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Constrained Smoothing Splines for the Term Structure of Interest Rates

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CONSTRAINED SMOOTHING SPLINES FOR THE TERM STRUCTURE OF INTEREST RATES

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Abstract. In this article we apply the constrained smoothing b-splines (COBS) to interpolate and construct measures associated with the term structure of interest rates. The COBS method has the useful advantage of incorporating important constraints observed in the term structure such as: monotonicity, non-negative values and robustness related to quantile regression methods. We compare COBS with some usual methods utilized in statistical term structure fitting: linear interpolation, smoothing splines and the parametric Nelson-Siegel and Svensson methods. We apply this technique to Brazilian daily term structure data and we show that the constrained smoothing spline is a competitive method to be used in term structure analysis, specially in the case of a low liquidity market like the Brazilian market.

Keywords: Term Structure, No-Arbitrage, Interpolation, Smoothing Splines.

1. Introduction

The use of term structure of interest rates in finance and macroeconomics has been an active line of research in the last 30 years. For macroeconomics, term structure curves carries information about expected future inflation rates and future GDP growth. For finance, the use of term structure is important for making investment decisions, pricing derivatives and performing hedging operations. However, available data does not provide us with a complete term structure curve, what we observe is an set of discrete points relating yields to different maturities. To overcome this problem is necessary some interpolation method to construct a continuos curve.

The literature of the term structure interpolation can be divided in parametric and nonparametric methods. Parametric methods have some advantages. First, they assume specification forms that are parsimonious and can give economic interpretation of their parameters. Second, they functional forms can be imposed in such a way to obey the relationships imposed by economic theory. Third, as pointed by Ait-Sahalia and Duarte (2003), parametric methods can be tested against nested models to test if imposed restrictions by the theory are valid. Some typical examples of parametric interpolation can be found in Nelson and Siegel (1987) and Svensson

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(1994). However, as pointed out by Hagan and West (2006), parametric methods are not immune to problems. First, they fail the requirement of positivity in the interpolated curves for the spot and forward rates, which is necessary to rule out arbitrage opportunities. Second, local stability of fitted curve is also compromised, a very noise curve can be very poorly fitted. Finally, the construction of the discount function, derived from the spot interest rates, can fail to be a decreasing function as required.

As pointed by Ait-Sahalia and Duarte (2003), nonparametric methods share many advantages against parametric methods. First, since they not assume a particular functional form, they are robust to misspecification errors. Second, nonparametric methods can be used as a first step in the analysis of data to guide the specification effort. Third, nonparametric estimation can be quite feasible when the sample size is small and appropriate shape restrictions are imposed. For nonparametric interpolation, the usual[1] methods employed are the quadratic and cubic piecewise approximation functions introduced by McCulloch (1971) and McCulloch (1975). Following this approach, Shaefer (1981) uses Bernstein polynomials andPham (1998) uses Chebyshev polynomials. Another examples are: Vasicek and Fong (1982) exponential splines, Barzanti and Corradi (1998) tension splines and Lin and Yu (2005) Bayesian formulation of spline methods. However, the nonparametric methods cited above share some number of operational problems: the choice of knot location and the number of knot points, instability on fitting the interpolating curve on extremes of maturity line and great sensibility to outliers, which makes the curve very unstable.

In this paper we apply the method of Constrained Smoothing B-Splines (herein after COBS) introduced by He and Ng (1999) to tackle those problems in parametric and nonparametric methods. First, our methodology is robust to outliers, since it formulates the B-Spline by a L_1 projection, it shares the properties of quantile regression methods of Koenker and Basset (1978). Second, it uses information criteria to select the knot points instead of an *ad hoc* procedure. In a nonparametric setting, the knot points can be interpreted as the selected functions used to approximate the term structure. Third, we rule out some arbitrage opportunities by constraining the signal and the format of the estimated term structure curve. More specifically, we impose positivity and monotonicity to the spot and forward rates and apply boundary conditions and a monotonically decreasing property to the discount functions.

In order to reinforce the advantages of the COBS against other approaches, we evaluate it against some usual methods utilized in statistical term structure fitting. Namely, we compare

[1]See Hagan and West (2006) and Anderson et al. (1996) for extensive reviews of methods utilized in term structure analysis.

COBS with linear interpolation, smoothing splines and the parametric Nelson-Siegel and Svensson methods. This paper is closer to the B-Splines methodology introduced by Shea (1984) and Steeley (1991). We also combine the B-splines methodology with the method of smoothing splines used by Fisher et al. (1995). Others related works are the kernel regression methods presented in Linton. et al. (2001) and the penalized spline approach of Jarrow et al. (2004).

The remainder of the paper is structured as follows. The next section describes the relationship among spot interest rates, forward rates and discount functions and points the restrictions imposed by the assumption of no arbitrage. Section 3 details the methodology of COBS. Section 4 compares our method with alternative methodologies. Section 5 concludes.

2. Term Structure Definitions

We define the *spot interest rate,* $y(m)$, as the rate of return applied to maturity of a bond or a contract expiring in m period. Today's price of receiving \$1.00 in m periods is given by the *discount function,* $d(m)$. Under continuous compounding, spot interest rates and the discount function are related by the following formula:

$$d(m) = e^{-y(m)m} \tag{2.1}$$

Therefore, from the discount function we can recover the interest spot rate, or yield by:

$$y(m) = -\frac{log(d(m))}{m} \tag{2.2}$$

From the equation above we have the restriction that the discount function need to be positive $d(m) > 0$.To rule out arbitrage opportunities we need the boundary conditions $d(0) = 1$ and $\lim_{m\to\infty} d(m) = 0$, and that the function is monotonically decreasing, $d'(m) < 0$[2].

A *forward rate,* $f(m)$, is the rate paid for a future investment arranged today and made at time m in the future. Using continuous compounding, the forward rate is given by:

[2]To see this, notice that if the discount function has $d(0) \neq 1$ one can make an instantaneous costless profit by selling the bond if $d(0) > 1$ or buying if $d(0) < 1$. Now, assume $\lim_{m\to\infty} d(m) = \varepsilon > 0$, then one can make a costless profit by adopting a roll-over selling strategy. For example, one can sell a bond with maturity L , where L is very large, receiving at the time of purchase $d(L) > 0$, when time L arrives, he or she can pay \$1 buy selling again a new set of bonds $\frac{1}{d(2L)}$ and so on. Furthermore, the amount of bonds sold at a determined time will never explode since $d(m)$ is assumed monotonically decreasing and this strategy will not be characterized as a doubling strategy, see Duffie (1996) pg. 104, for any maturity m, $\frac{1}{d(m)} < \frac{1}{d(+\infty)} < \frac{1}{\varepsilon} < +\infty$. Finally, if $d'(m) > 0$ for some interval $m \in (m_0, m_1)$, then one can make a costless profit by buying $d(m_0)$ and selling $d(m_1)$, at time zero he or she will have a profit of $d(m_1) - d(m_0) > 0$ and he or she can hold the money received at m_0 to pay the bond sold when time m_1 arrives.

$$e^{y(m)m} = e^{\int_0^m f(x)dx} \tag{2.3}$$

The relation above can also be written as:

$$y(m) = \frac{1}{m} \int_0^m f(x)dx \tag{2.4}$$

From the equation above and the first equation we can relate the discount and forward rates by the following formulas:

$$d(m) = exp\{-\int_0^m f(x)dx\} \tag{2.5}$$

$$f(m) = -\frac{d'(m)}{d(m)} \tag{2.6}$$

The last equation and the no arbitrage condition of $d(m) > 0$ and $d'(m) > 0$, imply the restriction that $f(m) > 0$.

All those relations show that the term structure of interest rates can be constructed from any of the three rates, spot, discount or forward. The relationship works in a similar way for discrete compounding rates.

3. Constrained Smoothing B-Splines (COBS)

To present the methodology of constrained smoothing spline of He and Ng (1999), we review the fundamental concepts of the method. A complete derivation can be found in the original article of He and Ng (1999), but related concepts of L_p fitting, quantile regression methods and the linear programming technics utilized can be found in Koenker (2005).

The initial concept is the concept of a *smoothing spline.* A smoothing spline can be defined as the of solution of the minimization problem of the following functional :

$$S_\lambda(g) = \sum_{i=1}^{n} (y_i - g(X_i))^2 + \lambda \int (g"(x))^2 dx \tag{3.1}$$

where g can be any curve, X_i is a data set and λ is the parameter controlling the smoothness of the adjusted curve. You may have notice that in this formulation there exists a trade-off between the residual minimization and the roughness of fit. According to Hardle (1990), this minimization problem has a single solution $\widehat{m}_\lambda(x)$, given by a cubic polynomial named a cubic spline.

The method of smoothing splines are extended by Bosch et al. (1995) to the problem of estimating a quantile smoothing spline, i.e. estimating a conditional quantile function specified by the choice of quantile τ:

$$\min_{g \in \mathbb{R}} \sum_{i=1}^{n} \rho_\tau \left(y_i - g(X_i)\right)^2 + \lambda \int \left(g^{,,}(x)\right)^2 dx \tag{3.2}$$

Using the methodology developed in quantile regression literature[3] Koenker et al. (1994) consider this problem a special case in L_p fitting, in special $L_1$and L_∞ , in the form[4]:

$$J(g) = ||g^{,,}||_p = \int (g^{,,}(x)^p)^{1/p} \tag{3.3}$$

The methodology of He and Ng (1999) can be viewed as a special case of 3.2, again formulating the smoothing problem using a conditional quantile function $g_\tau(x)$ which it is a function of x such as $P(Y < g_\tau(x)|X = x) = \tau$. Sorting the observations $\{(x_i, y_i)\}_{i=1}^n$ with $a = x_0 < x_1 < ... < x_n < x_{n+1} = b$, can be defined a smooth function g and a indicator function $\rho_\tau(u) = 2[\tau - I(u < 0)\}u = [1 + (2\tau - 1)sgn(u)]|u$.

Defining the concept of fidelity in the form:

$$fidelity = \sum_{i=1}^{n} \rho_\tau(y_i - g(x_i)) \tag{3.4}$$

He and Ng (1999) utilizes theL_p quantile smoothing spline of Koenker et al. (1994) $\widehat{g}_\tau L_p(x)$ as the solution of the problem:

$$\min_g fidelity + \lambda L_p roughness \tag{3.5}$$

The roughness measure can be defined to $L_1$and L_∞ problems as:

[3]See Koenker and Basset (1978) and Koenker (2005) for extensive references on quantile regression methods.
[4]See Koenker (2005) for a discussion on L_p fitting. This problems can be solved using standard linear programming methods, and again see Koenker (2005) for a discussion on computational aspects of this problems.

$$L_1 roughness = V(g') = \sum_{i=1}^{n-2} |g'(x_{i+1}^{+}) - g'(x_i^{+})| \tag{3.6}$$

$$L_\infty roughness = V(g') = ||g''||_\infty = max_x g''(x)] \tag{3.7}$$

and the fidelity measures as:

$$fidelity = \sum_{i=1}^{n} |y_i - g(x_i)| \tag{3.8}$$

$$s(x) = \sum_{j=1}^{N+m} a_j B_j(x) \tag{3.9}$$

Note the similarity with the problem in 3.1. The smoothing b-splines is a smoothing splines problem with the following structure:

$$\min_{\theta \in \mathbb{R}^{\mathbb{N}+\gt}} \sum_{i=1}^{n} \left| y_i - \sum_{j=1}^{N+m} a_j B_j(x_i) \right| + \lambda \sum_{i=1}^{N} \left| \sum_{j=1}^{N+m} a_j B_j'(t_{i+m-1}) \right| \tag{3.10}$$

He and Ng (1999) notes that this problem can be formulated as:

$$\min_{\theta \in \mathbb{R}^{\mathbb{N}+\gt}} \sum_{i=1}^{N+m} |\widetilde{y_i} - \widetilde{x_i}|\, \theta \tag{3.11}$$

$$\widetilde{y_i} = \begin{pmatrix} y \\ 0 \end{pmatrix} and\ \tilde{X} = \begin{bmatrix} B \\ \lambda C \end{bmatrix}$$

where $\theta = (a_1, a_2, ..., a_{N+m})$ are the parameters at knot x_i. The vector $\widetilde{y_i}$ is a pseudo response vector. The B matrix is given by:

|

$$B = \begin{bmatrix} B_1(x_1) & \dots & B_{N+m}(x_1) \\ \vdots & \dots & \vdots \\ B_1(x_n) & \dots & B_{N+m}(x_n) \end{bmatrix} \tag{3.12}$$

and the C matrix by the expression:

$$C = \begin{bmatrix} B_1'(t_{m+1}) - B_1'(t_m) & \dots & B_N'(t_{m+1}) - B_N'(t_m) \\ \vdots & \dots & \vdots \\ B_1'(t_{N+m}) - B_1'(t_{N+m-1}) & \dots & B_N'(t_{N+m}) - B_N'(t_{N+m-1}) \end{bmatrix} \tag{3.13}$$

The curve $\widehat{m}_\lambda, L_1(x) = \sum_{i=1}^{N+m} \widehat{a}_j B_j(x)$ is a linear median smoothing B-spline. The estimation is based on applying linear programming in

$$\min\{1'(u+v)|\widetilde{y_i} - \widetilde{x}_i\theta = u - v_i, (u', v') \in \mathbb{R}^{2(n+M)}\} \tag{3.14}$$

The quadratic smoothing B-spline is formulated in analogous way. The problem now is:

$$\min_{\theta \in \mathbb{R}^{N+>}} \sum_{i=1}^{n} \left| y_i - \sum_{j=1}^{N+m} a_j B_j(x_i) \right| + \lambda \max \sum_{i=1}^{N+m} \left| a_j B_j''(x) \right| \tag{3.15}$$

where $\theta = (a_1, a_2, ..., a_{N+m})$

And again can be formulated as:

$$\min_{\theta \in \mathbb{R}^{N+>}} \sum_{i=1}^{n} \left| y_i - \sum_{j=1}^{N+m} a_j B_j(x_i) \right| + \lambda\sigma \tag{3.16}$$

$$s.t. \ -\sigma \leq a_j B_j''(x)(t_{i+m-1}) \leq \sigma$$

for i=1,...N+1

The expression can be put in the form

$$\min_{\theta \in \mathbb{R}^{N+>}} \sum_{i=1}^{n+!} |\widetilde{y_i} - \widetilde{x}_i\theta| \tag{3.17}$$

$$s.t.\widetilde{D\theta} = \begin{bmatrix} D & 1 \\ -D & 1 \end{bmatrix} \theta \geq 0 \tag{3.18}$$

where

$$\widetilde{y} = \begin{pmatrix} y \\ 0 \end{pmatrix} \text{ and } \widetilde{X} = \begin{bmatrix} B & 0 \\ 0 & \lambda \end{bmatrix} \text{ (3.19)}$$

and

$$D = \begin{bmatrix} B_1''(t_m) & \dots & B_{N+m}''(t_m) \\ \vdots & \dots & \vdots \\ B_1''(t_{N+m}) & \dots & B_{N+m}''(t_{N+m}) \end{bmatrix} \tag{3.20}$$

The curve $\widehat{m}_\lambda, L_\infty(x) = \sum_{i=1}^{N+m} \widehat{a}_j B_j(x)$ is a quadratic median smoothing B-spline, where the estimation problem is solved by using linear programming in:

$$\min\{1'(u+v) | \widetilde{y_i} - \widetilde{x_i}\theta = u - v, \widetilde{D\theta} \geq 0, (u', v') \in \mathbb{R}^{2(n+M)}\} \tag{3.21}$$

As the problem is formulated as by a L_1 projection it share the properties of robustness related to quantile regression methods of Koenker and Basset (1978), and is less sensible to outliers in reduced samples that the methods of smoothing splines and other interpolation schemes. This property is especially attractive in the case of markets with little liquidity, what normally it occurs in emerging markets, and in special in the swap market that we will analyze.

Other attractive feature of the method is the possibility of incorporate general constraints of monotonicity. The constraints are imposed constructing a matrix H in the form:

$$H = \begin{bmatrix} B_1'(t_m) & \dots & B_{N+m}'(t_m) \\ \vdots & \dots & \vdots \\ B_1'(t_{N+m+!}) & \dots & B_{N+m}'(t_{N+m+1}) \end{bmatrix} \tag{3.22}$$

The monotonicity can be imposed for increasing functions making $H\theta \geq 0$ and $H\theta \leq 0$ for decreasing functions. In quadratic spline will be necessary a extra set of N+2 constraints is given by $\begin{bmatrix} H & 1 \end{bmatrix}\theta \geq 0$ for increasing functions $\begin{bmatrix} H & 1 \end{bmatrix}\theta \leq 0$ for decreasing functions. Convexity constraints also can be imposed, in the case of $\widehat{m_{L1}}$ the convexity is imposed making $C\theta \geq 0$ and for the case of and for $\widehat{m_{L\infty}}$ trough the use of $\begin{bmatrix} D & 0 \end{bmatrix}\theta \geq 0$, and concavity is obtained reverting the signals.

It is possible to incorporate pointwise constraints such as:

$$g(x) = y_i \tag{3.23}$$

$$g(x) \geq y_i \tag{3.24}$$

$$g(x) \leq y_i \tag{3.25}$$

$$g'(x) = y \tag{3.26}$$

as additional constraints in the linear programming problem. This restrictions will be specially useful for interpolating the discount function, as presented in Section 4.2.

The method of He and Ng (1999) also be formulated as an regression b-splines setting the λ in 3.5 equals to zero. In this case He and Ng (1999) shows that the linear median regression B-Spline is given by:

$$\min_{\theta \in \mathbb{R}^{\mathbb{N}+\gtrdot}} \Sigma(u+v) \tag{3.27}$$

$$s.t. y - \widetilde{X}\theta = u - v$$

$$u \in \mathbb{R}^{\ltimes}, v \in \mathbb{R}^{\ltimes}$$

$$\widetilde{X} = B$$

One recurrent problem in the term structure interpolation literature is the number and the location of knot points of splines. In general the choice is ad hoc in linear, quadratic and exponential splines, putting the knot points in some lattices of interest rate curve more important to fixed income and derivatives instruments. Some methods as the penalized smoothing splines of Jarrow et al. (2004) uses generalized cross validation. The knot selection and the smoothing parameter λ in the constrained smoothing method of He and Ng (1999) can be made using the Akaike Information Criteria (AIC) and the Schwartz Information Criteria (SIC). The Akaike Information Criteria is equivalent to use of generalized cross validation, and the Schwartz Information Criteria is a version of AIC which penalizes more heavily the number of parameters in the model. The AIC and SIC in constrained smoothing splines of He and Ng (1999) are given by:

$$SIC(\lambda) = log(\frac{1}{n}\rho_\tau(y_i - \widehat{m}_\lambda))) + \frac{1}{2}p_\lambda log(n)/n \tag{3.28}$$

$$AIC(\lambda) = log(\frac{1}{n}\rho_\tau(y_i - \widehat{m}_\lambda))) + 2(N+m)/n \tag{3.29}$$

This makes the knot and smoothing parameter choice a fully automatic procedure, removing the ad hoc procedures in the model specification. It notices that if necessary the choice of the number of knots and its localization can be imposed by the user. That is useful in Brazil, since the procedure of market marking the fixed income instruments uses a series of fixed points of the interest rate curve.

4. Applications

Deacon and Derry (1994) concluded that the B-Spline is the most preferred by practitioners and the survey of BIS Bank of International Settlements - “Zero Coupon Yield Curves: Technical Documentation 1999” reports that more used methods by Central Banks are the nonparametric Smoothing Splines and the parametric methods of Nelson and Siegel (1987) and Svensson (1994). Therefore, we use those methodologies as benchmarks to compare to our COBS method. We show the applications of the method in yield curve interpolation and the discount function and forward rate construction.

The data set used in our model are the spot interest rates for the Brazilian economy. Since the Brazilian government does not issue long maturity bonds, the spot rates are obtained from future swap contracts between floating interbank rates and fixed predetermined rates, DI x Pre. Those DI x Pre swap contracts are from the stock exchange future market in Brazil, the *BM&F - Bolsa de mercadoria e futuros*. We use daily data from January 1st, 2004 to January 30th, 2006, in a total of 1482 days. The goal of using Brazilian data was to illustrate how those methods work for liquidity markets.

4.1. **Yield Curve Interpolation.** To illustrate the application of the COBS methodology, we use our full sample to estimate four models: smoothing splines, the parametric family of Nelson-Siegel Nelson and Siegel (1987), the Svensson Svensson (1994) and finally the COBS.

The smoothing-spline is given by equation 3.1 and the Nelson-Siegel corresponds to :

$$y(m) = \beta_0 + \beta_1 \frac{1 - e^{-m/\tau}}{-m/\tau} + \beta_2 \left[\frac{1 - e^{-m/\tau}}{-m/\tau} - e^{-m/\tau} \right] \tag{4.1}$$

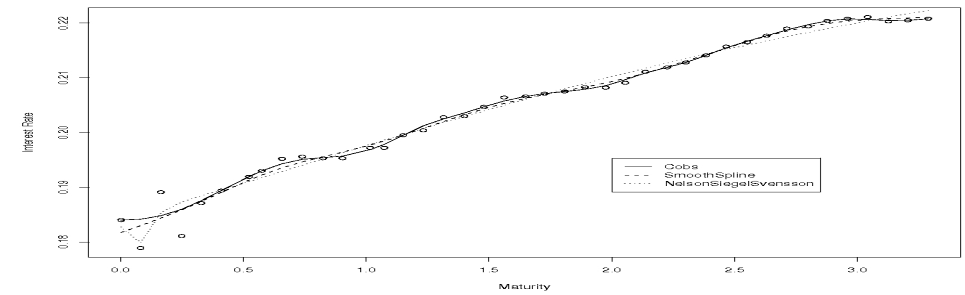
The method of Svensson (1994) is basically the addition of a extra term in 4.1:

$$y(m) = \beta_0 + \beta_1 \frac{1 - e^{-m/\tau_1}}{-m/\tau_2} + \beta_2 \left[\frac{1 - e^{-m/\tau_1}}{-m/\tau_1} - e^{-m/\tau_1} \right] + \beta_3 \left[\frac{1 - e^{-m/\tau_2}}{-m/\tau_2} - e^{-m/\tau_2} \right] \tag{4.2}$$

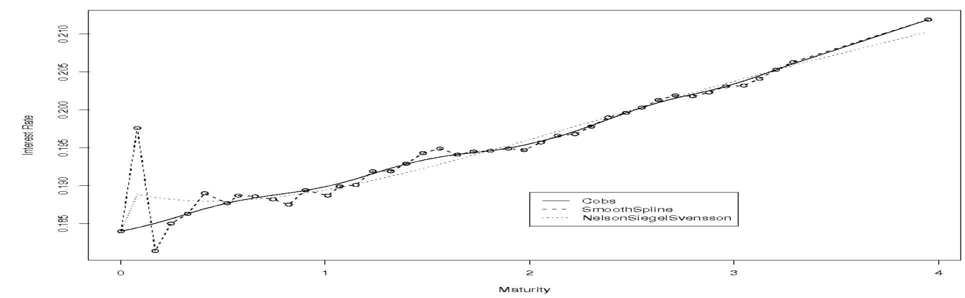
As the method of Svensson (1994) is more flexible and it has a better fitting than the method of Nelson and Siegel (1987), we show only the Svensson (named Nelson-Siegel-Svensson in figures) to facilitate the visualization. We show figures to some specific days to enhance the analysis and a perspective plot with results for all days summarize the results[5].

In Figure 4.1 we show the results for the spot rate curve, for the following dates 02/07/2000, 03/14/2000, 03/27/2004 and 05/03/2004[6]. The first two graphs, days 02/07/2000 and 03/14/2000, show the difficulty of Nelson-Siegel-Svensson to fit the initial maturities - the fitted curve is very noise in this part given the few points and the high variation. The robustness of COBS method is evident is both cases. In the last graph, day 03/27/2004, we can see the instability of Nelson-Siegel-Svensson at the long end of the curve, which is an known problem in the literature. In general, the fit of COBS and smoothing splines are very similar, since the positive constraint is not a binding constraint for the spot rates.

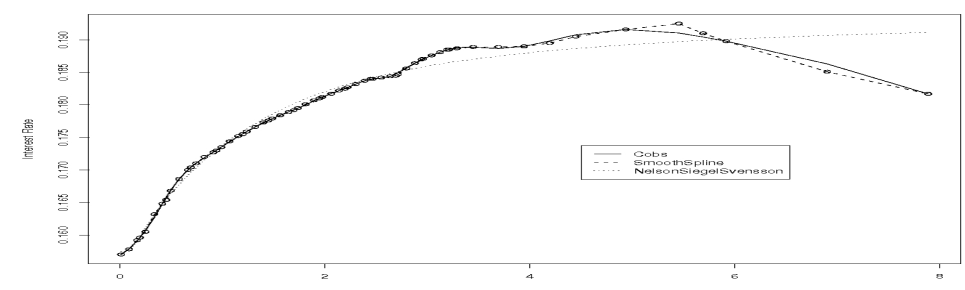
[5]We also compared the COBS with the linear, quadratic, cubic and exponential splines. Due to space limitations we do not put in this version. Those results are disponible up to request to the authors.
[6]All the maturities are measured in years.



(a) Day 07/02/2000



(b) Day 14/03/2000



(c) 27/03/2004

FIGURE 4.1. Spot Rate Interpolation

Figures 4.2, 4.3 and 4.4 shows the interpolation results for the full sample. The figures show a better fit for the COBS method. The smoothing spline has some few points of instability and the Nelson-Siegel-Svensson has instability problems at the short and long end of the curve.

4.2. **Discount Curve with monotonicity constraints.** The construction of the discount function it is a more challenging problem, since the no arbitrage restrictions are more binding and can effectively be violated by unconstrained methods. Recall from Section 2, that the no arbitrage conditions impose to the discount function the following restrictions:

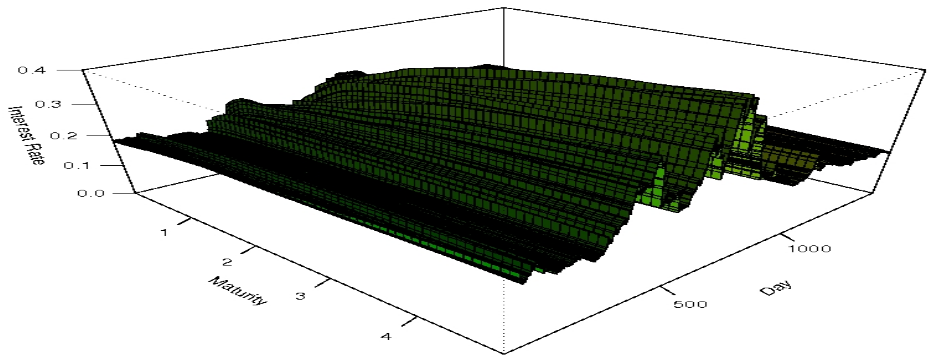


FIGURE 4.2. Constrained Spline Interpolation

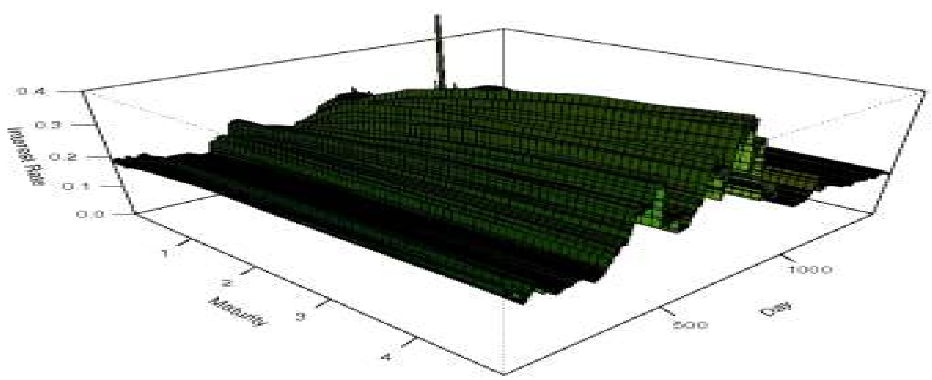


FIGURE 4.3. Smoothing Spline Interpolation

$$d(0) = 1 \tag{4.3}$$

$$d(m) > 0 \tag{4.4}$$

$$d'(m) < 0 \tag{4.5}$$

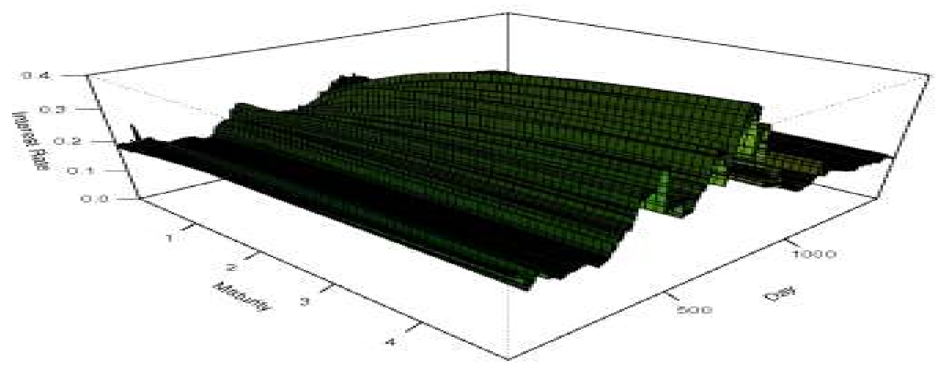


FIGURE 4.4. Nelson-Siegel-Svensson Interpolation

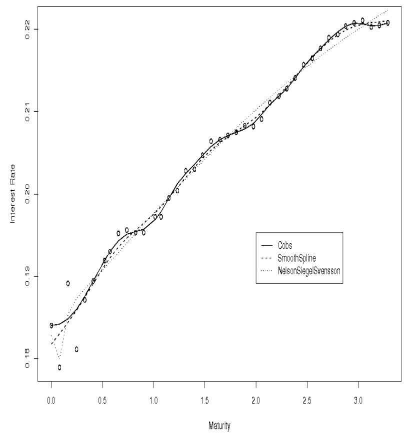
$$\lim_{m \to \infty} d(m) = 0 \tag{4.6}$$

When we use equation 2.2 to calculate the discount function and use a interpolating structure to complete the curve, the violation of these restrictions can show up if no constraint is imposed. The COBS method applies the restrictions 4.5 and 4.4 using the structures in equation 3.19.

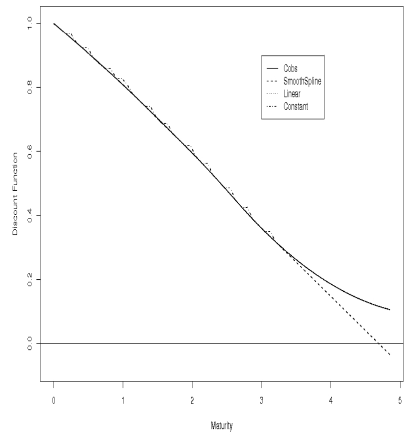
Figure shows the results of the discount function interpolation[7] for days 07/02/2000, 21/11/2000 and 09/02/2005. We also include the linear and piecewise constant interpolation in those methods since they are commonly used in practice. In the 3 days shown, we can observe noisy fits for other methods than the COBS. In day 07/02/2000 the smoothing spline create negative discount functions for long maturities. In day 21/11/2000 the discount function constructed by piecewise constant interpolation is very distant of the other methods. This is caused by the reduced number of points in the longer maturities in these day. Day 09/02/2005 shows a very unstable discount function fitted by the smoothing spline method. This problem can be caused by the local nature of adjust in smoothing spline. Again, the robustness properties of the COBS method of He and Ng (1999) prevents this type of extreme behavior.

In Figures 4.6, 4.7 and 4.8 we show the discount function estimated to the full sample using the COBS, smoothing splines and piecewise constant interpolation. The COBS method respect all

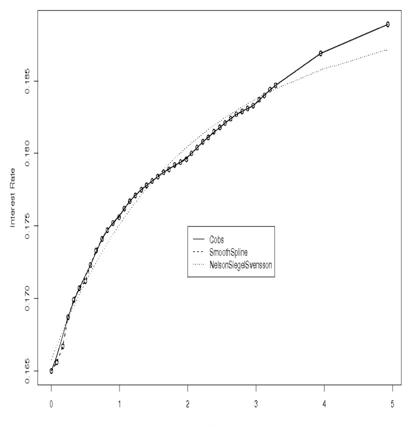
[7]We also show the interpolated spot rate for these days to facilitate the interpretation.



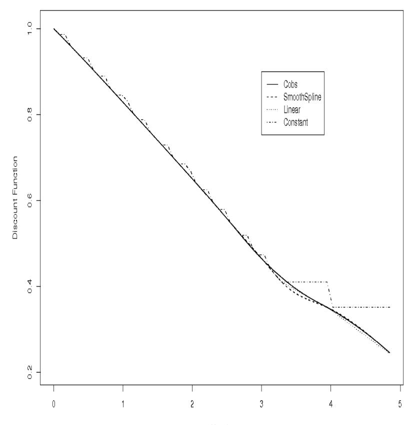
(a) Spot Rate 07/02/2000



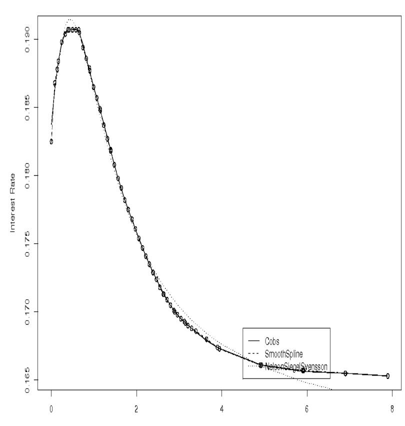
(b) Discount Function Day 07/02/2000



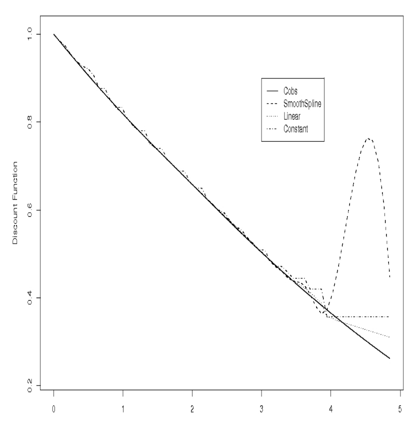
(c) Spot Rate 21/11/2000



(d) Discount Function Day 21/11/2000



(e) Spot Rate 09/02/2005



(f) Discount Function 09/02/2005

FIGURE 4.5. Discount Function Interpolation 4.2

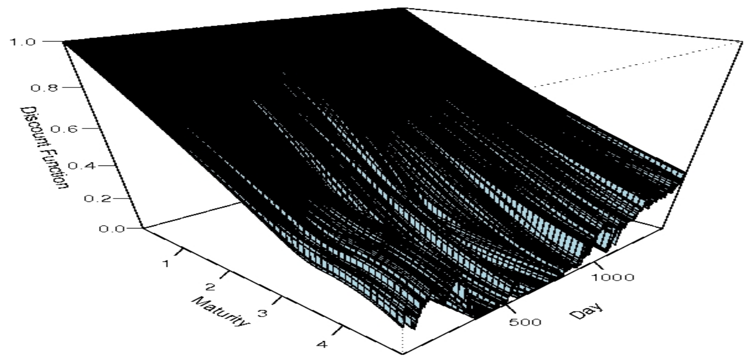


FIGURE 4.6. Discount Function - Constrained Spline

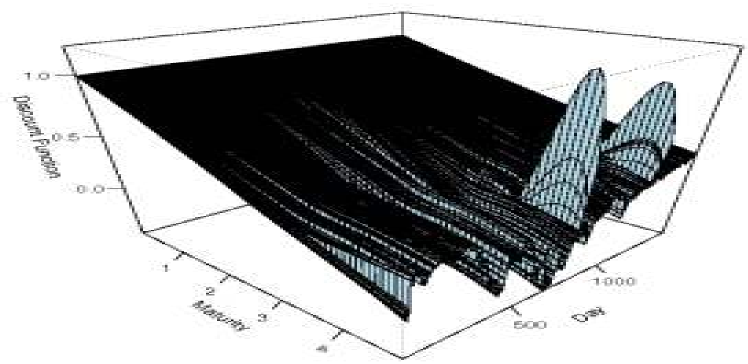


FIGURE 4.7. Discount Function - Smoothing Spline

the imposed restrictions, however the smoothing spline and piecewise methods calculates negative discount rates and the smoothing spline method is unstable in some regions of the curve.

4.3. **Forward Rate Construction.** We construct the forward rate according to 2.6 , and interpolate the forward rate using the COBS, the smoothing spline and the Nelson-Siegel-Svensson methods.

The Nelson-Siegel forward curve can be constructed as:

$$f(m) = \beta_0 + \beta_1 e^{-m/\tau} + \beta_2 m/\tau e^{-m/\tau} \tag{4.7}$$

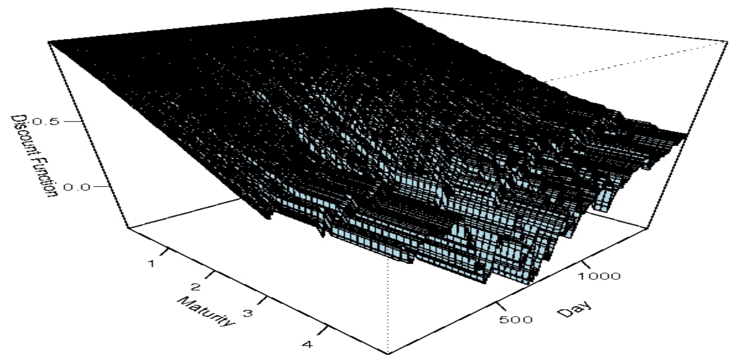


FIGURE 4.8. Discount Function - Piecewise Constant Function

The Nelson-Siegel-Svensson has the following form:

$$f(m) = \beta_0 + \beta_1 e^{-m/\tau_1} + \beta_2 m/\tau_1 e^{-m/\tau_1} + \beta_3 m/\tau_2 e^{-m/\tau_{22}} \tag{4.8}$$

Figure 4.9 shows the result for the dates 02/07/2000, 11/21/2000 and 02/09/2005. COBS and the smoothing spline displays a very similar curve, but, again, the Nelson-Siegel-Svensson do not correctly adjust the forward rate curve and shows instability at the short and long maturities. This pattern behavior of the Nelson-Siegel-Svensson method is due to the lack of robustness of the parametric methods when teh sample size is particularly noisy (short ende) and has small sample size (long end).

In Figures 4.10, 4.11 and 4.12 we show the estimated forward rates using the COBS, smoothing splines and the Nelson-Siegel-Svensson methods. The constrained splines adjusts the forward rate without any negative point and the forward curve is stable because of the robust nature of the method. . The smoothing splines has some problems of negative rates in the beginning of the sample, caused by the low liquidity in this periods. Finally, as in the other cases, the Nelson-Siegel-Svensson[8] is very problematic when used to interpolate a term structure curve with few observed maturities and therefore must be used with caution in markets with low liquidity.

[8]In Figure 4.12, to facilitate the visualization we truncate the extreme points, making the curve discontinuos, but the fitted curve is continuous.

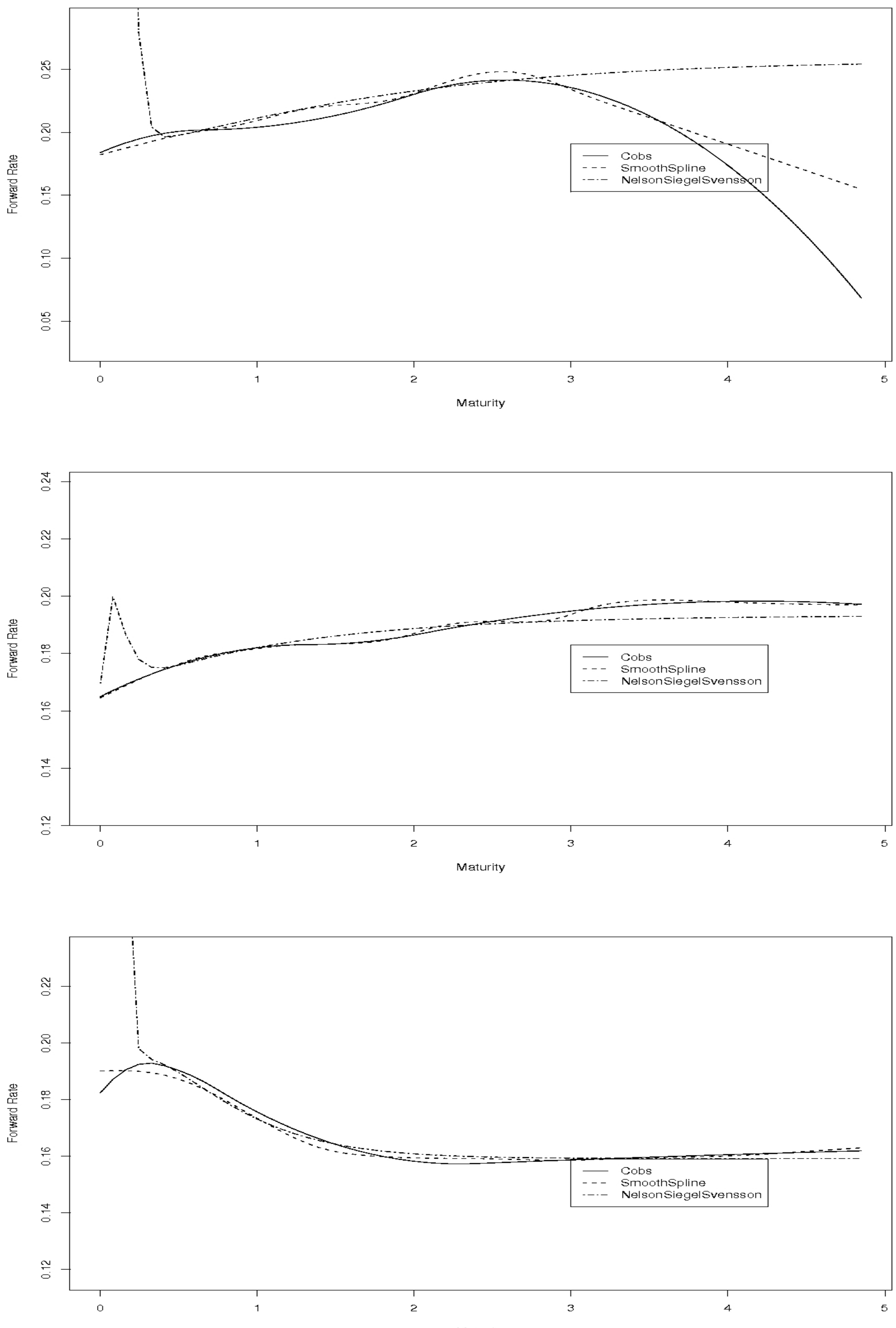


FIGURE 4.9. Forward Rate Interpolation

5. CONCLUSIONS

Looking at our results, we conclude that the COBS methodology of He and Ng (1999) is a very competitive method to fit the term structure of interest rates. Its two main characteristics

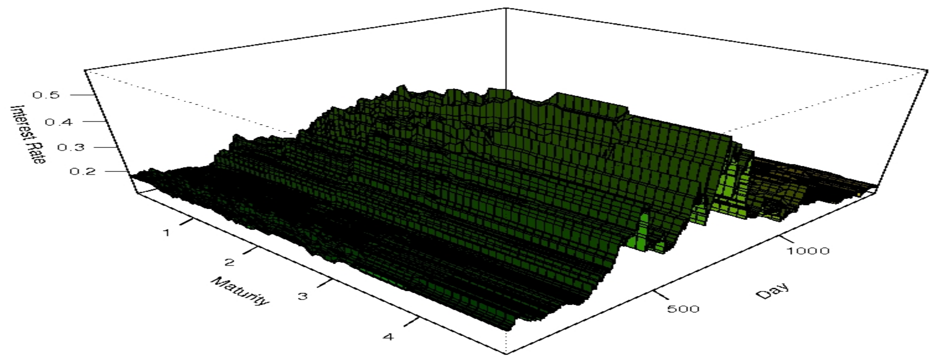


FIGURE 4.10. Forward Rate - Constrained Splines

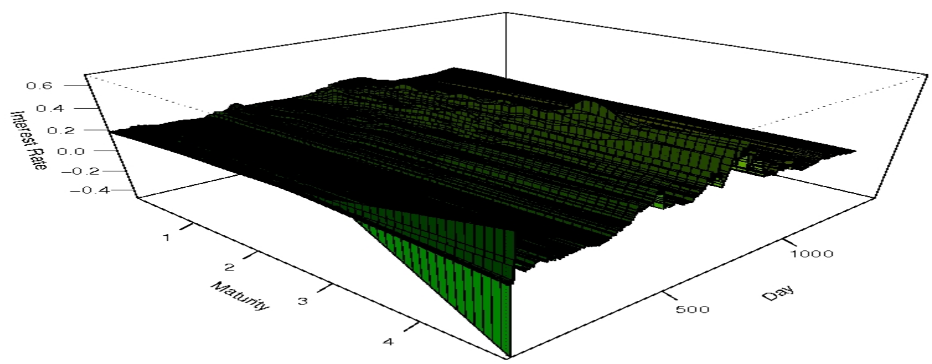


FIGURE 4.11. Forward Rate - Smoothing Splines

are the robustness to outliers, derived from the L_1 estimation, and the possibility to incorporate the necessary restrictions to the adjustment of the term structure of interest rates: positivity of the spot and forward rates, monotonicity and also pointwise constraints for the discount function. Violations of those no arbitrage conditions are not captured by usual fitting criteria like RMSE and can have very large costs, specially for hedging operations.

Ou metholdology is even more important for low liquidity markets like the Brazilian and other emerging economies. As mentioned in Ait-Sahalia and Duarte (2003), small sample problems are well adressed by using nonparametric constrained methods. In fact, our results show that other methods that do not incorporate such constraints show many violations of no arbitrage

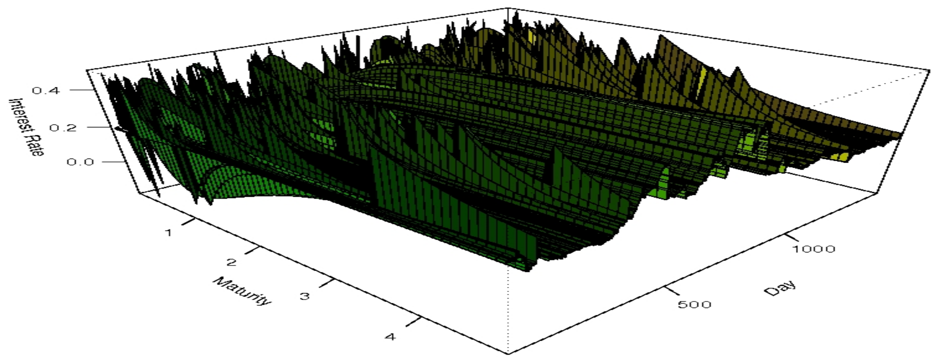


FIGURE 4.12. Forward Rate - Nelson-Siegel-Svensson

conditions when we run estimations for Brazil. Smoothing splines and the Nelson-Siegel-Svensson methods implied, in many cases, negative and inconsistent discount and forward rates. In this sense, our paper showed that the COBS method is, in that sense, superior to the two more used methodologies in term structure interpolation: the nonparametric method of smoothing splines and the parametric methodology of Nelson-Siegel-Svensson.

REFERENCES

Ait-Sahalia, Y., Duarte, J., 2003. Nonparametric option pricing under shape restrictions. Journal of Econometrics 116, 9–47.

Anderson, N., Breedon, F., Deacon, M., Derry, A., Murphy, G., 1996. Estimating and Interpreting the Yield Curve. Wiley.

Barzanti, L., Corradi, C., 1998. A note on interest rate term structure estimation using tension splines. Insurance: Mathematics and Economics 22, 139–143.

Bosch, R. j. Y., Ye, Y., Woodworth, G. G., 1995. A convergent algorithm for the quantile regression with smoothing splines. Computational Statistics & Data Analysis 19, 613–630.

Cochrane, J., Hansen, L. P., 1992. Asset pricing explorations for macroeconomists. In: Blanchard, O. J., Fisher, S. (Eds.), NBER Macroeconomics annual 1992. MIT Press, pp. 115–182.

Deacon, M., Derry, A., 1994. Estimating the term structure of interest rate, working Paper 24, The Bank of England.

Duffie, D., 1996. Dynamic Asset Pricing Theory, 2nd Edition. Princeton University Press.

Fisher, M., Nychka, D., Zervos, D., 1995. Fitting the term structure of interest rates using smoothing splines. Finance and Economics Discussion Series, Board of Governors of Federal Reserve System 1995-1, nd.

Hagan, P., West, G., 2006. Interpolation schemes for curve construction. Applied Mathematical Finance ((to appear)).

Hardle, W., 1990. Applied Nonparametrix Regression. Cambridge University Press.

He, X., Ng, P., 1999. Cobs: Qualitatively constrained smoothing via linear programming. Computational Statistics (14), 315337.

Jarrow, R., Ruppert, D., Yu, Y., 2004. Estimating the term structure of corporate debt with a semiparametric penalized spline model. Journal of the American Statistical Association 99, 57–66.

Koenker, R., 2005. Quantile Regression. Cambridge University Press.

Koenker, R., Basset, G., 1978. Regression quantiles. Econometrica 46, 33–50.

Koenker, R., Ng, P., Portnoy, S., 1994. Quantile smoothing splines. Biometrika 81, 673–680.

Lin, M., Yu, Y., 2005. Estimating the interest rate term structure of treasury and corporate debt with bayesian splines. Journal of Data Science 3, 233–240.

Linton., O., Mammen, E., Nielsen, J., Tanggard, C., 2001. Estimating yield curves by kernel smoothing methods. Journal Of Econometrics 105:1, 185–223.

McCulloch, J., 1971. Measuring the term structure of interest rates. Journal of Business (44), 19–31.

McCulloch, J., 1975. The tax-adjusted yield curve. Journal of Finance (30), 811–830.

Nelson, C. R., Siegel, A. F., 1987. Parsimonous modelling of yield curves. Journal of Business 60 (4), 473–489.

Pham, T. M., 1998. Estimation of the term structure of interest rates: an international perspective. Journal of Multinational Financial Management 8, 265–283.

Shaefer, S. M., 1981. Measuring a tax-specific term structure of interest rates in the market for british government securities. Economic Journal 91, 415–438.

Shea, G., 1984. Pitfalls in smoothing interest rate structure data: Equilibrium models and spline approximation. Journal of Financial and Quantitative Analysis 19, 253–269.

Steeley, J. M., 1991. Estimating the gilt-edged term structure: Basis splines and confidence intervals. Journal of Business Finance and Accounting 18, 513–529.

Svensson, L. E. O., 1994. Estimating and interpreting forward interest rates: Sweden 1992-1994. NBER Working Paper (4871).

Vasicek, O., Fong, H. G., 1982. Term structure modeling using exponential splines. Journal of Finance 37, 339–356.