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# Does It Really Matter Whether the Exchange Rate Floats or Not?

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# A Direct Proof of the First Welfare Theorem*†

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**Abstract:** The First Welfare Theorem is usually proved by contradiction. However, this type of proof should not be used whenever a direct argument can be applied instead. This note provides a direct proof of this classic theorem.

**Keywords:** First Welfare Theorem, direct proof.

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## 1 Introduction

A central result in general equilibrium theory is the First Welfare Theorem. This theorem shows that, under a relatively small set of assumptions, every competitive equilibrium allocation is Pareto efficient.

The most popular (and in fact the only widely known) way of establishing the First Welfare Theorem is to carry out a proof by contradiction. With few variations, this is the approach usually found in most (if not all) textbooks. Some examples are Aliprantis, Brown and Burkinshaw [1]; Arrow and Hahn [3]; Mas-Collel, Green and Whinston [7] and Takayama [9]. This proof was first presented by Kenneth Arrow in [2].

A proof by contradiction should not be used if a direct proof is available. For instance, Paul Halmos stated this (see Knuth, Larrabee and Roberts [6], page 111). Halsey Royden stated the same on page 3 of his classic textbook [8] on Real Analysis. This last author goes as far as to write "All students are enjoined in the strongest possible terms to eschew proofs by contradiction! The reason for this *prohibition* ..." (my emphasis).

There are at least four reasons to prefer direct proofs. The first is purely stylistic. A direct proof is usually more elegant than a proof by contradiction. The second reason is mentioned in Royden [8]. A proof by contradiction often does not illustrate the connection between the hypotheses and the statement to



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be proved. A direct proof clearly shows how the postulates lead to the desired conclusion. The third reason is also provided by Royden [8]. Mistakes are more common in proofs by contradiction. The fourth reason is the strongest one. Royden [8] states that "In proofs by contradiction, however, you are (assuming the theorem is true) in the unreal world where any statement can be derived, and so the falsity of a statement is no indication of an erroneous deduction."

The controversy about proofs by contradiction can be summarized as follows. A small number of researchers will say that a proof by contradiction is not a proof. Most of them will prefer a direct proof to a proof by contradiction, particularly if the direct proof illustrates in a more effective way how the postulates lead to the desired result.

Let $R$ and $S$ be any statements. Suppose that one wants to show that $[R \Rightarrow S]$. A direct proof consists of starting from $R$ and, after some logical steps, arriving at $S$. A proof by contradiction starts by assuming that the statement $[R \,\&\, (\neg S)]$ (where $\neg$ means 'not') is true. It finishes by showing that this leads to a contradiction. The contraposition approach is to show that $[(\neg S) \Rightarrow (\neg R)]$ using a direct proof. Since the statements $[R \Rightarrow S]$ and $[(\neg S) \Rightarrow (\neg R)]$ are equivalent, it does not matter which of the two implications is established by direct reasoning.

The First Welfare Theorem states that if a price system and an allocation is a competitive equilibrium, then this allocation is Pareto efficient. The standard proof starts by assuming that some competitive equilibrium allocation is not Pareto efficient. It finishes by obtaining a contradiction. A contraposition argument is used in this essay. It is shown that if an allocation is not Pareto efficient, then this allocation is not a competitive equilibrium allocation (regardless of the prevailing price system).

The proof presented in this note has the obvious advantage of not being affected by Royden's harsh criticism. It also establishes a more direct connection between the concepts of competitive equilibrium and Pareto efficiency. The proof clearly shows that any feasible allocation that is not Pareto efficient cannot be supported, as a competitive equilibrium, by any vector of prices. This is exactly the same argument used to convince an undergraduate student that an allocation that does not lay on the contract curve of an Edgeworth's box is not a competitive equilibrium allocation.

This paper is not the first one to establish the First Welfare Theorem without resorting to a contradiction argument. Debreu shows on pages 94 and 95 of his *Theory of Value* [4] that if an allocation is feasible it cannot Pareto dominate a competitive equilibrium allocation. On page 101 of [5], Debreu shows that any allocation that makes some agent better-off without harming another (when compared to a competitive equilibrium allocation) is not feasible. Compared to Debreu's proofs, the one presented in this paper has the advantage of being simpler and more intuitive.

This paper is organized as follows. For simplicity, the new approach to prove the First Welfare Theorem is initially used in a pure exchange economy in Section 2. The argument is generalized to an economy with production in Section 3. Section 4 concludes.

## 2 A Pure Exchange Economy

There exists a set $\mathcal{I} = \{1, 2, ..., I\}$ of consumers and a set $\mathcal{L} = \{1, 2, ..., L\}$ of commodities. The commodity space is $\mathbb{R}^L_+$. Each consumer $i \in \mathcal{I}$ has a preference relation $\succeq_i$ on her consumption set $X_i \subseteq \mathbb{R}^L_+$ and has an initial endowment $\varpi_i \in X_i$. As usual, $x_i \succ_i \tilde{x}_i$ means that $x_i \succeq_i \tilde{x}_i$ and $\tilde{x}_i \not\succeq_i x_i$. An allocation is a vector $x \in \mathbb{R}^{IL}_+$. It can be written as $x = (x_1, x_2, ..., x_I)$, where each $x_i \in X_i$. A price system is any vector $p \in \mathbb{R}^L_+$. Given a price system $p$, the budget set of consumer $i$ is the set $B_i(p) = \{x_i \in X_i : p \cdot x_i \leq p \cdot \varpi_i\}$.

The definitions that follow spell out the remaining introductory formalities.

**Definition 1** *A preference relation $\succeq_i$ is* locally non-satiable *if for every $\bar{x}_i \in X_i$ and all $\delta > 0$ there exists $\tilde{x}_i \in \{x_i \in X_i : \|x_i - \bar{x}_i\| < \delta\}$ satisfying $\tilde{x}_i \succ_i \bar{x}_i$.*

**Definition 2** *A bundle $x_i \in X_i$ is a* maximal element *for $\succeq_i$ on a set $\tilde{X}_i \subseteq X_i$ if $x_i \succeq_i \tilde{x}_i$ for all $\tilde{x}_i$ in $\tilde{X}_i$.*

**Definition 3** *An allocation $x$ is* feasible *if $\sum_{i=1}^I x_i = \sum_{i=1}^I \varpi_i$ and $x_i \in X_i$ for all $i$.*

**Definition 4** *An allocation $x$ is* Pareto efficient *if it is feasible and there is no feasible allocation $\bar{x}$ that satisfies $\bar{x}_i \succeq_i x_i$ for all $i$ and $\bar{x}_i \succ_i x_i$ for some $i$.*

**Definition 5** *A* competitive equilibrium *is a vector $(p, x)$ that satisfies:*

1. *$x$ is feasible;*
2. *for each $i$, $x_i$ is a maximal element for $\succeq_i$ on $B_i(p)$.*

**Theorem 1** *Suppose that each $\succeq_i$ is locally non-satiable. If $(p, x)$ is a competitive equilibrium, then $x$ is Pareto efficient.*

**Proof.** It is enough to show that if an allocation $x$ is not Pareto efficient, then there is no price system $p$ such that $(p, x)$ is a competitive equilibrium. Take any $x$ that is not Pareto efficient and any price system $p$. Either (i) $x$ is not feasible or (ii) $x$ is feasible and there exists another feasible allocation $\bar{x}$ that satisfies $\bar{x}_i \succeq_i x_i$ for all $i$ and $\bar{x}_i \succ_i x_i$ for some $i$, say $i = 1$. If (i) is true there is nothing to show. Consider the situation in which (ii) holds. Since $\bar{x}$ is feasible,

$$\sum_{i=1}^{I} \bar{x}_i = \sum_{i=1}^{I} \varpi_i \Rightarrow \sum_{i=1}^{I} p \cdot \bar{x}_i = \sum_{i=1}^{I} p \cdot \varpi_i \ . \tag{1}$$

Consider the inequality

$$p \cdot \bar{x}_i \leq p \cdot \varpi_i \tag{2}$$

If (2) holds for $i = 1$, $x_1$ cannot be a maximal element for $\succeq_1$ on the set $B_1(p)$. Thus, $(p, x)$ is not a competitive equilibrium. If (2) fails for $i = 1$, (1) implies that there exists a consumer $k$ such that $p \cdot \bar{x}_k < p \cdot \varpi_k$. Since $\succeq_k$ is locally non-satiable, there exists a $\tilde{x}_k \in B(p)$ satisfying $\tilde{x}_k \succ_k \bar{x}_k \succeq_k x_k$ . Therefore, $x_k$ cannot be a maximal element for $\succeq_k$ on the set $B_k(p)$, from which follows that $(p, x)$ is not a competitive equilibrium. □

Definition 3 assumed the absence of free disposal. This is not essential for the proof. The same reasoning would work for an economy with free disposal (if one bears in mind that with locally non-satiable preferences there is no competitive equilibrium with a negative price).

It should be clear that the approach used in the proof of theorem 1 will also show that a competitive equilibrium allocation belongs to the core.

# 3 A Production Economy

The environment builds on the one described in Section 2. There exists a set $\mathcal{I} = \{1, 2, ..., I\}$ of consumers, a set $\mathcal{L} = \{1, 2, ..., L\}$ of commodities and a set $\mathcal{J} = \{1, 2, ..., J\}$ of firms. The commodity space is $\mathbb{R}^L_+$. Each firm $j \in \mathcal{J}$ has a production set $Y_j \subseteq \mathbb{R}^L_+$. Each consumer $i \in \mathcal{I}$ has a preference relation $\succeq_i$ on her consumption set $X_i \subseteq \mathbb{R}^L_+$ and an initial endowment $\varpi_i \in X_i$. An allocation is a vector $(x, y) \in \mathbb{R}^{(I+J)L}_+$, where $x \in \mathbb{R}^{IL}_+$ and $y \in \mathbb{R}^{JL}_+$. It can be written as $(x, y) = (x_1, x_2, ..., x_I, y_1, y_2, ..., y_J)$, where $x_i \in X_i$ and $y_j \in Y_j$. A price system is any vector $p \in \mathbb{R}^L_+$. The matrix $\Theta = [\theta_{ij}]_{I \times J}$, where $\theta_{ij} \geq 0$, describes the share of firm $j$'s profit that is owned by consumer $i$. Of course, $\sum_{i=1}^I \theta_{ij} = 1$ for every $j$. The budget set of consumer $i$ is $B_i(p, y) = \left\{ x_i \in X_i : p \cdot x_i \leq p \cdot \varpi_i + \sum_{j=1}^J \theta_{ij} p \cdot y_j \right\}$.

Locally non-satiable preferences and maximal element are defined as in the previous section. The remaining definitions are:

**Definition 6** *An allocation $(x, y)$ is* feasible *if $\sum_{i=1}^I x_i = \sum_{i=1}^I \varpi_i + \sum_{j=1}^J y_j$, $x_i \in X_i$ for all $i$ and $y_j \in Y_j$ for all $j$.*

**Definition 7** *An allocation $(x, y)$ is* Pareto efficient *if it is feasible and there is no feasible allocation $(\bar{x}, \bar{y})$ that satisfy $\bar{x}_i \succeq_i x_i$ for all $i$ and $\bar{x}_i \succ_i x_i$ for some $i$.*

**Definition 8** *A* competitive equilibrium *is a vector $(p, x, y)$ that satisfies:*

1. *$(x, y)$ is feasible;*
2. *for each $j$, $p \cdot y_j \geq p \cdot \bar{y}_j$ for all $\bar{y}_j \in Y_j$;*
3. *for each $i$, $x_i$ is a maximal element for $\succeq_i$ on $B_i(p, y)$.*

**Theorem 2** *Suppose that each $\succeq_i$ is locally non-satiable. If $(p, x, y)$ is a competitive equilibrium, then $(x, y)$ is Pareto efficient.*

**Proof.** Take an allocation $(x, y)$ that is not Pareto efficient and a price system $p$. It suffices to show that $(p, x, y)$ is not a competitive equilibrium. Either (i)

$(x, y)$ is not feasible or (ii) $(x, y)$ is feasible and there exists another feasible allocation $(\bar{x}, \bar{y})$ that satisfies $\bar{x}_i \succeq_i x_i$ for all $i$ and $\bar{x}_i \succ_i x_i$ for some $i$, say $i = 1$. If (i) is true there is nothing to show. Consider the situation in which (ii) holds. If $p \cdot y_j < p \cdot \bar{y}_j$ for some $j$, $(p, x, y)$ is not a competitive equilibrium. If $p \cdot y_j \geq p \cdot \bar{y}_j$ for all $j$, the fact that $(\bar{x}, \bar{y})$ is feasible implies

$$\sum_{i=1}^{I} \bar{x}_i = \sum_{i=1}^{I} \varpi_i + \sum_{j=1}^{J} \bar{y}_j \Rightarrow \sum_{i=1}^{I} p \cdot \bar{x}_i = \sum_{i=1}^{I} p \cdot \varpi_i + \sum_{j=1}^{J} p \cdot \bar{y}_j =$$
$$\sum_{i=1}^{I} p \cdot \varpi_i + \sum_{j=1}^{J} \sum_{i=1}^{I} \theta_{ij} p \cdot \bar{y}_j \leq \sum_{i=1}^{I} p \cdot \varpi_i + \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} p \cdot y_j \Rightarrow \tag{3}$$
$$\sum_{i=1}^{I} p \cdot \bar{x}_i \leq \sum_{i=1}^{I} p \cdot \varpi_i + \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} p \cdot y_j.$$

If

$$p \cdot \bar{x}_i \leq p \cdot \varpi_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j \tag{4}$$

holds for $i = 1$, $x_1$ cannot be a maximal element for $\succeq_1$ on the set $B_1(p, y)$. Thus, $(p, x, y)$ is not a competitive equilibrium. If (4) fails for $i = 1$, the last inequality in (3) implies that there is a consumer $k$ such that $p \cdot \bar{x}_k < p \cdot \varpi_k + \sum_j \theta_{kj} p \cdot y_j$. Since $\succeq_k$ is locally non-satiable, there exists a $\tilde{x}_k \in B(p, y)$ satisfying $\tilde{x}_k \succ_k \bar{x}_k \succeq_k x_k$ . Therefore, $x_k$ cannot be a maximal element for $\succeq_k$ on the set $B_k(p, y)$. Hence, $(p, x, y)$ is not a competitive equilibrium. □

# 4 Conclusion

The standard approach to prove the First Welfare Theorem is to carry out a proof by contradiction. That technique was first presented in Arrow [2]. Despite the fact that Debreu provided two alternative proofs without using a contradiction argument, Arrow's proof is the only one that is widely known among economists.

Proofs by contradiction should not be used whenever a direct proof is available. This is partially due to stylistic and didactic reasons. But it is also (and most important) due to the fact that some of researchers do not accept an argument by contradiction as a valid proof.

In this essay the First Welfare Theorem was established, for both pure exchange and production economies, by a simple and direct reasoning. This classic economic result is then reconciled with mathematicians' distaste for proofs by contradiction and liking of simple and intuitive arguments.

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