

Optimal Saving with Additive and Multiplicative Background Risk*

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Abstract

We study optimal saving when incomes are certain and risk bears on consumption. A key finding is that, with CARA utility and additive uncertain consumption, or CRRA utility and multiplicative consumption, risk-averse individuals do not form precautionary saving as in the standard theory of saving under uncertainty.

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1 Introduction

The theory of consumption and saving under uncertainty establishes that risk-averse consumers form a precautionary saving to face an uninsurable risk on their future income if the marginal utility of future consumption is convex (Leland (1968), Sandmo (1970) and Drèze and Modigliani (1972)). The optimal consumption path therefore depends on the impatience of the consumer (discount factor), consumption smoothing (degree of concavity of the instantaneous utility function) and also on the degree of prudence. If the third derivative of the future instantaneous utility functions is positive and future incomes are uncertain, then the consumer behaves with prudence (Kimball 1990) in allocating his wealth over time. This means that he raises his optimal saving and, hence, decreases his current consumption. Whenever a shock on income occurs, the consumer is able to take out of his cash and sustain his desired consumption profile. Thus he can postpone the risk until the last period of his life when saving is optimally null. Saving is used as self-insurance.

What happens if the consumer must face another source of income risk, which is independent of the existence of other risks?¹ This type of risk is called "background risk". It cannot be insured and is therefore completely exogenous. The typical background risk is the risk on human capital that is not insurable because of asymmetric information on the worker's real capabilities and effort. The background risk can be either additive or multiplicative.²

The objective of this paper is to present two setups in which the presence of another source of risk, a background risk, does not modify the intertemporal allocation of consumption. In the first setup, the background risk is additive and the instantaneous utility functions exhibit constant absolute risk aversion. In the second setup, the background risk is multiplicative and the instantaneous utility functions exhibit constant relative risk aversion. In both setups, risk-averse consumers do not modify their optimal saving in the presence of the background risk.

Section 5.2 describes the model. Section 5.3 presents the results when the background risk is additive and the instantaneous utility functions exhibit CARA. Section 5.4 presents the results of the CRRA instantaneous utility functions in the presence of a multiplicative background risk.

2 A two-period model

We consider a simple two-period life-cycle model, in which households receive uncertain labor incomes \tilde{y}_1 in period 1 and \tilde{y}_2 in period 2. Moreover, we suppose that households

¹The independence assumption is restrictive but helpful to isolate the background risk effect.

²For a presentation of additive background risk, see Gollier (2001), and Franke, Schlesinger, and Stapleton (2003) on multiplicative background risk.

bear a "background risk", \tilde{x} , on their wealth in every period. This risk is exogenous, uninsurable and statistically independent from the risks on labor incomes. For future consumption, the households allocate a part of their initial wealth to saving. They do so before the two sources of risk are realized. Finally, we assume that there exists a representative consumer who optimally decides how to allocate the present value of total wealth to present and future consumption.

3 Additive background risk and constant absolute risk aversion (CARA)

We assume throughout this section that consumption preferences exhibit constant absolute risk aversion (CARA). This class of utility functions has the property that absolute risk aversion is independent of wealth.

3.1 Consumer preferences

We assume that intertemporal cardinal consumption preferences are represented by a function of the form:

$$E[U(c_1, c_2)] = E[u_1(c_1) + u_2(c_2)], \quad t = 1, 2 \quad (1)$$

such that, for each t , the instantaneous utility function is CARA with time-independent risk-aversion. E is the mathematical expectation operator, c_1 and c_2 are first-period and second-period consumption respectively.

CARA utility functions are characterized by:

$$u(c) = -\frac{\exp(-\alpha c)}{\alpha},$$

where α is some positive scalar. The properties of CARA utility functions are:

$$u'(c) = \exp(-\alpha c) > 0, \quad u''(c) = -\alpha \exp(-\alpha c) < 0 \quad \text{and} \quad u'''(c) = \alpha^2 \exp(-\alpha c) > 0,$$

the degree of absolute risk aversion is constant:

$$A(c) = \frac{-u''(c)}{u'(c)} = \alpha,$$

and the degree of absolute prudence is also constant and equal to the degree of absolute risk aversion:

$$P(c) = \frac{-u'''(c)}{u''(c)} = A(c) = \alpha.$$

Since $A' = 0$, the absolute risk aversion is constant with wealth.

The representative consumer maximizes (1) under the linear budget constraint:

$$\begin{aligned} c_1 + s &= \tilde{y}_1 + \tilde{x}_1, \\ c_2 &= (1 + r)s + \tilde{y}_2 + \tilde{x}_2, \end{aligned} \tag{2}$$

where s is saving and r is the risk-free interest rate. The random variable \tilde{x}_1 and \tilde{x}_2 stand for the additive background risks of the first and second period respectively. These risks are assumed to be exogenous and statistically independent.

3.2 Optimal saving

The optimal decision on how to allocate his total wealth to present and future consumption requires to solve the program:

$$s^* \in \arg \max_s Eu_1(\tilde{y}_1 - s + \tilde{x}_1) + Eu_2(\tilde{y}_2 + (1 + r)s + \tilde{x}_2), \tag{3}$$

where s^* is the optimal saving. The question is how an independent additive background risk affects the intertemporal consumption allocation. To answer it, we use the notion of indirect utility function.³ Let us define the following indirect utility function: for all z_1 and z_2 ,

$$v(z_1) = E[u(z_1 + \tilde{x}_1)],$$

$$v(z_2) = E[u(z_2 + \tilde{x}_2)],$$

where $z_1 = \tilde{y}_1 - s$ and $z_2 = \tilde{y}_2 + (1 + r)s$. The indirect utility function v for a consumer facing no background risk is equivalent to the expected instantaneous utility function Eu with background risk. The effect of the background risk, \tilde{x} , on the choice variable, s , is identical to the effect of a change in preference from u to v . This method allows to pinpoint the effect of the background risk on the intertemporal allocation.

³see Gollier (2001), p. 114.

The objective function then becomes

$$Ev_1(\tilde{y}_1 - s) + Ev_2(\tilde{y}_2 + (1 + r)s), \quad (4)$$

which is equivalent to (1).

The optimal intertemporal consumption is given by the first-order condition:

$$Eu'_1(z_1 + \tilde{x}_1) = (1 + r)Eu'_2(z_2 + \tilde{x}_2) \quad (5)$$

The first-order condition determines how the additive background risk affects optimal saving. The result is stated in the following proposition:

Proposition 1 *If the preferences of an agent are represented by a utility function with constant absolute risk aversion, the choice consisting in allocating uncertain incomes to present and future consumption is not affected by the presence of an additive background risk.*

Proof:

Exponential utility allows to rewrite (5) as:

$$Eu'_1(z_1) \cdot E\left[-\frac{\exp(-\alpha\tilde{x}_1)}{\alpha}\right] = (1 + r) Eu'_2(z_2) \cdot E\left[-\frac{\exp(-\alpha\tilde{x}_2)}{\alpha}\right].$$

Since $E\tilde{x}_1 = E\tilde{x}_2$, then it follows

$$u'_1(z_1) = (1 + r) u'_2(z_2),$$

which is equivalent to

$$Ev'_1(\tilde{y}_1 - s) = (1 + r)Ev'_2(\tilde{y}_2 + (1 + r)s). \blacksquare$$

The change in the shape of the utility function, from u to v , leaves the optimal decision unmodified. This means that the presence of the additive background risk does not affect the intertemporal allocation. In other words, the current consumption is not reduced by the independent additive background risk.

4 Multiplicative background risk and constant relative risk aversion (CRRA)

In this section, the representative consumer has CRRA preferences and the background risk is multiplicative. Examples and a comprehensive study of this type of risk are provided by Franke, Schlesinger, and Stapleton (2003). Here we suggest that the multiplicative background risk is the inflation risk bearing on nominal wealth. Alternatively, we could think of the multiplicative risk as an uncertain income tax rate due to an unstable or predatory political environment.

4.1 Consumer preferences

We assume that intertemporal cardinal consumption preferences are represented by a function of the form:

$$E[U(c_1, c_2)] = E[u_1(c_1) + u_2(c_2)], \quad t = 1, 2 \quad (6)$$

such that, for each t , the instantaneous utility function is CRRA with time-independent risk-aversion. E is the mathematical expectation operator, c_1 and c_2 are first-period and second-period consumption respectively.

CRRA utility functions are characterized by:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

where $\gamma \geq 0$. The properties of CRRA utility functions are:

$$u'(c) = c^{-\gamma} > 0, \quad u''(c) = -\gamma c^{-\gamma-1} < 0 \quad \text{and} \quad u'''(c) = \gamma(\gamma+1)c^{-\gamma-2} > 0,$$

the degree of absolute risk aversion is decreasing in c :

$$A(c) = \frac{-u''(c)}{u'(c)} = \frac{\gamma}{c},$$

as the degree of absolute prudence:

$$P(c) = \frac{-u'''(c)}{u''(c)} = \frac{\gamma+1}{c}.$$

Since $A' < 0$, the absolute risk aversion is decreasing with wealth.

The representative consumer maximizes (6) under the linear budget constraint:

$$\begin{aligned} c_1 &= (\tilde{y}_1 - s)\tilde{x}_1, \\ c_2 &= [(1+r)s + \tilde{y}_2]\tilde{x}_2, \end{aligned} \tag{7}$$

where s is saving and r is the risk-free interest rate. The random variables \tilde{x}_1 and \tilde{x}_2 stand for the multiplicative background risks of the first and second period respectively. These risks are assumed to be exogenous and statistically independent.

4.2 Optimal saving

The optimal decision on how to allocate his total wealth to present and future consumption requires to solve the program:

$$s^* \in \arg \max_s Eu_1[(\tilde{y}_1 - s)\tilde{x}_1] + Eu_2([(1+r)s + \tilde{y}_2]\tilde{x}_2), \tag{8}$$

where s^* is the optimal saving. The question is how an independent multiplicative background risk affects the intertemporal consumption allocation. To answer it, we use the same indirect utility function as previously: for all z_1 and z_2 ,

$$v(z_1) = E[u(z_1\tilde{x}_1)],$$

$$v(z_2) = E[u(z_2\tilde{x}_2)],$$

where $z_1 = \tilde{y}_1 - s$ and $z_2 = \tilde{y}_2 + (1+r)s$.

The objective function then becomes

$$Ev_1(\tilde{y}_1 - s) + Ev_2(\tilde{y}_2 + (1+r)s), \tag{9}$$

which is equivalent to (1).

The optimal intertemporal consumption is given by the first-order condition:

$$Eu'_1(z_1\tilde{x}_1) = (1+r)Eu'_2(z_2\tilde{x}_2) \tag{10}$$

The effect of the multiplicative background risk on optimal saving is stated in the following proposition:

Proposition 2 *If the preferences of an agent are represented by a utility function with constant relative risk aversion, the choice consisting in allocating uncertain incomes to present and future consumption is not affected by the presence of a multiplicative background risk.*

Proof:

The utility function with CRRA allows to rewrite (10) as:

$$Eu'_1(z_1) \cdot E \left[\frac{\tilde{x}_1^{1-\gamma}}{1-\gamma} \right] = (1+r) Eu'_2(z_2) \cdot E \left[\frac{\tilde{x}_2^{1-\gamma}}{1-\gamma} \right].$$

Since $E\tilde{x}_1 = E\tilde{x}_2$, then it follows

$$u'_1(z_1) = (1+r) u'_2(z_2),$$

which is equivalent to

$$Ev'_1(\tilde{y}_1 - s) = (1+r)Ev'_2(\tilde{y}_2 + (1+r)s). \blacksquare$$

Again, the change in the shape of the utility function, from u to v , leaves the optimal decision unmodified. This means that the presence of the multiplicative background risk does not affect the intertemporal allocation. In other words, the current consumption is not reduced by the independent multiplicative background risks.

The cases of utility with CRRA and multiplicative background risks and utility with CARA and additive background risks are in fact linked by the following relationship (Gollier 2002):

$$(z\tilde{x})^\sigma = \exp[\sigma(z + \tilde{x})],$$

where $z = \ln Z$ and $\tilde{x} = \ln \tilde{X}$.

5 Conclusion

Our both propositions point out that the presence of an independent and uninsurable background risk may not affect the intertemporal optimal choice of consumption. This is true for two types of settings:

- when consumption preferences exhibit constant absolute risk aversion and the background risk is additive;
- when consumption preferences exhibit constant relative risk aversion and the background risk is multiplicative.

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