Existence, Uniqueness, and Stablitiy of Equilibrium in an Overlapping Generation Model with Monopolistic Competition and Free Entry and Exit of Firms

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22-10-2001

Abstract

In this paper we have analyzed existence, uniqueness and stability of a steady-state equilibrium in an overlapping generations model with monopolistic competition and free entry and exit of firms. We establish a strengthened Inada condition that is sufficient to exclude global contraction for any given set of well-behaved preferences. We also establish sufficient conditions for a non-trivial steady-state equilibrium to exist, and also sufficient conditions for its uniqueness and global stability. We show that the size of mark-up over marginal cost and the particular mix of fixed costs play a crucial role in these conditions and consequently on the dynamic behavior of the economy.

Keywords: Equilibrium, Existence, Monopolistic Competition, Overlapping Generations, Stability, Uniqueness.

Journal of Economic Literature: D43, D90, E13, O40.

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1 Introduction

In this work we analyze existence, uniqueness and stability of stationary equilibrium in an overlapping generation model à la Diamond (1965) with productive capital. Galor and Ryder (1989) analyzed this model, but here we drop the assumption of perfect competition assuming monopolistic competition, implying that ¯rms set a mark-up of price over their marginal costs. Dixit and Stiglizt's (1997) monopolistic competition model is today widely used in macroeconomics. For this reason we have chosen Dixit Stigliz's framework to introduce imperfect competition in an overlapping generation model. We also assume increasing returns to scale in production due to ¯xed costs and that the number of ¯rms is determined by a free entry condition so that pro¯ts are zero. The free entry assumption has been already examined in Chamberlin (1933) who argued that ¯rms will go in or go out of the market until pro¯ts become zero. We have considered three kinds of ¯xed costs with the intention of being comprehensive: ¯xed costs on output, capital and labor. We show that the equilibrium properties depend on the particular combination of these three kinds of ¯xed costs and the size of the mark-up. It should be noted that Galor and Ryder's model constitutes a limit case of our model: when the mark-up tends to one and all ¯xed costs are zero.

Several empirical works support the assumptions of our model. Hall (1986, 1988, 1990) and Morrison (1993) have reported both signi¯cant increasing returns and mark-ups of price over marginal costs in various U. S. industries. They also ¯nd that the economic pro¯ts are roughly zero on average, suggesting an industrial structure along the classic lines of monopolistic competition.[1]

The paper is organized as follows. In the next section, we describe our model. In Section 3, equilibrium is characterized. In section 4, possibility of global contraction is proved. In Section 5, a strengthen Inada condition avoiding global contraction is established. In Section 6, we establish su±cient conditions for nonexistence of non-trivial steady state equilibrium. In Section 7, su±cient conditions for existence, uniqueness and stability of a non-trivial steady state equilibrium are given. Finally, Section 8 concludes.

2 The Economy

It is a two-period OLG economy, in the line of Diamond (1965), with monopolistic competition. The speci¯cation of the monopolistic competitive follows Woodford and Rotemberg (1995). Generation t is a continuum of individuals in the interval $[0; N_t]$, where N_t grows at the rate n. There is a unique ¯nal good, which serves as

[1] Rotemberg and Woodford (1995) discuss the empirical evidence on the size of mark-up of price over marginal cost and increasing returns.

both consumption and investment goods. It is produced by a representative final firm and sold in a competitive market. Its production technology has constant returns to scale and it is defined over a continuum of intermediate goods in the interval $[0; I_t]$. Each intermediate good is produced by a monopolistic competitive firm. There is free entry and exit of intermediate goods firms, implying that I_t is endogenous. There are increasing returns in the production of intermediate goods. The production factors of intermediate goods firms are capital and labor.

The production function of the final good firm is:

$$Y_t = G_t(Q_t),$$

where Q_t is a function $[0; I_t] \rightarrow R_+$ specifying the amount $Y_{j;t} \geq 0$ of each type $j \in [0; I_t]$ of intermediate good purchased. We assume that the production function, G_t, is an increasing, concave, symmetric, and homogeneous of degree one function of the measure Q_t.[2] The production function varies over time, as the set of inputs changes. We also assume:

$$G_t(M_t) = I_t \quad \forall t; \tag{1}$$

Where M_t is the uniform measure. Assumption (1) is a normalization of G_t in each period.

The producer of each intermediate good set a price for it. Let be P_t a function $[0; I_t] \rightarrow R_+$ specifying the price $p_{j;t}$ of each type $j \in [0; I_t]$ of input purchased. The firm will distribute its purchase over the inputs so as to maximize its profits, $Q_t \in \operatorname{argmax}\{G_t(Q_t) - P_t Q_t\}$. Because G_t is homogeneous of degree one, it must be satisfied that $Q_t = Y_t D_t(P_t)$, where D_t is a homogeneous of degree zero function of the measure P_t. Furthermore, because G_t is symmetric, the component $D_{j;t}(P_t)$ of $D_t(P_t)$ indicating purchases of intermediate good j must depends only on the price, p_{jt}, charged for that intermediate good and the overall distribution of intermediate goods prices. We will be concerned only with symmetric equilibria. We will thus consider situations where all firms charges a price p_t while firm j charges $p_{j;t}$. Therefore, since D is homogeneous of degree zero, the demand for intermediate good j is given by:

$$Y_{j;t} = \frac{Y_t}{I_t} \Phi_t\left(\frac{p_{j;t}}{p_t}\right) \tag{2}$$

Since G_t is symmetric, assumption (1) implies that $\Phi_t(1) = 1$ for all t. We futhermore assume that

$$\Phi_t \text{ is differentiable at one, and } \Phi_t'(1) < -1 \text{ is independent of } t; \tag{3}$$

[2] As Rotemberg and Woodford (1995), by a symmetric function we mean a function whose value is unchanged if one exchanges the quantities purchased of any of the individual goods, so that the value of Y_t depends only upon the distribution of quantities purchased of each intermediate good, and not upon the identities of the intermediate goods purchased.

and for each t,

$$\forall \alpha \geq 0; \; \Delta_t(\alpha) + \alpha \Delta'_t(\alpha) \text{ is a monotonically decreasing function of } \alpha, \qquad (4)$$

where α is the relative price of intermediate good j. Assumption (3) means that the degree of susbtitutability between different intermediate goods, evaluated in the case of equal purchases of all intermediate goods, remains the same as additional intermediate goods are added, and the common elasticity of substitution is greater than one. Assumption (4) implies the existence of a downward-sloping marginal revenue curve for each producer of intermediate goods. The result of these assumptions is that at a symmetric equilibrium, firms face a time-invariant elasticity of demand.

All intermediate goods firms have the same production function given by

$$Y_{j,t} = F(K_{j,t} - \Theta, L_{j,t} - \Lambda) - \Phi, \qquad (5)$$

where F is a homogeneous of degree one function, $K_{j,t}$ is the capital stock of intermediate good firm j at time t, $L_{j,t}$ is employment in intermediate good firm j at time t, and Φ, Θ and Λ are no negative parameters denoting the fixed costs on output, capital and labor respectively. The depreciation rate of capital is constant and equal for all inputs firms, $0 \leq \delta \leq 1$.[3] The endowment of capital at time $t+1$, $K_{t+1} = K_{j,t+1} l_{t+1}$, is equal to the resources not consumed in the preceding period,

$$K_{t+1} = Y_t + (1 - \delta) K_t - C_t.$$

Let be $x_{j,t} = \frac{K_{j,t} - \Theta}{L_{j,t} - \Lambda}$ the ratio capital-employment, both net of fixed costs,of firm j at time t,[4] given that F is homogeneous of degree one, production of each intermediate good firm is,

$$Y_{j,t} = f(x_{j,t})(L_{j,t} - \Lambda) - \Phi.$$

We assume that the function f is C^2, positive, increasing, and strictly concave:

$$f(x) > 0, \quad f'(x) > 0 \text{ y } f''(x) < 0, \; \forall x > 0.$$

[3] We ignore produced materials as productive inputs. As Rotemberg and Woodford (1995) pointed out, equation (5) would represent the production function for total added value (the total product net of the value of materials inputs) of imperfectly competitive firms using produced materials as inputs if we assume a fixed-coefficient technology taking the form

$$G(K_{j,t}, L_{j,t}, M_{j,t}) = \min \left\{ \frac{F(K_{j,t} - \Theta, L_{j,t} - \Lambda) - \Phi}{1 - s_M}, \frac{M_{j,t}}{s_M} \right\} \qquad (p)$$

where $M_{j,t}$ denotes the materials inputs of firm j at time t, and $0 < s_M < 1$ corresponds to the share of materials costs in the value of gross output in a symmetric equilibrium.

[4] Thereafter, this ratio will be called ratio capital-employment.

Inada conditions are satisfied at the origin,

$$\lim_{x \to 0} f(x) = 0, \qquad \lim_{x \to 0} f'(x) = \infty, \tag{6}$$

and there is an upper bound to the ratio capital-employment $\bar{x}$, such that[5]

$$f(\bar{x}) = (1 + n)\bar{x}, \tag{7}$$

where $n \geq -1$ is the population growth rate.

By monopolistic competition we mean that each intermediate good firm j takes as given aggregate demand, Y_t, and the price charged by the other intermediate goods firms, p_t, and chooses its own price, $p_{j;t}$, taking into account the effect of price $p_{i;t}$ on its demand indicated by (2).At a symmetric equilibrium, the first order conditions for factor demands take the forms

$$\mu r_t = f'(x_{j;t}) \tag{8}$$

$$\mu \omega_t = f(x_{j;t}) - x_t f'(x_{j;t}), \tag{9}$$

where $\mu = [1 + \Delta'(1)^{-1}]^{-1}$ is the degree of market power, ω_t represents the wage at time t and r_t is the rental price of capital at time t.[6]

In each period t L_t individuals are born. Population grows exogenously to the constant rate $n \geq -1$. Therefore,

$$L_t = (1 + n) L_{t-1}.$$

Individuals are identical within as well as across time. Individuals live two periods. In the firs they work and earn the competitive market wage ω_t, and in the second they are retired. During the first period of their lifetimes individuals supply their unit-endowments of labor inelastically and allocate the resulting income, ω_t, between first period consumption, c_{1t}, and savings, s_t,

$$s_t = \omega_t - c_{1;t}.$$

savings earn the return r_{t+1} in the following period and enable the cohort to consume during retirement. Second period consumption is therefore

$$c_{2;t+1} = (1 + r_{t+1} - \delta)s_t.$$

[5] Alternatively, we may assume that $\lim_{x \to \infty} f'(x) = 0$, which togheter with (6) suffices to assure (7).

[6] I we assume that intermediate goods firms use produced materials as productive inputs and the production function of the intermediate goods firms is given by (p) then

$$\mu = \frac{1 - s_M}{1 - s_M + \Delta'(1)^{-1}};$$

higher than the degree of market power. In this case we need assume that $\Delta'(1) + 1 > s_M$.to guarantee that the optimization problem of intermediate goods firms has an interior maximum.

Individuals born at time t are characterized by their intertemporal utility function $u(c_{1,t}, c_{2,t+1})$ defined over non-negative consumption during the first and second period of their lives. the intertemporal utility function is C^2 and strictly quasi-concave on the interior of the consumption set $\mathbb{R}^2_+$. The utility function is assumed to be increasing in both variables:[7]

$$
\begin{aligned}
u_1(c_1, c_2) > 0 & \quad \text{para } (c_1, c_2) \gg 0 \\
u_2(c_1, c_2) > 0 & \quad \text{para } (c_1, c_2) \gg 0.
\end{aligned}
$$

Future consumption is a normal good,

$$u_1 u_{12} > u_2 u_{11} \text{para } (c_1, c_2) \gg 0,$$

and starvation is avoided in both periods,

$$
\begin{aligned}
\lim_{c_1 \to 0} u_1(c_1, c_2) = \infty & \quad \text{para } c_2 > 0 \\
\lim_{c_2 \to 0} u_2(c_1, c_2) = \infty & \quad \text{para } c_1 > 0.
\end{aligned}
\tag{10}
$$

Individuals are rationals. Then, they made their choices in the first period to maximize the intertemporal utility function,

$$s_t = s(\omega_t, \hat{r}_{t+1}) = \arg\max u[\omega_t - s_t, (1 - \delta + \hat{r}_{t+1})s_t],$$

where $\hat{r}_{t+1}$ is the anticipated return on next period's capital. We assume perfect foresight,

$$\hat{r}_{t+1} = r_{t+1}.$$

The following section establishes conditions under which a unique self-fulfilling expectation exists and is interior for every positive level of initial condition.

3 Characterization of Equilibrium

At a symmetric equilibrium labor market clears, employment in all intermediate goods firms is the same, $L_{j,t} = \frac{L_t}{I_t}$, and the ratio capital-employment is also equal in all intermediate goods firms, $x_{j,t} = x_t$. Hence, the aggregate production per capita of intermediate goods is given by

$$y_t = Y_{j,t} i_t = f(x_t)(1 - \Lambda i_t) - \Phi i_t, \tag{11}$$

where $i_t = \frac{I_t}{L_t}$ is the number of intermediate goods firms per capita. Aggregate capital at time $t + 1$ equal savings at time t, $L_t s(\omega_t, r_{t+1}) = K_{t+1}$, and then,

$$s(\omega_t, \rho_{t+1}) = k_{t+1}(1 + n), \tag{12}$$

[7] The following assumptions on the intertemporal utility are standar and identical to that in Galor and Ryder (1989).

where $k_{t+1} = K_{j,t+1} l_{t+1}$ is aggregate capital per capita at time $t+1$.There is free entry and exit of intermediate goods firms. The number of intermediate goods firms adjusts so that aggregate profits are zero,[8]

$$l_t \left(Y_{j,t} - \omega_t \frac{L_t}{l_t} - r_t \frac{K_t}{l_t} \right) = 0, \tag{13}$$

Substituting from (8), (9) and (11) into the zero profits condition, (13), yields:

$$l_t = \frac{(\mu - 1) f(x_t)}{(\mu f(x_t) - x_t f'(x_t))\Lambda + \mu\Phi + f'(x_t)\Theta} \equiv l(x_t). \tag{14}$$

From the assumption made on f follow that

$$l(x_t) > 0 \quad \forall x_t > 0$$

and

$$\lim_{x \to 0} l(x) = 0, \qquad \lim_{x \to \infty} l(x) = \frac{\mu - 1}{(\mu - \varepsilon_1)\Lambda},$$

where $0 \leq \varepsilon_1 = \lim_{x \to \infty} \frac{f'(x)x}{f(x)} < 1$. From the definition of x_t follows the following relation between l_t, k_t and x_t at a symmetric equilibrium,

$$k_t = x_t - l_t(x_t \Lambda - \Theta). \tag{15}$$

The following two equations characterize the equilibrium of the economy for all $t \geq 0$:

$$s\left[\mu^{-1}(f(x_t) - x_t f'(x_t)); \mu^{-1} f'(x_t)\right] = k_{t+1}(1+n), \tag{16}$$

$$k_t = x_t + \frac{(\mu - 1) f(x_t)(\Theta - \Lambda x_t)}{(\mu f(x_t) - x_t f'(x_t))\Lambda + \mu\Phi + f'(x_t)\Theta} \equiv k(x_t). \tag{17}$$

Equation (16) has been obtained from substituting of (8) and (9) into (12), and equation (17) follows from (14) and (15).). We should note that when $\mu = 1$ and $\Theta = 0$, $\Lambda = 0$ and $\Phi = 0$ then the number of inputs firms, l_t, is undetermined and $k_t = x_t$. This limit case is analyzed by Galor and Ryder (1989). From the properties of the production function f follow the two limit properties of function k,

$$\lim_{x \to 0} k(x) = 0, \qquad \lim_{x \to \infty} k(x) = \infty. \tag{18}$$

The strictly monotony of $k(x)$ is crucial for the existence of only one self-fulfilling expectations Lemma 1 establishes that at a symmetric equilibrium there is a one-to-one relation between aggregate capital and the ratio capital-employment both net of fixed costs.

Lemma 1 $k_t = k(x_t)$ is an strictly increasing function of x_t, $\forall x_t > 0$:

[8] Profits by firm are also zero.

Proof. k is an strictly increasing function of x if only if $\forall x > 0$,

$$k'(x) = 1 - \Lambda i(x) - i'(x)(x\Lambda - \Theta) > 0. \tag{19}$$

From the properties of f follows that $1 - \Lambda i(x) > 0$ $\forall x > 0$, since,

$$\Lambda i(x) = \frac{(\mu - 1) f(x) \Lambda}{(\mu f(x) - x f'(x))\Lambda + \mu\Phi + f'(x)\Theta} < \frac{(\mu - 1) f(x) \Lambda}{(\mu f(x) - x f'(x))\Lambda} < 1.$$

Differentiating $i(x)$, after a little of algebra we have that

$$i'(x) = i(x) A(x), \tag{20}$$

where

$$A(x) = \frac{f'(x)}{f(x)} - \frac{(\mu - 1) f'(x)\Lambda - f''(x)(x\Lambda - \Theta)}{(\mu f(x) - x f'(x))\Lambda + \mu\Phi + f'(x)\Theta}. \tag{21}$$

If $x\Lambda - \Theta < 0$ then $A(x) > 0$, since

$$\begin{aligned} A(x) &> \frac{f'(x)}{f(x)} - \frac{(\mu - 1) f'(x)\Lambda}{(\mu f(x) - x f'(x))\Lambda + \mu\Phi + f'(x)\Theta} > \\ &> \frac{f'(x)}{f(x)} - \frac{(\mu - 1) f'(x)}{(\mu f(x) - x f'(x))} > 0. \end{aligned}$$

and therefore $k'(x) > 0$. If $x\Lambda - \Theta > 0$ and $A(x) < 0$ then $k'(x) > 0$. Hence, a necessary condition for $k'(x) < 0$ is that $x\Lambda - \Theta > 0$ and $A(x) > 0$. But, we can show that in this case $k'(x) > 0$. From (19) and (20), if $x\Lambda - \Theta > 0$ and $A(x) > 0$, a sufficient condition for $k'(x) > 0$ is

$$\Lambda + A(x) x\Lambda < \frac{1}{i(x)}, \quad \forall x > 0. \tag{22}$$

Substituting from (14) and (21) into (22), after a little of algebra yields,

$$\frac{f''(x)}{f'(x)}(x\Lambda - \Theta) < \left(\frac{D(x)}{(\mu - 1)\varepsilon_x \Lambda} - 1\right) C(x), \tag{23}$$

where $C(x) = (1 - \varepsilon_x)\Lambda + \frac{\mu}{f(x)}\Phi + \frac{f'(x)}{f(x)}\Theta$, $D(x) = (\mu - \varepsilon_x)\Lambda + \frac{\mu}{f(x)}\Phi + \frac{f'(x)}{f(x)}\Theta$ and $\varepsilon_x = \frac{x f'(x)}{f(x)} < 1$ $\forall x > 0$ since f is strictly concave and $\lim_{x \to 0} f(x) = 0$. Since $f''(x) < 0$, $f'(x) > 0$ and $f(x) > 0$ $\forall x > 0$; the left hand side of inequality (23) is negative for all $x > \frac{\Theta}{\Lambda}$ and the right hand side is always positive. Hence inequality (23) is hold for all $x > \frac{\Theta}{\Lambda}$ and therefore $k'(x) > 0$ for all $x > 0$. ■

From Lemma 1 and limit properties (18) follow that $k_t = k(x_t) > 0$ for all $x_t > 0$. Lemma 1 establishes that $k(x)$ is a strictly increasing function of for all $x > 0$, then there exists k^{-1}, the inverse function of k, such that $x_t = k^{-1}(k_t)$ and

$i_t = i(k^{-1}(k_t))$. Thus, given k_t, a level of k_{t+1} that is a self-fulfilling expectation satisfies:

$$\frac{s(\mu^{-1}(f(k^{-1}(k_t)) - k^{-1}(k_t)\, f'(k^{-1}(k_t)); \mu^{-1} f'(k^{-1}(k_{t+1}))}{1+n} = k_{t+1} \qquad (24)$$

The following lemma establishes a sufficient condition for uniqueness of equilibrium. The condition is the same that in the case of perfect competition.

Lemma 2 Given $k_t > 0$ there exists a unique $k_{t+1} > 0$ that is a self-fulfilling expectation, if saving is a no decreasing function of the interest rate, that is, if

$$\frac{\partial s(w_t; r_{t+1})}{\partial r_{t+1}} \geq 0 \qquad \forall r_{t+1}.$$

Proof. Consider the following equation:

$$s\left(\omega_t; \mu^{-1} f'\left(k^{-1}(k_{t+1})\right)\right) = (1+n)\, k_{t+1}. \qquad (25)$$

Consider Figure 1, where each side of (25) is plotted as a function of k_{t+1}. Since $s(\omega_t; r_{t+1}) \leq \omega_t$ for all $(\omega_t; r_{t+1})$; it follows that $0 \leq \lim_{k_{t+1} \to \infty} s(\omega_t; \mu^{-1} f'(k^{-1}(k_{t+1}))) \leq \omega_t$. Thus, given $k_t > 0$ (and therefore given $\omega_t > 0$), there exists $k_{t+1} > 0$ which satisfies (25) if $\lim_{k_{t+1} \to 0} s(\omega_t; \mu^{-1} f'(k^{-1}(k_{t+1}))) > 0$. Therefore, given (10), a sufficient condition for the existence of $k_{t+1} > 0$ is

$$\frac{\partial s(\omega_t; r_{t+1})}{\partial r_{t+1}} \geq 0; \forall r_{t+1}.$$

Given that k is a strictly increasing function of x and the derivative of the right hand side of (25) is negative with respect to x_{t+1}, uniqueness is satisfied. ■

From Lemma 2, it follows that if savings are a no decreasing function of the return rate, then there exists Γ, such that $k_{t+1} = \Gamma(k_t)$, where Γ is a function from $\Re_+$ to $\Re_+$, with $\Gamma(0) = 0$, and

$$\frac{dk_{t+1}}{dk_t} = \Gamma'(k_t) = \frac{-s_w k^{-1}(k_t)\, f''(k^{-1}(k_t))\,(k^{-1})'(k_t)}{(1+n)\,\mu - s_r f''(k^{-1}(k_{t+1}))\,(k^{-1})'(k_{t+1})}.$$

given that k^{-1} is a homeomorphism of k_t then variables k_t and x_t have the same dynamic behavior and it is indifferent to define equilibria in terms of a sequence of k_t or in terms of a sequence of x_t. Thus, there exists a function Ω from $\Re_+$ to $\Re_+$, with $\Omega(0) = 0$, such that $x_{t+1} = \Omega(x_t) = (k^{-1} \circ \Gamma \circ k)(x_t)$, whose derivatives

$$\frac{dx_{t+1}}{dx_t} = \Omega'(x_t) = \left(k^{-1} \circ \Gamma \circ k\right)'(x_t) = \frac{-s_w x_t f''(x_t)}{\mu(1+n)\, k'(x_{t+1}) - s_r f''(x_{t+1})},$$

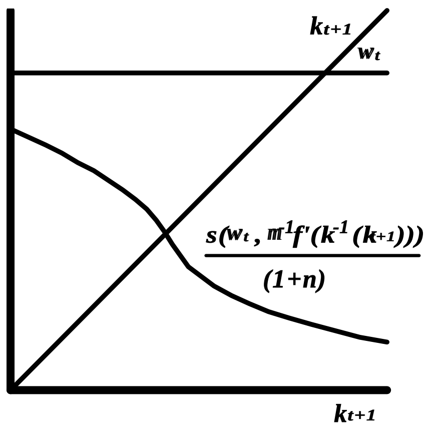

Figure 1: Existencia de un único k_{t+1}.

and such that a sequence $\{x_t\}_{t=0}^{\infty}$, which satisfies $x_{t+1} = \Psi(x_t)$ for all $t \geq 0$, has the same dynamic properties that a sequence $\{k_t\}_{t=0}^{\infty}$, which satisfies $k_{t+1} = \Gamma(k_t)$ for all $t \geq 0$. We define a dynamic equilibrium in terms of x_t.

Definition 1 A dynamic equilibrium is a sequence $\{x_t\}_{t=0}^{\infty}$, under which:

$$\frac{s(\mu^{-1}(f(x_t) - x_t f'(x_t)), \mu^{-1} f'(x_{t+1}))}{1+n} = k(x_{t+1}); \tag{26}$$

where x_0 is exogenously given.

Definition 2 A steady-state equilibrium is a stationary value of x_t, $\overline{x}$, under which

$$\frac{s(\mu^{-1}(f(\overline{x}) - \overline{x} f'(\overline{x})), \mu^{-1} f'(\overline{x}))}{1+n} = k(\overline{x}).$$

The following lemma establishes that condition (7) is sufficient to avoid explosive behaviors, $\lim_{t \to \infty} x_t = \infty$.

Lemma 3 If $x_t \geq \hat{x}$, being $\hat{x}$ such that $f(\hat{x}) = (1+n)\hat{x}$, then $x_{t+1} < x_t$.

Proof. from the definition of function k, it follows that

$$k(x) \geq x \frac{f(x) - x f'(x)}{\mu f(x) - x f'(x)} \equiv k_*(x) \qquad \forall x > 0,$$

where k_* is function k when only $*$ is strictly positive. Using (9), given that $\mu > 1$ and the properties of f, it follows that

$$\frac{\omega(x)}{(1+n)k(x)} \leq \frac{\omega(x)}{(1+n)k_*(x)} = \frac{1 - x\frac{f'(x)}{f(x)}\mu^{-1}}{(1+n)\frac{x}{f(x)}} < 1 \qquad \forall x \geq \hat{x}.$$

From the previous inequality and the assumptions made on the utility function, it follows that

$$k(x_{t+1}) = \frac{s(\omega_t; \rho_{t+1})}{1+n} \le \frac{\omega(x_t)}{1+n} < k(x_t) \qquad \forall x_t > 0.$$

Given that $k'(x) > 0 \ \forall x > 0$, the lemma therefore follows. ■

From the specification of the production and utility function, it follows that if $x_t = 0$ then $x_{t+1} = 0$. Therefore, in this economy there always, at least, the trivial steady-state $\bar{x} = 0$. From Lemma 3, it follows that all steady-state lies in the interval $[0; \hat{x})$. In the following section we show that the trivial steady-state could be the only one for any set of well-behaved preferences.

4 Global Contraction

A steady-state equilibrium must satisfy

$$\begin{aligned} \hat{\Delta}_1 &= \mu^{-1}[f(\hat{x}) - \hat{x}f'(\hat{x})] - (1+n)k(\hat{x}), \\ \hat{\Delta}_2 &= (1+n)k(\hat{x})[1 - \delta + \mu^{-1}f'(\hat{x})]. \end{aligned}$$

If the production function is specified so that

$$\mu^{-1}[f(x) - xf'(x)] - (1+n)k(x) < 0 \qquad \forall x > 0,$$

then irrespective of preferences the economy experiences global contraction, and $\hat{x} = 0$ is indeed the unique steady-state equilibrium, since

$$0 < k(x_{t+1}) \le \frac{\omega_t}{1+n} < k(x_t), \qquad \forall x_t > 0:$$

And given that k is an increasing function of x then $x_{t+1} < x_t \ \forall x_t > 0$.

Proposition 1 For any given set of well-behaved preferences and any set of fixed costs with Θ and/or Φ strictly positives, there exists a function $f(x)$ that satisfies the Inada conditions under which the only steady-state equilibrium is the trivial steady-state, $\bar{x} = 0$.

Proof. It is sufficient with an example to prove the proposition. Consider the function

$$f(x) = \begin{cases} 0 & x = 0 \\ \beta x - \alpha x \ln x & 0 < x \le e^{\frac{\beta - 1}{\alpha}} \end{cases}, \tag{27}$$

where $0 < \alpha < 1$ and $\beta > 0$. This function is used by Galor and Ryder (1989) to prove their Proposition 1, and as these authors show, it satisfies the Inada conditions. The economy undergoes a global contraction if

$$\omega(x) \equiv \mu^{-1}\alpha x < (1+n)k(x) \qquad \forall x \in \left(0; e^{\frac{\beta-1}{\alpha}}\right]. \tag{28}$$

If $\Theta \geq 0$, $\Lambda = 0$ and $\Phi \geq 0$, then $k(x) \geq x \; \forall x > 0$ and for some α sufficiently near to zero (28) is satisfied. If $\Theta > 0$, $\Lambda > 0$ and $\Phi \geq 0$ then $k(x) > x \iff x < \frac{\Theta}{\Lambda}$. If $\beta < 1$, for all α sufficiently near to zero, $\omega\left(e^{\frac{\beta-1}{\alpha}}\right) = \mu^{-1}\alpha e^{\frac{\beta-1}{\alpha}} < \frac{\Theta}{\Lambda}$, and since $\omega(x)$ is a strictly increasing function of x, (28) is satisfied. If $\Theta = 0$, $\Lambda > 0$ and $\Phi > 0$, then

$$\lim_{\alpha \to 0} k(x) = \frac{\mu\Phi x}{(\mu - 1)\beta\Lambda x + \mu\Phi} \equiv g(x),$$

where $g(0) = 0$, $\lim_{x\to\infty} g(x) = \frac{\mu\Phi}{(\mu-1)\beta\Lambda}$ and g is a strictly increasing and concave function for all $x \geq 0$. Given that $k(x)$ is a continuous function of α, if $\beta < 1$, for some α sufficiently near to zero (28) is satisfied. ■

5 A Strengthened Inada Condition

Galor and Ryder (1989) establish a strengthened Inada condition which rules out the kind of technology that would force contraction to the trivial steady-state equilibrium. We can also establish a strengthened Inada condition which will depend on the size of mark-up and the combination of fixed costs. Lemma 4 establishes the relation between the strengthened Inada condition under perfect competition given by Galor and Ryder and our strengthened Inada condition under monopolistic competition.

Proposition 2 Consider the overlapping generations economy. There exists $\overline{x} > 0$ such that

$$\lim_{t\to\infty} x_t = \overline{x}; \quad \forall x_t > 0;$$

only if

$$\lim_{x\to 0} \frac{-xf''(x)}{k'(x)} > (1+n)\mu. \tag{29}$$

Proof. If $\lim_{t\to\infty} x_t = \overline{x}$, $\forall x_t > 0$, then $x_{t+1} > x_t$, $\forall x_t \in (0; \overline{x})$. Given that k is a strictly increasing function of x, then

$$k(x_t) < k(x_{t+1}) \quad \frac{\omega(x_t)}{1+n} = \frac{\mu^{-1}(f(x_t) - x_t f'(x_t))}{1+n}, \quad \forall x_t \in (0; \overline{x}).$$

Rearranging,

$$\frac{f(x) - xf'(x)}{k(x)} > (1+n)\mu, \quad \forall x \in (0; \overline{x}).$$

In the limit, using l'Hôpital's rule,

$$\lim_{x\to 0} \frac{f(x) - xf'(x)}{k(x)} = \lim_{x\to 0}\frac{-xf''(x)}{k'(x)} > (1+n)\mu.$$

$\blacksquare$

Remark 1: Proposition 2 establishes a sufficient condition to avoid global contraction for any set of well-behaved preferences because condition (29) implies that $\mu^{-1}[f(x) - xf'(x)] > (1+n)\,k(x)$ for x sufficiently near to zero, since functions f and k are continuous. It should be note that the strengthened Inada condition (29) depends on both mark up over marginal cost and fixed costs.

The following lemma establishes the relation between the strengthened Inada condition under monopolistic competition, the Inada condition, and Galor and Ryder's strengthened Inada condition under perfect competition.

Lemma 4 (a) If $\Theta > 0$, condition $\lim_{x\to 0} \frac{-xf''(x)}{k'(x)} > (1+n)\,\mu$ implies Galor and Ryder's condition under $\lim_{x\to 0}(-xf''(x)) > (1+n)$ and the Inada condition $\lim_{x\to 0} f'(x) = \infty$.

(b) If $\Theta = 0$, and $\Phi > 0$, condition $\lim_{x\to 0} \frac{-xf''(x)}{k'(x)} > (1+n)\,\mu$ is satisfied if only if Galor and Ryder's condition $\lim_{x\to 0}(-xf''(x)) > (1+n)$ is satisfied, and it implies the Inada condition $\lim_{x\to 0} f'(x) = \infty$.

(c) If $\Theta = 0$, $\Lambda > 0$ and $\Phi = 0$, the Inada condition $\lim_{x\to 0} f'(x) = \infty$ implies condition $\lim_{x\to 0} \frac{-xf''(x)}{k'(x)} > (1+n)\,\mu$. Moreover, condition $\lim_{x\to 0} \frac{-xf''(x)}{k'(x)} > (1+n)\,\mu$ is satisfied if $\lim_{x\to 0} f'(x) > (1+n)\frac{\mu}{\mu-1}$.

Proof. If $\Theta > 0$, then $k(x) > x$, $\forall x < \frac{\Theta}{\Lambda}$; and therefore $\frac{f(x)-xf'(x)}{k(x)} < \frac{f(x)-xf'(x)}{x}$, $\forall x < \frac{\Theta}{\Lambda}$. Given that f and k are continuous functions, then

$$\lim_{x\to 0}\frac{f(x) - xf'(x)}{k(x)} = \lim_{x\to 0}\frac{-xf''(x)}{k'(x)} \qquad \lim_{x\to 0}\frac{f(x) - xf'(x)}{x} = \lim_{x\to 0}(-xf''(x)),$$

which, together Lemma 2 of Galor and Ryder (1989), implies (a) in Lemma 4. If $\Theta = 0$ and $\Phi > 0$, then

$$\begin{aligned}\lim_{x\to 0}\frac{-xf''(x)}{k'(x)} &= \lim_{x\to 0}\frac{f(x) - xf'(x)}{k(x)} = \lim_{x\to 0}\frac{\frac{f(x)-xf'(x)}{x}}{\frac{(f(x)-xf'(x))\Lambda+\Phi\mu}{(\mu f(x)-xf'(x))\Lambda+\Phi\mu}} = \\ &= \lim_{x\to 0}\frac{f(x) - xf'(x)}{x} = \lim_{x\to 0}(-xf''(x)),.\end{aligned}$$

which, together Lemma 2 of Galor and Ryder (1989), implies (b) in Lemma 4. If $\Theta = 0$; $\Lambda > 0$ and $\Phi = 0$, then

$$\begin{aligned}\lim_{x\to 0}\frac{-xf''(x)}{k'(x)} &= \lim_{x\to 0}\frac{f(x) - xf'(x)}{k(x)} = \lim_{x\to 0}\frac{\mu f(x) - xf'(x)}{x} = \\ &= \lim_{x\to 0}((\mu - 1)\,f'(x) - xf''(x)).\end{aligned}$$

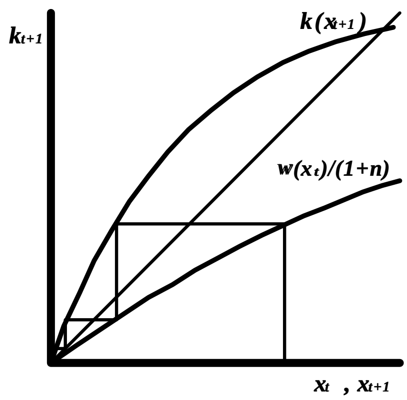

Figure 2: Nonexistence of Non-trivial Steady-state Equilibrium.

If $\lim_{x\to 0} f'(x) = \infty$ then $\lim_{x\to 0}((\mu - 1) f'(x) - x f''(x)) > (1+n)\mu$ and this last condition is satisfied if $\lim_{x\to 0} f'(x) > (1+n)\frac{\mu}{\mu-1}$. ■

Remark 2 From (c) in Lemma 4, it follows that the structure of fixed costs is such that there is only fixed costs on labor, $\Theta = 0; \Lambda > 0$ and $\Phi = 0$, a weaker condition that the Inada condition is enough to avoid global contraction for any set of well-behaved preferences. We can give an example: if $\Theta = 0; \Lambda > 0$ and $\Phi = 0$ and f satisfies the Inada conditions, then $\lim_{x\to 0} \frac{-xf''(x)}{k'(x)} = \infty$, hence, for all x sufficiently near to zero, $\frac{\mu^{-1}[f(x) - xf'(x)]}{1+n} > k(x)$, which, together. Lemma 3, implies that for Cobb-Douglas preferences with a marginal propension to save, s, sufficiently near to zero, there exists a non-trivial steady-state equilibrium.

6 Sufficient Conditions for the Non-existence of Non-trivial Steady-state equilibrium

Proposition 3 For any given set of well-behaved preferences, if the function f satisfies the Inada conditions, the unique steady-state equilibrium is the trivial equilibrium, $\overline{x} = 0$, if

(a) $\lim_{x\to 0} \frac{-xf''(x)}{k'(x)} < (1+n)\mu$

(b) $\frac{-xf''(x)}{k'(x)} < (1+n)\mu \ \forall x > 0$

Proof. Suppose that $s_t = \omega_t$ (i.e., there is no utility from first period consumption). Clearly, if global contraction is established under the above conditions for $s_t = \omega_t$, it can be established for all other feasible set of preferences under which $s_t \quad \omega_t$. Thus, modifying (26) $(1+n)k(x_{t+1}) = \omega(x_t) =$

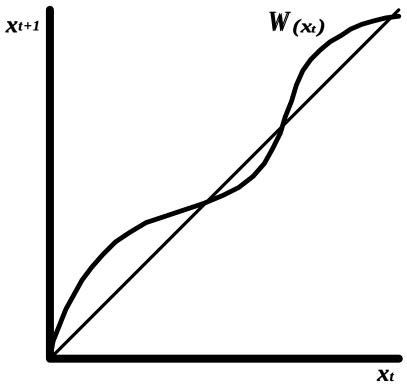

Figure 3: Existence of Non-trivial Steady-state Equilibria.

$^{1}{}^{i}{}^{1}(f(x_t) \;¡\; x_t f^{0}(x_t))$ and

$$\frac{dx_{t+1}}{dx_t} = \frac{¡\, x_t f^{00}(x_t)}{(1+n)\,{}^{1}k^{0}(x_{t+1})}.$$

Consider Figure 2. The unique steady-state is $\overline{x} = 0$ if function $\frac{!(x_t)}{1+n}$ intersects the function $k(x_{t+1})$ only at the origin. The proposition therefore follows from Figure 2. ■

Remark 3: Proposition 3 establishes su±cient conditions for global contraction. It should be note the importance of the mark up and the structure of ¯xed cost in the conditions of Proposition 3. So, identical economies except for the size of the mark up and/or the structure of ¯xed costs, could undergo completely different dynamic behaviors. One of them could irremediably converge to the trivial steady-state equilibrium while the other converges to a non-trivial steady-state equilibrium. It should be also note that if $£ = 0; ¤ > 0; © = 0$ and f satis¯es the Inada conditions, then the conditions of Proposition 3 are never satis¯ed, as it is followed from Lemma 4.

7 Existence, Uniqueness, and Stability of Non-trivial Steady-State Equilibrium

Existence of a non-trivial steady-state equilibrium is not guaranteed by the strengthened Inada condition. We need constraint the interactions between preferences and technology.

Proposition 4 There exists a non-trivial steady-state equilibrium if

(a) $s_{½}(!;r) \;¸\; 0 \; 8(!;r) \;¸\; 0.$

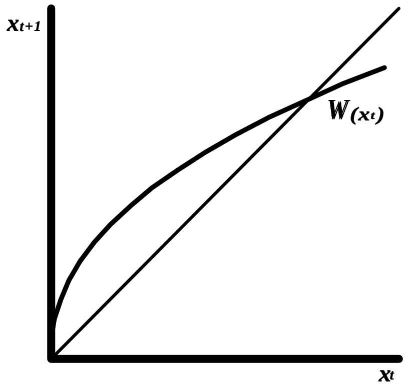

Figure 4: Existence of a Unique Non-trivial Steady-state Equilibrium.

(b) $\lim_{x \to 0} \frac{-s_\omega x f''(x)}{\mu(1+n)k'(x) - s_\rho f''(x)} > 1$.
(c) $\exists\, \tilde{x}$ such that $f'(\tilde{x}) = (1+n)\tilde{x}$.

Proof. Since:

$$\frac{dx_{t+1}}{dx_t} = \frac{-s_\omega x_t f''(x_t)}{(1+n)\mu k'(x_{t+1}) - s_\rho f''(x_{t+1})}.$$

Proposition 3 follows from Figure 3. The first condition guarantees the existence of ψ, the second condition guarantees that the shape of ψ is higher than one at the origin, and condition (c) implies that $\forall x_t \geq \tilde{x}\ x_{t+1} = \psi(x_t) < x_t$, as established in Lemma 3. Then, there exists $x > 0$, such that $\psi(x) = x$. ■

Proposition 5 There exists a unique globally stable non-trivial steady-state if

(a) $\lim_{x \to 0} \frac{-s_\omega x f''(x)}{(1+n)\mu k'(x) - s_\rho f''(x)} > 1$.
(b) $\exists\, \tilde{x}$ such that $f'(\tilde{x}) = (1+n)\tilde{x}$.
(c) $\psi'(x) \geq 0\ \forall x > 0$.
(d) $\psi''(x) \leq 0\ \forall x > 0$.
(e) $s_\rho(\omega; r) \geq 0\ \forall(\omega; r) \geq 0$.

Proof. Consider Figure 4. Uniqueness and global stability of the non-trivial steady-state equilibrium are satisfied if (i) function ψ exists, (ii) the curve $\psi(x)$ is strictly concave, (iii) $\lim_{x \to 0} \psi'(x) > 1$, and (iv) the curve intersects the bisectriz of the positive ortant at $x > 0$. Condition (e) is sufficient for (i). From Lemma 3 follows that condition (b) is sufficient to (iv). Moreover, (a) implies (iii) and (c) and (d) implies (ii). Therefore, the proposition is verified. ■

Corollary 1 A necessary condition for the existence of a unique globally stable non-trivial steady-state equilibrium is the strengthened Inada condition,

$$\lim_{x\,!\,0} \frac{i\,xf^{00}(x)}{k^{0}(x)} > (1+n)\,^{1}.$$

8 Conclusions

In this paper we have analyzed existence, uniqueness and stability of a steady-state equilibrium in an overlapping generations model with monopolistic competition and free entry and exit of ¯rms. The Galor and Ryder's (1989) results appear as a limit case of our analysis in which mark-up over marginal cost go to one, ¹ = 1, and there is constant returns to scale, £ = ¤ = © = 0.

Our analysis shows that for any given set of well-behaved preferences and any set of ¯xed costs with the ¯xed costs on output and/or the ¯xed costs on capital being strictly positive, there exists a production function that satis¯es the Inada conditions under which the only steady-state equilibrium is the trivial steady-state, characterized by production and consumption being zero We have established a strengthened Inada condition that is su±cient to exclude global contraction. However, we have also shown that if there is only ¯xed cost on labor then a weaker condition than the Inada conditions is su±cient to exclude global contraction.

We have established su±cient conditions for a non-trivial steady-state equilibrium to exist, and also su±cient conditions for its uniqueness and global stability. We show that the size of mark-up over marginal cost and the particular mix of ¯xed costs play a crucial role in these conditions. So, economies that only di®er in their mark-ups and/or in their mix of ¯xed costs could experiment radically di®erent dynamic behaviors. One of them could converge to the trivial steady-state equilibrium, and the other could converge to a strictly positive steady-state equilibrium.

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