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Abstract

The main goal of this work is to generalize the autoregressive conditional duration (ACD) model applied to times between trades to the case of time-varying parameters. The use of wavelets allows that parameters vary through time and makes possible the modeling of non-stationary processes without preliminary data transformations.

The time-varying ACD model estimation was done by maximum likelihood with standard exponential distributed errors. The properties of the estimators were assessed via bootstrap.

We present a simulation exercise for a non-stationary process and an empirical application to a real series, namely the TELEMAR stock. Diagnostic and goodness of fit analysis suggest that time-varying ACD model simultaneously modelled the dependence between durations, intra-day seasonality and volatility.

Keywords: ACD model, bootstrap, durations, non-stationarity, time-varying parameters, wavelet

1 Introduction

As computation power and data storage capacity grows, it becomes possible to gather and analyze large data sets, as for example high frequency financial data. One of the economic variables of main interest is the trade intensity of a given asset in a time interval, therefore we can be directed to the study of times between financial transactions. It is easy to notice that in periods of high liquidity, the times between transactions diminishe and in periods of low liquidity, these times become longer, in other words, longer times between transactions indicate less activity of the market. The behavior of times between transactions contains information on the intra-day activity of the market, being an important source of information of market's micro-structure.

Using similar concepts to the ones of the ARCH model for volatility, Engle and Russell [9] developed the autoregressive conditional duration (ACD) model to describe the evolution of the times between transactions (durations). This model was introduced to study data that occur irregularly in time, treating the time between event occurrences as a random process and considering a new class of point processes with dependent rates of occurrence. The properties of the quasi-likelihood estimators can be obtained as corollary of the results for ARCH models.

The stationarity is an essential assumption for the use of usual ACD models, being useful for derivation of the estimator's properties and for moment's calculation of the duration distribution. However, the majority of real processes are not stationary, presenting trends, seasonality or changes in volatility along time.

The idea of this work is to generalize the ACD models, allowing its estimation to be done without the stationarity assumption. Doing this, the durations, that are naturally non-stationary processes, can be modelled without the use of preliminary transformations in data, like removing intra-day seasonality, for example. This generalization will be done using wavelets, substituting each one of the usual ACD model parameters, constant in time, with its wavelet decomposition, allowing that ACD model coefficients vary in time. Once parameters vary in time, the stationarity assumption can be dropped and the coefficients should be capable to capture the process characteristics along time. We use the maximum likelihood estimation (MLE) method to obtain estimators and the parametric bootstrap method to evaluate the properties of the maximum-likelihood (ML) estimators from time-varying ACD model.

In Section 2 we present the usual ACD model and its properties, and briefly describe some concepts used in its generalization, as high frequency data, local stationary processes and wavelets. The generalization of ACD model with its wavelet decomposition is given in Section 3. In Sections

4 and 5 we have applications of time-varying ACD model for a simulated non-stationary process and for transaction durations of TELEMAR stock, respectively. We close the paper with some conclusions in Section 6.

2 Basic concepts

2.1 Locally stationary process

There exists no natural generalization from stationary processes to non-stationary processes. It is often not clear how to set down a reasonable asymptotic theory for non-stationary processes. If $X_t,\ t = 1, 2, \ldots, T$ is a non-stationary process, asymptotic considerations are contradictory since future observations of the non-stationary process may not contain any information at all on the probabilistic structure of the process observed in the present. To overcome this problem, we need different asymptotic considerations.

Dahlhaus [8] introduces a sequence of stochastic processes $\{X_{t,T},\ t = 1, \ldots, T\}$, called *locally stationary*, with transfer function A^0 and trend μ, through the representation

$$X_{t,T} = \mu\Big(\frac{t}{T}\Big) + \int_{-\pi}^{\pi} \exp(i\omega t) A^0_{t,T}(\omega) d\zeta(\omega), \tag{1}$$

where **(i)** $\zeta(\omega)$ is a stochastic process on $[-\pi, \pi]$ with $\overline{\zeta(\omega)} = \zeta(-\omega)$ and

$$cum\{d\zeta(\omega_1), \ldots, d\zeta(\omega_k)\} = \eta\left(\sum_{j=1}^{k} \omega_j\right) g_k(\omega_1, \ldots, \omega_{k-1}) d\omega_1 \ldots d\omega_k,$$

where $cum\{\ldots\}$ denotes the cumulant of k-th order, $g_1(\omega) = 0,\ g_2(\omega) = 1$, $|g_k(\omega_1, \ldots, \omega_{k-1})| \leq const_k$ for all k and $\eta(\omega) = \sum_{j=-\infty}^{\infty} \delta(\omega + 2\pi j)$ is the period 2π extension of the Dirac delta function.

(ii) There exists a constant K and a 2π-periodic function $A : [0, 1] \times \mathbb{R} \to \mathbb{C}$ with $A(u, -\omega) =$

$\overline{A(u,\omega)}$ and

$$\sup_{t,\omega} |A^0_{t,T}(\omega) - A(\frac{t}{T},\omega)| \leq KT^{-1},$$

for all T. The functions $A(u,\omega)$ and $\mu(u)$ are assumed to be continuous in u.

The smoothness of $A(u,\omega)$ in u guarantees that the process has (asymptotically) locally a stationary behavior, controlling the local variation of $A^0_{t,T}(\omega)$ as a function of t. The rescaling property inherent in the definition and the loss of t as time are an abstract setting for processes with evolutionary spectra, allowing for a meaningful asymptotic theory.

2.2 Wavelets

Wavelets are powerful mathematical tools used to decompose non-stationary time series and signals contaminated with noise. Its power comes from the fact that wavelets allow the series analysis in time and scale, simultaneously. For the basic facts on wavelets see [7,19,20,21].

Consider a function f on $L^2(\mathbb{R})$, the space of all square-integrable functions, and its expansion in terms of translations and dilations of a function $\psi(\cdot)$. The function $\psi(t)$ is called mother-wavelet and for some ψ's the generated wavelets $\psi_{j,k}(t) = 2^{\frac{j}{2}}\psi(2^j t - k)$ with $j,k \in \mathbb{Z}$ form an orthonormal basis of $L^2(\mathbb{R})$. This means that each basis function depends on two parameters, j (dilation, index for the scale 2^{j-1}, $j = 1,2,\ldots$) and k (translation, index for location in time), whereas the Fourier basis functions only depend on a single parameter, the frequency. To get this representation, we consider binary dilations 2^j and dyadic translations $k2^{-j}$ of ψ.

For such an f, we have

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{j,k}\psi_{j,k}(t),$$

where $c_{j,k} = \int_{-\infty}^{\infty} f(t)\psi_{j,k}(t)dt$ are the wavelet coefficients.

A function f in $L^2(\mathbb{R})$ can also be represented by an expansion in terms of dilations and translations of a scaling function $\phi(\cdot)$, called father-wavelet, namely $\phi_{j,k} = 2^{j/2}\phi(2^j t - k)$ with $j, k \in \mathbb{Z}$.

A way to get wavelets $\psi(\cdot)$ from the father-wavelet is $\psi(t) = \sqrt{2}\sum_k h_k \phi(2t-k)$, where $h_k = \sqrt{2}\int_{-\infty}^{\infty} \psi(t)\phi(2t-k)dt$, are high-pass filter coefficients.

Considering the orthonormal basis

$$\{\phi_{j_0,k}(t), \psi_{j,k}(t), j \geq j_0, k = 0, 1, \ldots, 2^j - 1\},$$

we can write a function $f \in L^2(\mathbb{R})$ as

$$f(t) = \sum_k c_{j_0,k}\phi_{j_0,k}(t) + \sum_{j \geq j_0}\sum_k d_{j,k}\psi_{j,k}(t),$$

where the coefficients are given by

$$c_{j_0,k} = \int_{-\infty}^{\infty} f(t)\phi_{j_0,k}(t)dt \text{ and } d_{j,k} = \int_{-\infty}^{\infty} f(t)\psi_{j,k}(t)dt,$$

and j_0 is the coarsest scale, usually taken as zero.

2.3 High frequency data

High-frequency data are those collected in small intervals of times, as seconds for example. The study of this type of data allows to get intra-day transactions structure information.

The high-frequency data used in this work are collected in irregular intervals of time, meaning that the transactions do not occur in a fixed time interval, nor the prices change regularly. The times between transactions become important and contain information about the intensity of transactions, see [15,16]. It is noticed the existence of an intra-daily periodic behavior in stock transactions: the transactions occur more frequently at the opening and closing of the market and occur less frequently at lunch hour, resulting in different transactions intensities throughout the day. Therefore, times between transactions present a daily cycle. For details see [1,11,18]. With this kind of data we can have multiple transactions at a given instant of time, even though with different prices. This occurs because time is measured in seconds, that can be a lengthy interval for periods of high liquidity. See [6,24] for further details.

2.4 Autoregressive conditional duration model

Engle and Russell [9] considered the autoregressive conditional duration (ACD) models class for the times between occurrences. The data are simply a list of times between occurrences and, possibly, characteristics associated with these times, as transaction volume or price. In ACD models the times between the occurrences of the transactions (durations) are treated as random variables and the conditional intensity is parameterized in terms of past events so that it is useful for trading processes.

Let t_i be the calendar time, measured in seconds from the midnight, when the i-th transaction took place. The durations are calculated by the differences between the times of trades occurrences, $x_i = t_i - t_{i-1}$, for $i = 1, \ldots, T$, where T is the number of durations.

In practice, most of the data sets can be fitted by the ACD(1,1) model. The representation of the usual ACD(1,1) model is given by

$$\begin{aligned} x_i &= \eta_i \epsilon_i, \qquad (2) \\ \eta_i &= \omega + \delta x_{i-1} + \gamma \eta_{i-1}, \end{aligned}$$

where η_i is the expectation of $i-th$ duration, given past information, ϵ_i are independent and identically distributed variables, with a non-negative standard distribution (e.g. exponential, Weibull, Gamma or Log-normal).

Assuming that the process x_i is weakly stationary, we obtain

$$\mu_x = E(x_i) = \frac{\omega}{1-\delta-\gamma}$$

and

$$Var(x_i) = \mu_x^2\Big(\frac{1-\gamma^2-2\delta\gamma}{1-2\delta^2-2\delta\gamma-\gamma^2}\Big).$$

Based on these results, the ACD(1,1) model must satisfy the conditions $\gamma+\delta<1$ and $\gamma^2+2\delta\gamma+2\delta^2<1$, so that the model has time-invariant unconditional mean and variance, respectively.

Modeling durations by ACD(1,1) model is analogous to volatility modeling by GARCH(1,1) model. As GARCH(1,1), ACD(1,1) is a good starting point to durations modeling. However, there are many alternatives for ACD model, as the logarithmic version of ACD model described in [2] that prevents restrictions of non-negativeness implied by the original ACD model specification, facilitating the hypothesis tests related to the market micro-structure, like the influence of the price on the durations. The stochastic conditional duration model, that has a random latent factor to capture the non observed randomness of the market current information, was considered by [3].

When the error distribution belongs to a parametric family, the ACD model parameters estimation can be made by maximum likelihood (ML). In the case of autoregressive conditional duration model of order (1,1) in (2), with standard exponential errors - EACD(1,1) - the log-likelihood is

$$l(\tilde{\theta};\tilde{x}) = \ln[L(\tilde{\theta};\tilde{x})] = \sum_{i=2}^{T}\Big[-\ln(\omega+\delta x_{i-1}+\gamma\eta_{i-1}) - \frac{x_i}{\omega+\delta x_{i-1}+\gamma\eta_{i-1}}\Big],$$

where $\tilde{\theta}=(\omega,\delta,\gamma)'$ and $\tilde{x}=(x_1,\ldots,x_T)'$.

The connection with GARCH model is very useful to the study of properties of ML estimators for ACD models, see [13,17]. The ML estimators are consistent and have well defined asymptotic covariance matrix. This result was proved by [12] for the exponential family with independent observations.

3 Time-varying autoregressive conditional duration model

We generalized the usual ACD model using wavelets, allowing that the model coefficients vary in time and modeling simultaneously the process characteristics, such as intra-day seasonality, volatility changes and duration dependence.

3.1 Time varying ACD(1,1)

For illustration we consider the particular case of ACD(1,1) model, but the general ACD(r,s) model is treated in the same way.

We write the time-varying ACD(1,1) model using the locally stationary processes notation given by (1), as

$$\begin{aligned} x_{i,T} &= \eta_{i,T}\,\epsilon_i, \\ \eta_{i,T} &= E(x_{i,T}|\mathcal{F}_{i-1}) = \omega\left(\frac{i}{T}\right)+\delta\left(\frac{i}{T}\right)x_{i-1,T} + \gamma\left(\frac{i}{T}\right)\eta_{i-1,T}, \end{aligned} \tag{3}$$

where $i = 1, 2, \ldots, T$, $\omega(\frac{i}{T}) > 0$, $\forall \dfrac{i}{T} \in (0, 1]$, ϵ_i are independent and identically distributed variables, with a non-negative standard distribution.

Using the decomposition in wavelets [19], the model is written replacing each parameter with its decomposition in wavelets:

$$\begin{aligned} x_{i,T} &= \Bigg\{\Big[\alpha_{00}^{(\omega)}\phi_{00}(u) + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\beta_{jk}^{(\omega)}\psi_{jk}(u)\Big] \\ &+ \Big[\alpha_{00}^{(\delta)}\phi_{00}(u) + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\beta_{jk}^{(\delta)}\psi_{jk}(u)\Big]x_{i-1,T} \\ &+ \Big[\alpha_{00}^{(\gamma)}\phi_{00}(u) + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\beta_{jk}^{(\gamma)}\psi_{jk}(u)\Big]\eta_{i-1,T}\Bigg\}\epsilon_i, \end{aligned}$$

where $u = \frac{i}{T} \in (0, 1]$ and J is the number of levels used in the wavelet decomposition.

3.2 Estimation

The parameter estimation is based on the log-likelihood function, supposing that the errors have standard exponential distribution,

$$
\begin{aligned}
l(\tilde{\theta};\tilde{x}) = & - \sum_{i=2}^{T}\Big\{\ln[\omega(u)+\delta(u)x_{i-1,T}+\gamma(u)\eta_{i-1,T}] \\
& + \frac{x_{i,T}}{\omega(u)+\delta(u)x_{i-1,T}+\gamma(u)\eta_{i-1,T}}\Big\},
\end{aligned} \tag{4}
$$

with

$$
\begin{aligned}
\omega(u) &= \alpha_{00}^{(\omega)}\phi_{00}(u)+\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\beta_{jk}^{(\omega)}\psi_{jk}(u),\\
\delta(u) &= \alpha_{00}^{(\delta)}\phi_{00}(u)+\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\beta_{jk}^{(\delta)}\psi_{jk}(u),\\
\gamma(u) &= \alpha_{00}^{(\gamma)}\phi_{00}(u)+\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\beta_{jk}^{(\gamma)}\psi_{jk}(u),
\end{aligned}
$$

$\tilde{x} = (x_1,\ldots,x_T)'$ and $\tilde{\theta} = (\alpha_{00}^{(\omega)},\ldots,\beta_{J-1,2^{J-1}-1}^{(\omega)},\alpha_{00}^{(\delta)},\ldots,\beta_{J-1,2^{J-1}-1}^{(\delta)},\alpha_{00}^{(\gamma)},\ldots,\beta_{J-1,2^{J-1}-1}^{(\gamma)})$ with dimension (3×2^J) $\times 1$.

It is important to notice that the log-likelihood function given by (4) is nonlinear in the parameters, needing iterative methods to be maximized. The BHHH method, proposed by [4] and consisting of a practical approach to find the maximum likelihood estimator using the gradient method as a tool, was used in this work.

The number of parameters to be estimated, 3×2^J, is related to the desired resolution level in the wavelet decomposition. The choice of the resolution level refers to the amount of details that will be used in the model, having an important role in an appropriate modeling of the studied process. If we choose a very low resolution level we lose information of the time series of interest, while a very high resolution level leads to estimators efficiency loss, reflected in its increase in variability.

The choice of the particular wavelet to be used is dictated by the features of the analyzed time series. During the estimation process, Haar, LA(8) and D(8) wavelets were used. Due to the smoothness of the simulated time-varying coefficient, LA(8) and D(8) wavelets performed better than the Haar wavelet. Since there was not much difference between LA(8) and D(8), the decision was to work with LA(8) wavelet, including the empirical application.

3.3 Estimator properties

To evaluate the maximum likelihood estimators properties, we use simulation procedures for the generation of duration process replicates. Next we describe briefly how the parametric bootstrap procedure was used.

Parametric bootstrap

The basic idea of the bootstrap procedure is to resample a data set directly, or via a fitted model, in order to create data replicas that allow the evaluation of biases, standard errors and confidence intervals without using analytical calculations.

A model is considered appropriate to describe the characteristics of interest when it is capable to reproduce these characteristics with high confidence. Thus, using the replicates, an envelope is constructed based on empirical quantiles of magnitude $100(1-\alpha/2)\%$ and $100(\alpha/2)\%$. If the estimated parameters lie inside these bootstrap confidence bands, we can conclude that the model is well fitted.

The applied method can be summarized in the following steps:

1. ML estimation of the wavelet coefficients, based on T observations, and reconstitution of the time-varying parameters, that will be used as data generating functions;

2. residuals calculation of the original model based on the difference between observed and

fitted values;

3. generation of B independent samples, with replacement, from the residuals;

4. generation of B samples of data, using the estimated wavelet coefficients obtained in step 1 and the residuals from step 3;

5. wavelet coefficients estimation by ML for each one of the B data samples and reconstitution of the time-varying parameters.

For details see [23].

4 A simulation exercise

In this section we present a simulation to analyze the behavior of estimators of the model (3) assuming that ϵ_i are exponentially distributed with mean one. In the simulation we used Monte Carlo and bootstrap methods aiming to evaluate the bias and variability of the estimators.

The simulated process was (omitting the dependence on T) the EACD(1,1) model,

$$\begin{aligned} x_i &= \eta_i \epsilon_i \\ \eta_i &= \omega(i) + \delta(i) x_{i-1} + \gamma(i) \eta_{i-1}, \end{aligned}$$

where $\omega(i) = 0.7$, for all i, $\delta(i) = 0.025 \cos(2\pi i/T) + 0.05$, $\gamma(i) = 0.92$, for all i, and $\epsilon_i \sim$ exponential(1), independent, $i = 1, \ldots, T$, with $T = 2048$.

Figure 1 presents the graph of the function $\delta(i)$. The simulated process and its autocorrelation function (ACF) are shown in Figure 2. The ACF presented a slow decline with several lags out of the 95% confidence interval, indicating that the simulated duration process is non-stationary. It is noticed that the simulated process had the same form of the function $\delta(i)$ used to generate the parameter of the past dependence of times between transactions.

(Figure 1)

(Figure 2)

The parameters of the model were estimated using the ML method, the $RATS$ program and the BHHH method, both for the time-varying case and the constant parameter case. LA(8) wavelet with $J = 4$ was chosen, resulting in 48 wavelet coefficients to be estimated for the time-varying ACD model.

In Figure 3 the time-varying ACD estimators are presented ($\hat{\omega}$, $\hat{\delta}$ and $\hat{\gamma}$) together with usual (constant parameters) ACD model estimators ($\tilde{\omega}$, $\tilde{\delta}$ and $\tilde{\gamma}$). The results for the time-varying ACD estimators are quite good. Moreover, $\hat{\omega}$ and $\hat{\gamma}$ present smaller biases than $\tilde{\omega}$ and $\tilde{\gamma}$. The standard deviations also shown in Figure 3 were obtained using 350 independent replicates of the process.

(Figure 3)

(Figure 4)

In Figure 4 we present some histograms for $\hat{\omega}$, $\hat{\delta}$ and $\hat{\gamma}$ at a number of fixed times, based on the 350 replicates and in Figure 5 we have the MLE with the 90% confidence bands obtained via bootstrap. We see that $\hat{\omega}$, $\hat{\delta}$ and $\hat{\gamma}$ are completely inside the confidence bands, indicating good estimation results.

(Figure 5)

Mean square errors (MSE) and residual mean square errors (RMSE) of the constant parameter estimators and its efficiency relative to ML time-varying estimator are shown in Table 1. For all parameters, the ML time-varying estimator was the most efficient.

(Table 1)

The residual analysis of fitted models is presented in Table 2. On the basis of Ljung-Box [14] tests, we conclude that the MLE captures the dependence structure of the simulated durations process for the selected lags ($q = 10$, 20). The MLE modelled well the volatility of durations, a fact that can be seen from the Ljung-Box tests for squared residuals.

The hypothesis that estimated models residuals follow an exponential distribution was not rejected by a Kolmogorov-Smirnov test, a fact that also can be verified based on the analysis of residuals in Figure 6. The good fitting of the residuals to the exponential distribution also can be verified in the QxQ plots, that plot residuals quantiles versus Exponential quantiles.

(Table 2)

(Figure 6)

In conclusion, the results of the time-varying ACD model estimators are significantly better than the estimator for the usual model with constant parameters. Even for the parameters not varying in time, better results were obtained for the estimators using wavelets. For further details, see [5].

5 An empirical application

In this section we apply the proposed methodology to the transaction durations of TELEMAR stock, traded at São Paulo Stock Exchange.

5.1 Descriptive analysis

The analyzed period was from 12:00 a.m. of August 20th to 5:00 p.m. of September 10th, 2004. Fifteen days of trading were observed in this period. The transactions that occurred during aftermarket were discarded, as well as the simultaneous transactions (durations equal zero). Finally, 16,384 observations were used for the modeling.

Figure 7 presents the durations graph, the sample ACF and the histogram. The durations vary throughout the day, indicating strong non-stationarity of the series and alterations in stock liquidity between days and throughout days. During lunch time (approximately between 1:00 p.m. and 2:30 p.m.) these durations are bigger and market liquidity is smaller. The ACF decays slowly to zero, indicating non-stationarity.

(Figure 7)

To verify the existence of intra-day seasonality, we studied the ACF of the number of transactions in 5 minutes periods. It is worth to note that for this part of the analysis we used a longer TELEMAR series, from August 4th to September 10th 2004 (27 days of trading). Notice that the ACF of the average durations, presented in Figure 8, also presents intra-day regularity with peaks in lags multiple of 84, corresponding to the number of 5 minutes interval in one trading day.

(Figure 8)

To better understand the intra-day seasonality, the graph of the averages of the number of transactions in periods of 5 minutes is presented, throughout the 27 trading days evaluated, in Figure 9. The graph presents a larger number of transactions in periods from 10:15 a.m. to 12:00 a.m. and from 2:40 p.m. to 4:55 p.m., corresponding to the periods before and after lunch. Also a substantial reduction in the number of transactions between 1:00 p.m. and 2:00 p.m. is noticed, during the lunch time.

(Figure 9)

5.2 Model fitting

The ML method was used, with programs written in RATS. Later, we implemented bootstrap method to study the properties of MLE.

In Table 3 we present the ML estimates of the wavelet coefficients and the standard errors in parentheses for the time-varying EACD(1,1) model. Using the Wald test, with level of significance of 5% , we had 36 significant parameters, which indicates that the parameters really vary in time and that the model with time-varying parameters is capable to fit the dependence structure of durations.

To estimate the usual EACD(1,1) model (last line of Table 3) it is necessary to transform the durations to obtain a stationary duration series. There is a variety of methodologies developed to fit trends, seasonality or to stabilize the variance of time series. Splines, wavelets, quadratic functions or sinusoids are some of these, used to model the intra-day seasonality for getting a stationary process. We divide the durations by the estimated seasonal pattern, f_l, to remove seasonality from data and to obtain a seasonally adjusted set of durations. The seasonality was estimated by a nonparametric function, see Figure 10, defined as the expected duration conditioned on time-of-day, where the expectation is computed by averaging the durations over five minutes intervals, namely,

$$\overline{x}_{lm} = \frac{\sum_{i=1}^{n_{lm}} x_{ilm}}{n_{lm}},$$

where l indicates the five minute interval of day, $l = 1, \ldots, 84$; m indicates the trading day, $m = 1, \ldots, 27$ and n_{lm} is the number of trades on lth five minute interval of the mth day. Then we calculate the average of $\overline{x}_{lm}$ for each five minute interval over the 27 trading days to obtain the seasonality function

$$f_l = \frac{\sum_{m=1}^{27} \overline{x}_{lm}}{27}, \text{ for } l = 1, \ldots, 84.$$

(Table 3)

In Figure 10 we present both estimators, MLE for the usual model and MLE for time-varying parameters model. The last entry of Figure 10 corresponds to the sum $\delta + \gamma$, giving an idea of the process persistence. The estimate of the level, $\hat{\omega}$, varies between 0 and 10.24. In the case of the parameter indicating dependence of past duration, δ, the estimate has values between 0.05 and 0.21. The estimate for the past conditional mean dependence, $\hat{\gamma}$, varies between 0.30 and 0.97.

(Figure 10)

In Figure 11 we have histograms of some wavelet coefficient estimators based on bootstrap simulations. The vertical dashed line corresponds to the MLE. The estimated coefficients show a reasonable symmetry and shapes similar to the normal distribution; 27% of wavelet coefficients passed the normality test.

(Figure 11)

Figure 12 shows the estimates and a 90% confidence interval, based on empirical quantiles of 5% and 95% , for the bootstrap estimates. We notice that $\hat{\omega}$, $\hat{\delta}$ and $\hat{\gamma}$ are practically contained inside the confidence bands, for both procedures, indicating that the time-varying model is well fitted to the data.

(Figure 12)

(Table 4)

To test if the models were able to capture the dependence presented in the durations we used the Ljung-Box test for lags 10 and 20 (Table 4). The results show that only the time-varying model estimated by ML could model appropriately the dependence in residuals and squared residuals. The residual analysis is presented in Figure 13, in which we verify a good residuals fitting to the exponential distribution, having little deviation only for bigger residuals.

(Figure 13)

Tables 5 and 6 show the mean, standard deviation and MSE for the residuals and the difference between observed and forecast duration, respectively. The time-varying model estimated by ML presented the smallest mean square error, residual variance and the best forecast.

(Table 5)

(Table 6)

6 Further Remarks

In this article we generalized the ACD model using the wavelet decomposition of its parameters, allowing the coefficients to vary throughout time. The use of a time-varying model allows the modeling of non-stationary series without the need of data preliminary transformations.

Using time-varying ACD model with exponentially distributed errors, proposed in Section 4, it was possible to model the TELEMAR's intra-day seasonality and the series's durations dependence structure together, as well as its volatility.

The estimators and respective standard deviations vary through time, confirming the non-stationarity of the series of durations used. The estimates obtained by fitting a time-varying parameter model have shown to outperform the estimates of constant parameter model. The bootstrap procedure is essential for building confidence intervals.

Preliminary analysis using MCMC methods indicate that these are viable alternative estimation procedures to the ML method. Due to difficulties in obtaining full conditional distributions, the Metropolis-Hastings and Griddy-Gibbs algorithms are some of the possibilities to be used here. This will be the subject of further research.

Another model of interest is the stochastic ACD model, where another error term is added to the second equation in (2). For the case of constant parameters, see for example [22].

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