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Abstract

This paper integrates endogenous fertility behavior into a model of the distribution of consumption and wealth in a competitive market economy. In particular, we examine under what conditions endogenous fertility leads to a non-degenerate distribution of consumption and wealth among dynasties in the long run when preferences are time-separable. We show that, if dynasties have a common discount factor and preferences over number of children satisfy a normality assumption, all steady states are characterized by equality of capital stocks and consumption among families. We also provide sufficient conditions for uniqueness of the steady state. In order to illustrate these results, we present an example in which preferences over number of children are logarithmic and the technology is Cobb-Douglas. For this combination of preferences and technology, there exists a unique egalitarian steady state. Moreover, the economy converges to this steady state in only one generation. We also show that, if dynasties are heterogeneous in their discount factors, more altruistic dynasties have higher fertility rates than less altruistic dynasties in all steady states. Also, assuming a normality condition for fertility and consumption, we show that more altruistic dynasties own higher capital stocks and consume more in steady state than less altruistic dynasties. Even though the stationary distribution of consumption and wealth is non-degenerate at each point in time, asymptotically all dynasties but the most altruistic one become an infinitesimal part of the population.

1. Introduction

This paper investigates the interaction between endogenous fertility behavior and the distribution of income and wealth in a competitive market economy. Specifically, we examine under what conditions endogenous fertility leads to a non-degenerate distribution of consumption and wealth among dynasties in the long run.

Stiglitz (1969) develops a general equilibrium model of the distribution of income and wealth among individuals, in the context of a neoclassical growth framework. Stiglitz analyzes an economy in which agents are divided into a finite number of long-lived families with mortal members in each generation. Different generations of a given family are linked through intergenerational transfers. In his model, the economy-wide capital stock is determined simultaneously with the distribution of capital among individuals. Stiglitz shows that, under the assumption that bequests are a linear function of family income, the steady state level of the economy-wide capital stock per capita is globally stable and wealth and income are asymptotically evenly distributed among families[1].

It has been shown, however, that the results in Stiglitz (1969) depend crucially on the assumption that families have linear bequest rules. If one assumes that bequests are determined endogenously by utility-maximizing agents, Stiglitz's results on the long-run equality of income and wealth among families do not hold in general, unless additional restrictions are imposed on preferences.

Becker (1980) analyzes a model that is similar to Stiglitz (1969) in all aspects, except for the fact that he assumes that infinitely-lived agents maximize a time-additive utility function with a constant rate of time preference. He shows that the household with the lowest discount rate owns all the capital in the long run. If households have equal discount rates, then the steady state distribution of income is indeterminate.

[1]In Stiglitz's model, different families receive the same wage rate and differ only in their per capita wealth holdings. The assumption that bequests are linear in income implies that an increase in per capita wealth by a given percentage raises bequests by a smaller percentage. As a result, the wealth per capita of the poorer families grows faster than that of the richer families.

Lucas and Stokey (1984) argue that the strong implications for the long-run distribution of income and wealth derived by Becker (1980) do not arise from any economic feature of the model, but are simply consequences of the assumption that preferences are additively separable over time. For this reason, they argue that it seems necessary to use a broader class of preferences in order to analyze the long-run distribution of wealth.

Lucas and Stokey (1984) study an optimal growth model in which preferences are recursive, but not necessarily additively separable over time. They show that, if preferences exhibit a property labeled increasing marginal impatience, which means that the consumer's discount factor is a decreasing function of steady state consumption, then there will exist a unique interior stationary distribution of consumption and wealth.

Sarte (1997) shows that, even if preferences are additively separable, progressive taxation may generate stationary equilibria with a non-degenerate wealth distribution in a competitive growth model. Sorger (2002) shows in the context of a Ramsey growth model with time-additive utility functions that, if households have market power on the capital market, there may exist a non-degenerate stationary distribution of capital.

This paper integrates endogenous fertility behavior into a model of the distribution of consumption and wealth in a competitive market economy. In particular, we examine under what conditions endogenous fertility leads to a non-degenerate distribution of consumption and wealth among dynasties in the long run, even if preferences are time-separable.

This paper considers a society divided into a finite number of dynasties, in which individuals from different generations are altruistically linked. All the members of a given dynasty have the same physical capital holdings, but dynasties differ in their per capita holdings and their size (number of members). Different dynasties interact in competitive markets for goods and factor services in each period. Parents are assumed to derive utility from their consumption, number of children and the well-being of each child.

If dynasties have a common discount factor, the model implies that they choose the same fertility rate in steady state, so the shares of each dynasty in the population are constant. We provide sufficient conditions on dynastic preferences and the costs of child rearing such that, in any steady state, all dynasties have the same capital stock. This egalitarian result follows from a normality assumption on fertility. We also provide sufficient conditions for uniqueness of the steady state.

If dynasties have different discount factors, we show that in any steady state more altruistic dynasties have higher fertility rates than less altruistic dynasties. Also, assuming a normality condition for fertility and consumption, we show that more altruistic dynasties own higher capital stocks and consume more in steady state than less altruistic dynasties. Even though the stationary distribution of consumption and wealth is non-degenerate at each point in time, asymptotically all dynasties but the most altruistic one become an infinitesimal part of the population.

The paper is organized as follows. Section 2 defines the setup of the model with common discount factors and defines a competitive equilibrium for the economy. Section 3 defines a steady state and an egalitarian steady state and provides sufficient conditions for uniqueness of an egalitarian steady state. Section 4 shows that, if preferences over number of children are logarithmic and if the technology is Cobb-Douglas, then there exists a unique egalitarian steady state, which is globally stable. Section 5 provides conditions for a non-degenerate steady state distribution of income and wealth when dynasties differ in their discount factors. Section 6 concludes.

2. The Model with Common Discount Factors

The setup of the model is the following. Society is divided into a finite number of dynasties. We define a dynasty or family line as a collection of agents composed of a parent and all his descendants. We assume that the economy starts with a finite number of parents, who in turn define a finite number of dynasties, indexed by $j = 1,...,M$. Agents live two periods, the first as children, in which they do not make any economic decisions, and the second as parents. Each period is taken to be a generation.

There is a large number of firms endowed with the same constant returns to scale technology, so we can assume, without loss of generality, that there is only one firm, which produces the only consumption good according to an aggregate constant returns to scale production function, described by $Y = F(K, N)$, where K and N denote aggregate capital and labor, respectively.[2]

Let y denote output per worker and $\bar{k}$ denote the aggregate capital-labor ratio.

Assumption 1. Assume that the production function satisfies the following properties:

$$y = f(\bar{k}) \equiv F(\bar{k},1) \qquad f(0)=0,\ f'(\bar{k})>0,\ f''(\bar{k})<0,\ \lim_{\bar{k}\to 0} f'(\bar{k})=\infty,\ \lim_{\bar{k}\to\infty} f'(\bar{k})=0$$

In each period, firms sell goods to the household sector and agents supply their labor at a wage w and rent their capital to firms at a rental rate r. The economy is assumed to be competitive, so both agents and firms take prices as given. Competition and profit maximization by firms together imply that factors are paid their marginal products and firms earn zero profits. This implies the following conditions:

$$r_t = f'(\bar{k}_t) \tag{1}$$

$$w_t = f(\bar{k}_t) - \bar{k}_t f'(\bar{k}_t) \tag{2}$$

Parents have identical preferences and supply inelastically one unit of labor. All the currently alive members of a given dynasty have the same stock of physical capital but dynasties differ in their per capita physical capital holdings. Moreover, dynasties may differ in size (number of members).

In this economy, agents are indexed by the dynasty to which they belong. Let k_t^i denote the capital stock of a member of dynasty i in period t and N_t^i the number of members

[2]Since each agent supplies one unit of labor, the number of hours supplied is equal to the size of the population.

of dynasty i. A typical member of dynasty i derives his income from the wage rate w_t and from capital k_t^i, which earns rent at the rate r_t. Capital depreciates at the rate d.

We assume that each child has a fixed cost f in units of the consumption good, so $f n_t^i$ is the total cost of child-rearing, where n_t^i is the fertility rate of dynasty i in period t. Parents choose a bequest k_{t+1}^i for each child, so total bequests equal $n_t^i k_{t+1}^i$. Parents also spend their resources on their own consumption c_t^i. The problem of the head of dynasty i is the following[3]:

$$\begin{aligned} &\max_{c_t^i \geq 0, n_t^i \geq 0, k_{t+1}^i \geq 0} \sum_{t=0}^{\infty} b^t u\left(c_t^i, n_t^i\right) \\ &\text{s.t.} \\ &c_t^i + n_t^i\left(f + k_{t+1}^i\right) = \left(1 - d + r_t\right) k_t^i + w_t \\ &k_0^i \text{ given} \end{aligned} \tag{3}$$

where $0 < b < 1$ is the discount factor, which is the same for all dynasties by assumption[4]. The constraint $k_{t+1}^i \geq 0$ states that dynasties cannot borrow to finance current consumption and expenditures on children.[5]

Assumption 2. Assume that $u(c,n)$ is strictly increasing, concave and twice continuously differentiable in both c and n, with $\lim_{c \to 0} u_c(c,n) = \infty$, $\lim_{n \to 0} u_n(c,n) = \infty$.

Definition 1. A competitive equilibrium is a sequence $E = \left(w_t, r_t, \overline{k}_t, y_t, \left\{\left(c_t^i, k_{t+1}^i, n_t^i\right) / i = 1,..., M\right\}\right)_{t=0}^{\infty}$ which satisfies the following conditions:

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[3] This formulation of the budget constraint incorporates the interaction between quantity and quality of children analyzed in Becker and Lewis (1973), Razin and Ben-Zion (1975) Becker and Barro (1988), Barro and Becker (1989), Benhabib and Nishimura (1993), Alvarez (1999) and Lucas (2002).

[4] Razin and Ben-Zion (1975) used a similar formulation in the context of an aggregative model of optimal growth with endogenous fertility. However, since in their model all agents have identical preferences and capital stocks, it is not suitable for an analysis of the long-run distribution of capital.

[5] Becker (1980) and Sorger (2002) also assume borrowing constraints in their models of the long-run distribution of wealth.

1) For each t, the pair (r_t, w_t) satisfies (1) and (2), respectively.

2) For each $i = 1,..., M$, the sequence $\left(c_t^i, k_{t+1}^i, n_t^i\right)_{t=0}^{\infty}$ solves the maximization problem (3).

3) The factor markets clear in every period, i.e., $\bar{k}_t = \sum_{i=1}^{M} a_t^i k_t^i \quad \forall t = 1,...,\infty$, where $a_t^i = \frac{N_t^i}{\sum_{j=1}^{M} N_t^j}$ and $\sum_{i=1}^{M} a_i = 1$.

4) The goods market clears in every period, i.e., $\sum_{i=1}^{M} a_i \left(c_t^i + n_t^i \left(f + k_{t+1}^i\right)\right) = f\left(\bar{k}\right) + (1-d)\,\bar{k} \quad \forall t = 1,...,\infty$.

In addition to (1) and (2), the budget constraint in (3) and condition 3 in definition 1, a competitive equilibrium is characterized by the first-order conditions of the dynasty's problem:

$$u_c\left(c_t^i, n_t^i\right) = \frac{b}{n_t^i}(1-d+r_{t+1})\,u_c\left(c_{t+1}^i, n_{t+1}^i\right) \tag{4}$$

$$u_n\left(c_t^i, n_t^i\right) = u_c\left(c_t^i, n_t^i\right)\left(f + k_{t+1}^i\right) \tag{5}$$

Equation (4) is the Euler equation with endogenous fertility and (5) is the first-order condition for the optimal number of children.

3. Steady State

Definition 2. A competitive equilibrium E is a steady state if it is a constant sequence $E = \left(w, r, \bar{k}, y, \left\{\left(c^i, k^i, n^i\right) / \, i = 1,..., M\right\}\right)$.

Definition 3. An egalitarian steady state is a steady state with $\left(c^i, k^i, n^i\right) = \left(c^j, k^j, n^j\right)\ \forall i, j = 1,...,M$

Let $R \equiv 1 - d + r$ be the steady state gross interest rate. We can characterize the steady state with the following system of equations:

$$b\,R = n^i \qquad\qquad i = 1,...,M \tag{6}$$

$$\frac{u_n\left(c^i, n^i\right)}{u_c\left(c^i, n^i\right)} = f + k^i \qquad\qquad i = 1,...,M \tag{7}$$

$$c^i + n^i\left(f + k^i\right) = Rk^i + w \qquad\qquad i = 1,...,M \tag{8}$$

$$R = 1 - d + f'\left(\bar{k}\right) \tag{9}$$

$$w = f\left(\bar{k}\right) - \bar{k}f'\left(\bar{k}\right) \tag{10}$$

$$\bar{k} = \sum_{i=1}^{M} a^i k^i \tag{11}$$

$$\sum_{i=1}^{M} a^i = 1 \tag{12}$$

Equations (8)-(12) imply:

$$f\left(\bar{k}\right) + \left(1 - d\right)\bar{k} = \sum_{i=1}^{M} a^i c^i + \left(f + \bar{k}\right)\sum_{i=1}^{M} a^i n^i \tag{13}$$

From (6), it can be seen that the effective discount factor is given by $\frac{b}{n^i}$, so that steady state fertility behaves as a time preference parameter. This implication will be crucial for the results below.

In the subsequent analysis, it will be assumed that a steady state exists (that is, there exists a solution to (6)-(12).[6] We will focus on the issue of uniqueness of the steady state.

In order to solve for the steady state, we will use the following strategy. First, we will postulate an economy-wide capital stock per capita $\bar{k}$. From (9) and (10), we can express the interest rate R and the wage rate w as functions of $\bar{k}$:

$$R = R\left(\bar{k}\right) \equiv 1 - d + f'\left(\bar{k}\right) \tag{14}$$

$$w = w\left(\bar{k}\right) \equiv f\left(\bar{k}\right) - \bar{k}f'\left(\bar{k}\right) \tag{15}$$

Using (6)-(8), we will solve for $c^i, k^i,$ and n^i for a given pair (R, w). Using (14) and (15), we will define a mapping from $\bar{k}$ to the individual capital stock $k_i, k_i = \Psi\left(\bar{k}\right)$. Then we will use (11) and (12) to solve for the equilibrium $\bar{k}$.

From (6), it is clear that in any steady state all families have the same fertility rate, given by

$$n_i = l = bR \qquad \forall i = 1,...,M \tag{16}$$

After substituting (16) into the left-hand side of (7), we can write the steady state marginal rate of substitution between children and consumption as a function of steady state consumption and fertility as follows:

[6] In section 4, we will provide an example of a combination of preferences and technology for which a steady state exists.

$$m(c_i, l) \equiv \frac{u_n(c^i, l)}{u_c(c^i, l)} \tag{17}$$

From (8), we can write k^i as a function of c^i and l (for given R and w):

$$k^i = \frac{c^i + fl - w}{R - l} \tag{18}$$

Notice that $R - l > 0$, since $l = bR$. If we substitute (17) and (18) into (7), we obtain

$$m(c^i, l) = \frac{fR + c^i - w}{R - l} \tag{19}$$

We want to find restrictions on $u(c, n)$ such that (19) defines c^i as a function of (R, w, l). Throughout this analysis, we will keep (R, w, l) constant and view both sides of (19) as functions of c^i.

Remark 1. From assumption 1, there exists a $\tilde{k} > 0$ such that $\bar{k} \leq f(\bar{k}) \leq \tilde{k}$, for all $0 \leq \bar{k} \leq \tilde{k}$ and $f(\bar{k}) < \bar{k}$, for all $\bar{k} > \tilde{k}$. Hence, $X = [0, \tilde{k}]$ is the set of maintainable capital stocks.

Assumption 3. Assume that $fR(\tilde{k}) - w(\tilde{k}) > 0$, where $\tilde{k}$ is the maximum sustainable capital stock per capita.

Remark 2. Since $w(\bar{k}) / R(\bar{k})$ is strictly increasing in $\bar{k}$, assumption 3 implies that $fR(\bar{k}) - w(\bar{k}) > 0 \;\; \forall \bar{k} \in X$.

Assumption 3 imposes a lower bound on the cost of child-rearing f, given by $\frac{w(\tilde{k})}{R(\tilde{k})}$. One way to interpret this assumption is that it requires the net cost of producing a descendant to be positive. An additional child costs f in the current period, which is worth Rf next period. Since an additional descendant will earn w next period, when he becomes an adult, the lifetime cost of an additional adult is $fR - w$, which is positive from assumption 3.

Assumption 4. $e\left(c^i, l\right) \equiv \frac{m_c\left(c^i, l\right) c^i}{m\left(c^i, l\right)} \geq 1$ for any $\left(c^i, l\right)$ satisfying (6)-(12).

Lemma 1 states that, in any steady state, $c^i = c^j \ \forall\, i, j = 1,\ldots, M$

Lemma 1. Let assumptions 1-4 hold. Then there is at most one c_i that solves (19) for given (R, w, l).

Proof: Let $\Omega(c_i, l) \equiv \frac{m(c_i, l)}{c_i}$

If we divide both sides of (19) by c^i and rearrange the terms, we obtain:

$$\Omega\left(c^i, l\right) = \frac{1}{R - l} + \frac{fR - w}{(R - l)c^i} \tag{20}$$

From (16) and assumption 3, the right-hand side of (20) is strictly decreasing in c^i. Differentiating $\Omega\left(c^i, l\right)$ with respect to c^i, we obtain:

$$\Omega_c\left(c^i, l\right) = \left(e\left(c_i, l\right) - 1\right) \frac{m\left(c^i, l\right)}{\left(c^i\right)^2} \tag{21}$$

From assumption 4 and (21), $\Omega\left(c^i, l\right)$ is weakly increasing in c^i at the steady state solution, so the left-hand side of (20) is weakly increasing in c^i at any such solution. Since the right-hand side of (20) is strictly decreasing in c^i, there exists at most one c^i satisfying (20). QED

The derivative of $\Omega\left(c^i, l\right)$ with respect to c^i can be related to the utility function $u(c,n)$ as follows:

$$\Omega_c\left(c^i, l\right) = \frac{[u_c u_{cn} - u_n u_{cc}]c^i - \dfrac{u_n}{u_c}}{\left(c^i\right)^2} \tag{22}$$

where all derivatives are evaluated at steady state values.

To gain some intuition on the restriction imposed on $e\left(c^i, l\right)$, consider the following problem:[7]

$$\begin{aligned} &\max_{c,n}\; u(c,n) + b\,v\left(k^{'}\right) \\ &\quad s.t. \\ &\quad c + n\left(f + k'\right) = I \end{aligned} \tag{23}$$

where v is utility per child and I is income. In this problem, $k^{'}$ is held constant.

Definition 4. n is normal if the maximizing value of n in (23) is increasing in I for all values of f and $k^{'}$. This is equivalent to the condition $u_c u_{cn} - u_n u_{cc} > 0$.[8]

[7]The following argument is motivated by a similar reasoning in Lucas (2002).

If we assume that n is normal in the sense defined above, the first term in the numerator of (22) will be positive at the steady state solution. Yet, these conditions are not enough to guarantee the existence of a unique c^i satisfying (20), since the second term in the numerator of (22) is also positive. Hence, we need the stronger condition $e\left(c^i, l\right) \geq 1$.

The intuition for this result is the following. Richer dynasties desire to have more children, since children are a normal good. Yet, they also face a higher price of children, because they invest more in each child. Hence, it might be possible to have a steady state in which two dynasties with different capital stocks and consumption find it optimal to have the same number of children. The condition $e\left(c^i, l\right) \geq 1$ requires the income effect on fertility to be strong enough, in the sense that the numerator of (22) has to be positive.

In light of Lemma 1 and using (16), we can define c^i as a function of (R,w), $c^i = p\,(R,w)$. The following proposition states that all steady states are egalitarian.

Proposition 1. Let assumptions 1-4 hold. Then all steady states are egalitarian, that is, they satisfy $\left(c^i, k^i, n^i\right) = \left(c^j, k^j, n^j\right)\ \forall i, j = 1,...,M$.

Proof. From equation (16) and Lemma 1, we obtain that, in any steady state, $\left(c^i, n^i\right) = \left(c^j, n^j\right)\ \forall i, j = 1,...,M$. Since, in any steady state, $c^i = p\,(R,w)$ and $l = bR$, we can use (18) to obtain:

$$k^i = \frac{p\,(R,w) + fbR - w}{R(1-b)} \tag{24}$$

[8]This condition can be obtained by differentiating implicitly the first-order conditions associated with (23) with respect to I and by requiring the derivative of the maximizing value of n with respect to I to be positive.

From (24), it is clear that $k^i = k^j \quad \forall i, j = 1,...,M$.   QED

From (14) and (16), we can define l as a function of $\bar{k}$, $l = bR(\bar{k}) \equiv l(\bar{k})$. Differentiating $l(\bar{k})$ with respect to $\bar{k}$, we obtain:

$$l'(\bar{k}) = bR'(\bar{k}) < 0 \tag{25}$$

since $R'(\bar{k}) = f''(\bar{k}) < 0$. Equation (25) states that when the economy-wide capital stock per capita is higher, the fertility rate is lower. The reason is that a higher $\bar{k}$ reduces the interest rate, so fertility has to be lower in order to reduce the effective rate of time preference.

From (14), (15), (20) and (25), we can define c^i as a function of $\bar{k}$, $c^i = C(\bar{k})$, where $C(\bar{k})$ satisfies:

$$\Omega\left(C(\bar{k}), bR(\bar{k})\right) = \frac{1}{R(\bar{k})(1-b)} + \frac{f - \Gamma(\bar{k})}{(1-b)C(\bar{k})} \tag{26}$$

where $\Gamma(\bar{k}) \equiv \dfrac{w(\bar{k})}{R(\bar{k})}$.

Definition 5. c is normal if the maximizing value of c in (23) is increasing in I for all values of f and k'. This is equivalent to the condition $u_n u_{cn} - u_c u_{nn} > 0$.

Assumption 5. $u_n u_{cn} - u_c u_{nn} > 0$ (c is normal).

Lemma 2. Let assumptions 1-5 hold. Then $C(\bar{k})$ is strictly decreasing in $\bar{k}$.

Proof. The appendix shows that, if we differentiate (26) implicitly with respect to $\bar{k}$, we obtain:

$$C'\left(\bar{k}\right)=\frac{\left[\frac{(1-b)\left[f\left(\bar{k}\right)+(1-d)\bar{k}-c^{i}\right]}{(1-b)^{2}R^{2}c^{i}}-\Omega_{n}b\right]}{\left[\Omega_{c}+\frac{f-\Gamma}{(1-b)\left(c^{i}\right)^{2}}\right]}R'\left(\bar{k}\right)<0 \qquad (27)$$

The denominator of (27) is positive, since $\Omega_c \geq 0$ (which follows from assumption 4), $f-\Gamma>0$ (which is equivalent to assumption 3) and $b \in (0,1)$. The term Ω_n in the numerator is the derivative of Ω with respect to n, which is equal to:

$$\Omega_{n}=\frac{u_{c}u_{nn}-u_{n}u_{cn}}{c^{i}}<0$$

which is negative from assumption 5. From the feasibility condition (13), we have $f\left(\bar{k}\right)+(1-d)\bar{k}>c^{i}$. Hence, the term inside brackets in the numerator of (27) is positive. Since $R'\left(\bar{k}\right)=f''\left(\bar{k}\right)<0$, we have established that $C'\left(\bar{k}\right)<0$. QED

The intuition for this result is the following. A higher $\bar{k}$ is associated with lower fertility, which raises the marginal rate of substitution between number of children and consumption, from the assumption that c is normal. The normality condition on fertility requires consumption to decline. A change in $\bar{k}$ also affects the cost of fertility through changes in w and R. By using the fact that, in equilibrium, R and w are related to marginal productivities, one can observe that the net (negative) effect of an increase in the wage rate on the cost of child rearing dominates, which reduces consumption further (this is captured by the first term in the numerator of (27)).

Substituting (14), (15), $c^i = C\left(\bar{k}\right)$ and $l = l\left(\bar{k}\right) \equiv b R\left(\bar{k}\right)$ in (18), we can write k^i as a function of $\bar{k}$, $k^i = \Psi\left(\bar{k}\right)$, which satisfies:

$$\Psi\left(\bar{k}\right) \equiv \frac{C\left(\bar{k}\right) + f l\left(\bar{k}\right) - w\left(\bar{k}\right)}{R\left(\bar{k}\right) - l\left(\bar{k}\right)} \tag{28}$$

Since $k^i = k^j \ \ \forall i, j = 1,...,M$, (11), (12) and (28) imply that the steady state economy-wide per capita capital stock $\bar{k}$ must satisfy the following condition:

$$\Psi\left(\bar{k}\right) = \bar{k} \tag{29}$$

Equation (29) equates the desired individual capital stock to the economy-wide capital stock per capita. The next proposition provides conditions under which (29) has a unique solution for $\bar{k}$.

Proposition 2. Let assumptions 1-5 hold. Then there exists at most one economy-wide capital stock per capita $\bar{k}$.

Proof. If we view both sides of (29) as functions of $\bar{k}$, the right-hand side is just the 45-degree line. If we differentiate (28) implicitly with respect to $\bar{k}$ and rearrange the terms, we obtain:

$$\Psi^{'}\left(\bar{k}\right) = \frac{C^{'}\left(\bar{k}\right) + \left(f + k_i\right) l^{'}\left(\bar{k}\right) + f^{''}\left(\bar{k}\right)\left(\bar{k} - k^i\right)}{\left(1 - b\right)\left(1 - d + f^{'}\left(\bar{k}\right)\right)} \tag{30}$$

When $k^i = \bar{k}$, (30) is reduced to

$$\Psi'\left(\bar{k}\right)=\frac{C'\left(\bar{k}\right)+\left(f+k^{i}\right)l'\left(\bar{k}\right)}{\left(1-b\right)\left(1-d+f'\left(\bar{k}\right)\right)}<0 \tag{31}$$

since $C'\left(\bar{k}\right)<0$ from (27), $l'\left(\bar{k}\right)<0$ from (25) and $\left(1-b\right)\left(1-d+f'\left(\bar{k}\right)\right)>0$, from assumption 1. Hence, the function $\Psi\left(\bar{k}\right)$ has a negative slope when it crosses the 45-degree line, which implies that it can cross it only once, establishing the desired result.

4. Example: Log Preferences[9]

Let $u\left(c,n\right)=g\log\left(c\right)+h\log\left(n\right) \qquad g>0 \quad h>0 \quad 0<b<1$

Let $f\left(\bar{k}\right)=A\bar{k}^{a} \qquad A>0 \quad 0<a<1 \qquad d=1$

4.1. Steady State

For this specification of preferences, the steady state marginal rate of substitution between number of children and consumption is

$$m\left(c^{i},l\right)=\frac{u_{n}\left(c^{i},l\right)}{u_{c}\left(c^{i},l\right)}=\frac{hc^{i}}{gl} \tag{32}$$

The function $\Omega\left(c^{i},l\right)\equiv\frac{m\left(c^{i},l\right)}{c^{i}}$ is given by

$$\Omega\left(c^{i},l\right)=\frac{h}{gl} \tag{33}$$

[9]Logarithmic preferences for children have been recently used in Lucas (2002).

which does not depend on c^i. From (32), $e\left(c^i, l\right)=1$, so assumption 4 is satisfied. It is straightforward to verify that assumption 3 requires $f > \frac{(1-a)}{a} A^{\frac{1}{1-a}}$. Assumption 5 amounts to $u_{nn} = -\frac{h}{l^2} < 0$, which holds in this example.

Since assumptions 1-5 are satisfied for this example, propositions 1 and 2 imply that there exist at most one steady state for this combination of preferences and technology, and it must be egalitarian.

For this example, we can solve the steady state equations (6)-(12) and obtain a closed-form solution for the steady state values of consumption, capital and fertility, which are given by

$$c^i = \frac{gA}{g+h}\left[\frac{fab\left(g+h\right)}{h-ab\left(g+h\right)}\right]^a$$

$$k^i = \bar{k} = \frac{fab\left(g+h\right)}{h-ab\left(g+h\right)}$$

$$l = abA\left[\frac{h-ab\left(g+h\right)}{fab\left(g+h\right)}\right]^{1-a}$$

We need the additional restriction $h-ab\left(g+h\right)>0$ to guarantee existence of a steady state with positive consumption, capital and fertility.

4.2. Stability

In this subsection, we analyze the stability of the steady state computed above. We assume that the economy starts at time $t=0$. Let superscripts index dynasties and subscripts denote the time period (assumed to be a generation). We assume that there are M dynasties, with initial capital stock $k_0^{\ i}$, $i = 1,...,M$. The dynamic system associated to the competitive equilibrium defined in section 2 is described by the following equations:

$$\frac{c_{t+1}^{\ \ i}}{c_t^{\ i}} = \frac{b\,R_{t+1}}{n_t^{\ i}} \tag{34}$$

$$\frac{c_t^{\ i}}{n_t^{\ i}} = \frac{g\left(f + k_{t+1}^{\ \ i}\right)}{h} \tag{35}$$

$$c_t^{\ i} + n_t^{\ i}\left(f + k_{t+1}^{\ i}\right) = R_t k_t^i + w_t \tag{36}$$

$$R_t = f^{'}\left(\overline{k}\right) = a\, A\overline{k}_t^{\,a-1} \tag{37}$$

$$w_t = f\left(\overline{k}_t\right) - \overline{k}_t f^{'}\left(\overline{k}\right) = \left(1 - a\right) A\overline{k}_t^{\,a} \tag{38}$$

$$\overline{k}_t = \sum_{i=1}^{M} a_t^{\ i} k_t^{\ i} \tag{39}$$

$$a_{t+1}^{\ \ i} = \frac{n_t^{\ i}}{\sum_{j=1}^{M} a_t^{\ j} n_t^{\ j}}\, a_t^{\ i} \tag{40}$$

$$\sum_{i=1}^{M} a_t^{\ i} = 1 \tag{41}$$

Equation (34) is the Euler equation, (35) is the first-order condition for fertility, (36) is the budget constraint, (37) and (38) relate factor prices to the marginal productivities, (39) is the market-clearing condition for capital, (40) describes the evolution of the population shares of each dynasty and (41) states that the sum of the shares must sum up to one.

From (34) and (35), we find that at $t = 1$, the following holds:

$$c_1^i = \frac{bg R_1}{h}\left(f + k_1^i\right) \tag{42}$$

From (35) and (36), we can obtain another expression for consumption at $t = 1$:

$$c_1^i = \frac{g\left(w_1 + R_1 k_1^i\right)}{g + h} \tag{43}$$

From (42) and (43), we obtain

$$k_1^i = \frac{fb\left(g + h\right)}{h - b\left(g + h\right)} - \frac{h}{h - b\left(g + h\right)}\left(\frac{w_1}{R_1}\right) \tag{44}$$

Equation (44) implies that $k_1^i = k_1^j \ \ \forall i, j$. Hence, at $t = 1$, capital stocks are equal among dynasties, independently of their initial capital stocks. From (34)-(41), we obtain that this common value of the capital stock is equal to the steady state $\bar{k}$:

$$k_1^i = k_1^j = \bar{k} = \frac{fab\left(g + h\right)}{h - ab\left(g + h\right)} \qquad \forall i, j = 1,...,M \tag{45}$$

To summarize, the economy converges to the unique egalitarian steady state in only one generation. The intuition behind the stability of the steady state is the following. From (34), we have

$$\frac{c_1^{\,i}}{c_0^{\,i}} = \frac{b\,R_1}{n_0^{\,i}} \tag{46}$$

From (35) and (42), we have

$$n_0^{\,i} = \frac{h\left(w_0 + R_0 k_0^{\,i}\right)}{\left(g + h\right)\left(f + \bar{k}\right)} \tag{47}$$

Equation (46) shows that high fertility dynasties discount more the utility of each child, so their consumption grow less than the consumption of low fertility dynasties, and so does their capital stock. From (47), it follows that richer parents (higher $k_0^{\,i}$) dilute their wealth by having more children than poorer parents. Hence, the capital stock of poorer parents grows faster than that of richer parents, leading to convergence of the per capita capital stock among dynasties. For log preferences and Cobb-Douglas technology, this convergence takes only one generation.

5. Model with Heterogeneity in Discount Factors

In this section, we present some results for a version of the model in which dynasties are heterogeneous in their discount factors. Specifically, we assume that dynasty i has discount factor b^i, where $0 < b^i < 1 \quad \forall i = 1,...,M$. Without loss of generality, we can order dynasties according to decreasing altruism, i.e., $1 > b^1 > b^2 > ... > b^M > 0$.

The steady state is still characterized by equations (7)-(12), but we have to replace (6) with[10]

$$b^i R = n^i \tag{48}$$

[10] It should be noted that, since fertility rates may be different across dynasties, the shares of each dynasty in the population may vary over time in a steady state.

From (48), we can observe that more altruistic dynasties have higher fertility rates than less altruistic dynasties. Hence, we have $n^1 > n^2 > ... > n^M$. In order to solve for the steady state consumption c^i, we can rewrite (20) as

$$\Omega\left(c^i, n^i\right) = \frac{1}{R-n^i} + \frac{fR-w}{\left(R-n^i\right)c^i} \tag{49}$$

From the analysis in section 3, we know that for each triple $\left(R, w, n^i\right)$ there exists a unique c^i that solves (49). Hence, we can define $c^i = g\left(R, w, n^i\right)$, where $g\left(R, w, n^i\right)$ satisfies:

$$\Omega\left(g\left(R, w, n^i\right), n^i\right) = \frac{1}{R-n^i} + \frac{fR-w}{\left(R-n^i\right)g\left(R, w, n^i\right)} \tag{50}$$

Lemma 3 states that steady state consumption c^i increases with the steady state fertility rate n^i.

Lemma 3. Let assumptions 1-5 hold. Then, $c^i = g^i\left(R, w, n^i\right)$ is strictly increasing in n^i $\forall i = 1,..., M$.

Proof. Differentiating both sides of (50) with respect to n^i and rearranging the terms, we obtain that $g_n^{\ i}$ satisfies:

$$\left[\Omega_c + \frac{fR-w}{\left(R-n^i\right)\left(g^i\right)^2}\right] g_n^{\ i} = -\Omega_n + \frac{1}{\left(R-n^i\right)^2} + \frac{fR-w}{g^i\left(R-n^i\right)^2} \tag{51}$$

Since $\Omega_c \geq 0$, $\Omega_n < 0$, $fR > w$ and $R > n^i \quad \forall i = 1,...,M$, it follows from (51) that $g_n{}^i > 0$. QED

The following proposition states that in all steady states more altruistic dynasties have higher fertility, consumption and capital stocks than less altruistic dynasties.

Proposition 4. Let assumptions 1-5 hold. Then $\left(c^1, k^1, n^1\right) > \left(c^2, k^2, n^2\right) > ...\left(c^M, k^M, n^M\right)$.

Proof. From (48), we obtain $n^1 > n^2 > ... > n^M$. From lemma 3, we obtain $c^1 > c^2 > ... > c^M$. From (8), we can write k^i as a function of c^i and n^i for given $\left(R, w\right)$:

$$k^i = \frac{c^i + f n^i - w}{R - n^i} \tag{52}$$

From (52) and the previous results, we conclude that $k^1 > k^2 > ... > k^M$. QED

Hence, in any steady state more altruistic dynasties have higher fertility, consumption and capital stocks than less altruistic dynasties. In particular, the more altruistic dynasties tend to grow in size relative less altruistic ones. In other words, altruism is a trait that tends to prevail through its effect on fertility.[11]

6. Conclusion

In this paper, we construct a competitive growth model in which altruistic dynasties are heterogeneous in their initial stocks of physical capital. Parents make choices of family size along

[11] Barro and Becker (1989) show that more altruistic societies tend to grow in size relative to less altruistic ones in an aggregative model of optimal growth with endogenous fertility.

with decisions about consumption and intergenerational transfers. We show that, if dynasties have common discount factors and preferences over number of children satisfy a normality assumption, then all dynasties have the same stock of physical capital per capita in the long run. Moreover, if consumption satisfy a normality assumption, this common level of the capital stock is unique. If preferences are logarithmic and the technology is Cobb-Douglas, the economy converges to the unique egalitarian steady state in one generation.

We also show that, if dynasties have different discount factors, more altrustic dynasties have higher fertility rates in a steady state. Also, assuming a normality condition for fertility and consumption, we show that more altruistic dynasties own higher capital stocks and consume more in a steady state than less altruistic dynasties. Even though the stationary distribution of consumption and wealth is non-degenerate at each point in time, asymptotically all dynasties but the most altruistic one become an infinitesimal part of the population.

Appendix

Proof of Lemma 2. If we differentiate (26) implicitly with respect to $\bar{k}$, we obtain

$$\left[\Omega_c + \frac{(f-\Gamma)}{(1-b)(c^i)^2}\right] C'(\bar{k}) = -\frac{R'(\bar{k})}{(1-b)R^2} - \frac{\Gamma'(\bar{k})}{(1-b)c^i} - \Omega_n b R'(\bar{k}) \tag{53}$$

Since $\Gamma(\bar{k}) \equiv \frac{w(\bar{k})}{R(\bar{k})}$, we have $\Gamma'(\bar{k}) = \frac{w'(\bar{k})R - wR'(\bar{k})}{R^2}$. From (14), we have $R'(\bar{k}) = f''(\bar{k})$. From (15), we have $w'(\bar{k}) = -\bar{k}R'(\bar{k})$

This implies that $\Gamma'(\bar{k}) = \frac{-R'(\bar{k})\left[w(\bar{k}) + \bar{k}R(\bar{k})\right]}{R^2}$

From (14) and (15), we obtain

$$\Gamma^{'}\left(\bar{k}\right)=\frac{-R^{'}\left(\bar{k}\right)\left[f\left(\bar{k}\right)+(1-d)\bar{k}\right]}{R^{2}} \tag{54}$$

If we substitute (54) into the right-hand side of (53) and rearrange the terms, we obtain

$$\left[\Omega_{c}+\frac{(f-\Gamma)}{(1-b)\left(c^{i}\right)^{2}}\right]C^{'}\left(\bar{k}\right)=\left[\frac{(1-b)\left[f\left(\bar{k}\right)+(1-d)\bar{k}-c^{i}\right]}{(1-b)^{2}R^{2}c^{i}}-\Omega_{n}b\right]R^{'}\left(\bar{k}\right)$$

which gives the expression in the text.

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