A Class of All-pay Auctions with Affiliated Information

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1 Introduction

This paper analyzes a class of all-pay auctions with affiliated signals. We consider an auction mechanism in which the winning bidder pays a mixture of her own bid and of the second highest bid. This mechanism combines the features of the all-pay auction and the war of attrition.

The main reason for studying the properties of a combined mechanism stems from the literature on *games of fair division*[1]. Güth (1986) outlines an axiomatic approach to derive allocation rules in that class of games. The author introduces the axiom that the resulting allocation should be envy-free with respect to the stated preferences : according to the evaluation expressed by her bid no agent among bidders should prefer another agent's net trade to her own one. This axiom implies that the object must be given to the highest bidder and the price must lie between the highest and the second highest. Thus the price can always be described as a convex combination of the highest and the second highest bid. This motivates the study of optimal bidding behavior when the price of the object is any convex combination of the highest and second-highest bid.

Our analysis is also motivated by Krishna and Morgan (1997). In their analysis of the war of attrition, those authors emphasize that the

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1 See, for instance, Crawford (1977), Güth (1986), Moulin (1984), Steinhaus (1948).

individual bidding equilibrium strategy becomes *unbounded* as a bidder's private signal approaches the upper bound of the support. This happens despite the expected value of the object is *finite*. We show, coherently with intuition, that if the winning bidder has to pay even a negligible amount of her own bid, the boundedness of her equilibrium bidding strategy is restored. That points out the fact that the war of attrition is a knife-edge case in the overall class of bidding games.

Lizzeri and Persico (2000) is also closely related to our paper. They prove existence and uniqueness of equilibrium for a large class of bidding games. Our combined mechanism is a special case of the payoff function specified in their paper. However, they prove existence and uniqueness of equilibrium for the two-player, asymmetric case with affiliated signals, whereas we deal with the N-bidder, symmetric case with affiliated signals. We give a sufficient condition for the existence of equilibrium strategies. We show that the combined mechanism raises a higher expected revenue for the seller than the all-pay auction, but lower than the war of attrition. Finally we stress that the condition that ensures the existence of an increasing, symmetric equilibrium rules out the full extraction of buyers' surplus from the seller.

2 The Combined All-Pay Auction Mechanism

We model our combined auction mechanism as follows. Before the auction starts, each bidder i receives a signal X_i, $i \in \{1, \dots, N\}$. The main assumption of this model is that private signals are affiliated real random variables. Intuitively, this assumption means that high values of one of the variables make it more likely that the other variables take on high values. More formally, let z and z' be points in $\mathbb{R}^N$. Let $z \vee z'$ denotes the component-wise maximum of z and z', and $z \wedge z'$ the component-wise minimum of z and z'. The variables of the model are *affiliated* if, for all z and z',

$$f(z \vee z')\, f(z \wedge z') \geqslant f(z)\, f(z') \tag{1}$$

Each bidder tenders a sealed offer b_i, and payoffs are:

$$W_i = \begin{cases} V_i - [\alpha b_i + (1-\alpha)\max_{j \neq i} b_j] & \text{if } b_i > \max_{j \neq i} b_j \\ -b_i & \text{if } b_i < \max_{j \neq i} b_j \\ \dfrac{1}{\#\{k : b_k = b_i\}} V_i - b_i & \text{if } b_i = \max_{j \neq i} b_j, \end{cases}$$

where $\alpha \in (0, 1)$, and $i \neq j$. According to our combined mechanism, the winning bidder pays a mixture of her own and of the second-highest bid. As usual, we assume that if $b_i = \max_{j \neq i} b_j$ a random device assigns the object among the winning bidders according to any fixed criterion.

Throughout the paper we use $Y_1 \equiv \max_{j \neq 1} X_j$, $f_{Y_1}(\cdot \mid x)$ as the conditional density function of Y_1 given that $X_1 = x$, and $F_{Y_1}(\cdot \mid x)$ the corresponding cumulative distribution function. Moreover, we define the modified hazard-rate function

$$\lambda_\alpha(y \mid x) \equiv \frac{f_{Y_1}(y \mid x)}{1 - (1-\alpha)F_{Y_1}(y \mid x)},$$

and we assume this function to be non-increasing in x.

Define $V(x, y) = E[V_1 \mid X_1 = x, Y_1 = y]$. Since X_1 and Y_1 are affiliated, $V(x, y)$ is a non-decreasing function of its arguments. As in Milgrom and Weber (1982), we assume that it is increasing.

We start our analysis of the auction game with the derivation of the necessary condition for the existence of a symmetric equilibrium strategy. Since the game is symmetric, we will focus on player 1.

Suppose that all bidders $j \neq 1$ follow an increasing strategy β. Furthermore, suppose that bidder 1 receives a signal $X_1 = x$ and bids b. Then bidder 1's expected payoff is

$$\begin{aligned}\Pi(b, x) = &\int_{-\infty}^{\beta^{-1}(b)} [V(x, y) - (\alpha b + (1-\alpha)\beta(y))] f_{Y_1}(y \mid x) dy \\ &- [1 - F_{Y_1}(\beta^{-1}(b) \mid x)]b \end{aligned} \quad (2)$$

From (2) the first-order condition writes

$$\begin{aligned}&\frac{1}{\beta'(\beta^{-1}(b))}[V(x, \beta^{-1}(b)) - b] f_{Y_1}(\beta^{-1}(b) \mid x) - \alpha F_{Y_1}(\beta_{-1}(b) \mid x) \\ &- [1 - F_{Y_1}(\beta_{-1}(b) \mid x)] + b f_{Y_1}(\beta_{-1}(b) \mid x)\frac{1}{\beta'(\beta_{-1}(b))} = 0 \end{aligned} \quad (3)$$

At a symmetric equilibrium, $b = \beta(x)$, so that (3) yields

$$\begin{aligned}\beta'(x) &= V(x, x)\frac{f_{Y_1}(x \mid x)}{1 - (1-\alpha)F_{Y_1}(x \mid x)} \\ &= V(x, x)\lambda_\alpha(x \mid x), \end{aligned} \quad (4)$$

and thus

$$\beta(x) = \int_{-\infty}^{x} V(u, u)\lambda_\alpha(u \mid u) du$$

The following proposition provides a sufficient condition for β to be a symmetric equilibrium.

Definition 1 *Let* $\phi_\alpha : \mathbb{R}^2 \to \mathbb{R}$ *be defined by* : $\phi_\alpha(x,y) = V(x,y)\lambda_\alpha(y \mid x)$.

Proposition 1 *Assume that, for all* y, $\phi_\alpha(\cdot, y)$ *is an increasing function. A symmetric equilibrium in the combined mechanism is given by the function* β *defined as*

$$\beta(x) = \int_{-\infty}^{x} V(u,u)\lambda_\alpha(u \mid u)du \tag{5}$$

Proof : Suppose that bidder 1 bids an amount $\beta(z)$, when the signal is x. Her expected payoff is

$$\begin{aligned}\Pi(z,x) &= \int_{-\infty}^{z} [V(x,y) - (\alpha\beta(z) + (1-\alpha)\beta(y))]f_{Y_1}(y \mid x)dy \\ &\quad - [1 - F_{Y_1}(z \mid x)]\beta(z) \\ &= \int_{-\infty}^{z} V(x,y)f_{Y_1}(y \mid x)dy + (1-\alpha)\beta(z)F_{Y_1}(z \mid x) - \beta(z) \\ &\quad - (1-\alpha)[\beta(z)F_{Y_1}(z \mid x) - \int_{-\infty}^{z} \beta'(y)F_{Y_1}(y \mid x)dy] \\ &= \int_{-\infty}^{z} V(x,y)f_{Y_1}(y \mid x)dy + (1-\alpha)\int_{-\infty}^{z} \beta'(y)F_{Y_1}(y \mid x)dy - \beta(z)\end{aligned}$$

Using (4) and (5) we get

$$\begin{aligned}\Pi(z,x) &= \int_{-\infty}^{z} V(x,y)f_{Y_1}(y \mid x)dy - \int_{-\infty}^{z} V(y,y)\lambda_\alpha(y \mid y) \\ &\quad \times [1 - (1-\alpha)F_{Y_1}(y \mid x)]dy \\ &= \int_{-\infty}^{z} [V(x,y)\lambda_\alpha(y \mid x) - V(y,y)\lambda_\alpha(y \mid y)] \\ &\quad [1 - (1-\alpha)F_{Y_1}(y \mid x)]dy \\ &= \int_{-\infty}^{z} [\phi_\alpha(x,y) - \phi_\alpha(y,y)][1 - (1-\alpha)F_{Y_1}(y \mid x)]dy\end{aligned}$$

Since we assumed $\phi_\alpha(\cdot, y)$ to be an increasing function, for all $y > x$, $[\phi_\alpha(x,y) - \phi_\alpha(y,y)] < 0$, and for all $y < x$, $[\phi_\alpha(x,y) - \phi_\alpha(y,y)] > 0$. Thus, $\Pi(z,x)$ is maximized at $z = x$.

The equilibrium expected payoff to bidder 1 is

$$\Pi(x,x) = \int_{-\infty}^{x} [\phi_\alpha(x,y) - \phi_\alpha(y,y)][1 - (1-\alpha)F_{Y_1}(y \mid x)]dy \tag{6}$$

which is non-negative due to the assumption on $\phi_\alpha(x,y)$. □

The expression for the equilibrium expected payoff reveals a feature of the combined mechanism which is worth emphasizing. The assumption that

$\phi_\alpha(\cdot \mid y)$ is an increasing function ensures the existence of a symmetric and strictly *increasing* equilibrium. The same assumption, however, guarantees that the equilibrium expected payoff to a bidder is *strictly* positive, that is, the seller cannot fully extract the surplus from buyers. Then, the existence of an increasing equilibrium strategy and full surplus extraction come into conflict.

We conclude this section by showing that the equilibrium bidding strategy in the combined mechanism is always bounded when a buyer's signal approaches the upper bound of the support, $\overline{x}$. Krishna and Morgan (1997) show that in the war of attrition each bidder's bid becomes *unbounded* as her signal approaches $\overline{x}$. This happens despite the expected value of the object at $\overline{x}$, $V(\overline{x},\overline{x})$, is finite. If the all-pay mechanism is such that the winner has to pay even a small fraction of her own bid, it is reasonable to expect the boundedness of the bidding function to be restored. The following proposition proves this claim.

Proposition 2 *Suppose that, for all y, $\lambda_\alpha(y \mid \cdot)$ is non-increasing and $\phi_\alpha(\cdot, y)$ is increasing. Then (i)* $\lim_{x\to\underline{x}}\beta(x) = 0$ *and (ii)* $\lim_{x\to\overline{x}}\beta(x) \leqslant \lim_{x\to\overline{x}} \dfrac{1}{\alpha}V(x,x)$.

Proof : *(i)* follows directly from (5).

To verify *(ii)*, we use (5) to write

$$\begin{aligned}
\beta(x) &= \int_{-\infty}^{x} V(t,t)\lambda_\alpha(t \mid t)dt \\
&\leqslant V(x,x)\int_{-\infty}^{x} \frac{f_{Y_1}(t \mid t)}{1-(1-\alpha)F_{Y_1}(t \mid t)}dt \\
&\leqslant V(x,x)\int_{-\infty}^{x} \frac{f_{Y_1}(t \mid \underline{x})}{1-(1-\alpha)F_{Y_1}(t \mid \underline{x})}dt \\
&\leqslant V(x,x)\frac{1}{1-(1-\alpha)F_{Y_1}(x \mid \underline{x})}\int_{-\infty}^{x} f_{Y_1}(t \mid \underline{x})dt,
\end{aligned}$$

where in the first inequality we have used the fact that $V(x,x)$ is increasing in both arguments, in the second the fact that $\lambda_\alpha(y \mid x)$ is non-increasing in x. Taking the limit for both sides yields

$$\lim_{x\to\overline{x}}\beta(x) \leqslant \lim_{x\to\overline{x}} \frac{1}{\alpha} V(x,x)$$

which completes the proof. □

This last result points out that the equilibrium bidding function in the combined mechanism has a continuity property with respect to the two extreme cases, that is, the all-pay auction and the war of attrition. Indeed, as the signal approaches the upper bound of the support the bidding function is bounded by a function which decreases as α tends to 1, and increases as α tends to zero. Thus the constant α captures the slope of the bidding function when each bidder's signal approaches the upper bound of the support.

3 Revenue Ranking

Krishna and Morgan (1997) prove that the revenue from the war of attrition is greater than that in the all-pay action. Since in the combined mechanism the winner pays a mixture of her own and of the second-highest bid, it seems intuitive that the combined mechanism raises a higher revenue than the all-pay action and a lower revenue than the war of attrition. This intuition is made rigorous in the following proposition.

Proposition 3 *Suppose that $\phi_\alpha(\cdot, y)$ is increasing. Then the expected revenue from the combined mechanism is (i) lower the expected revenue from the war of attrition, and (ii) greater than the expected revenue from the all-pay auction.*

Proof : Recall from Krishna and Morgan (1997) that in the war of attrition the expected payment by a bidder who receives a signal x is

$$e^W(x) = \int_{-\infty}^{x} V(y,y) f_{Y_1}(y \mid x) \left[\frac{\lambda(y \mid y)}{\lambda(y \mid x)}\right] dy, \tag{7}$$

where $\lambda(y \mid x) \equiv \frac{f_{Y_1}(y|x)}{1-F_{Y_1}(y|x)}$.

In the all-pay auction the expected payment by a bidder who receives a signal x is

$$e^A(x) = \int_{-\infty}^{x} V(y,y) f_{Y_1}(y \mid y) dy \tag{8}$$

Let us compute the expected payment in the combined mechanism when a bidder receives a signal x.

$$\begin{aligned}
e^{CM}(x) &= \int_{-\infty}^{x} [\alpha\beta(x) + (1-\alpha)\beta(y)] f_{Y_1}(y \mid x) dy + [1 - F_{Y_1}(x \mid x)]\beta(x) \\
&= (1-\alpha)\left[\beta(x) F_{Y_1}(x \mid x) - \int_{-\infty}^{x} \beta'(y) F_{Y_1}(y \mid x) dy\right] \\
&\quad + \alpha\beta(x) F_{Y_1}(x \mid x) + [1 - F_{Y_1}(x \mid x)]\beta(x) \\
&= \beta(x) - (1-\alpha)\int_{-\infty}^{x} \beta'(y) F_{Y_1}(y \mid x) dy \\
&= \int_{-\infty}^{x} V(y,y)\lambda_\alpha(y \mid y) dy - \int_{-\infty}^{x} V(y,y)\lambda_\alpha(y \mid y)(1-\alpha) F_{Y_1}(y \mid x) dy \\
&= \int_{-\infty}^{x} V(y,y)\lambda_\alpha(y \mid y) f_{Y_1}(y \mid x)\left[\frac{1-(1-\alpha)F_{Y_1}(y \mid x)}{f_{Y_1}(y \mid x)}\right] dy
\end{aligned}$$

$$= \int_{-\infty}^{x} V(y, y) f_{Y_1}(y \mid x) \left[\frac{\lambda_\alpha(y \mid y)}{\lambda_\alpha(y \mid x)} \right] dy \tag{9}$$

Since $F_{Y_1}(y \mid x)$ is non-increasing in x and $\alpha \in (0, 1)$, then

$$\frac{\lambda_\alpha(y \mid y)}{\lambda_\alpha(y \mid x)} < \frac{\lambda(y \mid y)}{\lambda(y \mid x)},$$

that is, $e^W > e^{CM}$. In order to establish point *(ii)* observe that

$$f_{Y_1}(y \mid y) < f_{Y_1}(y \mid x) \frac{\lambda_\alpha(y \mid y)}{\lambda_\alpha(y \mid x)} \Rightarrow 1 < \frac{1 - (1 - \alpha) F_{Y_1}(y \mid x)}{1 - (1 - \alpha) F_{Y_1}(y \mid y)}$$

which is always true due to the assumption on $F_{Y_1}(y \mid \cdot)$. □

4 Conclusion

The aim of this paper is to contribute to a very recent interest in combined auction mechanisms. We have analyzed an auction mechanism which is the convex combination of the all-pay auction and the war of attrition under the assumption that participants' signals are affiliated.

The equilibrium bidding strategy of the combined mechanism is always bounded even when a buyer's signal approaches the highest possible value. This result strengthens the idea that the war of attrition is a knife-edge case in the class of auction mechanisms.

We have proposed a sufficient condition for the existence of a symmetric and increasing bidding function. The same condition ensures that expected profit to a buyer is always positive at that equilibrium. Then full surplus extraction is incompatible with the equilibrium bidding strategy that we have derived. It is well known that the seller can extract the whole surplus from buyers in a common value environment. McAfee *and al.* (1989), for instance, show that the seller can construct an optimal (all-pay) mechanism which combines buyers' signals in a complicated way. This might explain why more standard all-pay auction mechanisms cannot achieve full surplus extraction.

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