Optimal Portfolio Strategies with Stochastic Wage Income: the Case of a Defined Contribution Pension Plan*

Paolo Battocchio†

May 6, 2002

Abstract

We consider a stochastic model for a defined-contribution pension fund in continuous time. In particular, we focus on the portfolio problem of a fund manager who wants to maximize the expected utility of his terminal wealth in a complete financial market. The fund manager must cope with a set of stochastic investment opportunities and with the uncertainty involved by the labor market. After introducing a stochastic interest rate, we assume a market structure characterized by three assets: a riskless asset, a bond and a stock. Moreover, we introduce a stochastic process for salaries, and develop the model according to the stochastic dynamic programming methodology.

We show that the optimal portfolio is formed by three components: a speculative component proportional to the market price of risk of the two risky assets through the relative risk aversion index, an hedging component proportional to the diffusion term of the interest rate, and a preference-free hedging component proportional to the volatilities of the salary process. Finally, after specifying a suitable functional form for the drift term of the salary process, we find a close form solution to the asset allocation problem.

JEL Classification: C61, G11, G23.

Keywords: defined-contribution pension plan, salary risk, stochastic optimal control, Hamilton-Jacobi-Bellman equation.

---

*I would like to thank R. Anderson, A. Attar, D. Blake, G. Deelstra, F. Menoncin, and O. Scaillet for helpful comments. I am grateful for the financial support from the Belgian French Community's program "Action de Recherches Concertée" 99/04-235.

†IRES, Université Catholique de Louvain, Place Montesquieu, 3, 1348 Louvain-la-Neuve, Belgium. Tel: 0032-10-474484, e-mail: battocchio@ires.ucl.ac.be

1 Introduction

There are two extremely different ways to manage a pension fund. On one hand, we find defined-benefit plans (DB), where benefits are fixed in advance by the sponsor and contributions are initially set and subsequently adjusted in order to maintain the fund in balance. On the other hand, there are defined-contribution plans (DC), where contributions are fixed and benefits depend on the returns on fund's portfolio. However, DC plans allow contributors to know, at each time, the value of their retirement accounts. Historically, fund managers have mainly proposed DB plans, which are definitely preferred by workers. In fact, in the case of DB plans, the associated financial risks are supported by the plan sponsor rather than the individual member of the plan. Nowadays, most of the pension plans proposed are based on DC schemes, which involve a considerable transfer of risks to workers. Accordingly, DC pension funds provide contributors with a service of saving management, without guaranteeing any minimum performance. As we have already highlighted, only contributions are fixed in advance, while the final retirement account depends fundamentally on the administrative and financial skill of the fund manager. Thereafter, an efficient financial management is essential to gain contributors' trust.

The goal of the fund manager is to invest the accumulated wealth in order to optimize the expected value of a suitable terminal utility function. The classical dynamic optimization model, initially proposed by Merton (1971), assumes a market structure with constant interest rates. In the case of pension funds, the optimal asset-allocation problem involves a quite long period, generally from 20 to 40 years. It follows that the assumption of constant interest rates is not good for our purpose. Moreover, the benefits proposed by DC pension plans require often the specification of the stochastic behavior of other variables, such as salaries. Thus, the fund manager must cope not only with financial risks, but also with background risks, where by "financial risk" we mean the risk involved by financial markets, and by "background risk" we mean all the risks outside the financial markets (e.g. salary and inflation).

Merton (1969,1971,1990), Duffie (1996), Karatzas and Shreve (1998) provide general treatments of optimal portfolio choice in continuous-time. The optimal portfolio problem becomes more and more complex when we allow for background risks. At this regard, it is important to distinguish between two different classes of background risks: the "level" background risks and the "ratio" background risks. The first set of risk affects the amount of wealth which can be invested, while the second set of risk affects only the wealth growth rate. In this work, we consider only a scalar dimensional "level" background risk given by the shareholder's salary.

In this work, we consider a stochastic model for pension fund dynamics in continuous time. We assume a financial market with stochastic interest rates and consisting in three assets: a riskless asset, a stock and a bond, which can be bought and sold without incurring any transaction costs or restriction on short sales. More precisely, we study an optimal investment problem related to the accumulation phase of a defined-contribution pension fund. We consider the case of a shareholder who, at each period $t \in [0, T]$, contributes a constant proportion of his salary to a personal pension fund. At the time of retirement T, the accumulated pension fund will be converted into an annuity. Similar models have been presented recently by Blake, Cairns, and Dowd (2000), Boulier, Huang, and Taillard (1999), and Deelstra, Grasselli, and Koehl (2001). Especially, Blake et al. (2000) assume a stochastic process for salaries including a non-hedgeable risk component and focus on the replacement ratio as the central quantity of interest. Boulier et al. (1999) assume a deterministic process for salaries and consider a guarantee on the benefits. Accordingly, they strongly support the real need for a downside protection of contributors who is more directly exposed to the financial risk borne by the pension fund. Deelstra et al. (2001) also allow for a minimum guarantee in order to minimize the randomness of the retirement account, but they describe the flow of contributions through a non-negative, progressive measurable and square-integrable process. A recent model for a DC pension scheme in discrete time is proposed by Haberman and Vigna (2001). In particular, they study both the "investment risk", that is the risk of incurring a poor investment performance during the accumulation phase of the fund, and the "annuity risk", that is the risk of purchasing an annuity at retirement in a particular negative economic scenario involving a low conversion rate.

The problem of optimal portfolio choice for a long-term investor in presence of a wage income is treated also by El Karoui and Jeanblanc-Picqué (1998), Campbell and Viceira (2001) and Franke, Peterson, and Stapleton (2001). Under a complete market with a constant interest rate, El Karoui and Jeanblanc-Picqué (1998) present the solution of a portfolio optimization problem for an economic agent endowed with a stochastic insurable stream of labor income. Thus, they assume that the income process does not involve a new source of uncertainty. Campbell and Viceira (2001) focus on some aspects of labor income risk in discrete-time. In particular, they look at individual's labor income as a dividend on the individual's implicit holding of human wealth. Franke et al. (2001) analyse the impact of labor income uncertainty resolution on portfolio choice. They show how the portfolio startegy of an investor changes when his labor income uncertainty is resolved early or late in life.

The methodological approach we use to solve the optimal asset-allocation problem of a pension fund is the stochastic dynamic programming. Alternative approaches (see for instance Deelstra et al. (2001), and Lioui and Poncet (2001)) are based on the Cox-Huang methodology, where the resulting partial differential equation is often simpler to solve than the Hamilton-Jacobi-Bellman equation coming from the dynamic programming.

In this work, we intend to proceed by steps in order to set up the basis for an analytical solution of the Hamilton-Jacobi-Bellman equation under a general framework. Moreover, we study some important properties of the optimal portfolio. In particular, we show that it is optimal to invest in a combination of three portfolios: a speculative portfolio proportional to the market price of risk of the risky assets through the relative risk aversion index, an hedging portfolio proportional to the diffusion term of the interest rate, and a preference-free hedging portfolio proportional to the diffusion terms of the salary process.

The work is organized as follows. Section 2 and Section 3 present respectively the financial market structure and the defined contribution process. In Section 4, the pension fund dynamics is derived. Section 5 develops the optimal asset allocation problem. The properties of the optimal portfolio are discussed in Section 6. In

Section 7, an explicit solution to the asset allocation problem is derived. Section 8 concludes.

2 The Financial Market

In this section we introduce the market structure of our optimal asset allocation model and we define the stochastic dynamics of the interest rate and asset values. We consider a complete and frictionless financial market which is continuously open over the fixed time interval $[0, T]$, where $T \in \mathbb{R}^+ - \{0\}$ denotes the retirement time of a representative shareholder. The uncertainty involved by the financial market is described by a 2-dimensional standard Brownian motion

$$W(t) = \begin{bmatrix} W^0(t) \\ W^1(t) \end{bmatrix}, t \in [0, T],$$

defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = \{\mathcal{F}(t)\}_{t\in[0,T]}$ is the filtration generated by the Brownian motion and $\mathbb{P}$ represents the historical probability measure. The natural interpretation of $\mathcal{F}(t)$ is to view it as the information available to the investor at time $t \in [0, T]$. The two Wiener processes $W^0(t)$ and $W^1(t)$ are supposed to be independent. We can impose this constraint without loss of generality. In fact, we can always shift from uncorrelated to correlated Wiener processes (and viceversa) via the Cholesky decomposition of the correlation matrix. A description of the Cholesky decomposition of the correlation matrix is provided in Appendix A.

We consider a general one-factor model for the forward interest rate $f(t, \tau)$, whose dynamics is given by:

$$\begin{aligned} &df\,(t, \tau) = \alpha\,(t, \tau)\,dt + \nu\,(t, \tau)\,dW^0\,(t)\,, \quad \tau \in [0, T]\,, \\ &f\,(0, \tau) = f_0, \end{aligned} \tag{1}$$

where we assume $\nu\,(t, \tau) > 0$.[1] Following Heath et al. (1992), from the forward interest rate we can derive the behaviour of the spot interest rate.

Proposition 1 *If $f(t, \tau)$ satisfies Equation (**??**), then the short rate satisfies*

$$dr(t) = \mathrm{a}(t)dt + \mathrm{b}(t)dW^0(t), \tag{2}$$

where

$$\begin{cases} \mathrm{a}(t) = \frac{\partial}{\partial\tau} f(t, \tau)|_{\tau=t} + \alpha\,(t, t)\,, \\ \mathrm{b}\,(t) = \nu\,(t, t)\,. \end{cases}$$

We assume that the fund manager can invest in three assets: a riskless asset, a bond, and a stock.

[1]We assume a strictly positive diffusion term for the forward interest rate, since it leads to a strictly negative volatility term in the bond's dynamics. In fact, when the interest rate increases, then the bond value decreases. Thus, the two diffusion terms must have opposite sign.

The price process $X^0(t,r)$ of the riskless asset is given by

$$dX^0(t,r) = X^0(t,r)r(t)dt, \qquad (3)$$
$$X^0(0) = 1,$$

where the dynamics of $r(t)$, under the real probability measure $\mathbb{P}$, is defined in Equation (**??**). The natural interpretation of the riskless asset is view it as the value of a bank account, where the instantaneous interest rate is given by $r(t)$ and there is no default risk.

The second investment opportunity we consider is a stock. For the sake of simplicity, we introduce in our model only one stock, which can be interpreted as an index of the stock market. Nevertheless, if we allow for a complete market with a finite number of stocks, no further difficulties are added to the model because the only source of troubles is the market incompleteness. Let $X^1(t,r)$ denote the price process of the stock. The dynamics of $X^1(t,r)$ is given by

$$dX^1(t,r) = X^1(t,r)\left[\mu_1(t,r)dt + \sigma_{1,0}(t,r)dW^0(t) + \sigma_{1,1}(t,r)dW^1(t)\right], \qquad (4)$$
$$X^1(0) = X_0^1,$$

where $\sigma_{0,1} \neq 0$ and $\sigma_{1,1} \neq 0$.

The third asset we introduce is a bond rolling over zero coupon bonds with maturity τ, where $\tau \in [0,T]$.

Given the forward interest rate (**??**), we assume that there exists a market for zero coupon bonds for every value of τ. It is known that a zero coupon bond with maturity τ, called τ-bond, is a contract which guarantees the holder 1 monetary unit (face value) to be paid on the maturity τ. We denote by $X^B(t,\tau)$ the price at time $t \in [0,\tau[$ of a zero coupon bond with maturity τ. Then, we have a market with an infinite number of bonds, where each bond is regarded as a derivative of the underlying riskless asset. Thus, each bond is characterized by the same market price of risk (see for example Björk, 1998). Now, when the market has specified the dynamics of a basic bond price process, say with maturity τ, the market has also indirectly specified the price of risk which is the same for each bond, as we have already noted. Then, the basic τ-bond and the forward interest rate fully determine the price of all bonds. Actually, assuming the existence of an infinite number of zero coupon bonds is quite unrealistic. However, since the forward rate dynamics has only one source of randomness, we only need one zero coupon bond to replicate the other ones. Following Heath et al. (1992), the bond dynamics is defined as follows:

Proposition 2 *If $f(t,\tau)$ satisfies Equation (**??**), then $X^B(t,\tau)$ satisfies*

$$\frac{dX^B(t,\tau)}{X^B(t,\tau)} = \mu_B(t,\tau)dt + \sigma_B(t,\tau)dW^0(t), \qquad (5)$$

where

$$\begin{cases} \mu_B(t,\tau) = r(t) - \int_t^\tau \alpha(t,s)\,ds + \frac{1}{2}\left(\int_t^\tau \nu(t,s)\,ds\right)^2, \\ \sigma_B(t,\tau) = -\int_t^\tau \nu(t,s)\,ds. \end{cases} \qquad (6)$$

In order to ease the following notation, we characterize the two risky assets, the stock and the bond, by the vector $X(t,r) = (X^1(t,r), X^B(t,\tau,r))'$. Then, we have

$$dX(t,r) = I_X(t,r) \ \left[\mu_X(t,r)dt + \Sigma(t,r)dW(t)\right], \tag{7}$$

where

$$I_X(t,r) \equiv \begin{bmatrix} X^1(t,r) & 0 \\ 0 & X^B(t,\tau,r) \end{bmatrix},$$

$$\mu(t,r) \equiv (\mu_1(t,r), \mu_B(t,\tau,r))',$$

and

$$\Sigma(t,r) \equiv \begin{bmatrix} \sigma_{0,1}(t,r) & \sigma_{1,1}(t,r) \\ \sigma_B(t,\tau,r) & 0 \end{bmatrix}.$$

As $\sigma_{1,1}(t,r) \neq 0$ and $\sigma_B(t,\tau,r) \neq 0$, it follows that

$$\det \Sigma(t,r) = -\sigma_{1,1}(t,r)\sigma_B(t,\tau,r) \neq 0,$$

thus, consistently with the assumption of complete market, the diffusion matrix $\Sigma(t,r)$ is invertible.

3 The Defined-Contribution Process

The introduction in the optimal portfolio problem of non-capital income causes several computational difficulties, although the underlying methodological approach is the same as for the no-wage income case. In general, when we introduce "level" background risks, which affect directly the level of wealth (e.g. salary), the solution of the partial differential equation (PDE) characterizing the stochastic optimal control problem becomes harder and harder to compute. However, since our goal is to analyse the optimal portfolio strategies for a DC pension fund during the accumulation phase, we cannot overlook the leading role of the salary process.

Since Merton (1971), in his classical dynamic optimization model, examines the effects of introducing a deterministic wage income in the consumption-portfolio problem. In the more recent literature, Boulier et al. (1999), and Deelstra et al. (2001) provides some models for DC pension fund in continuous time involving deterministic salaries. In particular, Boulier et al. (1999) strongly support the real need for a downside protection of contributors who are more directly exposed to the financial risk borne by the pension fund. Accordingly, they introduce a guarantee on the benefits. Deelstra et al. (2001) also allow for a minimum guarantee in order to minimize the randomness of the retirement account, but they describe the flow of contributions through a non-negative, progressive measurable and square-integrable process. Blake et al. (2000) consider a model for DC pension fund where salaries are modeled through a stochastic process including a non-hedgeable component. Then, they focus on the replacement ratio as the central quantity of interest. Haberman and Vigna (2001) provide a model for DC pension fund in discrete-time with a fixed contribution rate. They study both the "investment risk", that is the risk of incurring a poor investment performance during the accumulation phase of the fund, and the "annuity risk", that is the risk of purchasing an annuity at retirement in a particular negative economic scenario involving a low conversion rate. The problem of optimal portfolio choice for a long-term investor in presence of a wage income is treated also by El Karoui and Jeanblanc-Picqué (1998), Campbell and Viceira (2001) and Franke, Peterson, and Stapleton (2001). Under a complete market with a constant interest rate, El Karoui and Jeanblanc-Picqué (1998) present the solution of a portfolio optimization problem for an economic agent endowed with a stochastic insurable stream of labor income. While they assume that the income process does not involve a new source of uncertainty, in the present work we allow for a non-hadgeable salary risk. Campbell and Viceira (2001) focus on some aspects of labor income risk in discrete-time. In particular, they look at individual's labor income as a dividend on the individual's implicit holding of human wealth. Franke et al. (2001) analyse the impact of labor income uncertainty on portfolio choice. They show how the portfolio startegy of an investor changes when his labor income uncertainty is resolved early or late in life. While they add the labor income to the terminal value of the portfolio investments, in the present work the income process enters directly in the wealth dynamics at each time $t \in [0, T]$.

This paper is principally related to the work of Blake et al. (2000). Indeed, we characterize the salary process through a stochastic differential equation. Accordingly, we show how the optimal portfolio choices depend directly on the uncertainty involved by salary. The introduction of stochastic salaries, instead of deterministic, allows us to consider the effects due to the labor income uncertainty, and in particular to its resolution over time.

The dynamic evolution of salaries is given by

$$\frac{dS(t,r)}{S(t,r)} = \mu_S(t,r)dt + \sigma_{S,0}(t,r)dW^0(t) + \sigma_{S,1}(t,r)dW^1(t) + \sigma_S(t)dW^S(t), \tag{8}$$

where $W^S(t)$ is a one-dimensional standard Brownian motion independent of $W^0(t)$ and $W^1(t)$. As Blake et al. (2000) point out, the assumption of a time-dependent drift term $\mu_S(t,r)$ allows us to incorporate possible age-dependent salary growth. At this regard, it is well known that salaries grow faster at younger ages. This empirical evidence suggests a decreasing function of time for the drift term $\mu_S(t,r)$. The diffusion terms $\sigma_{S,0}(t,r)$ and $\sigma_{S,1}(t,r)$ allow us to model any link between salary growth and returns on bond and stock markets. We note again that the unique stock we have introduced in our simple model can be always interpreted as the index of the stock market. According to the stochastic salary process modeled by Blake et al. (2000), the term $\sigma_S(t)dW^S(t)$ allows us to incorporate non-hedgeable salary risks, that is risks associated properly to the labor market and not to the financial market.

Now, we assume that each employee puts a constant proportion γ of his salary into the personal pension fund. Then, the defined contribution process is characterized as follows

$$C(t,r) = \gamma S(t,r),$$

and

$$\frac{dC(t,r)}{C(t,r)} = \frac{dS(t,r)}{S(t,r)}.$$

Finally, the defined contribution process characterizing the pension fund is given by

$$\frac{dC(t,r)}{C(t,r)} = \mu_S(t,r)dt + \sigma'_{S,X}(t,r)dW(t) + \sigma_S(t)dW^S(t), \tag{9}$$

where $\sigma_{S,X}(t,r) = (\sigma_{S,0}(t,r), \sigma_{S,1}(t,r))'$.

4 The Pension Fund

The purpose of this section is to derive the budget equation characterizing the pension fund dynamics. First, let us summarize the whole market structure:

$$\begin{cases} dr(t) = \alpha(t,r)dt + \nu(t,r)dW^0(t) \\ \\ dX^0(t,r) = X^0(t,r)r(t)dt \\ \\ dX(t,r) = I_X(t,r)\ \left[\mu(t,r)dt + \Sigma(t,r)dW(t)\right] \end{cases}$$

to which we have to add the defined contribution process described in Equation (9).

Let $\theta_X(t,r) = (\theta_1(t,r), \theta_B(t,r))'$ and $\theta_0(t,r)$ denote the number of shares invested respectively in the two risky assets, the stock and the bond, and in the riskless asset. The accumulated wealth at any time $t \in [0,T]$ is given by

$$F(t,r) = \theta'_X(t,r)X(t,r) + \theta_0(t,r)X^0(t,r) + C(t,r). \tag{10}$$

Then, we can write the number of shares invested in the riskless asset at time t as follows

$$\theta_0(t,r) = \frac{F(t,r) - \theta'_X(t,r)X(t,r) - C(t,r)}{X^0(t,r)}.$$

Substituting for $\theta_0(t,r)$ into Equation (10) and differentiating, we obtain that

$$\begin{aligned} dF(t,r) = {} & \theta_X(t,r)dX(t,r) + \\ & + \left[F(t,r) - \theta'_X(t,r)X(t,r) - C(t,r)\right]\frac{dX^0(t,r)}{X^0(t,r)} + \\ & + dC(t,r). \end{aligned} \tag{11}$$

Now, it is advantageous to write the pension fund dynamics in terms of fraction of wealth instead of number of shares invested in each assets. Let $x(t,r) = (x_1(t,r), x_B(t,r))'$ be the vector of the fraction of wealth invested in the two risky assets at any time $t \in [0,T]$, where

$$x_1(t,r) \equiv \frac{\theta_1(t,r)X^1(t,r)}{F(t,r)},$$

$$x_B(t,r) \equiv \frac{\theta_B(t,r)X^B(t,\tau,r)}{F(t,r)}.$$

Thereafter, we can write the pension fund dynamics (11) as follows

$$\frac{dF(t,r)}{F(t,r)} = [1 - x(t,r)\mathbf{1} - c(t,r)] \frac{dX^0(t,r)}{X^0(t,r)} + + x(t,r) I_X^{-1}(t,r) dX(t,r) + + c(t,r) \frac{dC(t,r)}{C(t,r)}, \tag{12}$$

where $c(t,r) = \frac{C(t,r)}{F(t,r)}$ indicates the fraction of wealth given by wage income at time $t \in [0,T]$, and $\mathbf{1} = (1,1)^{'}$. Finally, substituting for $X^0(t,r)$, $X(t,r)$ and $C(t,r)$ from Equations (3),(4) and (5) into Equation (12), the wealth process becomes

$$\frac{dF(t,r)}{F(t,r)} = \left\{ r(t) + x^{'}(t,r) \left[\mu(t,r) - r(t)\mathbf{1} \right] + c(t,r) \left[\mu_S(t,r) - r(t) \right] \right\} dt + + \left(x^{'}(t,r)\Sigma(t,r) + c(t,r)\sigma_{S,X}^{'}(t,r) \right) dW(t) + + c(t,r)\sigma_S(t) dW^S(t).$$

For the sake of simplicity, in the following sections we will not indicate the functional dependences, unless it is necessary.

5 The Optimal Asset Allocation Problem

The goal of the fund manager is to choose a portfolio strategy in order to maximize the expected value of a terminal utility $K(F(T))$. We assume that the terminal utility K is an increasing and concave function in F. Then, we may formally state the stochastic optimal control problem as follows

$$\begin{cases} \underset{x}{Max} \quad \mathbb{E}_0 \left[K(F(T)) \right] \\ d \begin{bmatrix} r \\ F \end{bmatrix} = m dt + M \begin{bmatrix} dW \\ dW^S \end{bmatrix} \\ F(0) = F_0; \quad r(0) = r_0 \end{cases} \tag{13}$$

where

$$\underset{2\times 1}{m} \equiv \begin{bmatrix} \alpha \\ F \left[r + x^{'}(\mu_X - r\mathbf{1}) + c(\mu_S - r) \right] \end{bmatrix},$$

$$\underset{2\times 3}{M} \equiv \begin{bmatrix} \delta^{'} & 0 \\ F \left(x^{'}\Sigma_X + c\sigma_{S,X}^{'} \right) & Fc\sigma_S \end{bmatrix},$$

$$\delta^{'} \equiv [\nu \quad 0].$$

The scalar variables F and r represent the two state variables, while the elements of x represent the two control variables.

The methodology used to solve this optimal control problem is the stochastic dynamic programming. By the theory (e.g. Björk (1998)), we know that the original optimal control problem is equivalent to the problem of finding a solution to a suitable partial differential equation (PDE), known as the Hamilton-Jacobi-Bellman (HJB) equation. Under our assumption, the HJB equation provides a very nice solution to the optimal control problem in which we are considering only Markov processes. We will not describe rigorously the whole theoretical structure of this approach, but we will limit our analysis to the basic steps necessary to specify the HJB equation which characterizes our optimal control problem.

Let $J(t; F_0, r_0)$ denote the value function of the optimal control problem (13), it follows that

$$J(t; F_0, r_0) = \mathbb{E}_t \left[K(F(T)) \mid F(0) = F_0, r(0) = r_0 \right],$$

where $\mathbb{E}_t$ stands for $\mathbb{E}\left(\cdot | \mathcal{F}(t)\right)$.

The Hamiltonian corresponding to (13) results to be

$$\mathcal{H} = J_r \alpha + J_F F \left[r + x^{'} (\mu - r\mathbf{1}) + c(\mu_S - r) \right] + \frac{1}{2} tr \left\{ M M^{'} \left[\begin{array}{cc} J_{rr} & J_{rF} \\ J_{Fr} & J_{FF} \end{array} \right] \right\},$$

where we denote $J_r \equiv \frac{\partial J}{\partial r}$, $J_F \equiv \frac{\partial J}{\partial F}$, $J_{rr} \equiv \frac{\partial^2 J}{\partial r^2}$, $J_{FF} \equiv \frac{\partial^2 J}{\partial F^2}$ and $J_{rF} = J_{Fr} \equiv \frac{\partial^2 J}{\partial r \partial F}$.

By working out the Hamiltonian, we obtain that

$$\begin{aligned} \mathcal{H} = & J_r \alpha + J_F F \left[r + x^{'} (\mu - r\mathbf{1}) + c(\mu_S - r) \right] + \\ & + \frac{1}{2} J_{rr} \delta^{'} \delta + J_{rF} F \left(x^{'} \Sigma + c \sigma_{S,X}^{'} \right) \delta + \\ & + \frac{1}{2} J_{FF} F^2 \left[\left(x^{'} \Sigma + c \sigma_{S,X}^{'} \right) \left(x^{'} \Sigma + c \sigma_{S,X}^{'} \right)^{'} + c^2 \sigma_S^2 \right]. \end{aligned} \tag{14}$$

The first order condition (FOC) give us the following linear system of two equations and two unknowns:

$$\frac{\partial \mathcal{H}}{\partial x} = J_F F (\mu - r\mathbf{1}) + J_{Fr} F \Sigma \delta + J_{FF} F^2 \left(\Sigma \Sigma^{'} x + c \Sigma \sigma_{S,X} \right) = 0. \tag{15}$$

We note that J_{FF} must be strictly negative. Indeed, the second order condition (SOC) holds if the corresponding Hessian matrix H is negative definite, where

$$H = F^2 J_{FF} \Sigma \Sigma^{'}.$$

As $\Sigma \Sigma^{'}$ is a variance-covariance matrix, it is a positive definite matrix. Then, H is a negative definite matrix if and only if $J_{FF} < 0$. Under our assumptions, we can easily show that J_{FF} is effectively strictly negative. In fact, as the terminal utility K is concave in F, the value function J results to be a strictly concave function in F (see for example Stokey and Lucas (1989)). Moreover, the completeness of the market implies that the matrix $\Sigma \Sigma^{'}$ is invertible.

Let $x^*(t) = (x_1^*(t), x_B^*(t))^{'}$ denote the vector of optimal fractions of wealth invested in the stock and in the bond. From Equation (15), we obtain that

$$x^* = -\frac{J_F}{F J_{FF}} \left(\Sigma \Sigma^{'} \right)^{-1} (\mu - r\mathbf{1}) - \frac{J_{Fr}}{F J_{FF}} \left(\Sigma^{'} \right)^{-1} \delta - c \left(\Sigma^{'} \right)^{-1} \sigma_{S,X}, \tag{16}$$

where $J(t, F, r)$ solves the following PDE equation, called the Hamilton-Jacobi-Bellman equation:

$$\begin{cases} J_t + \mathcal{H}^* = 0 \\ J(T, F(T), r(T)) = K(F(T)) \end{cases} \tag{17}$$

where $J_t \equiv \frac{\partial J}{\partial t}$, and $\mathcal{H}^*$ denotes the value of the Hamiltonian with respect to the optimal proportions x^*. We note that the solution of this PDE is just the value function corresponding to our optimal control problem. The hard work of stochastic dynamic programming consists just in solving the highly nonlinear PDE involved by the optimal control problem. There is no general analytical method to solve a parabolic PDE. In this case, the strategy that is usually adopted is the following: first, we try to guess an a priori suitable parametrized form for the solution to the PDE, then we use the PDE itself in order to specify the parameters. However, this analytical procedure is hardly ever successful. Thus, we must often try for a numerical approximation.

6 The Optimal Portfolio

In this section, we will analyse some structural properties of the optimal portfolio $x^*(t)$. In particular, we show that the optimal portfolio can be interpreted as the sum of three different components in the following way:

$$x^*(t, r) = \phi_1(t, r) p_1(t, r) + \phi_2(t, r) p_2(t, r) + p_3(t, r).$$

Comparing this with Equation (16), we find that it is optimal to invest in a suitable combination of three portfolios:

1. a speculative portfolio $\phi_1 p_1$ proportional to the market price of risk corresponding to the two risky assets through the relative risk aversion index, and defined as follows

   $$\phi_1 p_1 \equiv -\frac{J_F}{F J_{FF}} \left(\Sigma \Sigma'\right)^{-1} (\mu - r\mathbf{1}),$$

   where

   $$\phi_1 \equiv -\frac{J_F}{F J_{FF}}$$

   represents the only part of the portfolio depending on individual preferences;

2. an hedging portfolio $\phi_2 p_2$ proportional to the diffusion term of the interest rate through the cross derivative of the value function J, and defined as follows

   $$\phi_2 p_2 \equiv -\frac{J_{Fr}}{F J_{FF}} \left(\Sigma'\right)^{-1} \delta,$$

   where

   $$\phi_2 \equiv -\frac{J_{Fr}}{F J_{FF}}$$

   represents the only part of the portfolio depending on individual preferences;

3. a preference-free hedging component p_3 proportional to the volatilities of the salary process corresponding to the risk involved by the interest rate and by the stock, and defined as follows:

$$p_3 \equiv -c \left(\Sigma'\right)^{-1} \sigma_{S,X}.$$

We note that p_1 and p_2 are two components depending only on the financial market structure. Accordingly, the investment policies corresponding to p_1 and p_2 do not require the knowledge of the preferences or the endowments of the fund's shareholders. It follows that p_1 and p_2 are the same for all participants. On the other hand, ϕ_1 and ϕ_2 depend directly on the individual preferences. The interpretation of this result in terms of pension fund management is that the optimal portfolio is set up by two purely financial components, p_1 and p_2, common to every shareholder, and which must be adjusted on the basis of the individual preferences through ϕ_1 and ϕ_2.

Moreover, we see that the speculative component of the optimal portfolio ($\phi_1 p_1$) does not depend directly on the volatility of the interest rate. Even though p_1 is independent of ν, the diffusion term of interest rate could affect ϕ_1 through the value function J. In general, we cannot rule out this dependence. This is possible only when the value function meets some suitable properties, in particular when the value function is separable in wealth by product. We will discuss further about the separability in wealth and its consequences in the next section.

Let us analyse the second preference-free component (p_2), it results that

$$p_2 = \begin{bmatrix} 0 \\ \frac{1}{\sigma_B}\nu \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \frac{\nu(t, r(t))}{\int_t^\tau \nu(s, r(s))ds}.$$

We can see that the stock does not contribute at all to the hedging component, while the proportion invested in the bond depends only on the volatility of the interest rate. Accordingly, when the interest rate volatility increases, the optimal proportions invested in the bond increases, while the investment in the riskless asset decreases by the same proportion.

Finally, let us analyse the third component of the optimal portfolio.

$$p_3 = -c \begin{bmatrix} \frac{1}{\sigma_{1,1}}\sigma_{S,1} \\ \frac{1}{\sigma_B}\sigma_{S,0} - \frac{\sigma_{0,1}}{\sigma_B \sigma_{1,1}}\sigma_{S,1} \end{bmatrix}.$$

As noted, p_3 represents an hedge component of the optimal portfolio x^*, and it is preference-free. It follows that p_3 is completely defined, without specifying the functional form of the value function J. Moreover, p_3 is independent of the non-hedgeable salary risk. We see that an increase in the volatility of salaries with respect to the risk of the stock market ($\sigma_{S,1}$) draws immediately in a fall, both in the proportion invested in the stock, and in the proportion invested in the bond (we recall that $\sigma_B < 0$). Therefore, an increase in $\sigma_{S,1}$ involves an increase in the proportion invested in the riskless asset. On the other hand, a rise in the volatility of salaries with respect to the risk of the interest rate ($\sigma_{S,0}$) does not affect the investment in the stock. In this case, we have a rise in the optimal proportion invested in the bond, and a corresponding decrease in the investment in the riskless asset.

It is important to observe that if we assume deterministic salaries, then the preference-free component (p_3) disappears because we have $\sigma_{S,0} = \sigma_{S,1} = \sigma_S = 0$. In fact, in the work by Boulier et al. (2001), where deterministic salaries are assumed, the optimal portfolio is characterized by only two components.

7 The Value Function: An Explicit Solution

In the previous section, we have highlight some interesting properties of the optimal portfolio $x^*(t,r)$, without specifying the value function J. In order to state precisely the value function J, we should solve the HJB equation. In general, there is no analytical method to solve an highly nonlinear PDE. As there is a very little hope of finding an analytic solution to the original problem, it is common practice, to some extent, to rig it in order to obtain a problem analytically solvable. Consistently with this practice, in this section, we study the functional form of the HJB Equation (17), we rig it in a suitable way, and finally we provide a particular solution to the optimal control problem.

Let assume that the salary risk is completely hedgeable, that is $\sigma_S(t) = 0$. We substitute the expression for $x^*(t,r)$ into Equation (14), giving us the Hamiltonian

$$\begin{aligned}\mathcal{H}^* &= J_r\alpha + J_F F r + J_F C\left[\mu_S - r - \sigma_{S,X}^{'}\Sigma^{-1}(\mu - r\mathbf{1})\right] + \\ &+ \frac{1}{2}J_{rr}\delta^{'}\delta - \frac{1}{2}\frac{(J_{Fr})^2}{J_{FF}}\delta^{'}\delta - \frac{J_{Fr}J_F}{J_{FF}}\delta^{'}\Sigma^{-1}(\mu - r\mathbf{1}) + \\ &- \frac{1}{2}\frac{(J_F)^2}{J_{FF}}(\mu - r\mathbf{1})^{'}\left(\Sigma\Sigma^{'}\right)^{-1}(\mu - r\mathbf{1}).\end{aligned}$$

Thus, the PDE equation we have to solve is the following:

$$\begin{cases} J_t + \mathcal{H}^* = 0 \\ J(T, F(T), r(T)) = K(F(T)). \end{cases} \tag{18}$$

The standard approach to solve this kind of PDE is to try for a separability condition. In the financial literature, since Merton (1969,1971), the condition of separability in wealth by product represents a common assumption in the attempt to solve explicitly optimal portfolio problems. Accordingly, we assume that our value function is given by the product of two terms: an increasing and concave function of the wealth F, and an exponential function depending on time and the interest rate r. Thus, the value function J can be written as follows

$$J(t,r,F) = U(F)e^{h(t,r)}.$$

Substituting for this expression into the HJB Equation (18), we obtain that

$$\begin{cases} J(t,r,F)h_t + \mathcal{H}^* = 0 \\ h(T, r(T)) = 0, \end{cases}$$

where

$$\begin{aligned}\mathcal{H}^* = & h_r\alpha + \frac{U_F}{U}C\left[\mu_S - r - \sigma_{S,X}^{'}\Sigma^{-1}(\mu - r\mathbf{1})\right] + \frac{1}{2}h_{rr}\delta^{'}\delta + \\ & -\frac{1}{2}\frac{(U_F)^2 h_r^2}{U_{FF}U}\delta^{'}\delta - \frac{(U_F)^2 h_r}{U_{FF}U}\delta^{'}\Sigma^{-1}(\mu - r\mathbf{1}) + \\ & -\frac{1}{2}\frac{(U_F)^2}{U_{FF}U}(\mu - r\mathbf{1})^{'}\left(\Sigma\Sigma^{'}\right)^{-1}(\mu - r\mathbf{1}) + \frac{U_F F}{U}r.\end{aligned}$$

In order to have our model consistent with the assumption of separability in wealth, we must impose that $\frac{U_F}{U}F$, $\frac{U_F}{U}$, and $\frac{(U_F)^2}{U_{FF}U}$ are constant with respect to the wealth F. How we expected, there is no utility function U such that this is true. How we have already highlighted, in order to provide a particular solution to the HJB, it is usual to rig the optimal control problem in some suitable way. In this case, we force the drift of the salary process to have a particular, but consistent, functional form.

Let assume that

$$\mu_S = r + \sigma_{S,X}^{'}\Sigma^{-1}(\mu - r\mathbf{1}).$$

This assumption can be supported by the standard macroeconomic theory. Indeed, in a standard stochastic monetary economy, it is possible to predict, around the steady state, a strong correlation between the interest rate and the inflation rate. As the real wages are constant along the steady state deterministic equilibrium, it follows that the nominal wages are also strongly correlated to the interest rate (see for example Stokey and Lucas (1987)). Accordingly, it seems consistent to approximate the drift of the salary process (μ_S) through the interest rate plus an adjustment term depending on the volatility of the salary and the Sharpe ratio.

Thus, the HJB equation becomes

$$\begin{aligned}0 = & h_t + h_r\alpha + \frac{1}{2}h_{rr}\delta^{'}\delta + \\ & -\frac{1}{2}\frac{(U_F)^2 h_r^2}{U_{FF}U}\delta^{'}\delta - \frac{(U_F)^2 h_r}{U_{FF}U}\delta^{'}\Sigma^{-1}(\mu - r\mathbf{1}) + \\ & -\frac{1}{2}\frac{(U_F)^2}{U_{FF}U}(\mu - r\mathbf{1})^{'}\left(\Sigma\Sigma^{'}\right)^{-1}(\mu - r\mathbf{1}) + \frac{U_F F}{U}r.\end{aligned} \tag{19}$$

According to the previous analysis, we must have $\frac{U_F}{U}F$, and $\frac{(U_F)^2}{U_{FF}U}$ constant with respect to the wealth F. It is easy to check that these two differential equations are solved by a power utility function. Thus, we have

$$U(F) = \frac{F^\lambda}{\lambda},$$

where $\lambda \neq 0$, and $\lambda < 1$ in order to have a concave value function. We note that the constant λ interprets the relative risk aversion index of our utility function.

Substituting for this expression into Equation (19), the HJB equation becomes

$$
\begin{aligned}
0 &= h_t + h_r \left[\alpha - \frac{\lambda}{\lambda - 1} \delta' \Sigma^{-1} (\mu - r\mathbf{1}) \right] + \\
&+ \frac{1}{2} h_{rr} \delta' \delta - \frac{1}{2} \frac{\lambda}{\lambda - 1} h_r^2 \delta' \delta + \\
&- \frac{1}{2} \frac{\lambda}{\lambda - 1} (\mu - r\mathbf{1})' \left(\Sigma \Sigma' \right)^{-1} (\mu - r\mathbf{1}) + \lambda r.
\end{aligned}
$$

Now, following the methodology used by Menoncin (2001), we look for a solution of the PDE such that $(h_r)^2 = h_{rr}$. Accordingly, after solving this differential equation, we obtain that $h(t, r)$ must have the following functional form:

$$
h(t, r) = A(t) - \ln \left[D(t) + G(t) r \right], \tag{20}
$$

where $A(t), D(t), G(t) \in \mathbb{R}$, and $[D(t) + G(t)r] > 0$.

Thus, the HJB equations can be written as follows

$$
\begin{cases} h_t + z(t, r) h_r + p(t, r) h_{rr} + q(t, r) = 0 \\ h(T, r(T)) = 0 \end{cases} \tag{21}
$$

where

$$
\begin{aligned}
z(t, r) &= \left[\alpha - \frac{\lambda}{\lambda - 1} \delta' \Sigma^{-1} (\mu - r\mathbf{1}) \right], \\
p(t, r) &= -\frac{1}{2 (\lambda - 1)} \delta' \delta, \\
q(t, r) &= -\frac{1}{2} \frac{\lambda}{\lambda - 1} (\mu - r\mathbf{1})' \left(\Sigma \Sigma' \right)^{-1} (\mu - r\mathbf{1}) + \lambda r.
\end{aligned}
$$

This PDE can be solved by applying the Feynman-Kač representation theorem (see for example Øksendal (2000)). Accordingly, we find the following solution to the HJB Equation (21):

$$
h(t, r) = \int_t^T E_t \left[q(t, \widetilde{r}) \right] ds,
$$

where the dynamics of the variable $\widetilde{r}$ is given by

$$
\begin{aligned}
d\widetilde{r} &= z(t, \widetilde{r}) ds + \left(-\frac{1}{\lambda - 1} \right) \nu(t, \widetilde{r}) dW^0, \\
\widetilde{r} &= \widetilde{r}_0.
\end{aligned}
$$

We note that, under the Feynman-Kač representation theorem, the interest rate dynamics is different from the original dynamics defined in Equation (**??**). In fact, the interest rate giving the exact solution to our optimal control problem has the same diffusion term as the interest rate in Equation (**??**), while the drift term is different. Let us analyze the new drift term:

$$
z(t, \widetilde{r}) = \alpha - \frac{\lambda}{\lambda - 1} \frac{\mu_B - \widetilde{r}}{\sigma_B} \nu.
$$

We see that $z(t,\widetilde{r})$ is equal to the original drift term (α) diminished by the bond market price of risk multiplied by the interest rate diffusion term and the relative risk aversion index. Accordingly, we note that the modified interest rate solving the HJB equation follows an extended Vasicek model as defined by Hull and White (1990).

Finally, the optimal portfolio x^* can be written as follows

$$\begin{aligned} x^* = & -\frac{1}{\lambda-1}\left(\Sigma\Sigma'\right)^{-1}(\mu - r\mathbf{1})+ \\ & -\frac{1}{\lambda-1}\left(\Sigma'\right)^{-1}\delta\int_t^T \frac{\partial}{\partial r}E_t\left[q(t,\widetilde{r})\right]ds+ \\ & -c\left(\Sigma'\right)^{-1}\sigma_{S,X}. \end{aligned} \tag{22}$$

We note that the second preference-free component (p_2) is the only one component depending explicitly on the retirement time T.

As Menoncin (2001) highlights, the closed form solution for the portfolio allocation problem we have obtained in Equation (22) is the exact solution if and only if the function $h(t,r)$ can be written as in Equation (20). Nevertheless, if the function $h(t,r)$ does not meet this condition, Equation (22) is still valid as an approximation of the true result. Indeed, we can develop in Taylor series the function $h(t,r)$ around a given value of r (say $\hat{r}$) in order to obtain a polynomial in r approximating the exact solution.

8 Conclusion

In this work, we have studied the optimal portfolio problem for a defined contribution pension fund. After introducing a stochastic interest rate, we have assumed a market structure characterized by three assets: a riskless asset, a bond and a stock. Moreover, we have introduced a stochastic process for salaries, and developed the model according to the stochastic dynamic programming methodology.

We have shown that the optimal portfolio is formed by three components: a speculative component proportional to the market price of risk of the two risky assets through the relative risk aversion index, an hedging component proportional to the diffusion term of the interest rate, and a preference-free hedging component proportional to the volatilities of the salary process.

Finally, after specifying a suitable functional form for the drift term of the salary process, we have found a close form solution to the asset allocation problem.

We plan to extend this model to a stochastic process for inflation.

A The Cholesky Decomposition of the Correlation Matrix

Let $\begin{bmatrix} W_x(t) & W_y(t) \end{bmatrix}'$ denote a vector of two independent standard Wiener processes. Then, we have

$$cov\left[dW_x dW_y\right] = \mathbb{E}\left[dW_x dW_y\right] = 0,$$

and variance-covariance matrix $\Sigma = tI_{(2)}$, where $I_{(2)}$ denotes the identity matrix of dimension two. We can transform $\begin{bmatrix} W_x & W_y \end{bmatrix}'$ into a vector of two correlated Wiener processes $\begin{bmatrix} \widetilde{W_x} & \widetilde{W_y} \end{bmatrix}'$ with the same mean (i.e. zero mean), but with variance-covariance matrix

$$\widetilde{\Sigma} = \begin{bmatrix} \sigma_x^2 & \varphi\sigma_x\sigma_y \\ \varphi\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix},$$

by applying to the original vector of uncorrelated processes the Cholesky decomposition as follows

$$\begin{bmatrix} \widetilde{W_x} \\ \widetilde{W_y} \end{bmatrix} = C'_{\widetilde{\Sigma}} \begin{bmatrix} W_x \\ W_y \end{bmatrix},$$

where $C_{\widetilde{\Sigma}}$ is just the Cholesky decomposition of the matrix $\widetilde{\Sigma}$. The matrix $C_{\widetilde{\Sigma}}$ is an upper-triangular matrix such that $\widetilde{\Sigma} = C'_{\widetilde{\Sigma}} C_{\widetilde{\Sigma}}$. Finally, we have

$$\begin{bmatrix} \widetilde{W_x} \\ \widetilde{W_y} \end{bmatrix} = \begin{bmatrix} \sigma_x & \varphi\sigma_y \\ 0 & \sigma_y\sqrt{1-\varphi^2} \end{bmatrix}' \begin{bmatrix} W_x \\ W_y \end{bmatrix} = \begin{bmatrix} \sigma_x W_x \\ \sigma_y\varphi W_x + \sigma_y\sqrt{1-\varphi^2}W_y \end{bmatrix}.$$

In conclusion, the following general result holds: given a set of Wiener processes, correlated or uncorrelated, it can always be represented as a vector of Wiener processes with the same drift of the initial processes, and diffusion term equal to the transpose of the Cholesky matrix calculated with respect to the variance-covariance matrix of the initial processes.

References

[1] Blake, D., Cairns, A.J.G., and K. Dowd (1999)."PensionMetrics: stochastic pension plan design and value at risk during the accumulation phase". *BSI-Gamma Foundation, Working Paper Series 11.*

[2] Blake, D., Cairns, A.J.G., and K. Dowd (2000). "Optimal dynamic asset allocation for defined-contribution plans". *The Pension Institute*, London, Discussion Paper PI 2003.

[3] Björk, T. (1998). "Arbitrage theory in continuous time". *Oxford University Press*, New York.

[4] Boulier, J-F., S.-J. Huang, and G. Taillard (2001). "Optimal Management Under Stochastic Interest". *Insurance: Mathematics and Economics*, 28, 173-189.

[5] Campbell, J.Y., and L.M. Viceira (2001). "Strategic asset allocation: portfolio choice for long-term investors". *Oxford University Press.*

[6] Cox, J.C., Ingersoll, J., and Ross, S. (1985). "A theory of the term structure of interest rates". *Econometrica*, Vol. 53, pp. 385-408.

[7] Deelstra, G., Grasselli, M., and P-F. Koehl (2000). "Optimal investment strategies in a CIR framework". *Journal of Applied Probability*, Vol. 37, pp. 936-946.

[8] Deelstra, G., Grasselli, M., and P-F. Koehl (2001)."Optimal design of the guarantee for defined contribution funds". *Preprint.*

[9] Duffie, D. (1996). "Dynamic asset pricing theory". *Princeton*, Princeton.

[10] El Karoui, N., and M. Jeanblanc-Picqué (1998). "Optimization of consumption with labor income". *Finance and Stochastics*, Vol. 2, pp. 409-440.

[11] Franke, G., Peterson, S., and R.C. Stapleton (2001). "Intertemporal portfolio behaviour when labor income is uncertain". *SIRIF Conference, "Dynamic Portfolio Strategies",* Edinburgh, May 2001.

[12] Haberman, S., and E. Vigna (2001). "Optimal investment strategy for defined contribution pension schemes". *Insurance: Mathematics and Economics*, Vol. 28, pp. 233-262.

[13] Heath, D., Jarrow, R., and A.J. Morton (1992). "Bond Pricing and the term structure of interest rates. A new methodology for contingent claims valuation". *Econometrica*, 60, pp. 77-106.

[14] Hull, J., and A. White (1990). "Pricing Interest-Rate Derivative Securities". *The Review of Financial Studies*, 3, pp. 573-592.

[15] Karatzas, I., and S. Shreve (1991). "Brownian motion and stochastic calculus". *Springer*, New York.

[16] Lioui, A., and P. Poncet (2001). "On Optimal Portfolio Choice under Stochastic Interest Rates". *Journal of Economic Dynamic and Control*, 25, 1841-1865.

[17] Lucas, R. E., and N.L. Stokey (1987). "Money and interest in a cash-in-advance economy". *Econometrica*, Vol. 55, pp. 491-513.

[18] Lucas, R. E., and N.L. Stokey (1989). "Recursive methods in economic dynamics". *Harvard University Press*, Cambridge, Mass.

[19] Menoncin, F. (2001). "Optimal portfolio and background risk: an exact and an approximated solution". *14th Annual Australasian Finance and Banking Conference*, Sydney, December 2001.

[20] Merton, R.C. (1969) "Lifetime portfolio selection under uncertainty: the continuous-time case". *Review of Economics and Statistics*, Vol. 51, pp. 247-257.

[21] Merton, R.C. (1971) "Optimum consumption and portfolio rules in a continuous time model". *Journal of Economic Theory*, Vol. 3, pp. 373-413.

[22] Merton, R.C. (1990) "Continuous-time finance". *Blackwell*, Cambridge, Mass.

[23] Øksendal, B. (1998). "Stochastic differential equation". *Springer-Verlag*, Berlin.

[24] Vasicek, O.E. (1977). "An equilibrium characterization of the term structure". *Journal of Financial Economics*, Vol. 5, pp.177-188.