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Jose A. Garcia-Martinez

WP 2010-5
July 2010

Departamento de Teoría e Historia Económica
Facultad de Ciencias Económicas y Empresariales
Universidad de Málaga
ISSN 1989-6908

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José A. García-Martínez

Departamento de Estudios Económicos y Financieros. Universidad Miguel Hernández

June 2010

E-mail address: Jose.Garciam@umh.es

Phone 1: +34 966658886
Phone 2: +34 645125308

Address for manuscript correspondence:
José A. García-Martínez
Departamento de Estudios Económicos y Financieros
Universidad Miguel Hernández
Avenida de la Universidad s/n, Edificio La Galia
E-03202, Elche, Alicante (SPAIN)

[1] This paper is based on Chapter 1 of my Ph.D. thesis at the University of Alicante. I wish to express my gratitude to Fernando Vega-Redondo, my thesis advisor, for his encouragement and invaluable comments. I also thank Dilip Mookherjee, Carlos Alós-Ferrer, Pablo Beker, Joseph Harrington, Francisco Marhuenda, Giovanni Ponti, and Frédéric Palomino for their comments and suggestions. My thanks also go out to the Institute for Economic Development and the Department of Economics at Boston University for their hospitality. I also acknowledge contributions by participants at seminars and conferences where the paper has been presented in Alicante, Boston, Marseilles and Ankara. Financial support from the Instituto Valenciano de Investigaciones Económicas (IVIE) and the Spanish Ministry of Education and Science through Grant EX2004-0807 is gratefully acknowledged.

Abstract

We consider a selection process and a hierarchical institution in a dynamic model as in Harrington [3], where agents are "climbing the pyramid" in a rank-order contest based on the "up or out" policy. Agents are ranked according to the quality of their performances in a particular environment that they face in groups, and a fraction of the highest ranked agents are promoted. The size of this fraction characterizes the selectivity of the process, and we distinguish between local and global selectivity. We study the role of the degree of local and global selectivity in the dynamic process where agents' types differ in their expected performances. Surprisingly, we find that an increase in the selectivity of the process can be detrimental to the agents with the highest expected performances. In fact, it does not matter how small the expected performance of a particular type of agent is. If the degree of selectivity is high enough, that type of agent will survive. However, if the selectivity decreases, the only survivor is the agent with the highest expected performance.

JEL classification: D00; D23; C73; D72

Keywords: Social hierarchy; Selection; Selectivity; Promotion

1 Introduction

Selection processes exist in all societies. In general terms, agents interact with one another, within organizations and institutions, and as a result of that interaction certain individuals are promoted over others and achieve higher status in the form of greater political or economic power, increased prestige, more responsibility and wider intellectual influence. In such selection processes, the characteristics of the individual agents obviously play an important role, but institutional factors such as the selectivity of the promotion system and hierarchical structures are also highly influential.

In this paper we present some interesting and counterintuitive properties of a family of selection systems in a hierarchical institution. These selection systems are characterized according to the amount of selectivity implemented. In fact, we parameterize the degree of selectivity and study the dynamic of the process for all possible values of the selectivity parameters.

In our model, the concept of degree of selectivity is determined by the fraction of agents promoted. For example, consider a set of agents that are ranked by the outcomes of their activities or by any other "agent's characteristic". We then select a fraction $\alpha \in (0,1)$ of the highest ranked agents to be promoted. The closer α is to zero, the greater the selectivity of the process. We can define the degree of selectivity of the selection process as $1-\alpha$.

We consider a selection process and a hierarchical structure in a dynamic model as in Harrington [3], where agents are "climbing the pyramid" in a rank-order contest based on the "up or out" policy. We generalize the selection process used in Harrington.

Like Harrington, we do not consider strategic interaction and each agent is endowed, at $t = 0$, with one of two actions or behavioral rules: A and B. Thus, we consider only two types[1] of agent: type A and type B. The environment where type A outperforms B is more frequent than the environment where B outperforms A. In this sense, A is a better type and has a higher expected success rate.

The population at any level of the hierarchy is matched in groups of n agents, and each group faces a particular environment. Agents are ranked according to the quality of their performances in this particular environment. The top k performing agents from each group become eligible for promotion: this is the *local selection* process. The eligible agents are pooled together and the top fraction θ of performing agents is promoted; this is the *global selection* process.

In the next period, the agents promoted compete with one another again on their new level for promotion to the following level, always under the "up or out" policy. Thus, the non-promoted agents are no longer considered for promotion.

In our model, A-agents have good performances more often than B-agents. Therefore, we might expect increases in selectivity to punish B-agents, and the proportion of B-agents promoted to decrease as selectivity

[1] Our main conclusion holds if more than two types are considered.

increases. However, we show that this is not always the case, and, in fact, the proportion of B-agents promoted increases if selectivity increases enough.

We obtain the following main results. If the level of selectivity is low enough, the selection process is not strong enough to overcome the inertia of the initial population. The dynamic depends on the initial conditions, and the population eventually becomes homogeneous, i.e. either type A or type B. If selectivity increases enough, the whole population will come to be type A for any initial mixed population. In that case, the selection process is strong enough eventually to select agents of type A, the best performers. Finally, if the selectivity is increased far enough, the behavior of the system depends only on local selectivity. If it is not too high, then only A-agents survive as before. However, surprisingly, if local selectivity is increased far enough, type B agents also survive, and the proportion for which they account at equilibrium increases as selectivity increases. Therefore, *no matter how low the success rate of a type is, if the selection process has a high enough selectivity, agents of this type survive in the long run.*

To understand why this happens, we must first observe that the probability of promotion of a particular type depends on three things: the initial proportions of each type of agent in the population, the selectivity of the selection process, and the probability of success of each type of agent. If, for example, B-agents are scarce, the rivals of a B-agent are mostly A-agents. In that case, almost the only way for that B-agent to get promoted is to be better than A-agents. Otherwise, he is at the bottom of the rank order. Therefore, the probability of promotion of a scarce type depends mainly on the probability of success of his type, and selectivity does not affect him too much. By contrast, if a particular type, e.g. A-agents, is abundant, the rivals of an A-agent are also mostly A-agents. Being successful is not too important for promotion because all A-agents are doing the same thing. The probability of promotion depends mainly on the selectivity, on how many agents are promoted.

Therefore, an increase in selectivity tends to punish the more common type of agents because it decreases their probability of promotion, but it does not affect the relatively scarce type. *Thus, we can favor or punish diversity by tuning the level of selectivity.* A deeper intuition is provided in Section 5.

The structure of our model is quite similar to the way in which some sports competitions are organized. There are different levels, players compete in separate groups at each level and the best players in each group are promoted to the next level. The final goal of each player is to get the top of the pyramid. In this example, there is only local competition in groups.

In many social systems, however, there is not only local competition for promotion but also broader competition throughout each level. In the academic world, for example, students first compete at their own universities (local competition), and once they have finished their courses some of the most brilliant students from all universities then compete for a position at a university department (global competition). If they are able to secure a position they then compete locally again, within their new departments, to obtain a Ph.D.. After obtaining their Ph.D., however, they must then compete general once again with other Ph.D. graduates from all other universities for a limited number of jobs in a given number of organizations. Similar processes exist among people who work in business. They first compete within their own groups or departments (locally) and the best among them become candidates for promotion. Those candidates, however, must sometimes compete again in a broader context with candidates from other departments for promotion to a higher status.

A stylized example would be that of a sales company that promotes people according to their success in selling. The company employs men and women and men sell better to men and women sell better to women. If the potential market has more men than women, men could be the A-agents and women the B-agents. In another example, we can think of the rules (A and B) as different available technologies: one of them is the best more often than the other, and agents are proficient in either technology A or B.

In political careers we could consider two kinds of politicians: demagogues and principled politicians. The demagogues could be the B-agents if they have the favor of the voters in some political environments that are less probable than those in which the principle agents (A-agents) have the best results.

The promotion of B-agents may or may not be desirable depending on the nature of the situation and the preferences of the institutions concerned.

It is not easy to find an application that fits all of the model's elements because the model seeks to represent a family of complex institutions in a very stylized manner to point out a very specific characteristic of a selection process. Obviously, in any real situation the selection process is influenced by many more factors. However, we believe that the property identified with our model is robust enough to play a role in more complex situations.

The rest of this article is presented as follows: Section 2 describes the model and the dynamic equations, Sections 3, 4 and 5 analyze the dynamics and provide some intuitions, and Section 6 gives the conclusion and discusses related literature.

2 The Model

As in Harrington [3], we consider a hierarchical system with a lowest level and no upper bound on the highest level. The initial population resides at the lowest level of the system, and comprises two types: A and B, which compete for promotion. The objective is to analyze how the proportions of A-agents and B-agents in this initial cohort of agents change as they migrate up through the hierarchy. This analysis is conducted for any degree of selectivity. If the hierarchy is to be kept "full" then at the end of each round a fresh cohort of agents must enter the lowest level to replace those who have moved on. Another structure that can be considered is a hierarchy with just T levels where new agents imitate the agents at the top, as described in Harrington [4]. Note that the two cases are equivalent if the new agents in this second case reproduce the profile of agents at the top.

Therefore, we consider that at level t there is a large enough population of agents (a continuum), where $a_t \in [0, 1]$ denotes the proportion of A-agents at level t, and b_t the proportion of B-agents (with $b_t = 1 - a_t$). We seek to specify a dynamic function $a_{t+1} = f(a_t)$ that relates the proportion of A-agents at level t to the proportion of A-agents after going through a selection process, i.e., the proportion of A-agents in the set of agents promoted to level $t+1$.

We consider that the agents at level t are **randomly matched** in groups of $n \geq 2$. We assume that the random matching process has the following properties: *First*, the probability with which a given agent is matched with agents of given types equals the product of the proportions of agents of the respective types in the population. *Second*, the proportion of a given class of grouping is equal to the probability (ex-ante) of such a grouping. The existence of a random matching process having these properties is proved in Alós-Ferrer [1][2].

Thus, the proportion of *groups* containing x A-agents (and $s = n - x$ B-agents) is equal to the probability of such a group [3], i.e., $\binom{n}{x} a_t^x (1-a_t)^{n-x}$. This is also the proportion of *agents* in groups with x A-agents with regard to the initial population (level t) because the groups are composed of equal number of agents.

These agents face a stochastic environment which is the same for all the members of a particular group. However, the environment of each group is stochastically independent of other groups. We categorize all the different possible environments into three types. In a type 1 environment, A-agents respond correctly to the environment and B-agents respond wrongly. In a type 2 environment, B-agents respond correctly and A-agents wrongly. Finally, in type 3, neither A nor B responds correctly[4]. The probability[5] of an environment of type i is P_i, with $i = 1, 2, 3$, and $\sum_{i=1}^{3} P_i = 1$. Therefore, each agent faces an uncertain future environment, but there is no aggregate uncertainty because of our assumptions. Therefore, at each level after the random matching, a proportion P_1 (P_2, P_3) of the groups has a type 1 $(2, 3)$ environment. This is assumed to be i.i.d. across levels, so that the probability of an agent facing a given environment is independent of the environment that he/she has faced in the past.

Therefore, the proportion of agents in groups with a number x of A-agents under a type 1 environment is $\binom{n}{x} a_t^x (1-a_t)^{n-x} P_1$. In such groups A-agents outperform B-agents. The system selects the k top performing agents from each group, with $k \leq n$. This process is called **local selection**. The selected agents from each group become *eligible* for promotion or survival. The proportion of eligible agents is $\frac{k}{n}$ with regard to the initial population (level t). This ratio measures the selectivity of the local selection. Thus, if a group in a type 1 environment has more A-agents than vacancies available (i.e., $x \geq k$), then all the eligible agents selected from this group are successful, and the proportion of eligible A-agents that have been successful is $\frac{k}{n} \binom{n}{x} a_t^x (1-a_t)^{n-x} P_1$. However, if $x < k$, then x successful A-agents are selected as eligible and some unsuccessful B-agents have to be randomly chosen to fill the $k - x$ vacancies, and the proportion of eligible A-agents that have been successful is $\frac{x}{n} \binom{n}{x} a_t^x (1-a_t)^{n-x} P_1$. Analogously, a fraction P_2 of groups will face a type 2 environment and similar reasoning can be used. Finally, a fraction P_3 of groups will face a type 3 environment. In this case, all the eligible

[2] Alós-Ferrer [1] gives a constructive existence proof for the case $n = 2$. The generalization to groups of n agents is straightforward.

[3] The x is distributed as a binomial distribution, $x \backsim B(n, a_t)$.

[4] A fourth environment can be considered in which both rules are right answers. This environment adds no new insights to the analysis, so we do not consider it.

[5] This probability P_i can also be seen as the expected success rate of an agents of type i, with $i = 1, 2$.

agents will be unsuccessful. Consequently, the total proportion of eligible A-agents that have been successful will be: $ES_t^a = \sum_{x=0}^{k-1} \frac{x}{n}\binom{n}{x} a_t^x (1-a_t)^{n-x} P_1 + \sum_{x=k}^{n} \frac{k}{n}\binom{n}{x} a_t^x (1-a_t)^{n-x} P_1 = \sum_{x=0}^{n} min[x,k]\frac{1}{n}\binom{n}{x} a_t^x (1-a_t)^{n-x} P_1$, and the proportion of eligible B-agents[6] that have been successful will be $ES_t^b = \sum_{s=0}^{n} min[s,k]\frac{1}{n}\binom{n}{s} a_t^{n-s}(1-a_t)^s P_2$. Clearly, the proportion of eligible agents who are successful will be $ES_t = ES_t^a + ES_t^b$. On the other hand, the proportion of eligible agents who are unsuccessful will be $EU_t = \dfrac{k}{n} - ES_t$. Finally, the proportion of eligible A-agents who are unsuccessful comprises two terms. First, the A-agents selected as eligible from the groups under the type 2 environment that do not have enough B-agents to fill all the k vacancies, i.e. $k-(n-x)$ A-agents; and second the A-agents selected from the groups under the type 3 environment. The latter will be a proportion $k\frac{x}{n}$ of type A-agents from groups with x A-agents: $EU_t^a = \sum_{x=n-k+1}^{n} (k-(n-x))\frac{1}{n}\binom{n}{x} a_t^x (1-a_t)^{n-x} P_2 + \sum_{x=1}^{n} k\frac{x}{n}\frac{1}{n}\binom{n}{x} a_t^x (1-a_t)^{n-x} P_3$

In the second step of the selection process, these eligible agents selected from each group are pooled together and a new rank-order is made according their previous performances in their groups, with the eligible successful agents on top and the unsuccessful ones after them. The top fraction θ is eventually promoted to the next level. This process is called **global selection**. The parameter θ measures the selectivity of the global selection process. Thus, the proportion of agents eventually promoted is $\theta\frac{k}{n}$.

The dynamic equation $a_{t+1} = f(a_t)$ is a piecewise function which has two different pieces because two cases can occur in global selection.

First case: the proportion of agents promoted ($\theta\frac{k}{n}$) is ES_t or less. In this case, the system selects a proportion $\theta\frac{k}{n}$ of agents randomly from among the eligible successful agents. Since the proportion of A-agents among the eligible successful agents is $\frac{ES_t^a}{ES_t}$, this will be the proportion of A-agents in the next level: $a_{t+1} = \frac{ES_t^a}{ES_t}$.

Second case: the proportion of agents promoted ($\theta\frac{k}{n}$) is greater than ES_t. The system now selects all eligible successful agents plus a randomly selected proportion $\theta\frac{k}{n} - ES_t$ of eligible unsuccessful agents. Note that the proportion of A-agents among the eligible unsuccessful agents is $\frac{EU_t^a}{EU_t}$. Consequently, the proportion of A-agents selected will be $ES_t^a + \left(\theta\frac{k}{n} - ES_t\right)\frac{EU_t^a}{EU_t}$ with regard to the initial population (level t). Finally, the proportion of A-agents with regard to the population of agents selected (level $t+1$) is $a_{t+1} = \frac{1}{\theta\frac{k}{n}}\left(ES_t^a + \left(\theta\frac{k}{n} - ES_t\right)\frac{EU_t^a}{EU_t}\right) = \frac{1}{\theta\frac{k}{n}}\left(ES_t^a + \frac{\theta\frac{k}{n} - ES_t}{\frac{k}{n} - ES_t} EU_t^a\right)$.

Therefore the dynamic equation has this form:

$$a_{t+1} = f(a_t) = \begin{cases} \frac{ES_t^a}{ES_t} & if \quad \theta\frac{k}{n} \le ES_t \\ \frac{1}{\theta\frac{k}{n}}\left(ES_t^a + \frac{\theta\frac{k}{n} - ES_t}{\frac{k}{n} - ES_t} EU_t^a\right) & if \quad \theta\frac{k}{n} > ES_t \end{cases} \tag{1}$$

The following property of the proportion of successful eligible agents (ES_t) is worth pointing out: ES_t has a lower bound, which is reached when the whole population of the level is type B, i.e. $b_t = 1$. As indicated above, B-agents have a lower success rate because the type 2 environment is less probable than type 1, ($P_1 > P_2$). Thus, the proportion of successful agents is smallest when the whole population of agents is type B. Note that

[6] The number of B-agents in a group is $s = n - x$

$\frac{k}{n}$ is the proportion of eligible agents (all type B if $b_t = 1$), a fraction P_2 of them from under environment type 2 and consequently successful. Therefore, if $b_t = 1$, then $ES_t = \frac{k}{n}P_2$. We can state that in all cases $ES_t \geq \frac{k}{n}P_2$. As a consequence, if the proportion of agents eventually promoted, $\theta\frac{k}{n}$, is lower than $\frac{k}{n}P_2$, then all the agents promoted are successful eligible agents. Note that $\theta\frac{k}{n} < \frac{k}{n}P_2 \Leftrightarrow \theta < P_2$, so if $\theta < P_2$ all agents promoted are successful eligible agents.

Another interesting property derived from this is the following: For all $\theta \in (0, P_2)$, the dynamic is the same, i.e. the proportion of A-agents in the set of promoted agents does not change for any $\theta \in (0, P_2)$. That is because the system randomly chooses agents from among the eligible successful agents, so the proportion of A-agents selected is always the same and equal to the proportion of A-agents among the eligible successful agents, i.e. $\frac{ES_t^a}{ES_t}$. Therefore, the effective selectivity of the global selection is limited because it is parameterized by θ. This result is related to our assumption of considering only two results: right and wrong. Consequently, in the pool of eligible agents there are only two kinds of agent: successful and unsuccessful. If we had considered more than two possible results[7], the role of global selection would have been more significant. At the beginning of Section 5, we show that if only global selection works, the composition in the long run is identical for any level of selectivity.

Remark 1 *The proportion of agents who are successful (ES_t) is always $\frac{k}{n}P_2$ or greater. If $\theta < P_2$, all the agents promoted are successful. For all $\theta \in (0, P_2)$, the proportion of A-agents among the agents promoted does not change, and the effect of global selection on the population profile is identical for any level of global selectivity in this interval. However, local selectivity always has an effect on the population profile.*

We use $S_{[n,k,\theta]}$ to denote the promotion system that selects k agents from groups of n agents in local selection and a proportion θ of eligible agents in global selection. We consider the following concept of equilibrium.

Definition 1 $a^* \in [0, 1]$ *is a globally stable equilibrium of the dynamic system given by* $a_{t+1} = f(a_t)$ *if for all* $a_0 \in (0, 1)$, $\lim_{t\to\infty} a_t = a^*$. *And,* $a^*\left[S_{[n,k,\theta]}\right]$ *denotes* a^* *for the promotion system* $S_{[n,k,\theta]}$.

In the following sections, we analyze the dynamics. First, we study the simple case where $n = 2$, i.e. $S_{[n=2,k=1,\theta]}$ and only global selection changes. After that, we fix global selection and local selection changes. Eventually, we focus on the general case $S_{[n,k,\theta]}$.

[7] For example, ten levels of success.

3 Selection in Pairwise Contest $S[n=2,k=1,\theta]$

To derive the dynamic equation, we detail the equation (1) for $n=2$ and $k=1$.

$$a_{t+1}=f(a_t)=\begin{cases} f_1(a_t) & if \quad \theta\frac{k}{n}\le ES_t \\ f_2(a_t) & if \quad \theta\frac{k}{n}> ES_t \end{cases}$$
$$=\begin{cases} \frac{2P_1a_t-P_1a_t^2}{P_2+2P_1a_t-(P_1+P_2)a_t^2} & if \quad \theta\le P_2+2P_1a_t-(P_1+P_2)a_t^2 \\ \frac{a_t}{\theta}\left(2P_1-P_1a_t+\frac{\theta-(P_2+2P_1a_t-(P_1+P_2)a_t^2)}{1-(P_2+2P_1a_t-(P_1+P_2)a_t^2)}(1-P_1-(1-a_t)P_2)\right) & if \quad \theta> P_2+2P_1a_t-(P_1+P_2)a_t^2 \end{cases} \tag{2}$$

The following proposition gives us the globally stable equilibrium of the system under the selection process $S_{[n=2,k=1,\theta]}$, i.e., $a^*[S_{[n=2,k=1,\theta]}]$. The expression represented by a^{**} is a root of the equation $a-f_2(a)=0$ and depends on P_1, P_2 and θ. The explicit expression can be found as a_3 in the proof of proposition 1 in the Appendix.

Proposition 1 *Let $P_1>P_2$, $\bar{\theta}_1=\frac{3P_1P_2}{P_1+P_2}$, and $\bar{\theta}_2=\frac{P_1^2+2P_2-P_1(1+P_2)}{P_2}$, and consider the selection system $S_{[n=2,k=1,\theta]}$ specified by the Eq. (2). The globally stable equilibrium is:*

- *If $P_1\ge 2P_2$ then* $\quad a^*[S_{[n=2,k=1,\theta]}]=1$
- *If $P_1<2P_2$ then*[8] $\quad a^*[S_{[n=2,k=1,\theta]}]=\begin{cases} \frac{2P_1-P_2}{P_1+P_2} & when \quad \theta\le\bar{\theta}_1 \\ a^{**} & when \quad \bar{\theta}_1<\theta<\bar{\theta}_2 \\ 1 & when \quad \bar{\theta}_2<\theta\le 1 \end{cases}$

  In addition, if $\theta\in[\bar{\theta}_1,\bar{\theta}_2]$, then $a^{**}\in[\frac{2P_1-P_2}{P_1+P_2},1]$. Moreover, a^{**} is increasing in θ.

The propositions outlined above establish that if A-agents are on average much better performers than B-agents ($P_1\ge 2P_2$), in the long run all agents will be type A. Thus, the value of θ (global selectivity) does not really matter, as we get the same result for any θ. The performance of A-agents is too strong.

On the other hand, if the gap between the two success rates P_1 and P_2 is not too large ($P_1<2P_2$), in the long run the population profile depends on the value of θ, i.e. the degree of selectivity of the system. Thus, if selectivity is low enough, i.e., θ is greater than $\bar{\theta}_2$, in the long run all agents will be type A. However, if selectivity increase and $\theta<\bar{\theta}_2$, B-agents survive, and their proportion increases as selectivity increases. This happens until θ drops below a certain threshold ($\bar{\theta}_1$). If $\theta<\bar{\theta}_1$ (selectivity is maximum) the proportion of each type of agent does not change with θ. This is consistent with remark 1. Thus, if selectivity is intense then B-agents, who follow the rule with the lower success rate, survive in a greater proportion than for any other value of θ.

The propositions outlined above therefore establish, surprisingly enough, that the more selective the system is (smaller θ), the higher the proportion of agents who follow the rule with the worse success rate is. We provide a comprehensive intuition at the end of the section 5.

[8] If $P_1<2P_2$, then $\bar{\theta}_1<\bar{\theta}_2$ (see proof of Proposition 1 in the Appendix).

In this section, local selectivity is fixed and only global selection changes. Therefore, the effective selectivity is limited because of remark 1. In next section, we fix global selection and change local selectivity by mean of n. The ratio $\frac{1}{n}$ determines the level of local selectivity. Thus, we can increases the effective selectivity as much as we want it.

4 Groups of n agents with $k = 1$, $S[n, k = 1, \theta]$

In this section, we assume $k = 1$ and consider n as a parameter: The agents now interact in groups of n agents. If we consider different group sizes, we can also consider a wide range of degrees of local selectivity. The level of local selectivity is characterized by the quotient $\frac{k}{n}$: in fact it is $1 - \frac{k}{n}$. By changing parameter n, we can thus consider levels of local selectivity[9] between $\dfrac{1}{2}$ and 1.

To derive the dynamic equation, we detail the equation (1) for $k = 1$.

$$a_{t+1}=\begin{cases} \frac{(1-(1-a_t)^n)P_1}{(1-(1-a_t)^n)P_1+(1-a_t^n)P_2} & if \quad (1-(1-a_t)^n)P_1+(1-a_t^n)P_2\geq\theta \\ \frac{1}{\theta}\left[(1-(1-a_t)^n)P_1+(a_t^nP_2+a_t(1-P_1-P_2))\,\frac{\theta-(1-(1-a_t)^n)P_1+(1-a_t^n)P_2}{1-(1-(1-a_t)^n)P_1+(1-a_t^n)P_2}\right] & if \quad (1-(1-a_t)^n)P_1+(1-a_t^n)P_2<\theta \end{cases} \tag{3}$$

To understand what happens in the long run, we consider two extreme cases, one in which there is no global selection, (i.e. $\theta = 1$), and another in which there is the strongest possible global selection (i.e. $\theta \leq P_2$).

4.1 The promotion system $S[n, k = 1, \theta = 1]$

In this case, there is no global selection, $\theta = 1$, and Eq. (3) becomes:

$$a_{t+1} = (1 - (1 - a_t)^n)P_1 + a_t^n P_2 + a_t(1 - P_1 - P_2) \tag{4}$$

Let a^*_{M1} be an inner root that belongs to the open interval $(0, 1)$ of the equation:

$$f_2(a_t) - a_t = (1 - (1 - a_t)^n)P_1 + a_t^n P_2 - a_t(P_1 + P_2) = 0 \tag{5}$$

This root exists and is unique if $P_1 < (n - 1)P_2$ (see the proofs of the result in the Appendix).

Proposition 2 *Let $P_1 > P_2$. Consider the selection process $S_{[n,k=1,\theta=1]}$ specified by the Eq. (4). The globally stable equilibrium is:*

$$a^*[S_{[n,k=1,\theta=1]}] = \begin{cases} a^*_{M1} & if \quad P_1 < (n-1)P_2 \\ 1 & if \quad P_1 \geq (n-1)P_2 \end{cases}$$

Proposition 3 *In addition, $a^*[S_{[n,k=1,\theta=1]}]$ is decreasing in n. Thus, $a^*[S_{[n,1,\theta=1]}] \geq a^*[S_{[n+1,1,\theta=1]}]$, and the inequality is strict if $P_1 \geq (n - 1)P_2$.*

The propositions outline above establish that if A-agents are on average much better performers than B-agents ($P_1 \geq (n - 1)P_2$), in the long run all agents will be type A. However, the previous propositions also

[9] Since we consider in this section $k = 1$, the fraction $\frac{k}{n}$ belongs to $(0, \frac{1}{2}]$, more specifically $\frac{k}{n} \in \{\frac{1}{n}, \forall n \in N\}$, thus the level of local selectivity will be in the interval $[\frac{1}{2}, 1)$.

demostrate that no matter how low the success rate of a type is, if the selection process has a high enough selectivity, agents of this type survive in the long run. If n increases enough, type B agents survive and increase in proportion. This might well seem paradoxical, but the intuition for it is in the section **??**.

To conclude this section, note that if n goes to infinity, Eq. (4) goes to the equation $a_{t+1} = P_1 + a_t(1-P_1-P_2)$, and therefore $a^*[S_{[n,1,\theta=1]}]$ goes to $\frac{P_1}{P_1+P_2}$. Thus, $a^*[S_{[n,1,\theta=1]}] \in (\frac{P_1}{P_1+P_2}, 1]$, and the proportion of B-agents has an upper bound.

4.2 The promotion system $S[n, k = 1, \theta \leq P_2]$

In this case, the global selectivity is intense, the proportion of agents selected in the global selection $(\frac{1}{n}\theta)$ is always smaller than the proportion of eligible successful agents (ES_t), (see remark (1)). Thus, we can rewrite the Eq. (3) as:

$$a_{t+1} = \frac{(1-(1-a_t)^n)P_1}{(1-(1-a_t)^n)P_1 + (1-a_t^n)P_2} \tag{6}$$

As in the previous section, we find two possible kinds of long-run behavior: The entire population is type A, and with both types of agent surviving. As before, it is not possible to find an explicit expression for the inner stady state, which we denote by a^*_{M2}. Where a^*_{M2} is a root of the equation:

$$a_{t+1} - a_t = \frac{(1-(1-a_t)^n)P_1}{(1-(1-a_t)^n)P_1 + (1-a_t^n)P_2} - a_t = 0 \tag{7}$$

This root belongs to the open interval $(0, 1)$, and it exists and is unique if $P_1 < nP_2$, (see the proof of the result below in the Appendix).

Proposition 4 *Consider the selection process $S_{[n,1,\theta \leq P_2]}$ specified by Eq. (6). The globally stable equilibrium is:*

$$a^*[S_{[n,1,\theta \leq P_2]}] = \begin{cases} a^*_{M2} & if \quad P_1 < nP_2 \\ 1 & if \quad P_1 \geq nP_2 \end{cases}$$

Note that Eq. (7) above can be written as $\frac{(1-(1-a_t)^{n+1})P_1 + a_t^{n+1}P_2 - a(P_1+P_2)}{(1-(1-a_t)^n)P_1 + (1-a_t^n)P_2} = 0$ and it has the same roots as the equation $(1-(1-a_t)^{n+1})P_1 + a_t^{n+1}P_2 - a(P_1+P_2) = 0$. The only difference between this last equation and Eq. (5) is that $n+1$ appears instead of n. We can therefore state that:

$$a^*[S_{[n+1,1,\theta=1]}] = a^*[S_{[n,1,\theta \leq P_2]}]$$

Therefore, if we increase the size of the groups by one individual, the effect in the long run is the same as imposing the highest degree of global selectivity. In other words, increasing local selectivity by increasing from n to $n+1$ is equivalent to increase the global selectivity by decreasing from $\theta = 1$ to $\theta \leq P_2$.

Moreover, since the solution is decreasing in n:

$$a^*[S_{[n,1,\theta=1]}] \geq a^*[S_{[n,1,\theta \leq P_2]}]$$

On the other hand, note that if n goes to infinity, Eq. (6) goes to the equation $a_{t+1} = \frac{P_1}{P_1+P_2}$, and therefore, $a^*[S_{[n,1,\theta \leq P_2]}]$ goes to $\frac{P_1}{P_1+P_2}$. Thus, $a^*[S_{[n,1,\theta \leq P_2]}] \in (\frac{P_1}{P_1+P_2}, 1]$ as before.

In this section, the level of local selectivity is specified by the expression $1 - \frac{1}{n}$, and thus we have so far only considered levels of selectivity that are higher than $\frac{1}{2}$. We now look at what happens when the level of selectivity is lower than $\frac{1}{2}$, i.e. the proportion of agents promoted is greater than $\frac{1}{2}$. To find the answer to this question we look in the following section at the more general family of promotion systems $S[n, k, \theta]$.

5 The General Case $S[n, k, \theta]$

First, we show that if there is no local selection, i.e. $\frac{k}{n} = 1$, then the global selection by itself is not enough to produce a result different from $a^*[S_{[n,k,\theta]}] = 1$, only A-agent survive. The effect of global selection is bounded as is shown in remark 1. The proof is in the Appendix.

Proposition 5 *Let $P_1 > P_2$, $k = n$, and consider the selection processes $S_{[n,k,\theta]}$ specified by the Eq. (1).*

The globally stable equilibrium is: $a^*[S_{[n,k,\theta]}] = 1$.

We now consider the general promotion system with $k < n$. The objective is to give a general picture of the dynamics involved. From now on, we simplify matters by considering that $P_3 = 0$ (i.e. $P_1 + P_2 = 1$). The expression that we obtain if we expand Eq. (1) is far too large and complex, which makes it impossible for us to work in the same way as we did in the previous sections. For this reason, we first show an analytical result that characterizes the local stability of the steady states $a = 0$ and $a = 1$. Then, we state a conjecture that we check by numerical analysis. After that, we can draw some conclusions.

Note that $g(a_t) = f(a_t) - a_t$ is a continuous function, the roots are the steady states, and the sign of $g(a_t)$ determines whether a_t increases or decreases. As $a = 0$ and $a = 1$ are always steady states of the system, $g(0) = g(1) = 0$. Obviously, if $g(a) > 0$ for any $a \in (0, \varepsilon)$ with some $\varepsilon > 0$, then $a = 0$ is unstable. Also, it is unstable if $g'(0) > 0$. Next, we present a result about the **local stability** of the steady states $a = 0$ and $a = 1$. The proof is in the Appendix.

Proposition 6 *Let $P_1 > P_2$, $P_3 = 0$, $\frac{k}{n} < 1$ and consider the selection process $S_{[n,k,\theta]}$ specified by the Eq. (1):*

$$\text{If } \theta\frac{k}{n} < P_2 \text{ and,} \begin{cases} \frac{k}{n} < \frac{P_2}{P_1} \text{ then } a = 0 \text{ and } a = 1 \text{ are unstable.} \\ \frac{k}{n} \geq \frac{P_2}{P_1} \text{ then } a = 0 \text{ is unstable and } a = 1 \text{ is locally stable.} \end{cases}.$$

If $\theta\frac{k}{n} \in [P_2, P_1]$, then $a = 0$ is unstable and $a = 1$ is locally stable.

If $\theta\frac{k}{n} > P_1$, then $a = 0$ and $a = 1$ are locally stable.

Therefore, only three types of behavior are possible. *Type I*, $a = 0$ and $a = 1$ are unstable. Note that, as $g(a_t)$ is continuous, there must be at least one inner steady state (a root) in this case[10]. If there was only one,

[10] This is true by Bolzano's Theorem. On the one hand, if $a = 0$ and $a = 1$ are unstable, then $g(a)$ is positive around $a = 0$ and negative around $a = 1$. On the other hand, $g(a)$ is continuous and $g(0) = g(1) = 0$.

this would be globally stable, and B-agents would also survive. *Type II*, $a = 0$ is unstable and $a = 1$ is stable. In this case, if there was no inner steady states, then $a = 1$ would be globally stable, and A-agents would be the only survivors. *Type III*, $a = 0$ and $a = 1$ are stable. As in the first case, there must be at least one inner steady state. If there was only one, then both $a = 0$ and $a = 1$ would be locally stable and the basins of attraction would be determined by that inner steady state.

Conjecture 1 *In the open interval* $(0, 1)$*,* $g(a)$ *has no roots in the second type described above and has only one root in the first and third types if* P_1 *is not very close to* P_2

We check the validity of this conjecture by numerical analysis. It is presented in further detail in the appendix. In order to conduct numerical analysis, values must be specified for a sizable subset of the function's four parameters[11]: (k, n, θ, P_1). After specifying values for (k, n, θ, P_1), the equation $g(a) = 0$ becomes an equation of just one variable, and it is a quotient of univariate polynomials. It is straightforward to prove that the denominator is always positive, thus, we only have to check the roots in the open interval $(0, 1)$ of just one univariate polynomial[12].

Our numerical results confirm the validity of the conjecture[13] if P_1 is not very close to P_2, see Appendix for details. Therefore, we can draw some conclusions from Proposition 6. Under low enough levels of selectivity ($\theta\frac{k}{n} > P_1$) in the promotion system, the selection process is not strong enough to overcome the inertia of the initial population. The dynamic depends on the initial conditions, and the population eventually becomes homogeneous, i.e. either type A or type B. We call this dynamics as *type III*. If selectivity is increased, the basin of attraction of the state $a = 1$ increases. If selectivity increases enough ($\theta\frac{k}{n} \in [P_2, P_1]$), the whole population will be type A for any initial population. In that case, the selection process is strong enough to eventually select the A-agents. We call this dynamic *type II*. Finally, if the selectivity is increased far enough ($\theta\frac{k}{n} < P_2$) the behavior of the system depends on local selectivity alone[14]. If that selectivity is not too high ($\frac{k}{n} \geq \frac{P_2}{P_1}$), then only A-agents survive as before, i.e., the dynamic is also *type II*. However, surprisingly, if the local selectivity is increased far enough ($\frac{k}{n} < \frac{P_2}{P_1}$), type B agents also survive. We call this dynamic *type I*. Therefore, *no matter how low the success rate of a type is, if the selection process has a high enough selectivity, agents of that type survive in the long run.*

To understand why this happens, we must first observe that the dynamic of the system depends on the probabilities of promotion of each type of agent[15]. For example, if the system is in a period t and the probability of an A-agent being promoted is greater than that of a B-agent, then the proportion of A-agents in period $t + 1$

[11] In this section, $P_2 = (1 - P_1)$.

[12] This simple numerical analysis does not present any difficulty and can be done with any mathematical software such as Mathematica or Matlab.

[13] When the performance of both actions (A and B) are very similar, and consequently P_1 and P_2 are close to $\frac{1}{2}$, the dynamic can be more complicated in some very few particular cases. However, if actions are too similar, selection becomes less interesting. More details are given in the Appendix.

[14] This is consistent with Remark 1.

[15] See proof of proposition 6.

is greater than in t. I.e., the proportion of A-agents increases and the proportion of B-agents decreases. To simplify matters we consider a promotion system with local selection only[16]. Let us now focus on a particular type of agent who faces one of the two following extreme scenarios:

- If agents of this particular type are scarce (let us say close to extinction), an agent of this type will generally match with agents of the other type[17]. Thus, in general, there will only be one agent of this particular type in a group, who will only be promoted if he/she is successful (responds in the right way to the environment). In this case, the rest of the agents in his/her group will be unsuccessful. Thus, if an agent of this scarce type is successful, he/she is almost sure to get promoted. In such a context, the probability of promotion of this particular type of agent is not influenced by an increase in the degree of selectivity in the system. His/her probability of promotion depends almost entirely on his/her probability of success, i.e. it is P_1 if the agent is type A and P_2 otherwise.

- However, when agents of this particular type abound (let us say the other type is close to extinction), an agent of this particular type will generally match with agents of his/her own type[17]. Thus, if all the agents in a group are of the same type, they respond in the same way to the same environment. The competitors of a particular agent in his/her own group are also successful (or unsuccessful) as he/she is. Thus, for purposes of promotion, it does not matter at all if this particular type of agent is successful or not: the probability of promotion depends almost entirely on how many people are promoted. Therefore, the probability of promotion of this particular type of agent is strongly influenced by an increase in the degree of selectivity in the system.

Therefore, an increase in selectivity tends to punish the more common type of agents because it decreases their probability of promotion, but it does not affect the relatively scarce type. If selectivity is high enough, no one type can be abundant enough to be the only survivor. Thus we can favor or punish diversity by tuning the level of selectivity.

To give a clearer picture of just what is happening we present the following particular case. We continue to consider only local selection ($\theta = 1$) and assume that the dynamic of the model is *type II*. In that case, the only global equilibrium is the whole population being type A ($a^* = 1$). Consequently, for any state of the system a_t, the probability of promotion of A-agents is greater than that of B-agents. In the rest of the paragraph, we focus on states in which $a_t \simeq 1$. When $a_t \simeq 1$, the probability of promotion of A-agents is approximately equal to the proportion of agents promoted ($\frac{k}{n}$), and the probability of promotion of B-agents is approximately equal to their probability of success (P_2). Obviously, if the dynamic is *type II*, then $\frac{k}{n} > P_2$. However, if $\frac{k}{n}$ is decreased (selectivity increases), the probability of promotion of A-agents decreases, while the probability of promotion of B-agents remains practically unchanged. Therefore, if $\frac{k}{n}$ decreases beyond P_2, then the probability of promotion

[16] The argument is similar, though more complicated, if we also consider global selection.

[17] This happens with a probability close to one.

of B-agents is greater than that of A-agents, and the proportion of A-agents will decrease in the next period. When this happens, the homogeneous equilibrium $a^* = 1$ becomes unstable, and the system converges to a stable globally mixed equilibrium in which there are agents of both types. The dynamic changes from *type II* to *type I*.

On the other hand, we can show by a similar argument that if $\frac{k}{n}$ increases beyond P_1, the state $a = 0$ becomes locally stable. In that case, for states of the system close to $a = 0$, the probability of promotion for B-agents is greater than for A-agents. In addition, the state $a = 1$ changes from globally to locally stable, and the dynamic changes from *type II* to *type III*. The less selective a system is, the easier it is for it to be dominated by one type of agent and for it to achieve homogeneity.

Therefore, if selectivity increases two forces work together. On the one hand, the more selective a system is, the more important an agent's success or failure in the promotion becomes and, thus, the less important the effect of the initial proportions of the different types of agent is. On the other hand, an increase in selectivity tends to punish the more common type of agent because it decreases their probability of promotion but does not affect that of the relatively scarce type. Thus, selectivity can encourage diversity.

In this section, obviously, if we consider $n = 2$ the results are consistent with proposition 1[18].

6 Conclusion

In this paper we show that the degree of selectivity in a selection process can play an important and counter-intuitive role even in the long run. By means of a dynamic model, we study the role of selectivity in a family of promotion systems within a hierarchy. The dynamic depends on the probability of promotion of each type of agent, which in turn depends on three factors: first, the composition of the population, i.e. the proportion of agents of each type; second, how strong the selection process is (which we measure with the level of selectivity); and third, the probability of success in the activity undertaken by agents within the organization. We show that an increase in selectivity tends to punish the more common type of agent because it decreases their probability of promotion, while it does not affect the relatively scarce type. Consequently, we must be careful about the degree of selectivity in the promotion mechanism within hierarchical social systems. As we show, if we wish to increase the presence in the social system of certain agents with a high expected success rate we may, in certain contexts, have to decrease the selectivity of the promotion mechanism rather than increasing it. By contrast, we may have to increase the selectivity if we want to increase the presence of agents with low performances.

Our result depends largely on one particular *critical assumption*: In a group, all agents of the same type are either better or worse than other types simultaneously, thus their successes (or failures) correlate perfectly with one another. If two agents are under the same environment and are following the same rule and one of them is successful, then the other will also be successful, or at least more successful than other types, with a probability

[18] Note that $\bar{\theta}_2 = 2P_2$ when $P_3 = 0$

of one. If that is the case, selectivity will have this paradoxical effect in the dynamic of the process. In the real world it is not easy to find a situation where this correlation is so strong. We expected that the stronger the correlation is, the more noticeable this paradoxical effect will be.

The parameters that we use to measure the level of selectivity in the promotion system can be adjusted to obtain the optimum population profile for an organization. Obviously, the first step is to define an institutional payoff function. We could consider this payoff function as the aggregate of the payoff of all the groups, and the group payoffs as the aggregate of individual performances. For example, this payoff function could depend on the one hand on the institutional cost of having groups with no agents responding in the right way to the environment. On the other hand, it could depend on the proportion of groups with at least one agent who responds in the right way to the environment. After defining institutional payoffs, we can calculate the optimum profile. By tuning[19] the level of selectivity, we can get the optimum profile if the optimum is within certain limits, though not all population profiles can be achieved by tuning the level of selectivity.

This paper is closely related to a number of papers. In Harrington [3,6], he uses a selection process in a hierarchical structure to compare the performance of rigid behavior with that of flexible behavior. Harrington considers only one particular selection process with a unique level of selectivity, while we consider a family of selection processes that we characterize according to the amount of selectivity implemented. In Harrington[4,7], he follows the same line of study but now introduces the concept of "social learning", i.e. young agents who observe the older ones at the top of the hierarchy and imitate them. Finally, in Harrington [5], he adopts a strategic approach. In these papers, the degree of selectivity is always $\frac{1}{2}$. In fact, the promotion system that he uses in his papers is $S_{[n=2,k=1,\theta=1]}$. Thus, the level of selectivity is fixed, and there is no global selection. By contrast, Vega-Redondo[11] employs only global selection, although with a different approach and purpose.

Although Harrington considers more kinds of agent behavior, our result can be applied to his model. Regarding Harrington [3] which was the main inspiration of the present paper, we could say that his results would change if the degree of selectivity increases or decreases. If the level of selectivity is increased enough, the stable equilibrium where the entire population follows the most successful rigid rule will lose its stability and there will be a heterogeneous population in the long run following different rules. And if the degree of selectivity is decreased enough, all the homogenous equilibria become locally stable equilibria. We can make this assertion because the reasoning in the previous section can be applied to his model.

The present paper is also related to the literature on tournaments developed since the seminal paper by Lazear and Rosen [9]. There are some papers which have also focused on the selection role of contests, e.g., Rosen ([10], Section V), and Hvide and Kristiansen [8].

An interesting extension would be to analyze the role of selectivity with a strategic approach, and how the level of selectivity can modify individual behavior.

[19] This makes sense if the system only selects according to performance and not to type. This can happen if the types are not observable, the types are related to gender, race, or nationality and it is therefore politically incorrect, and so on.

Appendix

PROOF OF PROPOSITION 1

We assume $P_1 > P_2 > 0$, $P_1 + P_2 \leq 1$, $\theta \in [0,1]$, $a_t \in [0,1]$, and we omit the time subscript wherever it is not confusing to do so.

The dynamic of the selection process $S_{[\theta]}$ is given by the equation:

$$a_{t+1} = f(a_t) = \begin{cases} f_1(a_t) & if \quad h(a_t) \geq 0 \\ f_2(a_t) & if \quad h(a_t) < 0 \end{cases}$$
$$= \begin{cases} \frac{2P_1 a_t - P_1 a_t^2}{P_2 + 2P_1 a_t - (P_1+P_2)a_t^2} & if \quad P_2+2P_1a_t-(P_1+P_2)a_t^2-\theta \geq 0 \\ \frac{a_t}{\theta}\left(2P_1 - P_1 a_t + \frac{\theta-(P_2+2P_1a_t-(P_1+P_2)a_t^2)}{1-(P_2+2P_1a_t-(P_1+P_2)a_t^2)}(1-P_1-(1-a_t)P_2)\right) & if \quad P_2+2P_1a_t-(P_1+P_2)a_t^2-\theta<0 \end{cases} \tag{8}$$

Note that to analyze the dynamics of $a_{t+1} = f(a_t)$, it suffices to study the function:

$$g(a_t) = f(a_t) - a_t = \begin{cases} g_1(a_t) = f_1(a_t) - a_t & if \quad h(a_t) \geq 0 \\ g_2(a_t) = f_2(a_t) - a_t & if \quad h(a_t) < 0 \end{cases} \tag{9}$$

Thus, if $\hat{a} \in [0,1]$ is a root of $g(a_t)$, i.e., $g(\hat{a}) = 0$, then $\hat{a}$ is a *steady state* of $a_{t+1} = f(a_t)$, i.e., $\hat{a} = f(\hat{a})$. In addition, if $g(a_t) \gtreqless 0$, then $a_t \lesseqgtr a_{t+1}$.

Obviously, $a = 0$ (all the population is type B) and $a = 1$ (all the population is type A) are always steady states, and, therefore roots of $g(a)$.

For example, in our model, if $g(a) > 0$ for all $a \in (0,1)$, then $a = 1$ is a globally stable equilibrium, i.e., for any initial condition belonging to $(0,1)$ the system converges to $a = 1$. On the other hand, if there is only one inner root $\hat{a}$, i.e. $g(\hat{a}) = 0$, and $g(a) > 0$ for all $a \in (0, \hat{a})$, and $g(a) < 0$ for all $a \in (\hat{a}, 1)$, then $\hat{a}$ is a globally stable equilibrium. An **inner root** is a root that belongs to the open interval $(0,1)$.

As the proof is long and involves several steps, it is useful first to present an outline of the proof: First, we show that $g(a)$ is continuous. Second, we calculate the roots of $g(a)$, which are the steady states of the dynamic equation. With this purpose, we calculate the roots of $g_1(a)$ and $g_2(a)$, and we prove that the function $g(a)$ has either no inner roots or only one. In fact, we prove that if $P_1 \geq 2P_2$, then there are no inner roots, and if $P_1 < 2P_2$, then there may be one inner root depending on the value of θ.

After that, we analyze the stability of the steady states, which can be done by checking the sign of $g(a)$ between the steady states. In fact, it suffices to check the sign of the first derivatives of $g(a)$ in $a = 0$ and $a = 1$. Note that on the one hand $g(a)$ is continuous, on the other hand $g(0) = 0$, $g(1) = 0$, and there is either one inner steady state or none. Thus, it can be stated that if $g\prime(0) > 0$ then $g(a)$ must be greater than zero between $a = 0$ and the second steady state. If there is no inner steady state, i.e. the second steady state is $a = 1$, it can be stated that $g(a)$ must be greater than zero for all $a \in (0,1)$. This means that for any initial condition $a \in (0,1)$ the system converges to $a = 1$, i.e., $a = 1$ is globally stable equilibrium. However, if there is an inner steady state, then $g(a)$ is positive between $a = 0$ and that inner steady state and negative between that inner steady state and $a = 1$. Consequently, the system converges to the said inner steady state.

We also need to prove that there are no periodic points. Note that we work with difference equations because we consider discrete time, and there may be periodic points.

We prove the proposition in several steps.

Lemma 1 *The function $g(a_t) = f(a_t) - a_t$ is continuous in* $[0,1]$.

It is straightforward to show that $f_1(a_t)$ and $f_2(a_t)$ are continuous: Both functions are quotients of polynomials and both denominators are strictly greater than zero. Furthermore, if $P_2 + 2P_1 a_t - (P_1 + P_2)a_t^2 = \theta$, then $f_1(a_t) = f_2(a_t)$. Since $f(a_t)$ is continuous, $g(a_t) = f(a_t) - a_t$ is also continuous.■

The following lemma determines the inner roots of the function $g(a)$ for $a \in (0,1)$.

Lemma 2 *a) If $P_1 \geq 2P_2$, then $g(a)$ has no inner roots.*

b) If $P_1 < 2P_2$, then $g(a)$ has either one or inner root or none depending on the value of θ.

b.1) If $\theta < \bar{\theta}_1 = \frac{3P_1P_2}{P_1+P_2}$, then the inner root will be $\bar{a}_3 = \frac{2P_1-P_2}{P_1+P_2}$, which is the only possible inner real root of $g_1(a)$.

b.2) If $\theta \in (\bar{\theta}_1, \bar{\theta}_2)$, with $\bar{\theta}_2 = \frac{P_1^2+2P_2-P_1(1+P_2)}{P_2}$, then the inner root will be a_3, which is the only possible real root of $g_2(a)$ in the interval $(0,1)$.

b.3) Finally, when $\theta \geq \bar{\theta}_2$, there is no inner root.

Several steps are also needed to prove this lemma. As the roots of $g(a)$ have to be roots of either $g_1(a)$ or $g_2(a)$, we first calculate the roots of $g_1(a)$, which reveals that there is only one root (called $\bar{a}_3$) in addition on $a = 0$ and $a = 1$. Second, we calculate the roots of $g_2(a)$, which shows that there are two roots (called a_3 and a_4) in addition on $a = 0$ and $a = 1$. Then we prove that $a_4 \geq 1$, and $a_3 > \frac{P_1}{P_1+P_2} > 0$. We also prove that $a_3 < 1$ if and only if $\theta \in [0, \bar{\theta}_2)$. Consequently, the only candidates for inner roots of $g(a)$ are $\bar{a}_3$ and a_3. We show when either $\bar{a}_3$ or a_3 is the root of $g(a)$. We also show that it is not possible for both $\bar{a}_3$ and a_3 simultaneously to be roots of $g(a)$. We prove the above lemma using Claims (2.1)-(2.6).

Claim 2.1 *a) $g_1(a)$ has a root, $\bar{a}_3$ in addition on $a = 0$ and $a = 1$.*

b) $\bar{a}_3 = \frac{2P_1-P_2}{P_1+P_2} \leq 1 \iff P_1 \leq 2P_2$

Note that $g_1(a_t) = f_1(a_t) - a_t = \frac{2P_1a_t - P_1a_t^2}{P_2+2P_1a_t-(P_1+P_2)a_t^2} - a_t = a_t\,(a_t-1)\,\frac{(P_1+P_2)a_t+(P_2-2P_1)}{P_2+2P_1a_t-(P_1+P_2)a_t^2}$, and the denominator is greater than zero. The equation $g_1(a_t) = 0$ has the following roots:

$$\bar{a}_1 = 0 \quad \bar{a}_2 = 1 \quad \bar{a}_3 = \frac{2P_1-P_2}{P_1+P_2}$$

The root $\bar{a}_3 = \frac{2P_1-P_2}{P_1+P_2} > 0$, and it is straightforward to show that $\frac{2P_1-P_2}{P_1+P_2} < 1 \iff P_1 < 2P_2$.

Claim 2.2 *a) $g_2(a)$ has two roots, a_3 and a_4 in addition on $a = 0$ and $a = 1$.*

b) $a_4 \notin (0,1)$

c) $a_3 \in (\frac{P_1}{P_1+P_2}, 1)$ if and only if $\theta \in [0, \bar{\theta}_2)$. And, $\frac{\partial\, a_3}{\partial\, \theta} > 0$

$$\begin{aligned} \text{As } g_2(a_t) &= f_2(a) - a_t = \frac{a_t}{\theta}\left(2P_1 - P_1a_t + \frac{\left(\theta-(P_2+2P_1a_t-(P_1+P_2)a_t^2)\right)(1-P_1-(1-a_t)P_2)}{1-(P_2+2P_1a_t-(P_1+P_2)a_t^2)}\right) - a_t \\ &= \frac{a_t\ \ (a_t-1)\ \ \left(a_t^2\left(P_2^2-P_1^2\right)+a_t(2P_1(P_1-P_2)+(P_1+P_2)(1-\theta))+P_2(1-P_2)-P_1(2-\theta-P_2)\right)}{(1-(P_2+2P_1a_t-(P_1+P_2)a_t^2))\theta} \end{aligned}$$

It is straightforward to show that the denominator $\left(1 - 2a_tP_1 - P_2 + a_t^2(P_1+P_2)\right)\theta > 0$. Thus, the equation $g_2(a_t) = 0$ has four roots. Two of them are $a_1 = 0$ and $a_2 = 1$. The other two, a_3 and a_4, are given by the polynomial:

$$pol[a_t] = a_t^2\left(P_2^2 - P_1^2\right) + a_t\left(2P_1\left(P_1-P_2\right) + \left(P_1+P_2\right)\left(1-\theta\right)\right) + P_2(1-P_2) - P_1\left(2-\theta-P_2\right)$$

Let $pol[a_t] = \alpha a_t^2 + \beta a_t + \gamma$. It is straightforward to show the sign of the coefficients, α, β, and γ:

$$\begin{aligned} \alpha &= \left(P_2^2 - P_1^2\right) < 0 \\ \beta &= 2P_1\left(P_1-P_2\right) + \left(P_1+P_2\right)\left(1-\theta\right) > 0 \\ \gamma &= P_2(1-P_2) - P_1\left(2-\theta-P_2\right) < 0 \end{aligned} \tag{10}$$

It is tedious but straightforward to prove that the discriminant is:

$$\beta^2 - 4\alpha\gamma = \left(P_1+P_2\right)^2\left(1-\theta\right)^2 - 4\left(P_1-P_2\right)^2\left(\left(P_1+P_2\right)\left(1-P_2\right) - P_1^2\right) \tag{11}$$

If the discriminant $\beta^2 - 4\alpha\gamma > 0$, then a_3 and a_4 are real roots, and since $\alpha < 0$ and $\beta > 0$ it results that:

$$a_3 = \frac{-\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha} < \frac{-\beta-\sqrt{\beta^2-4\alpha\gamma}}{2\alpha} = a_4$$

Now we prove by contradiction that if a_4 is a real root it is not smaller than 1.

$a_4 = \frac{-\beta-\sqrt{\beta^2-4\alpha\gamma}}{2\alpha} < 1 \Leftrightarrow -\beta - \sqrt{\beta^2 - 4\alpha\gamma} > 2\alpha \Leftrightarrow -\sqrt{\beta^2 - 4\alpha\gamma} > 2\alpha + \beta$

Therefore, if $a_4 < 1$, then $2\alpha + \beta < 0$ is necessary condition. Next we show that it is not possible for both conditions to hold simultaneously; $\beta^2 - 4\alpha\gamma \geq 0$ (real a_4) and $2\alpha + \beta < 0$.

Using (10) and (11) it is straightforward to show that:

First, $2\alpha + \beta < 0 \Longleftrightarrow (P_1 + P_2)(1 - \theta) - 2P_2(P_1 - P_2) < 0$

$\Longleftrightarrow (P_1 + P_2)^2 (1 - \theta)^2 < (2P_2 (P_1 - P_2))^2,$

and second, $\beta^2 - 4\alpha\gamma \geq 0 \Longleftrightarrow (P_1 + P_2)^2 (1 - \theta)^2 - 4(P_1 - P_2)^2 ((P_1 + P_2)(1 - P_2) - P_1^2) \geq 0$

$\Longleftrightarrow (P_1 + P_2)^2 (1 - \theta)^2 \geq 4(P_1 - P_2)^2 ((P_1 + P_2)(1 - P_2) - P_1^2)$

Combining the above expressions:

$4(P_1 - P_2)^2 ((P_1 + P_2)(1 - P_2) - P_1^2) \leq (P_1 + P_2)^2 (1 - \theta)^2 < (2P_2 (P_1 - P_2))^2$

Therefore, the following expression is a necessary condition for $2\alpha + \beta < 0$ and $\beta^2 - 4\alpha\gamma > 0$ to hold simultaneously:

$4(P_1 - P_2)^2 ((P_1 + P_2)(1 - P_2) - P_1^2) < (2P_2 (P_1 - P_2))^2$

$\Leftrightarrow (P_1 + P_2)(1 - P_2) - P_1^2 - P_2^2 < 0$

We obtain a contradiction because it is straightforward to show that the above expression is always greater than zero. Thus, $2\alpha + \beta < 0$ and $\beta^2 - 4\alpha\gamma \geq 0$ cannot hold simultaneously, and a_4 cannot be smaller than one ($a_4 \notin (0, 1)$). Consequently, a_4 cannot be an inner root of $g(a)$. In addition, if $\beta^2 - 4\alpha\gamma = 0$, $a_3 = a_4 \notin (0, 1)$.

Now, we prove part *c*) of the Claim (2.2): $a_3 \in (\frac{P_1}{P_1+P_2}, 1)$ if and only if $\theta \in [0, \bar{\theta}_2)$.

Initially, we prove that $\theta \in [0, \bar{\theta}_2) \Longrightarrow a_3 \in (\frac{P_1}{P_1+P_2}, 1)$. We first prove that a_3 is a real number if $\theta \in [0, \bar{\theta}_2)$. Then we prove that if a_3 is real, then a_3 is greater than $\frac{P_1}{P_1+P_2}$, increasing in θ, and 1 when $\theta = \bar{\theta}_2$.

Using expression (11), $\frac{\partial(\beta^2-4\alpha\gamma)}{\partial\theta} = 2(P_1 + P_2)^2 (\theta - 1) < 0$. Thus, $\beta^2 - 4\alpha\gamma$ is decreasing in θ. On the other hand if $\theta = \bar{\theta}_2 = \frac{P_1^2+2P_2-P_1(1+P_2)}{P_2}$, it is tedious but straightforward to show that then: $\beta^2 - 4\alpha\gamma\big|_{\theta=\bar{\theta}_2} = \frac{1}{P_2^2}\left(P_1^3 - P_1^2 + P_1P_2^2 - 2P_2^3 + P_2^2\right)^2 > 0$. As a_3 is real if the discriminant is non-negative, i.e. $\beta^2 - 4\alpha\gamma \geq 0$, the root a_3 has to be real if $\theta \in [0, \bar{\theta}_2)$.

We now prove that $a_3 > \frac{P_1}{P_1+P_2}$ if a_3 is real:

$a_3 = \frac{-\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha} > \frac{P_1}{P_1+P_2} \Longleftrightarrow \sqrt{\beta^2 - 4\alpha\gamma} < 2\alpha\frac{P_1}{P_1+P_2} + \beta$. It is straightforward to show that $2\alpha\frac{P_1}{P_1+P_2} + \beta = (P_1 + P_2)(1 - \theta) > 0$. Thus, $a_3 > \frac{P_1}{P_1+P_2} \Longleftrightarrow \beta^2 - 4\alpha\gamma < (P_1 + P_2)^2 (1 - \theta)^2$

$\Leftrightarrow (P_1 + P_2)^2 (1 - \theta)^2 - 4(P_1 - P_2)^2 ((P_1 + P_2)(1 - P_2) - P_1^2) < (P_1 + P_2)^2 (1 - \theta)^2$

$\Leftrightarrow -4(P_1 - P_2)^2 ((P_1 + P_2)(1 - P_2) - P_1^2) < 0$. It straightforward to show that the above expression is always negative.

We next prove that a_3 is increasing in θ when a_3 is real. We use expressions (10) and (11).

$\frac{\partial\, a_3}{\partial\, \theta} = \frac{\partial\, (\frac{-\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha})}{\partial\, \theta} = \frac{1}{2\alpha}(-\frac{\partial\, \beta}{\partial\, \theta} + \frac{1}{2}(\beta^2 - 4\alpha\gamma)^{-\frac{1}{2}}\frac{\partial(\beta^2-4\alpha\gamma)}{\partial\theta}) > 0 \Leftrightarrow -\frac{\partial\, \beta}{\partial\, \theta} + \frac{1}{2}(\beta^2 - 4\alpha\gamma)^{-\frac{1}{2}}\frac{\partial(\beta^2-4\alpha\gamma)}{\partial\theta} < 0$

$\Leftrightarrow (P_1 + P_2) + \frac{1}{2}(\beta^2 - 4\alpha\gamma)^{-\frac{1}{2}} 2(P_1 + P_2)^2 (\theta - 1) < 0 \Leftrightarrow (\beta^2 - 4\alpha\gamma) < (P_1 + P_2)^2 (1 - \theta)^2$

$\Leftrightarrow (P_1 + P_2)^2 (1 - \theta)^2 - 4(P_1 - P_2)^2 ((P_1 + P_2)(1 - P_2) - P_1^2) < (P_1 + P_2)^2 (1 - \theta)^2$

$\Leftrightarrow 4(P_1 - P_2)^2 ((P_1 + P_2)(1 - P_2) - P_1^2) > 0$. As expression $(P_1 + P_2)(1 - P_2) - P_1^2$ is clearly greater than zero, it can be stated that $\frac{\partial\, a_3}{\partial\, \theta} > 0$.

We now prove that if a_3 is real then $a_3 = 1$ when $\theta = \bar{\theta}_2$:

$a_3 = \frac{-\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha} = 1 \Leftrightarrow \sqrt{\beta^2 - 4\alpha\gamma} = 2\alpha + \beta$

$\Leftrightarrow \left(\sqrt{\beta^2 - 4\alpha\gamma}\right)^2 = (2\alpha + \beta)^2 \Leftrightarrow 0 = \alpha + \beta + \gamma$

$\Leftrightarrow 0 = \left(P_2^2 - P_1^2\right) + (2P_1(P_1 - P_2) + (P_1 + P_2)(1 - \theta)) + (P_2(1 - P_2) - P_1(2 - \theta - P_2))$

$\Leftrightarrow 0 = 2P_2 - P_1 + P_1^2 - \theta P_2 - P_1P_2$

$\Leftrightarrow \theta = \frac{P_1^2+2P_2-P_1(1+P_2)}{P_2} = \bar{\theta}_2$

Therefore, we have proved that if $\theta \in [0, \bar{\theta}_2)$ then a_3 is real. And if a_3 is real then a_3 has to be greater than $\frac{P_1}{P_1+P_2}$, increasing in θ, and 1 when $\theta = \bar{\theta}_2$. Consequently, if $\theta \in [0, \bar{\theta}_2)$ then a_3 belongs to $(\frac{P_1}{P_1+P_2}, 1)$.

We next prove part $c)$ of Claim (2.2) in the other direction: $a_3 \in (\frac{P_1}{P_1+P_2}, 1) \Longrightarrow \theta \in [0, \bar{\theta}_2)$.

If $a_3 \in (\frac{P_1}{P_1+P_2}, 1)$ then a_3 is real. We have shown above that if a_3 is real then $\frac{\partial\, a_3}{\partial\, \theta} > 0$ and $a_3 = 1$ if and only if $\theta = \bar{\theta}_2$. Thus, if $a_3 \in (\frac{P_1}{P_1+P_2}, 1)$ then $\theta \in [0, \bar{\theta}_2)$.

Claim 2.3 *If $a' \notin \{\bar{a}_3, a_3\}$, then a' is not an inner root of $g(a)$.*

This claim is true because, first, $g(a) = \begin{cases} g_1(a) & if \quad h(a) \geq 0 \\ g_2(a) & if \quad h(a) < 0 \end{cases}$; and, second, $g_1(a)$ has only one possible root in the interval $a \in (0, 1)$, which is $\bar{a}_3$, and $g_2(a)$ has only one possible root in the interval $a \in (0, 1)$, which is a_3.

The following result defines a property of thresholds $\bar{\theta}_1$ and $\bar{\theta}_2$ which will be useful later.

Claim 2.4 *Let $\bar{\theta}_1 = \frac{3P_1P_2}{P_1+P_2}$ and $\bar{\theta}_2 = \frac{P_1^2+2P_2-P_1(1+P_2)}{P_2}$.* $\quad \bar{\theta}_1 \leq \bar{\theta}_2 \Longleftrightarrow P_1 \leq 2P_2$

This claim is easy to prove:
$\frac{3P_1P_2}{P_1+P_2} \leq \frac{P_1^2+2P_2-P_1(1+P_2)}{P_2} \Leftrightarrow (P_1 - 2P_2)(P_2 + P_1(1 - P_1 - 2P_2)) \leq 0$
Since $(P_2 + P_1(1 - P_1 - 2P_2) > 0$, the sign depends on $(P_1 - 2P_2)$.

Claim 2.5 *If $P_1 \geq 2P_2$, then $g(a)$ has no inner roots.*

By Claim (2.3), only $\bar{a}_3$ and a_3 can be inner roots of $g(a)$. By Claim (2.1), if $P_1 \geq 2P_2$, then $\bar{a}_3 \geq 1$. Therefore, $\bar{a}_3$ cannot be an inner root if $P_1 \geq 2P_2$.

We now show that a_3 cannot be an inner root of $g(a)$ if $P_1 \geq 2P_2$. Note that if $h(a_3) \geq 0$ then a_3 cannot be an inner root of $g(a)$, see expression (9). Thus, we show that if $P_1 \geq 2P_2$ then $h(a_3) \geq 0$.

Consider $h(a_3) = P_2 + 2P_1a_3 - (P_1 + P_2)a_3^2 - \theta$ as a function of θ, i.e. $h(a_3(\theta), \theta)$. The derivative is:
$\frac{d\,h(a_3(\theta),\theta)}{d\,\theta} = 2P_1\frac{d\,a_3}{d\,\theta} - 2(P_1+P_2)a_3\frac{d\,a_3}{d\,\theta} - 1 = 2\frac{d\,a_3}{d\,\theta}(P_1 - (P_1+P_2)a_3) - 1 < 0$. It is smaller than zero because $\frac{d\,a_3}{d\,\theta} > 0$ and $a_3 > \frac{P_1}{P_1+P_2}$, see Claim (2.2). Therefore, $\frac{d\,h(a_3(\theta),\theta)}{d\,\theta} < 0$, i.e. if θ decreases, $h(a_3)$ increases.

As shown in Claim (2.2), $a_3 = 1$ if and only if $\theta = \bar{\theta}_2$. Consequently, if $h(a_3(\bar{\theta}_2), \bar{\theta}_2) \geq 0$ then a_3 is not an inner root of $g(a)$. It is straightforward to show that, $h(a_3(\bar{\theta}_2), \bar{\theta}_2) = h(1, \bar{\theta}_2) = P_1 - \bar{\theta}_2$. And, $P_1 - \bar{\theta}_2 \geq 0 \Leftrightarrow P_1 - \frac{P_1^2+2P_2-P_1(1+P_2)}{P_2} \geq 0 \Leftrightarrow \frac{1}{P_2}(P_1 - 2P_2)(1 - P_1) \geq 0 \Leftrightarrow P_1 \geq 2P_2$.

The following result shows when $\bar{a}_3$ and a_3 are roots of $g(a)$. Note that the sign of $h(a)$ determines whether $g(a) = g_1(a)$ or $g(a) = g_2(a)$, see expression (9)

Claim 2.6 *Let $P_1 < 2P_2$.*

a) $[\bar{a}_3 \in (0, 1)$ and $h(\bar{a}_3) \geq 0] \Leftrightarrow \theta \leq \bar{\theta}_1$. Consequently, if $\theta > \bar{\theta}_1$ then $\bar{a}_3$ cannot be an inner root of $g(a)$.

b) $[a_3 \in (0, 1)$ and $h(a_3) < 0] \Leftrightarrow \theta \in (\bar{\theta}_1, \bar{\theta}_2)$. Consequently, if $\theta \notin (\bar{\theta}_1, \bar{\theta}_2)$ then a_3 cannot be an inner root of $g(a)$.

We prove the first part of the claim, $h(\bar{a}_3) \geq 0 \Leftrightarrow \theta \leq \bar{\theta}_1 = \frac{3P_1P_2}{P_1+P_2}$:

By Claim (2.1), $\bar{a}_3 < 1 \Longleftrightarrow P_1 < 2P_2$. From expression (8), $h(\bar{a}_3) = 2P_1\bar{a}_3 - (P_1 + P_2)(\bar{a}_3)^2 + P_2 - \theta = 2P_1\frac{2P_1-P_2}{P_1+P_2} - (P_1 + P_2)\left(\frac{2P_1-P_2}{P_1+P_2}\right)^2 + P_2 - \theta \geq 0 \Longleftrightarrow \frac{3P_1P_2}{P_1+P_2} - \theta \geq 0$.

The proof of part $b)$ of the claim ($h(a_3) < 0 \Leftrightarrow \theta \in (\bar{\theta}_1, \bar{\theta}_2)$) is a little longer. First note that, by Claim (2.4), $\bar{\theta}_1 < \bar{\theta}_2 \Longleftrightarrow P_1 < 2P_2$.

We need a preliminary result: $h(a_3) = h(\bar{a}_3) = 0$ when $\theta = \bar{\theta}_1$.It is simple to show that if $\theta = \bar{\theta}_1$, then $a_3 = \frac{P_1+P_2+2P_1^2-5P_1P_2-\sqrt{\left(P_1-2P_1^2+P_2+P_1P_2-2P_2^2\right)^2}}{2(P_1^2-P_2^2)} = \frac{2P_1-P_2}{P_1+P_2}$. In the radicand, the expression $P_1 - 2P_1^2 + P_2 + P_1P_2 - 2P_2^2$ is positive if $P_1 < 2P_2$. Thus, if $\theta = \bar{\theta}_1$ then $a_3|_{\theta=\bar{\theta}_1} = \frac{2P_1-P_2}{P_1+P_2} = \bar{a}_3$

Therefore, if $\theta = \bar{\theta}_1$ then $h(a_3) = h(\bar{a}_3) = 0$, see part $a)$ of this Claim. On the other hand, it is proven in the previous Claim that $h(a_3)$ is *strictly decreasing* in θ. Thus, $h(a_3) > 0$ when $\theta < \bar{\theta}_1$, and $h(a_3) < 0$ when $\theta > \bar{\theta}_1$. In the case when $\theta \geq \bar{\theta}_2$, Claim (2.2) shows that $a_3 \geq 1$.

Consequently, if $[P_1 < 2P_2$ and $\theta < \bar{\theta}_1]$ then $\bar{a}_3$ is the unique inner root of $g(a)$. If $[P_1 < 2P_2$ and $\theta \in (\bar{\theta}_1, \bar{\theta}_2)]$ then a_3 is the unique inner root of $g(a)$. If $[P_1 < 2P_2$ and $\theta \geq \bar{\theta}_2)]$ then $g(a)$ has no inner roots.■

Before analyzing the stability of the steady states, we prove that there are no periodic points[20] of $a_{t+1} = f(a_t)$.

Lemma 3 *$a_{t+1} = f(a_t)$ has no periodic points.*

By Claim (2), the function $f(a_t) = \begin{cases} f_1(a_t) & if \quad h(a_t) \geq 0 \\ f_2(a_t) & if \quad h(a_t) < 0 \end{cases}$ has either only one inner steady state in $(0,1)$ or none at all. If the function has none, there are obviously no periodic points because either $a_t > a_{t+1}$ for all $a_t \in (0,1)$ or $a_t < a_{t+1}$ for all $a_t \in (0,1)$. In the case in which there is one steady state in $(0,1)$, if the function is increasing in the interval between the inner steady state and one, any possibility of there being periodic points completely disappears because for all a_t equal to or greater than the inner steady state $(\hat{a})$ either $a_t < a_{t+1}$ for all $a_t \in (\hat{a}, 1)$ or $a_t > a_{t+1}$ for all $a_t \in (\hat{a}, 1)$, and always $a_{t+1} \geq \hat{a}$. Thus, it suffices to prove that the function $f(a_t)$ is increasing from values that are higher than the inner steady state. We study the derivatives of $f_1(a_t)$ and $f_2(a_t)$.

The function $f_1(a_t)$ is increasing in a_t:

$$f_1(a_t) = \frac{2P_1 a_t - P_1 a_t^2}{P_2 + 2P_1 a_t - (P_1 + P_2) a_t^2}; \quad f_1'(a_t) = \frac{2(1 - a_t + a_t^2) P_1 P_2}{\left(P_2 + 2P_1 a_t - (P_1 + P_2) a_t^2\right)^2} > 0$$

We now prove that the function $f_2(a_t)$ is increasing in $a_t \in [\frac{P_1}{P_1+P_2}, 1]$

$$f_2(a_t) = \frac{a_t}{\theta}\left(2P_1 - a_t P_1 + (1 - P_1 - (1 - a_t)P_2))\frac{\theta - (P_2 + 2P_1 a_t - (P_1 + P_2)a_t^2)}{1 - (P_2 + 2P_1 a_t - (P_1 + P_2)a_t^2)}\right)$$

Let $w(a_t) = P_2 + 2P_1 a_t - (P_1 + P_2)a_t^2$, which is a concave function with a maximum in $a_t = \frac{P_1}{P_1+P_2}$. Note that $h(a_t) = w(a_t) - \theta$, therefore $f(a_t) = f_2(a_t)$ if $w(a_t) < \theta$. We can rearrange $f_2(a_t)$ in the following way:

$$f_2(a_t) = \frac{1}{\theta}\left(2P_1 a_t - P_1 a_t^2 + \left((1 - P_1 - P_2)\, a_t + P_2 a_t^2\right)\frac{\theta - w(a_t)}{1 - w(a_t)}\right)$$

$$f_2'(a_t) = \frac{1}{\theta}\left((2P_1 - 2P_1 a_t) + ((1 - P_1 - P_2) + 2P_2 a_t)\frac{\theta - w(a_t)}{1 - w(a_t)} + \left((1 - P_1 - P_2)\, a_t + P_2 a_t^2\right)\frac{w'(a_t)(\theta - 1)}{(1 - w(a_t))^2}\right)$$

All the terms in the previous expression are always positive except $(\theta - 1)$ (which is negative) and $w'(a_t)$ (which may be either positive or negative). The function $w(a_t)$ is concave, with a maximum in $a_t = \frac{P_1}{P_1+P_2}$. Thus, $w'(a_t) < 0$ when $a_t > \frac{P_1}{P_1+P_2}$, and $f_2(a_t)$ will be increasing in $a_t \in [\frac{P_1}{P_1+P_2}, 1]$.

As $f_1(a_t)$ and $f_2(a_t)$ are increasing in $a_t \in [\frac{P_1}{P_1+P_2}, 1]$, $f(a_t)$ is also increasing in $a_t \in [\frac{P_1}{P_1+P_2}, 1]$.

By Claim (2.2), $a_3 > \frac{P_1}{P_1+P_2}$. On the other hand, $\bar{a}_3 = \frac{2P_1 - P_2}{P_1+P_2} > \frac{P_1}{P_1+P_2} \Leftrightarrow P_1 > P_2$. As $\bar{a}_3$ and a_3 are the only candidates for inner steady states, it can stated that the function $f(a_t)$ is increasing in the interval between the inner steady state and one. Therefore, there are no periodic points.■

From now on, it can considered that there are no periodic points, and there is no need to mention them further.

We now study the stability of the steady states of $f(a)$, which are the roots of $g(a)$. We need the following results, which show the sign of the derivative of $g(a)$ in $a = 0$ and $a = 1$.

Lemma 4 *a)* $g'(0) > 0$

b) $\left[\theta < \hat{\theta}_2 \text{ and } 2P_2 > P_1\right] \Rightarrow g'(1) > 0$

[20] It is possible in difference equations for a solution not to be a steady point. Thus, point b is called a *periodic point* of $x_{t+1} = f(x_t)$ if $f^k(b) = b$ for some positive integer k, i.e b is again reached after k iterations. See Elaydi [2].

If $g_1'(0) > 0$ and $g_2'(0) > 0$ then $g'(0) > 0$. As $g_1'(a_t) = f_1^{'}(a_t) - 1$, it is straightforward to show that $g_1'(0) = \frac{2P_1}{P_2} - 1$. Since $P_1 > P_2$, we can state that $g_1'(0) > 0$. On the other hand, as $g_2'(a_t) = f_2^{'}(a_t) - 1$, it is straightforward to show that $g_2'(0) = \frac{P_2 - P_2^2 + P_1(P_2+\theta-2)}{(P_2-1)\theta}$, the denominator is negative $(P_2 - 1)\theta < 0$, and it is easy to show that the numerator is also negative, $P_2 - P_2^2 + P_1(P_2 + \theta - 2) = P_2(1 - P_2) + P_1(P_2 + (\theta - 2)) < 0$, therefore $g_2'(0) > 0$.

Part b).

$g'(1)$ is equal to either $g_1'(1)$ or $g_2'(1)$.

$g_1'(1) = \frac{2P_2}{P_1} - 1$, and this expression is greater than zero if and only if $2P_2 > P_1$.

$g_2'(a_t) = f_2^{'}(a_t) - 1$, and it is straightforward to prove that $g_2'(1) = \frac{P_1(1+P_2)+P_2(\theta-2)-P_1^2}{(P_1-1)\theta}$. The denominator is negative, $(P_1 - 1)\theta < 0$, and the numerator is negative if and only if $\theta < \frac{P_1^2+2P_2-P_1(1+P_2)}{P_2} = \hat{\theta}_2$. Therefore $g_2'(1) > 0$ if and only if $\theta < \hat{\theta}_2$.

Therefore, if either $2P_2 > P_1$ and $\theta \notin (\bar{\theta}_1, \bar{\theta}_2)$ or $2P_2 \leq P_1$, then there are no inner roots by Lemma (2), and as $g'(0) > 0$ the system converges to $a = 1$. If $2P_2 > P_1$ and $\theta < \bar{\theta}_2$, by Lemma (2) there is one inner root, and as $g'(1) > 0$ the system converge to the inner root. This inner root is either $\bar{a}_3$ if $\theta \leq \bar{\theta}_1$ or $\bar{a}_3$ if $\theta \in (\bar{\theta}_1, \bar{\theta}_2)$.■

PROOF OF PROPOSITION 2

We assume $P_1 > P_2$

The dynamics of the selection process $S_{[n,k=1,\theta=1]}$ is given by the equation:

$$a_{t+1} = f(a_t) = (1 - (1 - a_t)^n)P_1 + a_t^n P_2 + a_t(1 - P_1 - P_2)$$

Let $g(a_t) = f(a_t) - a_t = (1 - (1 - a_t)^n)P_1 + a_t^n P_2 - a_t(P_1 + P_2)$. The solutions of the equation $g(a_t) = 0$ are the steady points.

As that equation does not have an explicit solution, we study the first and the second derivatives:

$$g'(a_t) = n\,(1 - a_t)^{n-1}P_1 + n\,a_t^{n-1}P_2 - (P_1 + P_2)$$
$$g''(a_t) = -n\,(n-1)\,(1 - a_t)^{n-2}P_1 + n\,(n-1)\,a_t^{n-2}P_2$$

We now calculate the inflection points:

$g''(\hat{a}) = 0 \iff -n\,(n-1)\,(1-\hat{a})^{n-2}P_1 + n\,(n-1)\,\hat{a}^{n-2}P_2 = 0 \iff (1-\hat{a})^{n-2}P_1 = \hat{a}^{n-2}P_2 \iff (1-\hat{a})P_1^{\overline{\frac{1}{n-2}}} = \hat{a}P_2^{\overline{\frac{1}{n-2}}}$

$$\hat{a} = \frac{P_1^{\overline{\frac{1}{n-2}}}}{P_1^{\overline{\frac{1}{n-2}}} + P_2^{\overline{\frac{1}{n-2}}}} = \frac{1}{1 + (\frac{P_2}{P_1})^{\overline{\frac{1}{n-2}}}}$$

Therefore, if $n > 2$ we can then state that the function $g(a_t)$ has no more than one inflection point in $a_t \in (0, 1)$. As $\hat{a}$ must be smaller than one, $(\frac{P_2}{P_1})^{\overline{\frac{1}{n-2}}}$ must be positive (no negative roots).

Obviously, $a = 0$ and $a = 1$ are roots of the equation $g(a_t) = 0$. The second derivative at these points are $g''(0) = -n(n-1)P_1 < 0$ and $g''(1) = n(n-1)P_2 > 0$. Therefore, the function $g(a_t)$ is concave at $a_t = 0$ and convex at $a_t = 1$. As $g(a_t)$ is continuous in $a_t \in [0, 1]$, the function $g(a_t)$ is concave in $a_t \in [0, \hat{a})$ and convex in $a_t \in (\hat{a}, 1]$.

On the other hand the first derivatives of $g(a_t)$ at $a = 0$ and $a = 1$ are:

$g'(0) = (n-1)P_1 - P_2 > 0$, thus, the function $g(a_t)$ is increasing at $a = 0$.

$g'(1) = (n-1)P_2 - P_1$, thus, the function $g(a_t)$ at $a = 1$ is either increasing if $(n-1)P_2 > P_1$ or decreasing if $(n-1)P_2 < P_1$

Therefore, if $(n-1)P_2 > P_1$, the function $g(a_t)$ is necessarily equal to zero only in one point in the open interval $(0, 1)$. Calling this point a^*_{M1}, we can state that the function $g(a_t)$ is greater than zero in $a_t \in (0, a^*_{M1})$ and smaller that zero in $a_t \in (a^*_{M1}, 1)$. Thus, $a^*\left[S_{[n,k=1,\theta=1]}\right] = a^*_{M1}$ if $(n-1)P_2 > P_1$.

On the other hand, if $(n-1)P_2 \leq P_1$, then the function $g(a_t)$ can not have a value of zero in the interval $(0,1)$. Moreover, $g(a_t)$ will be positive in $a_t \in (0,1)$, thus $a^*\left[S_{[n,k=1,\theta=1]}\right] = 1$ if $(n-1)P_2 \leq P_1$.

To complete the proof we need only note that periodical points are not possible. This is because the first derivative of the function $f(a_t)$ is positive and, as such, $f(a_t)$ is increasing: $f(a_t) = (1-(1-a_t)^n)P_1 + a_t^n P_2 + a_t(1-P_1-P_2)$ and $f'(a_t) = n\,(1-a_t)^{n-1}P_1 + n\,a_t^{n-1}P_2 + (1-P_1-P_2) > 0$ Periodical points are therefore not possible.■

PROOF OF PROPOSITION 3

It is only necessary to prove that $a^*_{M1}|_n > a^*_{M1}|_{n+1}$, if $P_1 \geq (n-1)P_2$

Let $g(a;\,\bar{n})$ be the function $g(a_t)$ evaluates in $a_t = a$, with the parameter $n = \bar{n}$.

Let $a^*_{M1} = \tilde{a}$ if $n = \bar{n}$, and let $a^*_{M1} = \breve{a}$ if $n = \bar{n}+1$. Thus, $g(\tilde{a};\,\bar{n}) = 0$ and $g(\breve{a};\,\bar{n}+1) = 0$

We wish to prove that $\breve{a} < \tilde{a}$ if $(\bar{n}-1)P_2 > P_1$.

Note that proving the previous statement is the same as proving that $g(\tilde{a};\bar{n}+1) < 0$ because of the characteristics of the function $g(a_t)$. Therefore, it suffices to prove that $g(\tilde{a};\bar{n}+1) < 0$.

As $g(\tilde{a};\,\bar{n}) = 0$,

$$g(\tilde{a};\,\bar{n}) = (1-(1-\tilde{a})^{\bar{n}})P_1 + \tilde{a}^{\bar{n}}P_2 - \tilde{a}(P_1+P_2) = 0 \Leftrightarrow$$

$$\Leftrightarrow (1-\tilde{a})P_1 - (1-\tilde{a})^{\bar{n}}P_1 + \tilde{a}^{\bar{n}}P_2 - \tilde{a}P_2 = 0 \tag{12}$$

$$\Leftrightarrow \frac{\tilde{a}-\tilde{a}^{\bar{n}}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}}} = \frac{P_1}{P_2}$$

On the other hand, $g(\tilde{a};\,\bar{n}+1) = (1-(1-\tilde{a})^{\bar{n}+1})P_1 + \tilde{a}^{\bar{n}+1}P_2 - \tilde{a}(P_1+P_2)$ is smaller than 0 if and only if:

$$(1-\tilde{a})P_1 - (1-\tilde{a})^{\bar{n}+1}P_1 + \tilde{a}^{\bar{n}+1}P_2 - \tilde{a}P_2 < 0$$

Given Eq. (12), the previous expression will be true if and only if:

$$\tilde{a}^{\bar{n}}P_2 - \tilde{a}^{\bar{n}+1}P_2 > (1-\tilde{a})^{\bar{n}}P_1 - (1-\tilde{a})^{\bar{n}+1}P_1 \Leftrightarrow \frac{\tilde{a}^{\bar{n}}-\tilde{a}^{\bar{n}+1}}{(1-\tilde{a})^{\bar{n}}-(1-\tilde{a})^{\bar{n}+1}} > \frac{P_1}{P_2}\left(= \frac{\tilde{a}-\tilde{a}^{\bar{n}}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}}}\right)$$
$$\Leftrightarrow \frac{\tilde{a}^{\bar{n}}-\tilde{a}^{\bar{n}+1}}{(1-\tilde{a})^{\bar{n}}-(1-\tilde{a})^{\bar{n}+1}} > \frac{\tilde{a}-\tilde{a}^{\bar{n}}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}}} \Leftrightarrow \frac{\tilde{a}^{\bar{n}-2}}{(1-\tilde{a})^{\bar{n}-2}} > \frac{1-\tilde{a}^{\bar{n}-1}}{1-(1-\tilde{a})^{\bar{n}-1}} \Leftrightarrow \frac{\tilde{a}^{\bar{n}-2}}{1-\tilde{a}^{\bar{n}-1}} > \frac{(1-\tilde{a})^{\bar{n}-2}}{1-(1-\tilde{a})^{\bar{n}-1}}$$

It is straightforward to show that the left hand of the previous expression is increasing in $\tilde{a}$, and the right hand is decreasing in $\tilde{a}$. Moreover, if $\tilde{a} = \dfrac{1}{2}$ the two terms are equal. Therefore, if $\tilde{a} > \dfrac{1}{2}$ then the previous expression will be true.

Note that: $g(\frac{1}{2};\,\bar{n}) = (1-(1-\frac{1}{2})^{\bar{n}})P_1 + \left(\frac{1}{2}\right)^{\bar{n}}P_2 - \frac{1}{2}(P_1+P_2) = \left(\frac{1}{2}-\frac{1}{2^{\bar{n}}}\right)(P_1-P_2) > 0$

Since the previous expression is greater than zero, we can state that $\tilde{a} > \dfrac{1}{2}$■

PROOF OF PROPOSITION 4

The dynamics of the selection process $S_{[n,k=1,\theta\leq P_2]}$ is given by the equation:

$$a_{t+1} = \frac{(1-(1-a_t)^n)P_1}{(1-(1-a_t)^n)P_1+(1-a_t^n)P_2}$$

Let be $a_{t+1} = f(a_t)$ and

$$g(a_t) = f(a_t) - a_t = \frac{(1-(1-a_t)^n)P_1}{(1-(1-a_t)^n)P_1+(1-a_t^n)P_2} - a_t = \frac{P_1 - P_1(1-a_t)^n - a_tP_1 + a_tP_1(1-a_t)^n - a_tP_2 + a_t^{1+n}P_2}{P_1+P_2-P_1(1-a_t)^n - P_2a_t^n} =$$
$$\frac{(1-(1-a_t)^{n+1})P_1 + a_t^{n+1}P_2 - a_t(P_1+P_2)}{P_1+P_2-P_1(1-a_t)^n-P_2a_t^n}$$

Since the denominator is greater than zero, the sign and the roots of the previous expression depend on the numerator. Notice that the only difference between this numerator and the function $g(a_t)$ in the previous proposition **2** is that $n+1$ appears instead of n. We can therefore state that:

$$a^*[S_{[n+1,1,\theta=1]}] = a^*[S_{[n,1,\theta\leq P_2]}]$$

Moreover, since the equilibrium is decreasing in n (provided that $P_1 > P_2$):

$$a^*[S_{[n,1,\theta=1]}] \geq a^*[S_{[n,1,\theta\leq P_2]}]$$

$\blacksquare$.

PROOF OF PROPOSITION 6

Let $P_1 > P_2$, $P_3 = 0$, $\frac{k}{n} < 1$ and consider the selection process $S_{[n,k,\theta]}$ specified by the Eq. (1).

We must first observe that the dynamic of the system depends on the probabilities of promotion of each type of agent. If the system is in a period t (or level t) and the probability of an A-agent being promoted ($P_A(prom)$) is greater than that of a B-agent ($P_B(prom)$), then the proportion of A-agents in period $t+1$ is greater than in t. I.e. the proportion of A-agents increases and the proportion of B-agents decreases.

It is easy to prove the previous statement. Note first that we can obtain the dynamic equation from those probabilities.

$$\begin{aligned} a_{t+1} &= \frac{\text{A-agents promoted}}{\text{agents promoted}} = \frac{a_t\mathcal{P}_{A,t}(prom)}{a_t\mathcal{P}_{A,t}(prom)+b_t\mathcal{P}_{B,t}(prom)} = \frac{a_t\mathcal{P}_{A,t}(prom)}{\frac{k}{n}\theta} \\ \text{Analogously, } b_{t+1} &= \frac{b_t\mathcal{P}_{B,t}(prom)}{\frac{k}{n}\theta} \end{aligned}$$

Lemma 5 *If $P_{A,t}(prom) > P_{B,t}(prom)$, then $a_{t+1} > a_t$*

Note that, $a_{t+1} = \frac{a_t\mathcal{P}_{A,t}(prom)}{\frac{k}{n}\theta} \Leftrightarrow P_{A,t}(prom) = \frac{a_{t+1}}{a_t}\frac{k}{n}\theta$,

and $b_{t+1} = \frac{b_t\mathcal{P}_{B,t}(prom)}{\frac{k}{n}\theta} \Leftrightarrow P_{B,t}(prom) = \frac{b_{t+1}}{b_t}\frac{k}{n}\theta$. Therefore:

$$\begin{aligned} &P_{A,t}(prom) > P_{B,t}(prom) \Leftrightarrow \frac{a_{t+1}}{a_t}\frac{k}{n}\theta > \frac{b_{t+1}}{b_t}\frac{k}{n}\theta \Leftrightarrow \frac{a_{t+1}}{a_t} > \frac{b_{t+1}}{b_t} \\ &\Leftrightarrow \frac{a_{t+1}}{a_t} > \frac{(1-a_{t+1})}{(1-a_t)} > 0 \Leftrightarrow \frac{a_{t+1}-a_t}{a_t b_t} > 0 \Leftrightarrow a_{t+1} > a_t \qquad \blacksquare \end{aligned}$$

To find when equilibria $a=0$ and $a=1$ are either stable or unstable, we can study the promotion probabilities when the system are close to these equilibria. Thus, if we prove that the probability of promotion for an A-agent is greater (smaller) than it is for a B-agent, provided that the system is indefinitely very close to $a=0$, then we can state that these equilibria are unstable (stable). We can do the same thing with $a=1$. Thus, we obtain the following result, which explains when equilibria $a=0$ and $a=1$ are either (locally) stable or unstable.

We first obtain the probability of promotion. An agent can be promoted in two different events, i.e. he is either eligible and successful ($E.S.$) or he is eligible and but unsuccessful ($E.U.$). Therefore, the probability of an agent's following the rule i being promoted (in period t) can be written as:

$$P_{i,t}(prom) = P_t(prom/E.S.)P_{i,t}(E.S.) + P_t(prom/E.U.)P_{i,t}(E.U.)$$

Thus, the probability of promotion for an agent who follows rule i is a linear combination of the agent's probability of being in event $E.S.$ (i.e. $P_{i,t}(E.S.)$) and in $E.U.$ (i.e. $P_{i,t}(E.U.)$), where the weights are given by the probability of being promoted in each event (i.e. $P_t(prom/E.S.)$ and $P_t(prom/E.U.)$).

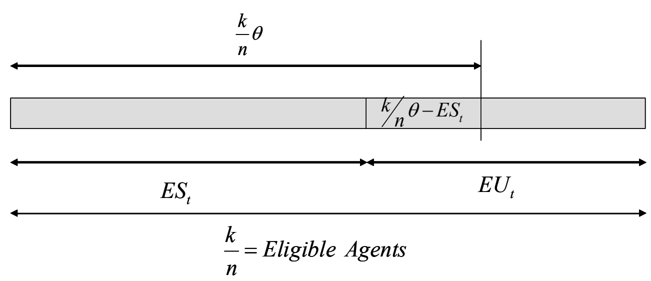

Figure 1: The grey rectangle represents the Eligible Agents

Using Figure 1, it is straightforward to derive the following probabilities:

$$
\begin{array}{|l|l|}
\hline
P_t(prom/E.S.) = \min\{\frac{\frac{k}{n}\theta}{ES_t}, 1\} & P_t(prom/E.U.) = \max\{\frac{\theta\frac{k}{n} - ES_t}{EU_t}, 0\} \\
\hline
P_{A,t}(E.S.) = \frac{ES_t^a}{a_t} & P_{A,t}(E.U.) = \frac{EU_t^a}{a_t} \\
\hline
P_{B,t}(E.S.) = \frac{ES_t^b}{(1-a_t)} = \frac{(ES_t - ES_t^a)}{(1-a_t)} & P_{B,t}(E.U.) = \frac{EU_t^b}{(1-a_t)} = \frac{(EU_t - EU_t^a)}{(1-a_t)} \\
\hline
\end{array} \quad (13)
$$

For example, $P_{A,t}(E.S.) = \dfrac{ES_t^a}{a_t} = \frac{\text{Proportion of A-agents that are eligible and successful}}{\text{Proportion of A-agents that can be chosen}}$.

If we want to analyze the stability[21] of $a = 1$, then it is sufficient to find which probability of promotion is greater when the state of the system is very close to one ($a_t \simeq 1$). If the promotion probability of an A-agent is greater(smaller) than that of a B-agent and this is true as much as the state closes to one is, then the state $a = 1$ will be locally stable(unstable).(Analogously to $a = 0$).

Notice that,

- if $a_t \simeq 1$ then:

$$
\begin{aligned}
ES_t &\simeq ES_t^a \simeq \frac{k}{n} P_1 \\
EU_t &\simeq EU_t^a \simeq \frac{k}{n} P_2
\end{aligned}
$$

  Note that ES_t, ES_t^a, EU_t, and EU_t are polynomials, see the expressions in Section 2. It is straightforward to derive that $\lim\limits_{a_t \to 1} ES_t = \frac{k}{n} P_1$. Analogously to ES_t^a, EU_t, and EU_t.

The probabilities $P_{B,t}(E.S.)$ and $P_{B,t}(E.U.)$ are straightforward to derive. We first replace the ES_t, ES_t^a, EU_t, and EU_t in expression 13 by the expressions in Section 2 and simplify them, afterwards, we calculate the limits of the expressions when a_t goes to 1.

Therefore:

$P_t(prom/E.S.) \simeq \min\{\frac{\theta}{P_1}, 1\}$	$P_t(prom/E.U.) \simeq \max\{\frac{\theta - P_1}{P_2}, 0\}$
$P_{A,t}(E.S.) \simeq \frac{k}{n} P_1$	$P_{A,t}(E.U.) \simeq \frac{k}{n} P_2$
$P_{B,t}(E.S.) \simeq P_2$	$P_{B,t}(E.U.) \simeq 0$

The promotion probability for an A-agent when the state is close to one, $a_t \simeq 1$, is:

$$
\begin{aligned}
P_{A,a_t \simeq 1}(prom) &\simeq \min\{\frac{\theta}{P_1}, 1\}\frac{k}{n} P_1 + \max\{\frac{\theta - P_1}{P_2}, 0\}\frac{k}{n} P_2 = \min\{\theta, P_1\}\frac{k}{n} + \max\{\theta - P_1, 0\}\frac{k}{n} \\
&= \frac{k}{n}(\min\{\theta, P_1\} + \max\{\theta - P_1, 0\}) = \left\{ \begin{array}{cc} \frac{k}{n}\theta \quad \text{if} \quad \theta < P_1 \\ \frac{k}{n}(P_1 + \theta - P_1) = \frac{k}{n}\theta \quad \text{if} \quad \theta \geq P_1 \end{array} \right\} = \frac{k}{n}\theta
\end{aligned}
$$

And for a B-agent (if $k < n$):

$$
\mathcal{P}_{B,a_t \simeq 1}(prom) \simeq \min\{\frac{\theta}{P_1}, 1\} P_2 + \max\{\frac{\theta - P_1}{P_2}, 0\} 0 = \min\{\frac{\theta}{P_1}, 1\} P_2 = \left\{ \begin{array}{c} \frac{\theta}{P_1} P_2 \quad \text{if} \quad \theta < P_1 \\ P_2 \quad \text{if} \quad \theta \geq P_1 \end{array} \right.
$$

Thus;

$$
\begin{aligned}
\mathcal{P}_{A,a_t \simeq 1}(prom) &\simeq \frac{k}{n}\theta \\
\mathcal{P}_{B,a_t \simeq 1}(prom) &\simeq \left\{ \begin{array}{c} \frac{\theta}{P_1} P_2 \text{ if } \theta < P_1 \\ P_2 \text{ if } \theta \geq P_1 \end{array} \right.
\end{aligned}
$$

We know that if $P_{A,a_t \simeq 1}(prom) \geq (<) P_{B,a_t \simeq 1}(prom)$ then $a = 1$ is stable (unstable). Thus, if $\theta < P_1$, then $a = 1$ will be stable (unstable) provided that $\frac{k}{n}\theta \geq (<) \frac{\theta}{P_1} P_2 \Leftrightarrow \frac{k}{n} \geq (<) \frac{P_2}{P_1}$. On the other hand, if $\theta \geq P_1$ then $a = 1$ will be stable (unstable) provided that $\frac{k}{n}\theta \geq (<) P_2$.

[21] In this proof, the stability we refer to is always the local stability

Analogously, we can study the state $a = 0$ and derive the promotion probabilities in this case:

$$\begin{aligned} \mathcal{P}_{B,a_t\simeq 0}(prom) &\simeq \frac{k}{n}\theta \\ \mathcal{P}_{A,a_t\simeq 0}(prom) &\simeq \begin{cases} \frac{\theta}{P_2}P_1 \text{ if } \theta < P_2 \\ P_1 \text{ if } \theta \geq P_2 \end{cases} \end{aligned}$$

If $\theta < P_2$ then $a = 0$ will be stable (unstable) provided that $\frac{k}{n}\theta \geq (<) \frac{\theta}{P_2}P_1 \Leftrightarrow \frac{k}{n} \geq (<) \frac{P_1}{P_2}$. Since $P_1 > P_2$, the state $a = 0$ will be unstable provided that $\theta < P_2$. On the other hand, if $\theta \geq P_2$, then $a = 0$ will be stable (unstable) provided that $\frac{k}{n}\theta \geq (<)P_1$. Since $\frac{k}{n}\theta \geq P_1 \Rightarrow \theta \geq \frac{n}{k}P_1 > P_1 > P_2$, the state $a = 0$ will be unstable if $\frac{k}{n}\theta < P_1$ and this state will be stable if $\frac{k}{n}\theta \geq P_1$.

In short:

if $\theta < P_1$ then $\begin{cases} a = 1 \text{ stable if } \frac{k}{n} \geq \frac{P_2}{P_1} \\ a = 1 \text{ unstable if } \frac{k}{n} < \frac{P_2}{P_1} \end{cases}$	if $\theta < P_2$ then $a = 0$ unstable
if $\theta \geq P_1$ then $\begin{cases} a = 1 \text{ stable if } \frac{k}{n}\theta \geq P_2 \\ a = 1 \text{ unstable if } \frac{k}{n}\theta < P_2 \end{cases}$	if $\theta \geq P_2$ then $\begin{cases} a = 0 \text{ stable if } \frac{k}{n}\theta \geq P_1 \\ a = 0 \text{ unstable if } \frac{k}{n}\theta < P_1 \end{cases}$

It is straightforward to derive that:

- If $\theta < P_1$ and,

  $\frac{k}{n} < \frac{P_2}{P_1}$ then $a = 0$ and $a = 1$ are unstable (*type I*).

  $\frac{k}{n} \geq \frac{P_2}{P_1}$ then $a = 0$ is unstable and $a = 1$ is stable (*type II*).

- If $\theta \geq P_1$ and,

  $\frac{k}{n}\theta < P_2$ then $a = 0$ and $a = 1$ are unstable (type I) (*type I*).

  $P_2 \leq \frac{k}{n}\theta \leq P_1$ then $a = 0$ is unstable and $a = 1$ is stable(*type II*).

  $\frac{k}{n}\theta > P_1$ then $a = 0$ and $a = 1$ are stable (*type III*).

The Figure 2 shows this result in graphic form. There are two cases[22].The above result in a more compact

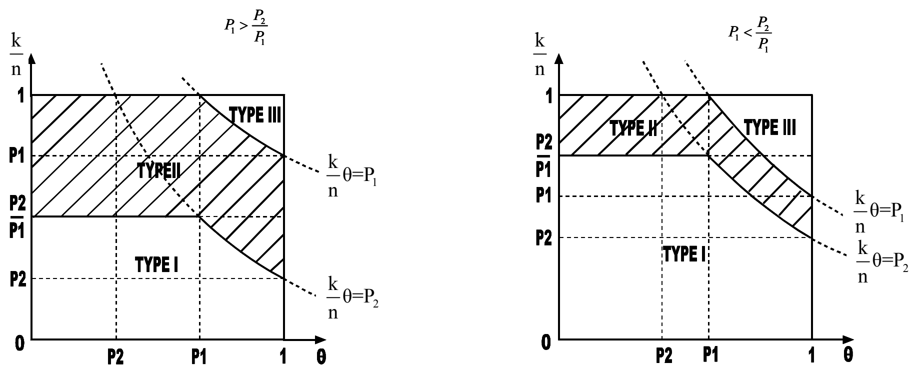

Figure 2: The vertical axes of these graphs represent $\frac{k}{n}$ and the horizontal axes represent θ. The right graph is the case of $P_1 < \frac{P_2}{P_1}$ and the left graph $P_1 > \frac{P_2}{P_1}$.

way:

If $\theta\frac{k}{n} < P_2$ and, $\begin{cases} \frac{k}{n} < \frac{P_2}{P_1} \text{ then } a = 0 \text{ and } a = 1 \text{ are unstable.} \\ \frac{k}{n} \geq \frac{P_2}{P_1} \text{ then } a = 0 \text{ is unstable and } a = 1 \text{ is stable.} \end{cases}$.

If $P_2 \leq \theta\frac{k}{n} \leq P_1$, then $a = 0$ is unstable and $a = 1$ is stable.

If $P_1 < \theta\frac{k}{n}$, then $a = 0$ and $a = 1$ are stable.

■

[22] It is straightforward to show that $P_2 < \frac{P_2}{P_1}$ is always true, and $P_1 > \frac{P_2}{P_1} \Leftrightarrow P_1 > \frac{(1-P_1)}{P_1} \Leftrightarrow P_1^2 + P_1 - 1 > 0 \Leftrightarrow P_1 > \frac{\sqrt{5}-1}{2} \simeq 0.618$

NUMERICAL ANALYSIS

We check the validity of the conjecture by numerical analysis. In order to conduct numerical analysis, values must be specified for the function's four parameters[23]: (k, n, θ, P_1). Numerical analysis was conducted for a sizable subset of $(k, n, \theta, P_1) \in \{k < n \mid k \in \{1, 2, .., 100\}\} \times \{3, 4, ..., 19, 20, 50, 100\} \times \{\frac{1}{50}, \frac{2}{50}, ..., \frac{50}{50}\} \times \{\frac{51}{100}, \frac{52}{100}, ..., \frac{99}{100}\}$. When $n = 100$, a wide range of local selectivity is considered: $\frac{k}{n} \in \{\frac{1}{100}, ..., \frac{99}{100}\}$. In addition, we consider other values of n. After specifying values for (k, n, θ, P_1), the equation[24] $g(a) = 0$ becomes an equation of just one variable, and it is a quotient of univariate polynomials. It is easy to see that because ES is defined by a univariate polynomial, and the product of two polynomials in a is a polynomial in a. It is straightforward to prove that the denominator of that quotient is always positive, thus, we only have to check the roots in the open interval $a = (0, 1)$ of just one univariate polynomial. This simple numerical analysis does not present any difficulty and can be done with any mathematical software such as Mathematica or Matlab. Given the parameter values that we considered, we calculated the roots of this polynomial for 825650 different parameter combinations.

The set of parameters considered is divided into four parts according to proposition 6. We obtained the following results:

If $\theta\frac{k}{n} < P_2$ and $\frac{k}{n} < \frac{P_2}{P_1}$, then there is only one root in the open interval $a = (0, 1)$. Therefore, if selectivity is strong enough there is only one inner root and by proposition 6 it will be globally stable.

If $\theta\frac{k}{n} < P_2$ and $\frac{k}{n} > \frac{P_2}{P_1}$, then there is not any root in the open interval $a = (0, 1)$. Therefore, by proposition 6, $a = 1$ will be globally stable.

If $\theta\frac{k}{n} \in [P_2, P_1]$, as in the previous case, there is not any root in the open interval $a = (0, 1)$ when $P_1 > 0.57$. However, there are some few isolated cases in which we found another kind of dynamic behavior. When this happens, we observe that $P_1 < 0.57$ and $\frac{1}{2} < \frac{k}{n} < \frac{P_2}{P_1}$. As it is stated above, if selectivity is high enough, there is only one inner steady state. When selectivity decreases, the steady state $a = 1$ can become locally stable before the inner steady state disappear but in some very few particular cases. As a result two inner steady states appear for an exceedingly small set of parameters. This is discussed below.

If $\theta\frac{k}{n} > P_1$, there is only one root in the open interval open interval $a = (0, 1)$ and by proposition 6 $a = 0$ and $a = 1$ will be locally stable. However, we found just one parameter combination where the conjecture does not hold in this interval. In this case, both $a = 0$ and $a = 1$ are locally stable and the inner steady state remains. Consequently, there are three steady states. When this happens, P_1 is very close to $\frac{1}{2}$ ($P_1 \leq 0.51$) and $\frac{1}{2} < \frac{k}{n} < \frac{P_2}{P_1}$.

Although we cannot state that the conjecture is generically true because the set of parameter (in which the conjecture is not true) does not have measure zero, we observe that this parameter set is exceedingly small. For example, if $n = 50$ we compute the equation $g(a) = 0$ for 242550 different parameter combinations and we only found three cases where the conjecture does not hold. When $n = 100$, the conjecture always holds for the parameter set considered.

Anyway, we found that the conjecture always hold if $P_1 > 0.57$. In the case $P_1 < 0.57$ we found a few cases where the conjecture does not hold, and only when $\frac{1}{2} < \frac{k}{n} < \frac{P_2}{P_1}$ and $\theta\frac{k}{n} > P_2$. This suggests that θ has to be greater than $\frac{n}{k}P_2$ which is in turn greater than $\frac{1}{2}$. The interval of θ where the conjecture does not hold is extremely small and belongs to the open interval $(\frac{n}{k}P_2, 1)$. For example, if $n = 50$, $k = 26$, $P_1 = 51$ and $P_2 = 49$, the interval of θ where the conjecture does not hold belongs to (0.9423, 0.9454). As P_1 increases we observed that this interval decreases. For example, if $n = 50$, $k = 26$, $P_1 = 57$ and $P_2 = 43$ the interval of θ where the conjecture does not hold belongs to (0.826923, 0.826928).

[23] We assumed that, $P_2 = (1 - P_1)$

[24] $g(a) = 0 \iff f(a) - a = 0$, and $f(a)$ is defined in expression (1)

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