A simple proof of existence of equilibrium in a one sector growth model with bounded or unbounded returns from below[1]

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Summary: We analyze a Ramsey economy when net investment is constrained to be non negative. We prove existence of a competitive equilibrium when utility need not be bounded from below and the Inada-type conditions need not hold. The analysis is carried out by means of a direct and technically standard strategy. This direct strategy (a) allows us to obtain detailed results concerning properties of competitive equilibria, and (b) is amenable to be easily adapted for the analysis of analogous models often found in macroeconomics.

Keywords: Ramsey model, One sector growth model, Non negative net investment, Competitive equilibrium.

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1 Introduction

Ever since the seminal work of Cass (1965) and Koopmans (1965), optimal capital accumulation models have played a central role in growth theory and macroeconomics. The development of dynamic equilibrium macroeconomics required further the integration of optimal capital accumulation and infinite horizon general equilibrium models: aggregative models of capital accumulation had to be reinterpreted as decentralized market economies and optimal accumulation paths as competitive equilibrium allocations. In this paper we prove existence of a competitive equilibrium for a version of the Ramsey (one sector) model in which net investment is constrained to be non negative.

Following the early work of Peleg and Yaari (1970) and Bewley (1972) the literature has often follow an abstract general approach to the question of existence. After describing the commodity space, the question of existence reduces to find a price system in a suitable (interpretable) subset of the space dual to the commodity space. An alternative related strategy first establishes core equivalence (the set of competitive equilibria coincides with the core of the economy) so that no restrictions on the nature of price systems are made a priori. Aliprantis, Brown, and Burkinshaw (1990) and Becker and Boyd (1997) are modern expositions of these approaches. Aliprantis, Border, and Burkinshaw (1997) follow the first approach and analyze a Ramsey economy in which gross investment k_{t+1} at period t is required to verify $k_{t+1} \geq (1-\delta)k_t$, where $k_t \geq 0$ is current stock of capital and $\delta \in (0,1)$ is the depreciation rate of capital. Their method of proof is appealing when analyzing complex models: it constitutes a general and elegant approach to the question of existence. Nevertheless, it remains at a high level of abstraction and does not provide much results that could help characterizing competitive equilibria. Here we propose a simpler strategy that does not demand a strong investment in mathematical techniques. In simple economies like the one considered in that paper, it is possible to display a price system and prove afterwards that it is a competitive equilibrium price system.

In the present paper we tackle the question of existence in Aliprantis, Border, and Burkinshaw's (1997) model using simple standard techniques and obtaining more detailed results concerning the behavior of equilibrium allocations and prices. Because of the standard techniques utilized, our approach is amenable to be adapted by practitioners to similar models without requiring a strong investment in sophisticated techniques.

The planner's problem is first analyzed and optimal paths characterized. The multipliers system associated with an optimal path is proven to be the supporting price system of a competitive equilibrium when the competitive allocation is the planner's optimal path. Our strategy of proof relies on a closer look at the model that allows us: (a) to obtain more general results dropping the assumption that utility is bounded from below and the Inada conditions on the one period utility function; (b) to prove more detailed results concerning properties of optimal (equilibrium) paths (monotonicity, convergence to a steady state, and so on) and actually display the price system explicitly. Our assumptions are those of Aliprantis, Border, and Burkinshaw (1997) except for that of differentiability of the one period utility function. This assumption is adopted for the sake of clarity and does not entail any loss of generality because: first, concave functions are differentiable almost everywhere; second, the proofs below can be reproduced as they stand substituting gradients by subgradients.

The next section describes the planner's problem and proves existence of optimal paths when the Inada condition on the one period utility function holds. These optimal paths are analyzed in section 3 and their properties studied. Section 4 proves existence of a competitive equilibrium while section 5 proves existence when the Inada-type conditions do not hold.

2 The planner's problem

This is a one sector growth model with net investment constrained to be non negative. Time is discrete so t ranges over the integers from zero to infinity. Production possibilities are represented by a gross production function f and a physical depreciation rate δ. Preferences are time additively separable and described by a one period reward function u and a discount factor β. The planner of this economy maximizes

$$U(\mathbf{c}) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

over non negative sequences $(\mathbf{k}, \mathbf{c})$ subject to the feasibility constraints

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t \quad \text{and} \quad (1-\delta)k_t \leq k_{t+1} \tag{1}$$

for $t = 0, 1, ...$ with $0 \leq k_0 \leq x_0$ and $x_0 \geq 0$ given. This model is also described and analyzed in Aliprantis, Border, and Burkinshaw (1997).

We introduce now some notation. Under the monotonicity assumptions below, at maxima $k_0 = x_0$ so that we will work with k_0 as the initial condition. For any initial condition $k_0 \geq 0$ when $\mathbf{k} = (k_1, k_2, ...)$ is such that $(1-\delta)k_t \leq k_{t+1} \leq f(k_t) + (1-\delta)k_t$ for all t we say it is feasible from k_0 and the class of all feasible accumulation paths is denoted $\Pi(k_0)$. A consumption sequence $\mathbf{c} = (c_0, c_1, ...)$ is feasible from $k_0 \geq 0$ when exists $\mathbf{k} \in \Pi(k_0)$ with $0 \leq c_t \leq f(k_t) + (1-\delta)k_t - k_{t+1}$ and the class of feasible from k_0 consumption sequences is denoted $\Sigma(k_0)$. The value function associated to this problem is denoted by v.

Assumption 1 The one period reward function $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is continuous, strictly increasing, and strictly concave. At zero either $u(0)$ is finite or $u(0) = -\infty$. Further, $\beta \in (0,1)$.

Remark 1 If $u(0)$ is finite we do not loose any generality if we assume further that $u(0) = 0$. Otherwise one could add and substract $(1-\beta)^{-1}u(0)$, ignore the positive constant, and work with $\tilde{u}(c) = u(c) - u(0)$ for all $c \geq 0$. Hence, hereafter u is assumed at zero to be either $u(0) = 0$ or $u(0) = -\infty$.

Strict monotonicity implies $u(c) > -\infty$ for all $c > 0$. Continuity is explicitly assumed to ensure it is continuous at zero because concavity implies continuity in the interior of its domain (and differentiability almost everywhere). Since the objective function is additive and $\beta < 1$ we do not loose any generality assuming $u(0) = 0$ when u is bounded from below. We assume some Inada-type condition to prove that optimal consumption is positive:

Assumption 2 The one period reward function u is differentiable in the interior of its domain. If $u(0) = 0$ then $u'(0) = \infty$.

The assumption that $u'(0) = \infty$ when $u(0) = 0$ is relaxed in section 5. Note that $u'(0) = \infty$ is already ensured when $u(0) = -\infty$: for any $c > 0$ and by concavity of u we have $u'(c) \geq (u(c) - u(c'))/(c - c')$ for all $c' > 0$. Taking the limit as $c \to 0$ the property follows. The production function is assumed to meet the standard properties of the neoclassical per capita production function.

Assumption 3 The gross production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing, strictly concave, $f(0) = 0$, and $\delta \in (0,1)$.

As in the case of u, continuity is assumed to obtain continuity at zero because concavity already implies continuity in the interior of its domain. The production function should satisfy the Inada conditions so as to make it feasible that optimal consumption is positive. Sustained growth of the stock of physical capital is ruled out assuming zero returns to physical capital asymptotically.

Assumption 4 The gross production function is differentiable in the interior of its domain with $f'(0) = \infty$ and $f'(\infty) = 0$.

For all $k_0 > 0$, and since $f'(0) > \delta$, there must be some $0 < k' \leq k_0$ such that $f(k') + (1 - \delta)k' > k'$. Hence, for all $k_0 > 0$ there is a feasible interior stationary consumption-accumulation plan described by $k' > 0$ and $c' = f(k') - \delta k' > 0$. Further, $f'(\infty) < \delta$ implies existence of a maximum sustainable capital stock: some $\bar{k} > 0$ for which $f(k) + (1 - \delta)k < k$ for all $k > \bar{k}$. In order to save notation we shall often write $F(k)$ for $f(k) + (1 - \delta)k$.

To prove existence of an optimal path we follow the classical strategy using continuity of both u and F. While the latter will ensure that $\Pi(k_0)$ is compact, the former will ensure that U is upper semicontinuous in which case Weierstrass theorem applies. The following has a standard proof.

Lemma 1 *For all $k_0 \geq 0$, (a) exists $A(k_0) > 0$ such that $\mathbf{k} \in \Pi(k_0)$ implies $k_t \leq A(k_0)$ for all t, (b) $\Pi(k_0)$ is compact in the product topology, (c) U is well defined and bounded from above over $\Sigma(k_0)$, and (d) if $k_0 > 0$ and $(\mathbf{c}, \mathbf{k})$ is optimal then $U(\mathbf{c}) > -\infty$ and $c_t = F(k_t) - k_{t+1}$ for all t.*

Observe that (a) follows for $A(k_0) = \max\{k_0, \bar{k}\}$ where $\bar{k}$ is the maximum sustainable capital stock. Then (b) follows from this bound and Tychonov theorem while (c) is a consequence of $A(k_0)$ also bounding feasible consumption. Finally, since for all $k_0 > 0$ there is an interior stationary feasible path, total utility at the optimum must be at least the (finite) value of this stationary plan while monotonicity of u does the rest. Hereafter we will concentrate on non wasting consumption paths and use the notation

$$U(k_0, \mathbf{k}) = \sum_{t=0}^{\infty} \beta^t u(F(k_t) - k_{t+1})$$

for any $k_0 \geq 0$ and $\mathbf{k} \in \Pi(k_0)$, the associated consumption path understood to be $c_t = F(k_t) - k_{t+1}$ for all t. Existence of an optimal path is ensured if $U(k_0, .)$ is upper semicontinuous over $\Pi(k_0)$.

Lemma 2 *For all $k_0 \geq 0$, $U(k_0, .)$ is upper semicontinuous over $\Pi(k_0)$ with respect to the relative product topology.*

Proof: Let $k_0 \geq 0$ and $\mathbf{k} \in \Pi(k_0)$ we have

$$\sum_{t=T+1}^{\infty} \beta^t u(F(k_t) - k_{t+1}) \leq \beta^{T+1} \max\{0, u(A(k_0))\}$$

for any $T \geq 0$ and converging to zero as $T \to \infty$. Note that the left hand expression does not depend on the particular $\mathbf{k}$ considered: if $(\mathbf{k}^n) \subset \Pi(k_0)$ with $\mathbf{k}^n \to \mathbf{k}^0$, for any $\varepsilon > 0$ exists T with

$$U(k_0, \mathbf{k}^n) = \sum_{t=0}^{\infty} \beta^t u(F(k_t^n) - k_{t+1}^n) \leq \sum_{t=0}^{T} \beta^t u(F(k_t^n) - k_{t+1}^n) + \varepsilon$$

for all $n \in \mathbb{N}$. Then

$$\begin{aligned} \limsup_{n\to\infty} U(k_0, \mathbf{k}^n) &\leq \lim_{n\to\infty} \sum_{t=0}^{T} \beta^t u(F(k_t^n) - k_{t+1}^n) + \varepsilon \\ &= \sum_{t=0}^{T} \beta^t u(F(k_t^0) - k_{t+1}^0) + \varepsilon \end{aligned}$$

where the equality is legitimate because u and F are continuous. This inequality holds for all T: then, $\limsup_{n\to\infty} U(k_0, \mathbf{k}^n) \leq U(k_0, \mathbf{k}^0) + \varepsilon$. Since ε was arbitrary $\limsup_{n\to\infty} U(k_0, \mathbf{k}^n) \leq U(k_0, \mathbf{k}^0)$. ■

Hence, an optimal path exists. Because u and f are strictly concave, it is straightforward to check that $U(k_0, .)$ is strictly concave while $\Pi(k_0)$ is convex so that:

Proposition 1 *For all $k_0 \geq 0$ there is a unique optimal accumulation path.*

One way to make any further analysis easier is to work with the value function. We should ensure first, however, that it solves the Bellman equation. From the properties of f and u it is clear that the value function is strictly increasing. The proof of the previous lemma will help us also establishing strict concavity.

Lemma 3 *The value function $v : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ associated to the planner's problem is (a) well defined, (b) finite valued in the interior of its domain, (c)*

$v(0) = 0$ when $u(0) = 0$ and $v(0) = -\infty$ when $u(0) = -\infty$, (d) strictly increasing, (e) strictly concave, and (f) continuous.

Proof: Claims (a) to (c) are direct consequences of lemma 1 and proposition 1. Concavity (e) follows from a standard argument and implies continuity in the interior of its domain.

Proof of (d): let $k_0 \geq 0$ and $\mathbf{k}$ be optimal. For any $k_0' > k_0$ define $\mathbf{k}'$ as $k_t' = \max\{k_t, (1-\delta)k_{t-1}'\}$ for all t, a feasible plan from k_0'. Consider the first period's returns: if $k_1 \geq (1-\delta)k_0'$ we have

$$u(F(k_0') - k_1') = u(F(k_0') - k_1) > u(F(k_0) - k_1)$$

because $k_0' > k_0$; when $k_1 < (1-\delta)k_0'$ we have

$$u(F(k_0') - k_1') = u(f(k_0')) > u(f(k_0)) \geq u(F(k_0) - k_1)$$

again for $k_0' > k_0$ and because $k_1 \geq (1-\delta)k_0$. For further periods analogous but weak inequalities hold so that $U(\mathbf{k}') \geq U(\mathbf{k})$. Then

$$\begin{aligned} v(k_0') \geq U(k_0', \mathbf{k}') &= u(F(k_0') - k_1') + \beta U(\mathbf{k}') \\ &> u(F(k_0) - k_1) + \beta U(\mathbf{k}) = U(k_0, \mathbf{k}) = v(k_0) \end{aligned}$$

and therefore $v(k_0') > v(k_0)$.

Proof of (f): concavity implies continuity in the interior of its domain. To see continuity at zero let $(k_0^n) \subset \mathbb{R}_{++}$ with $k_0^n \to 0$ and let for all $n \in \mathbb{N}$ be $\mathbf{k}^n \in \Pi(k_0^n)$ the associated optimal path. Observe that $k_t^n \leq A(k_0^n)$ for all n and t but since $k_0^n \to 0$ there is some $N \in \mathbb{N}$ with $k_t^n \leq \bar{k}$ for t and all $n \geq N$. Choose any $n \geq N$ and $T \geq 1$:

$$v(k_0^n) = \sum_{t=0}^{\infty} \beta^t u(F(k_t^n) - k_{t+1}^n) \leq \sum_{t=0}^{T} \beta^t u(F(k_t^n)) + \beta^{T+1} \frac{u(F(\bar{k}))}{1-\beta}.$$

When $u(0) = 0$ we have $v(0) = 0$ while for all $\varepsilon > 0$ exists T with

$$0 \leq v(k_0^n) \leq \sum_{t=0}^{T} \beta^t u(F(k_t^n)) + \varepsilon.$$

By feasibility $k_t^n \to 0$ for $t = 0, ..., T$ as $n \to \infty$. Then $0 \leq \lim_{n\to\infty} v(k_0^n) \leq \varepsilon$. Since ε was arbitrary it must be the case that $\lim_{n\to\infty} v(k_0^n) = 0$. When $u(0) =$

$-\infty$ an identical argument follows because $\lim_{n\to\infty} \sum_{t=0}^{T} \beta^t u(F(k_t^n)) = -\infty$. ■

The principle of optimality is formally stated in the following proposition. It will help characterizing basic properties of optimal paths.

Proposition 2 *The value function solves the Bellman equation and for all* $k_0 \geq 0$ *a feasible path* $\mathbf{k}$ *is optimal if and only if*

$$v(k_t) = u(F(k_t) - k_{t+1}) + \beta v(k_{t+1}) \tag{2}$$

holds for all t.

Proof: Our case meet the hypothesis of Stokey and Lucas (1989, theorem 4.2) so that the value function solves the Bellman equation. If $\mathbf{k}$ is optimal from k_0 then (2) holds by Stokey and Lucas (1989, theorem 4.4). Finally, suppose (2) holds for all t for some $\mathbf{k}$ feasible from $k_0 > 0$. Then

$$v(k_0) = \sum_{t=0}^{T} \beta^t u(F(k_t) - k_{t+1}) + \beta^{T+1} v(k_{T+1})$$

for all T. It rests to prove that $\beta^{T+1} v(k_{T+1}) \to 0$. Define

$$\Pi'(k_0) = \{\mathbf{k} \in \Pi(k_0) : U(k_0, \mathbf{k}) > -\infty\},$$

a non empty class (because at least a stationary interior path exists). Since U is uniformly bounded from above we know that

$$\limsup_{T\to\infty} \beta^{T+1} v(k_{T+1}) \leq 0.$$

because v is bounded from above and $\beta < 1$. Moreover, for any $\mathbf{k} \in \Pi'(k_0)$

$$U(k_0, \mathbf{k}) \leq \sum_{t=0}^{T} \beta^t u(F(k_t) - k_{t+1}) + \beta^{T+1} v(k_{T+1})$$

for all T so that

$$0 = \lim_{T\to\infty} \left\{ U(k_0, \mathbf{k}) - \sum_{t=0}^{T} \beta^t u(F(k_t) - k_{t+1}) \right\} \leq \liminf_{T\to\infty} \beta^{T+1} v(k_{T+1}).$$

and the result follows: $\lim_{T\to\infty} \beta^{T+1} v(k_{T+1}) = 0$ and $v(k_0) = U(k_0, \mathbf{k})$. ■

3 Properties of optimal paths

In this section we review important properties of optimal paths. Later they will show useful to prove existence of a supporting price system (planner's solutions will turn out to be competitive equilibrium allocations). The following lemma has several standard proofs and is stated here for further reference.

Lemma 4 *If $k_0 > 0$ and $\mathbf{k}$ is optimal then $\mathbf{k}$ is monotone (either $k_t \leq k_{t+1}$ for all t or $k_t \geq k_{t+1}$ for all t) and consumption positive ($c_t = F(k_t) - k_{t+1} > 0$ for all t).*

Several proofs of monotonicity can be found in the literature (e.g.: see Amir (1996) for the case $u(0) = 0$). That consumption is positive is a direct consequence of the Inada condition verified by u, implied in the case $u(0) = -\infty$ and assumed when $u(0) = 0$ (assumption 2).

In the standard Ramsey model $(1-\delta)k_t \leq k_{t+1}$ need not be verified for all t. In other words, in the present case we face the possibility that the non negativity constraint of net investment is binding at certain periods. The constraint, however, cannot be always binding in the long run.

Lemma 5 *If $k_0 > 0$ and $\mathbf{k}$ is optimal there cannot be an integer T such that $k_{t+1} = (1-\delta)k_t$ for all $t \geq T$.*

Proof: Let $k_0 > 0$ and $\mathbf{k}$ be optimal but assume such T exists. Since $k_t \to 0$, under assumption 4 we can choose some integer $T' \geq T$ such that $\beta f'(k_{T'+1}) > 1$. Lemma 4 implies that $k_{t+1} < f(k_t) + (1-\delta)k_t$ for all t so that there is $\varepsilon > 0$ small enough to verify

$$(1-\delta)k_{T'} < k_{T'+1}(1+\varepsilon) < f(k_{T'}) + (1-\delta)k_{T'}.$$

Define $\mathbf{k}'$ as $k'_t = k_t$ for $t = 1, ..., T'$ and $k'_t = k_t(1+\varepsilon)$ for $t \geq T'+1$. Up to date $T'+1$ the plan $\mathbf{k}'$ is obviously feasible in regard of the choice of ε. For $t \geq T'+2$ we have

$$(1-\delta)k'_t = (1-\delta)(1+\varepsilon)k_t = (1+\varepsilon)k_{t+1} = k'_{t+1}$$

because $k_{t+1} = (1-\delta)k_t$ for all $t \geq T$. The same equality implies

$$k'_{t+1} = (1-\delta)k'_t < f(k'_t) + (1-\delta)k'_t$$

because $f > 0$. Hence, $\mathbf{k}'$ is feasible from k_0. We next show that $\mathbf{k}'$ dominates $\mathbf{k}$ for some ε small enough. First observe that

$$f(k_{T'}) + (1-\delta)k_{T'} - k'_{T'+1} = f(k_{T'}) - \varepsilon k_{T'+1}.$$

Then define $\varphi(\varepsilon) = U(k_0, \mathbf{k}') - U(k_0, \mathbf{k})$, we have:

$$\begin{aligned}
\varphi(\varepsilon) &= \beta^{T'}[u(f(k_{T'}) - \varepsilon k_{T'+1}) - u(f(k_{T'}))] \\
&+ \beta^{T'+1}[u(f(k_{T'+1}(1+\varepsilon))) - u(f(k_{T'+1}))] \\
&+ \sum_{t>T'+1}^{\infty} \beta^t[u(f(k_t(1+\varepsilon))) - u(f(k_t))].
\end{aligned}$$

where we have used again that $k_{t+1} = (1-\delta)k_t$ for all $t \geq T$. Since the last term is positive we can write

$$\begin{aligned}
\varphi(\varepsilon) > & -\beta^{T'} u'(f(k_{T'}) - \varepsilon k_{T'+1})\varepsilon k_{T'+1} \\
& +\beta^{T'+1} u'(f(k_{T'+1}(1+\varepsilon)))f'(k_{T'+1}(1+\varepsilon))\varepsilon k_{T'+1}
\end{aligned}$$

for ε small enough where we have used that u and f are concave and differentiable. Then

$$\frac{\varphi(\varepsilon)}{\beta^{T'}} > \varepsilon k_{T'+1}\left[-u'(f(k_{T'}) - \varepsilon k_{T'+1}) + \beta u'(f(k_{T'+1}(1+\varepsilon)))f'(k_{T'+1}(1+\varepsilon))\right].$$

When $\varepsilon \to 0$ the term in brackets converges to

$$u'(f(k_{T'+1}))f'(k_{T'+1})\beta - u'(f(k_{T'})) > u'(f(k_{T'+1})) - u'(f(k_{T'})) > 0$$

where the first inequality is true because T' was chosen so that $\beta f'(k_{T'+1}) > 1$ and the second because $k_{T'+1} = (1-\delta)k_{T'} < k_{T'}$ while u' is strictly decreasing. In short, $\varphi(0) = 0$ and $\varphi(\varepsilon) > 0$ for some ε small enough: a contradiction. ■

The previous result allows us to prove that $k_t \to 0$ cannot be optimal, an essential step proving convergence to an interior steady state (together with monotonicity, lemma 4).

Proposition 3 *If $k_0 > 0$ and $\mathbf{k}$ is optimal then k_t cannot converge to zero.*

Proof: Assume the contrary: $k_0 > 0$ and $\mathbf{k}$ is optimal but $k_t \to 0$. By lemma 4 it must do so monotonically. That is, $k_t \geq k_{t+1}$ for all t. The rest of the proof follows in two steps.

Step 1: We claim that there is some T with $(1-\delta)k_t < k_{t+1}$ for all $t \geq T$. Suppose the claim is false: for any integer T exists $T' \geq T$ such that $(1-\delta)k_{T'-1} = k_{T'}$. Observe that T' can always be chosen so that $(1-\delta)k_{T'} < k_{T'+1}$ because by lemma 5 such equality cannot hold residually. Further, since $k_t \to 0$, T' can be chosen so that $\beta F'(k_{T'}) > 1$.

By lemma 4, $k_{T'} < F(k_{T'-1})$ so that we can choose $\varepsilon > 0$ small enough so that $k_{T'} + \varepsilon < F(k_{T'-1})$ and $(1-\delta)(k_{T'} + \varepsilon) < k_{T'+1}$. Then $\mathbf{k}'$ defined as $k'_t = k_t$ for all $t \neq T'$ and $k'_{T'} = k_{T'} + \varepsilon$ is feasible. If $\mathbf{k}$ is optimal

$$\varphi(\varepsilon) = u(F(k_{T'-1}) - k_{T'} - \varepsilon) + \beta u(F(k_{T'} + \varepsilon) - k_{T'+1}).$$

must have a maximum at zero. Since $(1-\delta)k_{T'-1} = k_{T'}$ we have

$$\varphi(\varepsilon) = u(f(k_{T'-1}) - \varepsilon) + \beta u(F(k_{T'} + \varepsilon) - k_{T'+1}).$$

Differentiate with respect to ε to obtain

$$\varphi'(0) = -u'(f(k_{T'-1})) + \beta u'(F(k_{T'}) - k_{T'+1})F'(k_{T'})$$

and use $(1-\delta)k_{T'} < k_{T'+1}$ and the fact that u' is decreasing to conclude

$$\begin{aligned} \varphi'(0) &> -u'(f(k_{T'-1})) + \beta u'(f(k_{T'}))F'(k_{T'}) \\ &> -u'(f(k_{T'-1})) + u'(f(k_{T'})) \geq 0. \end{aligned}$$

The second inequality uses $\beta F'(k_{T'}) > 1$ while the third follows from the hypothesis that $k_{T'} \geq k_{T'+1}$ and from u' being decreasing. Hence, $\varphi'(0) > 0$ thus contradicting that $\mathbf{k}$ is optimal. The claim must be therefore true.

Step 2: From the first step we know that there is some T with $(1-\delta)k_t < k_{t+1}$ for all $t \geq T$. Since lemma 4 ensures that $k_{t+1} < F(k_t)$ for all t, the Euler equation implies that

$$u'(F(k_t) - k_{t+1}) = \beta u'(F(k_{t+1}) - k_{t+2})F'(k_{t+1})$$

for all $t \geq T$. If $k_t \to 0$ exists $T' \geq T$ with $\beta F'(k_{t+1}) > 1$ for all $t \geq T'$ in which case

$$u'(F(k_t) - k_{t+1}) > u'(F(k_{t+1}) - k_{t+2})$$

for all $t \geq T'$. But this implies $c_t < c_{t+1}$ for all $t \geq T'$ and, in particular, $c_t \geq c_{T'}$ for all $t \geq T'$ while $k_t \to 0$ implies $c_t \to 0$ by feasibility: an absurdity. ■

We can now prove that the Euler equations do hold from some period on.

Proposition 4 *If $k_0 > 0$ and $\mathbf{k}$ is optimal, exists T with $(1-\delta)k_t < k_{t+1} < F(k_t)$ for all $t \geq T$.*

Proof: Let $k_0 > 0$ and $\mathbf{k}$ be optimal. Lemma 4 (consumption positive) established $k_{t+1} < F(k_t)$ for all t. Suppose the proposition is not true: then there is a subsequence (t_n) such that $(1-\delta)k_{t_n} = k_{t_n+1}$ for all $n \in \mathbb{N}$. Since $k_{t_n} > k_{t_n+1}$ lemma 4 (monotonicity of optimal plans) implies $k_t \to 0$ thus contradicting proposition 3. ■

Continuous behavior of optimal paths under changes in initial conditions will afterwards ensure that the equilibrium price system will be a continuous function of initial conditions.

Proposition 5 *Let $(k_0^n) \subset \mathbb{R}_+$ with $k_0^n \to k_0 > 0$ and let $\mathbf{k}^n$ denote the optimal path associated with the nth element of the sequence and $\mathbf{k}$ be optimal from k_0. Then $k_t^n \to k_t$ for all t.*

Proof: By proposition 2 we have

$$v(k_0^n) = u(F(k_0^n) - k_1^n) + \beta v(k_1^n).$$

for all $n \in \mathbb{N}$. Since $k_0^n \to k_0$ we can contain (k_1^n) in a fixed compact set. Then there is a convergent subsequence $k_1^{n_j} \to k_1'$ in which case continuity of u, F, and v ensure that

$$v(k_0) = u(F(k_0) - k_1') + \beta v(k_1')$$

but $k_1' = k_1$ because of uniqueness of the solution (proposition 1). Hence, $k_1^{n_j} \to k_1$ and the result follows from this argument repeated for all periods. ■

The last result uses these properties of optimal paths, specially monotonicity, to prove convergence of optimal paths to a steady state

Proposition 6 *There is some $k^s > 0$ with $F(k^s) - k^s > 0$ and $\beta F'(k^s) = 1$ such that for all $k_0 > 0$, if $\mathbf{k}$ is optimal, then $k_t \to k^s$.*

Proof: Let $k_0 > 0$ and $\mathbf{k}$ be optimal. By lemma 1 the sequence $\mathbf{k}$ is bounded from above, by proposition 3 is bounded away from zero, and by lemma 4 is monotone. It must therefore converge to some $k^s > 0$. By continuity of u, F, and v and by proposition 2 it must be the case that

$$v(k^s) = u(F(k^s) - k^s) + \beta v(k^s).$$

Hence, proposition 2 also implies that the stationary plan every period equal to k^s is optimal from k^s in which case $F(k^s) - k^s > 0$ is a consequence of lemma 4. Proposition 4 implies that the Euler equation holds along the stationary path so

$$u'(F(k^s) - k^s) = \beta u'(F(k^s) - k^s)F'(k^s)$$

and therefore $\beta F'(k^s) = 1$ because $0 < u'(F(k^s) - k^s) < \infty$. ■

4 Existence of a competitive equilibrium

In the standard Ramsey model, when net investment is not constrained to be non negative, equilibrium prices are given by the valuation of output in the margin at each period. That is, the Arrow-Debreu price of output in period t relative to period zero is given by $p_t = \beta^t u'(c_t)$ where $\mathbf{c}$ is the equilibrium consumption allocation. Such relation between present value prices and discounted marginal utility should continue to hold in the present case; the reason is that the consumer's maximization problem is not directly affected by the non negativity constraint. Whatever happens to the firm's problem, the consumer's decision remains unaltered. This intuition is formalized in this section proving that such price system is indeed an equilibrium price system when $\mathbf{c}$ is the planner's optimal consumption choice.

In the decentralized economy the household owns the firm and the initial stock of savings x_0. If $q > 0$ denotes the price of the initial good's market and π total benefits from the firm the household's income is given by $qx_0 + \pi$. The price of output in all periods will be a sequence $\mathbf{p} \in \ell_1^+ - \{\mathbf{0}\}$ so that the household's problem is

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ s.t. } \sum_{t=0}^{\infty} p_t c_t \leq qx_0 + \pi$$

where the maximum is taken over ℓ_∞^+. The firm's problem is to maximize profits π over production plans $(k_0, \mathbf{k}) \in \mathbb{R}_+ \times \ell_\infty^+$ subject to the feasibility constraints:

it solves

$$\begin{aligned}
\max \quad & \pi = -qk_0 + \sum_{t=0}^{\infty} p_t(F(k_t) - k_{t+1}) \\
\text{s.t.} \quad & (1-\delta)k_t \leq k_{t+1} \leq f(k_t) + (1-\delta)k_t \text{ for all } t.
\end{aligned} \tag{3}$$

A competitive equilibrium is a collection composed of: an initial stock $x_0 > 0$, a consumption plan $\mathbf{c} \in \ell_\infty^+$ for the household, a production plan $(k_0, \mathbf{k}) \in \mathbb{R}_+ \times \ell_\infty^+$ for the firm, and a price system $(q, \mathbf{p}) \in \mathbb{R}_{++} \times \ell_1^+ - \{\mathbf{0}\}$ such that markets clear

$$\begin{aligned}
x_0 &= k_0 \\
c_t + k_{t+1} &= F(k_t) \text{ for all } t,
\end{aligned}$$

the plan $\mathbf{c}$ solves the household's problem at prices $(q, \mathbf{p})$, and the production plan $(k_0, \mathbf{k})$ solves the firm's problem at prices $(q, \mathbf{p})$. We claim that the planner's solution is in fact a competitive equilibrium allocation when prices are given by discounted marginal utility. (Observe that this characterization of equilibrium prices is the same as in the standard Ramsey model: the non negativity constraint on net investment only affects the firm's problem; the household still makes decisions equalizing relative prices to the marginal rate of substitution.) More formally:

Proposition 7 *Let $x_0 > 0$, then $k_0 = x_0$, $(\mathbf{c}, \mathbf{k})$ optimal from k_0, $\mathbf{p}$ defined as $p_t = \beta^t u'(c_t)$ for all t, and $q = p_0 F'(k_0)$ is a competitive equilibrium.*

The remaining of this section is devoted to the proof. First observe that $c_t > 0$ and $c_t \to c^s = F(k^s) - k^s > 0$ by lemma 4 and proposition 6 respectively. Then $u'(c_t) > 0$ is uniformly bounded from above so that $0 < \beta < 1$ implies $\mathbf{p} \in \ell_1^+ - \{\mathbf{0}\}$. Further, since $p_0 > 0$ we have $q = p_0 F'(x_0) > 0$. It is straightforward to see that if $(k_0, \mathbf{k})$ solve the firm's problem, then $\mathbf{c}$ should maximize utility over all sequences $\mathbf{c}'$ with

$$\sum_{t=0}^{\infty} p_t c_t' \leq \sum_{t=0}^{\infty} p_t c_t. \tag{4}$$

To see that this is the case let $\mathbf{c}' \in \ell_\infty^+$ verify (4) and use concavity of u to write

$$\sum_{t=0}^{\infty} \beta^t (u(c_t) - u(c_t')) \geq \sum_{t=0}^{\infty} \beta^t u'(c_t)(c_t - c_t') = \sum_{t=0}^{\infty} p_t(c_t - c_t') \geq 0.$$

In short, the planner's consumption path solves the consumer's problem at prices

$(q, \mathbf{p})$ when $(k_0, \mathbf{k})$ solves the firm's problem. It only rests to prove that the production plan indeed solves the firm's problem.

From proposition 4 there is some T for which $(1-\delta)k_t < k_{t+1} < F(k_t)$ for all $t \geq T$. Since $\mathbf{k}$ is optimal, $(k_1, ..., k_T)$ must solve

$$\begin{array}{ll} \max & \sum_{t=0}^{T} \beta^t u(F(k'_t) - k'_{t+1}) \\ \text{s.t.} & (1-\delta)k'_t \leq k'_{t+1} \leq F(k'_t) \text{ for } t = 0, ..., T, \\ & k_0 \text{ and } k'_{T+1} = k_{T+1} \text{ given.} \end{array}$$

Associate the multiplier ρ_t to the constraint $(1-\delta)k'_t \leq k'_{t+1}$ and γ_t to $k'_{t+1} \leq F(k'_t)$ for $t = 0, ..., T$. Since $k_{T+1} < F(k_T)$, the Slater condition is verified: there is a set of multipliers $\rho_t, \gamma_t \geq 0$ for $t = 0, ..., T$ such that $(k_t, \rho_t, \gamma_t)_{t=0}^T$ maximizes the associated Lagrangian. By lemma 4 we know that $\gamma_t = 0$ for all $t = 0, ..., T$. Hence, Kuhn-Tucker first order conditions are

$$-\beta^t u'(F(k_t) - k_{t+1}) + \beta^{t+1} u'(F(k_{t+1}) - k_{t+2})F'(k_{t+1}) + \rho_t - \rho_{t+1}(1-\delta) = 0 \quad (5)$$

for $t = 0, ..., T-1$ while the Euler equations hold

$$-\beta^t u'(F(k_t) - k_{t+1}) + \beta^{t+1} u'(F(k_{t+1}) - k_{t+2})F'(k_{t+1}) = 0 \quad (6)$$

for $t \geq T$. For any $\mathbf{k}' \in \Pi(k_0)$ and any $T' \geq T$ define

$$\begin{aligned} \varphi(T', \mathbf{k}') &= \sum_{t=0}^{T'} p_t(F(k_t) - k_{t+1}) - p_t(F(k'_t) - k'_{t+1}) \\ &= \sum_{t=0}^{T'} \beta^t u'(c_t)[(F(k_t) - k_{t+1}) - (F(k'_t) - k'_{t+1})]. \end{aligned}$$

If $\mathbf{k}$ solves the firm's problem at prices $(q, \mathbf{p})$ it must be the case that $\lim_{T' \to \infty} \varphi(T', \mathbf{k}') \geq 0$. First note that F is concave so that

$$\varphi(T', \mathbf{k}') \geq \sum_{t=0}^{T'} \beta^t u'(c_t)[F'(k_t)(k_t - k'_t) - (k_{t+1} - k'_{t+1})].$$

Rearranging terms we can write

$$\begin{aligned} \varphi(T', \mathbf{k}') \geq\ & u'(c_0)F'(k_0)(k_0 - k'_0) - u'(c_0)(k_1 - k'_1) \\ & + \beta u'(c_1)F'(k_1)(k_1 - k'_1) - \beta u'(c_1)(k_2 - k'_2) \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&+\beta^{T'}u'(c_{T'})F'(k_{T'})(k_{T'}-k'_{T'}) - \beta^{T'}u'(c_{T'})(k_{T'+1}-k'_{T'+1})
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\varphi(T',\mathbf{k}') \;\geq\; & [-u'(c_0)+\beta u'(c_1)F'(c_1)](k_1-k'_1) \\
& +[-\beta u'(c_1)+\beta^2 u'(c_2)F'(c_2)](k_2-k'_2) \\
& \vdots \\
& +[-\beta^{T'-1}u'(c_{T'-1})+\beta^{T'}u'(c_{T'})F'(c_{T'})](k_{T'}-k'_{T'}) \\
& -\beta^{T'}u'(c_{T'})(k_{T'+1}-k'_{T'+1}).
\end{aligned}
$$

For $t \geq T$ the Euler equation (6) holds so that terms between T and T' vanish while (5) allows us to write

$$
\varphi(T',\mathbf{k}') \geq -\beta^{T'}u'(c_{T'})(k_{T'+1}-k'_{T'+1}) + \sum_{t=0}^{T-1}(-\rho_t+\rho_{t+1}(1-\delta))(k_{t+1}-k'_{t+1}).
$$

Rearraging terms

$$
\begin{aligned}
\varphi(T',\mathbf{k}') \;\geq\; & -\beta^{T'}u'(c_{T'})(k_{T'+1}-k'_{T'+1}) \\
& -\rho_0 k_1+\rho_1(1-\delta)k_1+\rho_0 k'_1-\rho_1(1-\delta)k'_1 \\
& -\rho_1 k_2+\rho_2(1-\delta)k_2+\rho_1 k'_2-\rho_2(1-\delta)k'_2 \\
& \vdots \\
& -\rho_{T-1}k_T+\rho_T(1-\delta)k_T+\rho_{T-1}k'_T-\rho_T(1-\delta)k'_T
\end{aligned}
$$

but $k'_{t+1}-(1-\delta)k'_t \geq 0$ and $k_{t+1}-(1-\delta)k_t \geq 0$ by feasibility while in this expression they appear with negative sign and multiplied by $\rho_t \geq 0$. Hence, the inequality is not altered if we suppress these terms:

$$
\begin{aligned}
\varphi(T',\mathbf{k}') \;\geq\; & -\beta^{T'}u'(c_{T'})(k_{T'+1}-k'_{T'+1}) \\
& -\rho_0 k_1+\rho_0 k'_1+\rho_T(1-\delta)k_T-\rho_T(1-\delta)k'_T.
\end{aligned}
$$

Since $k_1=(1-\delta)k_0$ and $k'_1=(1-\delta)k_0$, add and substract $\rho_0(1-\delta)k_0$ at the right hand side of this inequality and note that $\rho_T=0$ because $k_{T+1}>(1-\delta)k_T$ to conclude that

$$
\varphi(T',\mathbf{k}') \geq -\beta^{T'}u'(c_{T'})(k_{T'+1}-k'_{T'+1}) \geq -\beta^{T'}u'(c_{T'})k_{T'+1}
$$

but $u'(c_{T'})$ and $k_{T'+1}$ are bounded from above while $\beta^{T'}\to 0$. Then $\varphi(\infty,\mathbf{k}') \geq 0$

as was to be shown.

5 Existence without the Inada-type condition

The Inada-type condition on the utility function are not necessary for the existence of an equilibrium. In this section we substitute assumption 2 by:

Assumption 5 The one period reward function u is differentiable in the interior of its domain. Further, $u(0) = 0$ and $u'(0) < \infty$.

Fix some $\theta \in (0, 1)$ and for all $\varepsilon > 0$ define $u_\varepsilon(c) = u(c) + \varepsilon c^\theta$ for all $c \geq 0$ so that u_ε converges pointwise to u as $\varepsilon \to 0$. Clearly, u_ε does meet assumption 2: fixed $x_0 > 0$, for all $\varepsilon > 0$ proposition 7 ensures existence of an competitive equilibrium $(\mathbf{c}^\varepsilon, \mathbf{k}^\varepsilon, q^\varepsilon, \mathbf{p}^\varepsilon)$. Observe that $A(k_0)$ does not depend on preferences: $\mathbf{k}^\varepsilon$ and $\mathbf{c}^\varepsilon$ are contained in the compact set $[0, A(k_0)]^\infty$. Then there is a subnet (denote it again ε) and $(\mathbf{c}^0, \mathbf{k}^0)$ such that $\mathbf{k}^\varepsilon \to \mathbf{k}^0$ and $\mathbf{c}^\varepsilon \to \mathbf{c}^0$ pointwise. As $\Pi(k_0)$ is closed, $(\mathbf{c}^0, \mathbf{k}^0)$ is feasible. Finally observe that k^s is implicitly defined as $\beta F'(k^s) = 1$ (proposition 6) so that $k_t^\varepsilon \to k^s$ as $t \to \infty$ for any ε. The following proposition ensures that the net of price systems also converges.

Proposition 8 *Let $k_0 > 0$, then $u'_\varepsilon(c_t^\varepsilon)$ is uniformly bounded from above.*

Proof: Suppose that $0 < k_0 < k^s$. Since $F^t(k_0) \to \bar{k} > k^s$ we can fix some T such that $F^{T+1}(k_0) > k^s$. For any $\varepsilon > 0$ we know (lemma 4 and proposition 6) that $k_t^\varepsilon \leq k_{t+1}^\varepsilon$ and $k_t^\varepsilon \to k^s$. Hence, the Euler equations hold and

$$u'_\varepsilon(c_t^\varepsilon) = \beta u'_\varepsilon(c_{t+1}^\varepsilon) F'(k_{t+1}^\varepsilon) > u'_\varepsilon(c_{t+1}^\varepsilon)$$

because $k_{t+1}^\varepsilon < k^s$ so that $\beta F'(k_{t+1}^\varepsilon) > 1$. Then, $u'_\varepsilon(c_0^\varepsilon) > u'_\varepsilon(c_t^\varepsilon)$ for all $t \geq 1$. If $u'_\varepsilon(c_0^\varepsilon)$ is bounded from above uniformly over ε the result follows. Otherwise it must be the case that $c_0^\varepsilon \to 0$ in which case the Euler equation

$$u'(c_0^\varepsilon) + \varepsilon (c_0^\varepsilon)^{\theta - 1} = \beta (u'(c_1^\varepsilon) + \varepsilon (c_1^\varepsilon)^{\theta - 1}) F'(k_{t+1}^\varepsilon)$$

requires $c_1^\varepsilon \to 0$ (because the remaining elements of the equality converge to finite numbers). Proceed recursively up to date T to obtain $c_0^\varepsilon, ..., c_T^\varepsilon \to 0$ and therefore $k_{T+1}^\varepsilon \to F^{T+1}(k_0)$. Then, for ε small enough $k_{T+1}^\varepsilon > k^s$. A contradiction.

The case $k_0 = k^s$ is trivial because $u'_\varepsilon(c_t^\varepsilon) = u'_\varepsilon(c^s) \to u'(c^s)$.

Finally let $k_0 > k^s$. Since $k_t^\varepsilon \geq k_{t+1}^\varepsilon$ and $k_t^\varepsilon \to k^s$ there must be some T with $k_{t+1}^\varepsilon > (1-\delta)k_t^\varepsilon$ for all $t \geq T$. For $0 \leq t < T$ we have $c_t^\varepsilon = f(k_t^\varepsilon)$ so that consumption is decreasing (because k_t^ε is so). From T on, the Euler equations hold and

$$u_\varepsilon'(c_t^\varepsilon) = \beta u_\varepsilon'(c_{t+1}^\varepsilon)F'(k_{t+1}^\varepsilon) < u_\varepsilon'(c_{t+1}^\varepsilon)$$

because $k_{t+1}^\varepsilon > k^s$ so that $\beta F'(k_{t+1}^\varepsilon) < 1$. Then $c_t^\varepsilon > c_{t+1}^\varepsilon$ for all $t \geq T$. In short, $c_t^\varepsilon > c_{t+1}^\varepsilon > c^s$ and therefore $u_\varepsilon'(c_t^\varepsilon) < u_\varepsilon'(c^s) \to u'(c^s) < \infty$. ■

By the previous proposition there is some $M > 0$ such that $0 < u_\varepsilon'(c_t^\varepsilon) \leq M$ so that there must be a subsubnet (again denoted ε) and an element $\mathbf{p}^0 \in \ell_1^+ - \{\mathbf{0}\}$ such that $\mathbf{p}^\varepsilon \to \mathbf{p}^0$ pointwise. Write then $q^0 = p_0^0 F'(k_0) > 0$.

Proposition 9 *Let* $(\mathbf{c}^0, \mathbf{k}^0, \mathbf{p}^0, q^0)$ *be as defined above. Then it is a competitive equilibrium when preferences are represented by the one period utility function* u.

Proof: The proof of proposition 7 established optimality of $\mathbf{k}^\varepsilon$ for the firm's problem at prices $(\mathbf{p}^\varepsilon, q^\varepsilon)$ for all $\varepsilon > 0$. Then, for all $\varepsilon > 0$ and any $\mathbf{k}' \in \Pi(k_0)$ we have $\pi(\mathbf{k}', \mathbf{p}^\varepsilon, q^\varepsilon) < \pi(\mathbf{k}^\varepsilon, \mathbf{p}^\varepsilon, q^\varepsilon)$. Taking the limit as $\varepsilon \to 0$ yields $\pi(\mathbf{k}', \mathbf{p}^0, q^0) \leq \pi(\mathbf{k}^0, \mathbf{p}^0, q^0)$ so that $\mathbf{k}^0$ solves the firm's problem at prices $(\mathbf{p}^0, q^0)$.

With this result at hand the same argument as in the proof of proposition 7 ensures that $\mathbf{c}^0$ solves the household's problem at prices $(\mathbf{p}^0, q^0)$. The goods markets clear because F is continuous and $(\mathbf{c}^\varepsilon, \mathbf{k}^\varepsilon) \to (\mathbf{c}^0, \mathbf{k}^0)$ pointwise. ■

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