cancel the identical terms out one obtains

$$\lambda(s^0) E(s^0) [\bar{B}_H^* + \bar{B}_G^*] = - \sum_{t=0}^{k} \sum_{s^t} \lambda(s^t) E(s^t) p_t^* x(s^t) +$$

$$\sum_{s^k} \lambda(s^k)E(s^k)q_k^*[B_H^*(s^k) + B_G^*(s^k)] \ . \tag{36}$$

For a while, assume that

$$\lim_{t\to\infty} \sum_{s^t \in S_k^t} \lambda(s^t)E(s^t)q_t^* B_H^*(s^t) = 0 \ . \tag{37}$$

So, making $k \to \infty$, applying the transversality conditions in (37) and (31), and using (22) one obtains (11). To show that (37) holds, let $\bar{q}^* = \sup Q^*$ and $\bar{p}^* = \sup P^*$. Then,

$$0 \leq \left| \sum_{s^t \in S_k^t} \lambda(s^t)E(s^t)q_t^* B_H^*(s^t) \right| \leq \bar{q}^* \bar{p}^* \left\| \frac{B_G^*}{p^*} \right\|_\infty \sum_{s^t \in S_k^t} \lambda(s^t)E(s^t) \ .$$

Since $\sum_{s^t \in S_k^t} \lambda(s^t)E(s^t) \leq \sum_{s^t \in S^t} \lambda(s^t)E(s^t)$

$$0 \leq \left| \sum_{s^t \in S_k^t} \lambda(s^t)E(s^t)q_t^* B_H^*(s^t) \right| \leq \bar{q}^* \left\| \frac{B_G^*}{p^*} \right\|_\infty \bar{p}^* \sum_{s^t \in S^t} \lambda(s^t)E(s^t) \ .$$

It is now enough to show that $\sum_{s^t \in S^t} \lambda(s^t)E(s^t) \to 0$ as $t \to \infty$. Sum both sides of (28) over s^t. This yields

$$\bar{q}^* \sum_{s^t \in S^t} \lambda(s^t)E(s^t) \geq \sum_{s^{t+1} \in S^{t+1}} \lambda(s^{t+1})E(s^{t+1}) \ .$$

Since $\bar{q}^* \in (0,1)$, (37) is established.

It will now be shown that (12) and (13) hold. Fix s^t. Divide both sides of (22) by p_t^*. Then, forward it by one period, add over s_{t+1} and combine the resulting equation to (28) and (22) to obtain (12). For (13), divide (22) by (24) and combine the resulting equation to item (ii) of definition 1.

To obtain (15) and (14), divide (25) by (24). Then, use item (ii) of definition 1. This procedure yields

$$[1 - \tau(s^t)] \frac{\alpha^N \theta_t^N}{[l^N(s^t)]^{1-\alpha^N}} = -\frac{u_l(s^t)}{u_2(s^t)} \ .$$

The above expression becomes (15) when valuated at s^0. Concerning (14), solve the above equation for $[\tau(s^t) - 1]$, evaluate it at the histories (s^t, s_{t+1}) and $(s^t, \bar{s}_{t+1})$, and use (2).

It will now be shown that (12) and (13) hold. Fix s^t. Divide both sides of (22) by p_t^*. Then, forward it by one period, sum over s_{t+1} and combine the resulting equation to (28) and (22) to obtain (12). For (13), divide (22) by (24) and combine the resulting equation to item (ii) of definition 1.

Constraint (16) is obviously satisfied. Concerning (17), divide (23) by (24) to obtain

$$\frac{u_1(s^t)}{u_2(s^t)} = 1 + \frac{\xi(s^t)}{\lambda(s^t)} \geq 1 \ ,$$

where the inequality follows from the fact that $\lambda(s^t), \xi(s^t) \geq 0$.

For (18), note that

$$\sum_{s^t} \lambda(s^t) M(s^t) = \sum_{s^t} \sum_{s_{t+1}} M(s^t)[\lambda(s^t, s_{t+1}) + \xi(s^t, s_{t+1})] =$$

$$\beta^{t+1} \sum_{s^t} \sum_{s_{t+1}} \mu(s^t, s_{t+1}) M(s^t) \frac{u_1(s^t, s_{t+1})}{p^N(s^t, s_{t+1})} \geq$$

$$\beta^{t+1} \sum_{s^t} \sum_{s_{t+1}} \mu(s^t, s_{t+1}) \left[p^N(s^t, s_{t+1}) c_1^N(s^t, s_{t+1}) \frac{u_1(s^t, s_{t+1})}{p^N(s^t, s_{t+1})} \right] =$$

$$\beta^{t+1} \sum_{s^{t+1}} \mu(s^{t+1}) u_1(s^{t+1}) c_1^N(s^{t+1}) \geq 0 \ . \tag{38}$$

Now make $t \to \infty$ and apply (31) to obtain (18).

To obtain (19), proceed exactly as done to obtain (36). However, instead of summing from date zero to k, sum from some generic date j to k. This procedure yields

$$\lambda(s^j) E(s^j)[B_H^*(s^j) + B_G^*(s^j)] = -\sum_{t=j}^{k} \sum_{s^t \in S_j^t} \lambda(s^t) E(s^t) p_t^* x(s^t) +$$

$$\sum_{s^k \in S_j^k} \lambda(s^k) E(s^k) q_k^* [B_H^*(s^k) + B_G^*(s^k)] \ . \tag{39}$$

From (31) and (37), the second term in the right hand side goes to zero as $k \to \infty$. Hence, combine (22) and (39) to obtain

$$\sup_j \sup_{s^j \in S^j} \left| \frac{1}{u_T(s^j)} \sum_{t=j}^{\infty} \sum_{s^t \in S^t} \beta^{t-j} \mu(s^t | s^j) u_T(s^t) x(s^t) \right| \leq$$

$$\frac{\sup P^*}{\inf P^*} \left(\left\| \frac{B_H^*}{p^*} \right\|_\infty + \left\| \frac{B_G^*}{p^*} \right\|_\infty \right) < \infty \ .$$

Regarding (20), sum (34) over s^t and then from date j to date k. With some manipulation, the result is

$$\frac{1}{u_2(s^j)} \left| \sum_{t=j}^{\infty} \sum_{s^t \in S^t} \beta^{t-j} \mu(s^t | s^j) W(s^t) - u_1(s^j) c_1^N(s^j) \right| \leq$$

$$\frac{p^N(s^{j-1})}{p^N(s^j)}\left(\left|\frac{M(s^{j-1})}{p^N(s^{j-1})}\right|+\left|\frac{B(s^{j-1})}{p^N(s^{j-1})}\right|\right)+\left|c_1^N(s^j)\right|+\frac{E(s^j)p_j^*}{p^N(s^j)}\frac{p_{j-1}^*}{p_j^*}\left|\frac{B_H^*(s^{j-1})}{p_{j-1}^*}\right|\leq$$

$$\frac{1}{\varepsilon}\left(\left\|\frac{M}{p^N}\right\|_\infty+\left\|\frac{B}{p^N}\right\|_\infty\right)+\sup\Theta^N+\frac{E(s^j)p_j^*}{p^N(s^j)}\frac{\sup P^*}{\inf P^*}\left\|\frac{B_H^*}{p^*}\right\|_\infty . \tag{40}$$

On the other hand,

$$\frac{E(s^j)p_j^*}{p^N(s^j)}=\frac{\alpha^N\theta_j^N[l^T(s^j)]^{1-\alpha^T}}{\alpha^T\theta_j^T[l^N(s^j)]^{1-\alpha^N}}\leq\frac{\alpha^N\sup\Theta^N}{\alpha^T\inf\Theta^T[l^N(s^j)]^{1-\alpha^N}} .$$

But $\inf G^N>0$. So, $l^N(s^t)$ is bounded away from zero. This implies that the right hand side of (40) is bounded by some real number. As a consequence, (20) holds.

The "if" part of the proof will be concluded by showing that (21) is satisfied. Assume that there exists a uniform $\bar{\varepsilon}>0$ such that

$$\bar{\varepsilon}\leq\frac{\beta}{\mu(s^t)u_2(s^t)}\sum_{\hat{s}_{t+1}}\mu(s^t,\hat{s}_{t+1})u_1(s^t,\hat{s}_{t+1})\leq\frac{1}{\bar{\varepsilon}} , \tag{41}$$

$$\bar{\varepsilon}\leq\frac{c_1^N(s^t,\hat{s}_{t+1})}{c_1^N(s^t,s_{t+1})}\leq\frac{1}{\bar{\varepsilon}} . \tag{42}$$

Fix a history s^{t+1}. Pick $\bar{s}_{t+1}$ and $\tilde{s}_{t+1}$ so that $\frac{c_1^N(s^t,\bar{s}_{t+1})}{c_1^N(s^t,\tilde{s}_{t+1})}$ is the smallest value of $\frac{c_1^N(s^t,\hat{s}_{t+1})}{c_1^N(s^t,s_{t+1})}$ over all possible pairs $(s_{t+1},\hat{s}_{t+1})$. Therefore,

$$\bar{\varepsilon}^2\leq\frac{\beta}{\mu(s^t)u_2(s^t)}\sum_{\hat{s}_{t+1}}\mu(s^t,\hat{s}_{t+1})u_1(s^t,\hat{s}_{t+1})\frac{c_1^N(s^t,\bar{s}_{t+1})}{c_1^N(s^t,\tilde{s}_{t+1})}\Rightarrow$$

$$\bar{\varepsilon}^2\leq\frac{\beta}{\mu(s^t)u_2(s^t)}\sum_{\hat{s}_{t+1}}\mu(s^t,\hat{s}_{t+1})u_1(s^t,\hat{s}_{t+1})\frac{c_1^N(s^t,\hat{s}_{t+1})}{c_1^N(s^t,s_{t+1})} .$$

This establishes the first inequality in (21). Similar reasoning yields the second one. It only remains to show that there exists a $\bar{\varepsilon}$ as in (41) and (42). For the left inequality in (41), note that

$$\varepsilon\leq\frac{p^N(s^t,s_{t+1})}{p^N(s^t)}\leq\frac{1}{\varepsilon}\Rightarrow\varepsilon^2\leq\frac{p^N(s^t,\bar{s}_{t+1})}{p^N(s^t,\hat{s}_{t+1})}\leq\frac{1}{\varepsilon^2} .$$

Equations (23), (24) and (26) together imply

$$\varepsilon\leq\frac{p^N(s^t,s_{t+1})}{p^N(s^t)}=\frac{\beta}{\mu(s^t)u_2(s^t)}\sum_{\hat{s}_{t+1}}\mu(s^t,\hat{s}_{t+1})u_1(s^t,\hat{s}_{t+1})\frac{p^N(s^t,s_{t+1})}{p^N(s^t,\hat{s}_{t+1})}\leq\frac{1}{\varepsilon} .$$

Combine the last two expressions to obtain

$$\varepsilon\leq\frac{\beta}{\mu(s^t)u_2(s^t)}\sum_{\hat{s}_{t+1}}\mu(s^t,\hat{s}_{t+1})u_1(s^t,\hat{s}_{t+1})\frac{1}{\varepsilon^2}\Rightarrow$$

$$\varepsilon^3 \le \frac{\beta}{\mu(s^t)u_2(s^t)} \sum_{\hat{s}_{t+1}} \mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1}) \ .$$

Similar reasoning shows that the right inequality in (41) holds. To establish (42), it will be shown that if that condition fails then the optimality by households will be violated. Without loss of generality, assume that right inequality in (42) fails. Hence, by taking a subsequence $\{t_k\}_{k=0}^{\infty}$ if necessary, for each t one can find histories (s^t, s_{t+1}) and $(s^t, \hat{s}_{t+1})$ such that $\frac{c_1^N(s^t,\hat{s}_{t+1})}{c_1^N(s^t,s_{t+1})} \to \infty$. Since c_1^N is bounded above, $c_1^N(s^t, s_{t+1}) \to 0$, from which follows that $u_1(s^t, s_{t+1}) \to \infty$. On the other hand, the ratio $\frac{p^N(s^t,\hat{s}_{t+1})}{p^N(s^t,s_{t+1})}$ is bounded away from zero. Thus,

$$\frac{p^N(s^t, \hat{s}_{t+1})c_1^N(s^t, \hat{s}_{t+1})}{p^N(s^t, s_{t+1})c_1^N(s^t, s_{t+1})} \to \infty \Rightarrow \frac{M(s^t)}{p^N(s^t, s_{t+1})c_1^N(s^t, s_{t+1})} \to \infty \ .$$

So, for t sufficiently large, $M(s^t) > p^N(s^t, s_{t+1})c_1^N(s^t, s_{t+1})$. But not to spend cash holdings fully can not be an optimal choice when $u_1(s^t, s_{t+1}) \to \infty$.

For the "only if" part of the proposition, take an initial price $p^N(s^0) > 0$ and an object $\{[c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t), l^T(s^t), l^N(s^t), x(s^t)]_{s^t \in S^t}\}_{t=0}^{\infty}$ satisfying (1) and (10)-(21). It must be shown that there exist arrays $\{[B_G^*(s^t), M(s^t), B(s^t), B_H^*(s^t)]_{s^t \in S^t}\}_{t=0}^{\infty}$, $\{[E(s^t), p^N(s^{t+1}), w(s^t), q(s^t)]_{s^t \in S^t}\}_{t=0}^{\infty}$, and $\{[\tau(s^{t+1})]_{s^{t+1} \in S^{t+1}}\}_{t=0}^{\infty}$ that satisfy all conditions of a competitive equilibrium of bounded inflation.

Recall that $p^N(s^0)$ is given. Thus, it is possible to define $p^N(s^{t+1})$ recursively. Set those prices according to

$$p^N(s^t, s_{t+1}) = \frac{\beta p^N(s^t)}{\mu(s^t)u_2(s^t)} \sum_{\hat{s}_{t+1}} \mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1}) \frac{c_1^N(s^t, \hat{s}_{t+1})}{c_1^N(s^t, s_{t+1})} \ . \tag{43}$$

Define tax rates according to

$$\tau(s^{t+1}) - 1 = \frac{u_l(s^{t+1})}{u_2(s^{t+1})} \frac{[l^N(s^{t+1})]^{1-\alpha^N}}{\alpha^N \theta_{t+1}^N} \ . \tag{44}$$

Set $\lambda(s^t)$ as in (24), $\xi(s^t)$ as in (23), $E(s^t)$ as in (22), $q(s^t)$ as in (27) and $w(s^t)$ as in (25).

From (43), $p^N(s^t, s_{t+1})c_1^N(s^t, s_{t+1}) = p^N(s^t, \hat{s}_{t+1})c_1^N(s^t, \hat{s}_{t+1})$. Thus, one can define cash holdings as $M(s^t) = p^N(s^t, s_{t+1})c_1^N(s^t, s_{t+1})$. Let $B_H^*(s^t) = 0$. Define $B(s^1)$ to balance household's budget constraint at state s^1. The entire array $\{[B(s^t)]_{s^t \in S^t}\}_{t=0}^{\infty}$ is constructed in this recursive way, while the array $\{[B_G^*(s^t)]_{s^t \in S^t}\}_{t=0}^{\infty}$ is defined recursively to balance the budget constraint of the government.

It remains to show that the proposed (φ, χ, ζ) is a competitive equilibrium of bounded inflation. Combining (43) and (21) it is easy to check that inflation is bounded. Consider item (iii) of definition 1. The feasibility conditions in (1) trivially hold. Except for the condition $\|B_G^*/p^*\|_{\infty} < \infty$ (which will be established at the end of the proof), the budget constraint (3) is clearly satisfied. To conclude that (2) holds, it is enough to combine (44) and (14).

For item (i) it is enough to prove that (22)-(32) are satisfied. The variables were defined so that (22)-(25) hold. Concerning (26), from (43) one obtains

$$\frac{\beta^t \mu(s^t) u_2(s^t)}{p^N(s^t)} = \beta^{t+1} \sum_{\hat{s}_{t+1}} \frac{\mu(s^t,\hat{s}_{t+1}) u_1(s^t,\hat{s}_{t+1}) c_1^N(s^t,\hat{s}_{t+1})}{p^N(s^t,s_{t+1}) c_1^N(s^t,s_{t+1})} =$$

$$\beta^{t+1} \sum_{\hat{s}_{t+1}} \frac{\mu(s^t,\hat{s}_{t+1}) u_1(s^t,\hat{s}_{t+1}) c_1^N(s^t,\hat{s}_{t+1})}{p^N(s^t,\hat{s}_{t+1}) c_1^N(s^t,\hat{s}_{t+1})} \Rightarrow$$

$$\frac{\beta^t \mu(s^t) u_2(s^t)}{p^N(s^t)} = \beta^{t+1} \sum_{\hat{s}_{t+1}} \frac{\mu(s^t,\hat{s}_{t+1}) u_1(s^t,\hat{s}_{t+1})}{p^N(s^t,\hat{s}_{t+1})} \ .$$

The last equality combined to (24) generates (26).

Debt prices $q(s^t)$ were defined so that (27) holds. Combining (22) and (12) one obtains (28). Concerning (29), (16) implies that it holds in state s^0 and cash holdings were defined so that $M(s^{t-1}) = p^N(s^t) c_1^N(s^t)$ for $t \geq 1$. The definition of $B(s^t)$ guarantees that (30) holds.

Recall that $B_H^*(s^t) = 0$. Thus, the last limit in (31) holds. Concerning the first limit, variables were constructed so that (38) is satisfied, with the first inequality holding as equality. So, (18) implies that $\sum_{s^t} \lambda(s^t) M(s^t) \to 0$ as $t \to \infty$. For the second limit, observe that (35) can be derived exactly as before. Plus, (10) ensures that $\sum_{t=1}^{\infty} \sum_{s^t} \beta^t \mu(s^t) W(s^t)$ converges in $\mathbb{R}$. So, making $k \to \infty$ in (35) and using the fact that the other two transversality conditions in (31) hold one concludes that $\lim_{k\to\infty} \sum_{s^k} \lambda(s^k) q(s^k) B(s^k) = 0$.

With the exception of $\xi(s^t) \geq 0$ and $\left\|B/p^N\right\|_\infty < \infty$, all inequalities in (32) and (33) are trivially true. To show that former holds, divide (23) by (24) and use (17). With respect to the latter, the same procedure used to obtain (40) yields

$$\frac{p^N(s^j)}{p^N(s^{j-1})} \frac{1}{u_2(s^j)} \left| \sum_{t=j}^{\infty} \sum_{s^t \in S^t} \beta^{t-j} \mu(s^t|s^j) W(s^t) - u_1(s^j) c_1^N(s^j) \right| = \left| \frac{B(s^{j-1})}{p^N(s^{j-1})} \right| \ .$$

Constraints (20) and (21) together imply that the right hand side is bounded. Therefore, $\left\|B/p^N\right\|_\infty < \infty$.

The focus is now on item (ii) of definition 1. If $t \geq 1$, combine (24), (25) and (44) to obtain the first equality. If $t = 0$, use (15) instead of (44). Divide (22) by (24), combine the resulting equality to (13) and use the fact $\alpha^N \theta_t^N [l^N(s_t)]^{\alpha^N - 1} = w(s^t)/p^N(s^t)$ to obtain the second equality.

To finish the proof it only remains to show that $\|B_G^*/p^*\|_\infty < \infty$. Combine (22) and (39) to get

$$\frac{p_j^*}{p_{j-1}^*} \left| \frac{1}{u_T(s^j)} \sum_{t=j}^{\infty} \sum_{s^t \in S^t} \beta^{t-j} \mu(s^t|s^j) u_T(s^t) x(s^t) \right| = \left| \frac{B_G^*(s^{j-1})}{p_{j-1}^*} \right| \ .$$

Since P^* is a finite set, an appeal to (19) concludes. ■

6.3 Model's National Accounts

The nominal devaluation rate can be computed according to the formula

$$\frac{E(s^t, s_{t+1})}{E(s^t)} = \frac{\beta}{u_2(s^t, s_{t+1})} \sum_{\hat{s}_{t+1}} \frac{\mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1}) c_1^N(s^t, \hat{s}_{t+1})}{\mu(s^t) c_1^N(s^t, s_{t+1})} ,$$

while the tax rate τ is given by

$$\tau(s^t) = 1 + \frac{u_l(s^t)}{u_2(s^t)} \frac{[l^N(s^t)]^{1-\alpha^N}}{\alpha^N \theta_t^N} .$$

Let e denote the real exchange rate. In this essay,

$$e(s^t) = \frac{E(s^t) p_t^*}{p^N(s^t)} = \frac{u_T(s^t)}{u_2(s^t)} ,$$

where the first equality is the definition of real exchange rate and the second comes from households' first order conditions. The current account is identical to the trade balance x. The real wage in each sector can be evaluated by the marginal productivities. Let c and y denote, respectively, consumption and output. These variables are quantified according to

$$c(s^t) = c_1^N(s^t) + c_2^N(s^t) + e(s^t) c^T(s^t) ,$$

$$y(s^t) = y^N(s^t) + e(s^t) y^T(s^t) = \theta_t^N [l^N(s^t)]^{\alpha^N} + e(s^t) \theta_t^T [l^T(s^t)]^{\alpha^T} .$$

In this essay the term fiscal deficit refers to primary deficit. As usual, the deficit is the difference between the expenditures $g_t^N + e(s^t) g_t^T$ and the fiscal revenue $\tau(s^t) \theta_t^N [l^N(s^t)]^{\alpha^N - 1} l(s^t)$.

6.4 Examples' Solutions

All Ramsey examples use proposition 2. In each exercise, the lifetime utility $\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(s^t)$ is maximized subject to the constraints (1) and (10)-(15). Since the solution will also satisfy (16)-(21) the solution will be a Ramsey allocation. A proper choice for $p^N(s^0)$ will ensure that (16) holds. A numerical verification shows that (17) holds. Concerning constraints (18)-(21), they will be surely satisfied because in all examples the allocations take only finitely many values. Knowing the allocations, one can compute the policies as done in the second part of the proof of proposition 1.

It is a well known fact in the Ramsey policies the government uses distorting taxation only after using all available lump-sum revenues. Particularly, if the public expenditures are high enough the government will always be willing to raise all possible lump-sum revenue at date zero. This implies that the date zero cash-in-advance constraint will hold as equality. Otherwise, the money holdings left over would consist on wealth not taxed away through inflation in a lump-sum fashion.

The above property will be used in all examples. Assuming that the date zero cash-in-advance hold as equality, the right hand side of (10) can be simplified. Since in all solutions the government will use the inflation tax, the assumption in question is justified.

All examples share the same general structure. Therefore, a general approach to solve them was adopted.. Define $V(s^t)$ by $V(s^t) = W(s^t) + (1-\alpha^T)x(s^t)u_T(s^t)$. Hence, $\sum_{t,s^t} \beta^t \mu(s^t)V(s^t) = \sum_{t,s^t} \beta^t \mu(s^t)W(s^t) + (1-\alpha^T)\sum_{t,s^t} \beta^t \mu(s^t)u_T(s^t)x(s^t)$.

Since it was assumed that $\bar{B}^*_G = \bar{B}^*_H = 0$, it is possible to combine the above equation to (11) to conclude that $\sum_{t=0}^{\infty}\sum_{s^t\in S^t} \beta^t\mu(s^t)V(s^t) = \sum_{t=0}^{\infty}\sum_{s^t\in S^t} \beta^t\mu(s^t)W(s^t)$. As a consequence, constraint (10) can be replaced by an equivalent one, namely $\sum_{t=0}^{\infty}\sum_{s^t\in S^t} \beta^t\mu(s^t)V(s^t) = u_1(s^0)c_1^N(s^0)$.

After using the properties of u and the assumptions on the stochastic processes, one can write the Lagrange function as

$$\mathcal{L} = u(s^0) - \nu^N(s^0)\left[c_1^N(s^0) + c_2^N(s^0) + g_0^N - \theta_0^N\left(l^N(s^0)\right)^{\alpha^N}\right] -$$

$$\nu^T(s^0)\left[c^T(s^0) + g_0^T + x(s^0) - \theta_0^T\left(l^T(s^0)\right)^{\alpha^T}\right] - \nu^l(s^0)\left[l^N(s^0) + l^T(s^0) -\right.$$

$$\left.l(s^0)\right] - \delta(s^0)\left[\gamma^T\alpha^T\theta_0^T c_2^N(s^0)\left(l^N(s^0)\right)^{1-\alpha^N} - \gamma_2\alpha^N\theta_0^N c^T(s^0)\left(l^T(s^0)\right)^{1-\alpha^T}\right] -$$

$$r\left[(1-\tau_0)\gamma_2\alpha^N\theta_0^N\left(1-l^N(s^0)\right) - \gamma^l c_2^N(s^0)\left(l^N(s^0)\right)^{1-\alpha^N}\right] +$$

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^{t+1}\left(\mu(s^t,s)\left\{\sum_{s\in\{a,b\}} u(s^t,s) - \nu^N(s^t,s)\left[c_1^N(s^t,s) + c_2^N(s^t,s) + g_s^N -\right.\right.\right.$$

$$\left.\theta_s^N\left(l^N(s^t,s)\right)^{\alpha^N}\right] - \nu^T(s^t,s)\left[c^T(s^t,s) + g_s^T + x(s^t,s) - \theta_s^T\left(l^T(s^t,s)\right)^{\alpha^T}\right] -$$

$$\nu^l(s^t,s)\left[l^N(s^t,s) + l^T(s^t,s) - l(s^t,s)\right] -$$

$$\delta(s^t,s)\left[\gamma^T\alpha^T\theta_s^T c_2^N(s^t,s)\left(l^N(s^t,s)\right)^{1-\alpha^N} -\right.$$

$$\left.\left.\gamma_2\alpha^N\theta_s^N c^T(s^t,s)\left(l^T(s^t,s)\right)^{1-\alpha^T}\right]\right\} -$$

$$\mu(s^t)\left\{\kappa(s^t)\left[u_T(s^t) - \sum_{s\in\{a,b\}}\mu(s|s_t)u_T(s^t,s)\right] -\right.$$

$$\left.\left.\eta(s^t)\left[\frac{c_2^N(s^t,a)\left(l^N(s^t,a)\right)^{1-\alpha^N}}{\theta_a^N\left(l(s^t,a)-1\right)} - \frac{c_2^N(s^t,b)\left(l^N(s^t,b)\right)^{1-\alpha^N}}{\theta_b^N\left(l(s^t,b)-1\right)}\right]\right\}\right) -$$

$$\Lambda\left\{V(s^0)+\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\sum_{s\in\{a,b\}}\beta^{t+1}\mu(s^t,s)V(s^t,s)-(1-\sigma)\gamma_1 u(s^0)\right\}+$$

$$\Gamma\left\{u_T(s^0)x(s^0)+\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\sum_{s\in\{a,b\}}\beta^{t+1}\mu(s^t,s)u_T(s^t,s)x(s^t,s)\right\}\ .$$

There is no endogenous state variable. This suggests that a solution must be stationary. To solve the problem, compute the first order conditions. Then, define auxiliary variables $\hat{\kappa}$ and $\hat{\eta}$ according to

$$\hat{\kappa}(s^t,s)=\kappa(s^t)-\beta\kappa(s^t,s) \text{ and } \hat{\eta}(s^t,s)=\frac{\eta(s^t,s)}{\mu(s|s_t)}\ . \tag{45}$$

This allows to drop κ and η from the problem. Then, guess that the for $t\geq 1$ the allocations and the variables $\nu^N,\nu^T,\nu^l,\delta,\hat{\kappa}$ and $\hat{\eta}$ depend only on the last two events in of a history s^t. That will lead to a non linear system of equations with the same number of variables and equations. Note that the definition of $\hat{\eta}$ implies

$$\mu(a|s_t)\hat{\eta}(s^t,a)=\mu(b|s_t)\hat{\eta}(s^t,b)\ . \tag{46}$$

This last constraint generates two additional equations that must be added to the original system.

Most of the variables depend on the last two events because the structure of the maximization problem. An intuitive way to address this issue is to consider the optimal behavior of the tax rate τ. Suppose that the economy yesterday was at state a. Denote the optimal value of the tax rate by τ_a. Therefore, there will be two possible states tomorrow, (τ_a,a) and (τ_a,b). Hence, the optimal allocations will depend on the current state and the tax rate, which in its turn depends on the previous state.

It would already possible to compute the optimal allocations at this stage. However, the problem can be further simplified. With stationary allocations, constraint (12) can be written as $u_T(s^{t+1})=u_T(s^0)$. As a consequence, constraint (11) becomes equivalent to $\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\mu(s^t)x(s^t)=0$. So, rewrite the Lagrange function as

$$\mathcal{L}=u(s^0)-\nu^N(s^0)\left[c_1^N(s^0)+c_2^N(s^0)+g_0^N-\theta_0^N\left(l^N(s^0)\right)^{\alpha^N}\right]-$$

$$\nu^T(s^0)\left[c^T(s^0)+g_0^T+x(s^0)-\theta_0^T\left(l^T(s^0)\right)^{\alpha^T}\right]-\nu^l(s^0)\left[l^N(s^0)+l^T(s^0)-\right.$$

$$\left.l(s^0)\right]-\delta(s^0)\left[\gamma^T\alpha^T\theta_0^T c_2^N(s^0)\left(l^N(s^0)\right)^{1-\alpha^N}-\gamma_2\alpha^N\theta_0^N c^T(s^0)\left(l^T(s^0)\right)^{1-\alpha^T}\right]-$$

$$r\left[(1-\tau_0)\gamma_2\alpha^N\theta_0^N\left(1-l^N(s^0)\right)-\gamma^l c_2^N(s^0)\left(l^N(s^0)\right)^{1-\alpha^N}\right]+$$

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^{t+1}\left(\mu(s^t,s)\left\{\sum_{s\in\{a,b\}}u(s^t,s)-\nu^N(s^t,s)\left[c_1^N(s^t,s)+c_2^N(s^t,s)+g_s^N-\right.\right.\right.$$

$$\theta_s^N \left(l^N(s^t,s)\right)^{\alpha^N}\Big] - \nu^T(s^t,s)\left[c^T(s^t,s) + g_s^T + x(s^t,s) - \theta_s^T\left(l^T(s^t,s)\right)^{\alpha^T}\right] -$$

$$\nu^l(s^t,s)\left[l^N(s^t,s) + l^T(s^t,s) - l(s^t,s)\right] -$$

$$\delta(s^t,s)\Big[\gamma^T\alpha^T\theta_s^T c_2^N(s^t,s)\left(l^N(s^t,s)\right)^{1-\alpha^N} -$$

$$\gamma_2\alpha^N\theta_s^N c^T(s^t,s)\left(l^T(s^t,s)\right)^{1-\alpha^T}\Big] - \kappa(s^t,s)\left[u_T(s^t,s) - u_T(s^0)\right]\Big\} -$$

$$\mu(s^t)\eta(s^t)\left[\frac{c_2^N(s^t,a)\left(l^N(s^t,a)\right)^{1-\alpha^N}}{\theta_a^N\left(l(s^t,a)-1\right)} - \frac{c_2^N(s^t,b)\left(l^N(s^t,b)\right)^{1-\alpha^N}}{\theta_b^N\left(l(s^t,b)-1\right)}\right]\Bigg) -$$

$$\Lambda\left\{V(s^0) + \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\sum_{s\in\{a,b\}}\beta^{t+1}\mu(s^t,s)V(s^t,s) - (1-\sigma)\gamma_1 u(s^0)\right\} +$$

$$\Gamma\left\{x(s^0) + \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\sum_{s\in\{a,b\}}\beta^{t+1}\mu(s^t,s)x(s^t,s)\right\} .$$

Compute the first order conditions. Define $\hat{\eta}$ as in (45), add equation (46) to the system and drop η from the problem. Use the fact that $\Gamma = \nu^T(s^0) = \nu^T(s^t)$ to eliminate ν^T from the system. This procedure yields a non linear system of 61 equations and 61 variables that can be solved by Newton's method.

As final comment, standard properties of geometric series and Markov processes allow to infer that for any history contingent variable $k(s^t)$ that depends only on the last two events of s^t, the following is true:

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\sum_{s\in\{a,b\}}\beta^{t+1}\mu(s^t,s)k(s^t,s) =$$

$$z_a[(1-\mu_{ab})k_{aa} + \mu_{ab}k_{ab}] + z_a[\mu_{ba}k_{ba} + (1-\mu_{ba})k_{bb}] ,$$

where, if $s^0 = a$,

$$\begin{aligned}
z_a &= \beta + \frac{\beta^2}{\mu_{ab}+\mu_{ba}}\left[\frac{\mu_{ba}}{1-\beta} + \frac{\mu_{ab}(1-\mu_{ab}-\mu_{ba})}{1-\beta(1-\mu_{ab}-\mu_{ba})}\right] \\
z_b &= \frac{\beta^2}{\mu_{ab}+\mu_{ba}}\left[\frac{\mu_{ab}}{1-\beta} - \frac{\mu_{ab}(1-\mu_{ab}-\mu_{ba})}{1-\beta(1-\mu_{ab}-\mu_{ba})}\right]
\end{aligned}$$

and, if $s^0 = b$,

$$\begin{aligned}
z_a &= \frac{\beta^2}{\mu_{ab}+\mu_{ba}}\left[\frac{\mu_{ba}}{1-\beta} - \frac{\mu_{ba}(1-\mu_{ab}-\mu_{ba})}{1-\beta(1-\mu_{ab}-\mu_{ba})}\right] \\
z_b &= \beta + \frac{\beta^2}{\mu_{ab}+\mu_{ba}}\left[\frac{\mu_{ab}}{1-\beta} + \frac{\mu_{ba}(1-\mu_{ab}-\mu_{ba})}{1-\beta(1-\mu_{ab}-\mu_{ba})}\right]
\end{aligned}$$

and $k_{s\bar{s}} = k(s^t, s, \bar{s})$. Thus, it is possible to eliminate all infinite sums that show up in the problem.

Preferences and technology are parametrized to make the this economy resemble the one considered in Rebelo and Végh [17] and Rebelo [16]. Labor income shares $\alpha^T = 0.48$ and $\alpha^N = 0.63$ are borrowed from these authors. They adopted a quarterly value of 0.99 for β. Converting this figure to monthly units, one obtains $\beta = 0.99^{1/3}$, which is the value adopted in the incoming examples. The share parameter γ^l as set equal to 2/3, as in Kydland and Prescott [10]. This last values implies $\gamma^T + \gamma_1 + \gamma_2 = 1/3$. In Rebelo and Végh [17] and Rebelo [16], tradables and non tradables have the same share. Thus, $\gamma^T = \gamma_1 + \gamma_2$. The condition $\gamma_1/\gamma_2 = 2/3$ was imposed arbitrarily on those shares. Solving those three equations, one obtains $\gamma^T = 5/30$, $\gamma_1 = 2/30$ and $\gamma_2 = 3/30$. The results are robust to changes in the parameters. Observe that with this parametrization, in a steady state with low inflation around 40% of the household expenditures with non-tradables will be paid cash. This is a relatively high number. Hence, the adopted parametrization leads to a demand for cash holdings larger than one should expect. On the other hand, given the fiscal policy, a large money demand will lead to smaller inflation rates. So, the adopted parametrization is reducing the model's ability to generate high inflation and devaluation rates.

All solutions were computed with a maximum error of 10^{-9}. As a consequence, the allocations were evaluated with several decimals. However, the space constraint required them to be present with four decimal places only.

Example 1. The optimal allocations are

$$\begin{bmatrix} \chi(s^0) \\ \chi(s^t, a, a) \\ \chi(s^t, a, b) \\ \chi(s^t, b, a) \\ \chi(s^t, b, b) \end{bmatrix} = \frac{1}{10^4} \begin{bmatrix} 2943 & 763 & 1341 & 1947 & 1182 & 765 & 31 \\ 2947 & 768 & 1332 & 1931 & 1179 & 751 & -60 \\ 2818 & 687 & 1212 & 2174 & 1401 & 772 & 107 \\ 2947 & 768 & 1332 & 1929 & 1179 & 750 & -62 \\ 2818 & 687 & 1212 & 2172 & 1401 & 771 & 105 \end{bmatrix}.$$

Example 2. The competitive equilibrium allocations for this example are

$$\begin{bmatrix} \chi(s^0) \\ \chi(s^t, a) \\ \chi(s^t, b) \end{bmatrix} = \frac{1}{10^4} \begin{bmatrix} 2843 & 871 & 1306 & 2035 & 1234 & 801 & 133 \\ 2864 & 838 & 1257 & 1883 & 1175 & 708 & -58 \\ 2741 & 761 & 1141 & 2130 & 1404 & 726 & 100 \end{bmatrix}.$$

Example 3. The respective competitive equilibrium allocations are

$$\begin{bmatrix} \chi(s^0) \\ \chi(s^t, a, a) \\ \chi(s^t, b, a) \\ \chi(s^t, b) \end{bmatrix} = \frac{1}{10^4} \begin{bmatrix} 2951 & 678 & 1367 & 1907 & 1139 & 768 & 33 \\ 2972 & 857 & 1316 & 1977 & 1232 & 745 & -98 \\ 2949 & 670 & 1369 & 1905 & 1135 & 769 & -30 \\ 2825 & 631 & 1235 & 2161 & 1376 & 785 & 124 \end{bmatrix}.$$

Example 4. The respective competitive equilibrium allocations are

$$\begin{bmatrix} \chi(s^0) \\ \chi(s^t,a,a) \\ \chi(s^t,a,b) \\ \chi(s^t,b,a) \\ \chi(s^t,b,b) \end{bmatrix} = \frac{1}{10^4} \begin{bmatrix} 3332 & 352 & 1484 & 1642 & 995 & 647 & -645 \\ 3262 & 389 & 1667 & 2081 & 1147 & 933 & -58 \\ 3125 & 367 & 1507 & 2334 & 1382 & 952 & 109 \\ 3262 & 390 & 1667 & 2081 & 1148 & 933 & -60 \\ 3126 & 368 & 1507 & 2334 & 1382 & 952 & 108 \end{bmatrix}.$$

Example 5. The optimal allocations are

$$\begin{bmatrix} \chi(s^0) \\ \chi(s^t,a,a) \\ \chi(s^t,a,b) \\ \chi(s^t,b,a) \\ \chi(s^t,b,b) \end{bmatrix} = \frac{1}{10^4} \begin{bmatrix} 2978 & 796 & 1334 & 1948 & 1200 & 748 & -98 \\ 2969 & 801 & 1353 & 1998 & 1218 & 780 & -30 \\ 2840 & 720 & 1232 & 2243 & 1442 & 801 & 137 \\ 2970 & 801 & 1352 & 1997 & 1217 & 779 & -32 \\ 2840 & 720 & 1232 & 2241 & 1441 & 800 & 135 \end{bmatrix}.$$

Example 6. The optimal allocations are

$$\begin{bmatrix} \chi(s^0) \\ \chi(s^t,a,a) \\ \chi(s^t,a,b) \\ \chi(s^t,b,a) \\ \chi(s^t,b,b) \end{bmatrix} = \frac{1}{10^4} \begin{bmatrix} 3097 & 885 & 1464 & 1732 & 1003 & 729 & -25 \\ 3088 & 890 & 1485 & 1785 & 1021 & 764 & -179 \\ 2700 & 644 & 1122 & 2519 & 1693 & 826 & 321 \\ 3089 & 890 & 1483 & 1780 & 1020 & 761 & -185 \\ 2701 & 645 & 1120 & 2514 & 1692 & 823 & 315 \end{bmatrix}.$$

Example 7. The optimal allocations are

$$\begin{bmatrix} \chi(s^0) \\ \chi(s^t,a,a) \\ \chi(s^t,a,b) \\ \chi(s^t,b,a) \\ \chi(s^t,b,b) \end{bmatrix} = \frac{1}{10^4} \begin{bmatrix} 2993 & 874 & 1582 & 1967 & 1230 & 737 & -133 \\ 3003 & 879 & 1553 & 1916 & 1216 & 700 & -212 \\ 2659 & 539 & 967 & 2173 & 1341 & 831 & 373 \\ 3003 & 879 & 1553 & 1916 & 1216 & 700 & -212 \\ 2659 & 539 & 967 & 2172 & 1341 & 832 & 372 \end{bmatrix}.$$

Example 8. The optimal allocations are

$$\begin{bmatrix} \chi(s^0) \\ \chi(s^t,a,a) \\ \chi(s^t,a,b) \\ \chi(s^t,b,a) \\ \chi(s^t,b,b) \end{bmatrix} = \frac{1}{10^4} \begin{bmatrix} 3138 & 916 & 1535 & 1768 & 994 & 774 & -63 \\ 3132 & 921 & 1549 & 1803 & 1006 & 797 & -13 \\ 2697 & 598 & 1052 & 2477 & 1733 & 744 & 32 \\ 3133 & 921 & 1546 & 1797 & 1004 & 793 & -22 \\ 2698 & 598 & 1050 & 2472 & 1732 & 740 & 25 \end{bmatrix}.$$

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