Quality of Knowledge Technology, Returns to Production Technology and Economic Development[*]

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November 2, 2001

Abstract

Presenting a discrete time version of the Romer (1986) model, this paper analyzes optimal paths in a one-sector growth model when the technology is not convex. We prove that for a given quality of knowledge technology, the countries could take-off if their initial stock of capital are above a critical level; otherwise they could face a poverty-trap. We show that for an economy which wants to take-off by means of knowledge technology requires three factors: large amount of initial knowledge, small fixed costs and a good quality of knowledge technology.

Keywords: Optimal growth, optimal path, value function, poverty-trap, increasing returns.

Journal of Economic Literature: C61, O12, O32, O41.

[*] We thank Raouf Boucekkine for comments and suggestions. The second author has been supported by a grant "Actions Recherches Concertées" of the Ministry of the Belgian French Speaking Community.

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1 Introduction

Convex structures of the technology and prefences have played an important role in economic analysis of optimal one-sector growth models. They guarantee that the sequence of optimal stocks moves monotonically towards a unique steady state. (as in Cass (1965) and Koopmans (1965)). In these models, per-capita output should converge to a steady state due to the assumption of diminishing returns to per-capita capital in the production of per-capita output. However these studies were unable to explain the non-convergence of countries whose potential causes could be the different time preferences, technologies, demographies, market structures or economic policies.

In a model of endogenous technological change in which the knowledge accumulated by the agents is the basic form of capital, Romer (1986) relaxing this usual assumption of diminishing returns showed that per-capita output can grow without bound and the level of per-capita output accross different countries need not converge. In this analysis, new technology created by a single firm which has a positive external effect on the other firms is assumed to be the product of a research technology that exhibits diminishing returns. Thus, whereas production as a function of the firm exhibits diminishing returns, production as a function of the stock of knowledge in the economy is assumed to exhibit increasing returns.

On the other hand, Majumdar and Mitra (1982), Dechert and Nishimura (1983) analyzed an optimal growth model with a non-convex technology. Their key result was that the sequence of capital stocks is necessarily monotonic and under some assumptions they exhibit a poverty trap. Extending the analysis to an open country, Askenazy and Le Van (1999) in a continous time framework and Dimaria and Le Van (2001) in a discrete time framework also showed that if the debt constraint is hard, it could be optimal for a poor country to collapse while a rich country to converge to a high level of steady state.

In this paper, following Le Van, Morhaim and Dimaria (2001), we present a discrete time version of the Romer (1986) model and relax a fundamental hypothesis: nonconcavity of the production function. We analyze the case of a developing country with a production technology that exhibits linear production-capital ratio at the early stages of industrialization. Then for higher capital stocks, the production function becomes concave as in the case of a developed country. We prove the existence of solutions to the social-planner problem and characterize the properties of the optimal paths. We show that a for a given quality of knowledge technology, the countries could take-off if their initial stock of capital are above a critical level; otherwise they could face a poverty-trap. We show that even a developed country may face a poverty-trap if endowed with a low quality of knowledge technology. Between two countries with the same production function and the same level of initial capital stock, when one faces a poverty-trap, the other could take-off if it is endowed with a higher quality of knowledge technology. We show that the differences in the quality of knowledge technology between countries can provide an explanation for the non-convergence of the countries.

The paper is organized as follows. In Section 2, presenting the model and its assumptions, we study the existence of solutions to the social-planner problem and analyze the properties and the convergence of optimal paths. Finally, Section 3 concludes.

2 The Infinite-Horizon Growth Model

We consider a closed economy in which the preferences of the S identical consumers are globally represented by a strictly concave utility function of consumption, $u(c)$: The assumption is that:

(U1) $u(c)$ is twice continously differentiable, $u'(c) > 0$; $u''(c) < 0$; $\forall c > 0$ and $u(0) = 0$; $u'(0) = +\infty$:

The instantaneous production of output for a firm is given by $\tilde{F}(k_t; K_t; x_t)$; which depends on the firm specific knowledge (k_t), the aggregate knowledge (K_t), and the level of all other factors such as physical capital, labour, etc. To simplify and to have per-firm and per-capita values coincide, we restrict our attention to an equilibrium in which the number of firms and the number of consumers are equal by assuming that $S = N = 1$: Following Romer (1986) and Le Van, Morhaim and Dimaria (2001), we assume that the additional factors are fixed in supply so that the optimal solution for x is $\bar{x}$. Dropping $\bar{x}$ from the production function, let f; h and F be:

$$\tilde{F}(k; K; \bar{x}) = f(k)\,h(K)$$
$$F(k) := \tilde{F}(k; k; \bar{x}) = f(k)\,h(k):$$

We consider two cases:

i) a developed country where the production function is concave:

$$f(k) = k^{\alpha}; \ \alpha \in]0;1[;$$
$$h(k) = k^{\rho}; \ \rho > 0$$

ii) a developing country where the production function is a linear function in an initial phase and concave afterwards:

$$f(k) = \begin{cases} \delta k; & k \le \bar{k} \\ A + k^{\mu}; & k \ge \bar{k} \end{cases} \quad \text{and}$$
$$h(k) = k^{\rho}; \ \rho > 0$$

with $\delta - \mu \bar{k}^{\mu-1} < 0$; $0 < \delta < 1$; $0 < \mu < 1$ and $1 < \mu + \rho$. Note that $A < 0$ as $A = \delta\bar{k} - \bar{k}^{\mu} = \bar{k}\left(\delta - \bar{k}^{\mu-1}\right) < \bar{k}\left(\mu\bar{k}^{\mu-1} - \bar{k}^{\mu-1}\right) = \bar{k}^{\mu}(\mu - 1) < 0$.

The influence of the fixed costs on the output can be measured through two indicators which are $\bar{k}$ and δ. The fixed costs are more important if $\bar{k}$ increases or/and if δ decreases.

Investing an amount I_t of forgoing consumption, a firm with a current stock of private knowledge k_t produces additional knowledge which induces a rate of growth

$$k_{t+1} - k_t = G(I_t; k_t)$$

Assume that:

(G1) G is concave and homogenous of degree one. Then:

$$\frac{k_{t+1} - k_t}{k_t} = G\left(\frac{I_t}{k_t}; 1\right) = g\left(\frac{I_t}{k_t}\right)$$

(G2) $g(0) = 0$; $g'(0) = \frac{1}{\lambda} < +\infty$

(G3) $0 \le g(y) \le \alpha$.

For an arbitrary path K, the social optimization problem maximizes the utility of a representative consumer subject to the technology implied by the path K.

$$\text{Maximize} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t.:} \quad 0 \le \frac{k_{t+1} - k_t}{k_t} \le g\left(\frac{F(k_t) - c_t}{k_t}\right)$$
$$k_0 > 0; \text{given.}$$

Note that in social optimization problem, the production function exhibits an initial phase of increasing returns and a second phase with decreasing returns. Note also that this problem is equivalent to:

$$\text{Maximize} \sum_{t=0}^{\infty} \beta^t u\left(F(k_t) - k_t \gamma\left(\frac{k_{t+1} - k_t}{k_t}\right)\right)$$
$$\text{s.t.: } k_t \leq k_{t+1} \leq k_t g\left(\frac{F(k_t)}{k_t}\right) + k_t$$
$$k_0 > 0, \text{ given.}$$

where $\gamma := g^{-1}$.

Following from (G3), note that $k_t \leq k_{t+1} \leq k_t(1+\alpha)$. Then we have $k_t \leq k_0(1+\alpha)^t$ inducing that, $F(k_t) \leq F(k_0(1+\alpha)^t) \leq \left[\beta k_0^{\rho+1}(1+\alpha)^{(\rho+1)}\right]^t$.

In what follows we assume that:

(P1) $0 < \beta < 1$ and $\beta(1+\alpha)^{(\rho+1)} < 1$.

Denoting $g\left(\frac{F(\bar{k})}{\bar{k}}\right)$ by $\bar{g} < \alpha$, we also assume that:

(P2) $\gamma(x) = \lambda x$, when $x \in [0, \bar{g}]$.

Note that at zero an increase of one percent in the amount of investment induces an increase of $1/\lambda$ percent in the stock of private knowledge which enables us to interpret that $1/\lambda$ reflects the quality of the knowledge technology.

2.1 Existence of a Solution

A sequence $\vec{k} = (k_t)_t$ is called feasible from k_0 if it satisfies the constraints of the social optimization problem:

$$\forall t, k_t \leq k_{t+1} \leq k_t g\left(\frac{F(k_t)}{k_t}\right) + k_t$$

In this section, we first prove that every feasible sequence from k_0 belongs to a compact set for the product topology; second we show that the objective function is continous for this topology. Existence of solutions follows from these results.

2.1.1 Compactness

Let $\vec{k}$ be a feasible path from k_0. Then by assumption (G3), for every t:

$$k_t \leq k_{t+1} \leq k_t g\left(\frac{F(k_t)}{k_t}\right) + k_t \leq (1+\alpha)k_t$$

so, $k_t \le (1+\alpha)^t k_0$; that is:

$$\tilde{k} \in \prod_{t=0}^{+\infty} \left[k_0; (1+\alpha)^t k_0\right]:$$

Thus, $\tilde{k}$ belongs to a compact set for the topology. Since g and F are continous, the feasible set from k_0 is compact for the product topology.

2.1.2 Continuity of the objective function

The objective function is:

$$U: \tilde{k} \to \sum_{t=0}^{+\infty} \beta^t u\left(F(k_t) - k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right):$$

By means of the proof available in Le Van, Morhaim and Dimaria (2001), we know that U is continous for the product topology.

Thus, with the objective function being continous and the feasible path set being compact for the product topology, the problem has a solution.

2.2 Value function, Bellman equation

Let $V(k_0)$ denote the value function of the social optimization problem. It is clear that the value function verifies the Bellman equation:

$$V(k_0) = \max_{k_0 \le y \le k_0 g\left(\frac{F(k_0)}{k_0}\right) + k_0} \left\{ u\left(F(k_0) - k_0\gamma\left(\frac{y-k_0}{k_0}\right)\right) + \beta V(y)\right\}$$

V is the unique continous solution to Bellman equation. Let

$$\varphi(k_0) = \operatorname*{argmax}_{k_0 \le y \le k_0 g\left(\frac{F(k_0)}{k_0}\right) + k_0} \left\{ u\left(F(k_0) - k_0\gamma\left(\frac{y-k_0}{k_0}\right)\right) + \beta V(y)\right\}:$$

By the maximum theorem, φ is upper semi-continous.

2.3 Properties and convergence of optimal paths

In this section we derive some properties of optimal paths. First, we show the non-nullity of optimal consumption and capital. Second, we prove that there exists a critical value k_c which may be interpreted as the "top" of the poverty trap.

Lemma 1 $\frac{F(k)}{k}$ and $F'(k)$ are increasing functions in $[0;\bar{k}[$ and $]\bar{k};+\infty[$.

Proof. i) $\left(\frac{F(k)}{k}\right)' = \frac{1}{k^2}[F'(k)k - F(k)]$. What follows from the definition of $F(k)$ is that:

$$F'(k)k - F(k) = \begin{cases} \delta\rho k^{\rho+1}; & k \in [0;\bar{k}[\\ A(\rho-1)k^{\rho} + \mu k^{\mu+\rho} + (\rho-1)k^{\mu-\rho}; & k \in]\bar{k};+\infty[. \end{cases}$$

There are two cases to be checked when $k \in]\bar{k};+\infty[$:

$\rho \leq 1 \Rightarrow F'(k)k - F(k) = k^{\rho}[A(\rho-1) + (\mu+\rho-1)k^{\mu}] > 0$ as $A = \delta\bar{k} - \bar{k}^{\mu} < 0$ and

$\rho > 1 \Rightarrow F'(k)k - F(k) = k^{\rho}[\mu k^{\mu} + (\rho-1)(k^{\mu}+A)] > 0$ as $k^{\mu} + A \geq \bar{k}^{\mu} + A = \delta\bar{k}$. Thus $\frac{F(k)}{k}$ is an increasing function in $[0;\bar{k}[$ and $]\bar{k};+\infty[$.

ii)

$$F''(k) = \begin{cases} \delta(\rho+1)\rho k^{\rho-1}; & k \in [0;\bar{k}[\\ A\rho(\rho-1)k^{\rho-2} + (\mu+\rho)(\mu+\rho-1)k^{\mu+\rho-2}; & k \in]\bar{k};+\infty[. \end{cases}$$

It is clear that $F''(k) > 0$ for $k \in [0;\bar{k}[$. There are two cases to be checked when $k \in]\bar{k};+\infty[$:

$\rho \leq 1 \Rightarrow F''(k) = k^{\mu+\rho-2}[(\mu+\rho)(\mu+\rho-1) + A\rho(\rho-1)k^{-\mu}] > 0$ and
$\rho > 1 \Rightarrow F''(k) = k^{\rho-2}[(k^{\mu}+A)(\rho-1)\rho + (\mu+\rho)\mu k^{\mu} + \mu(\rho-1)k^{\mu}] > 0$.
Thus, $F'(k)$ is an increasing function in $[0;\bar{k}[$ and $]\bar{k};+\infty[$. ■

Lemma 2 $\frac{\partial^2 W(k;y)}{\partial k \partial y} > 0$ where $W(k;y) = u\left(F(k) - k^{\gamma}\left(\frac{y-k}{k}\right)\right)$.

Proof. See Le Van, Morhaim and Dimaria (2001). ■

Let $B(x) = xg(\frac{F(x)}{x}) + x$.

Lemma 3 Our optimal growth model can be written as:

$$\max \sum_{t=0}^{\infty} \beta^t W(x_t; x_{t+1})$$
$$\text{s.t. } x_t \leq x_{t+1} \leq B(x_t)$$
$$x_0 > 0 \text{ is given}$$

which leads to the value function verifying the Bellman equation given as

$$V(x_0) = \max_{x_0 \leq y \leq B(x_0)} [W(x_0;y) + \beta V(y)]$$

where B is an increasing function. Let $\{x_t\}$ and $\{x_t^0\}$ be optimal paths starting from x_0 and x_0^0 respectively. If $x_0 < x_0^0$ then $x_1 \leq x_1^0$.

Proof. The result comes from the fact that $\frac{\partial^2 W(k,y)}{\partial k \partial y} > 0$. See Benhabib and Nishimura (1985). ∎

Proposition 1 If $\{k_t, c_t\}$ is an optimal path from $k_0 > 0$, then $\forall t \geq 0$ $k_t > 0$, $c_t > 0$. Similarly if $k_1 \in \varphi(k_0)$ then $F(k_0) - k_0 \gamma\left(\frac{k_1 - k_0}{k_0}\right) > 0$.

Proof. The proof is based on Inada condition of the utility function. See Le Van, Morhaim and Dimaria (2001). ∎

Proposition 2 Let $\lambda > 0$ be given. There exists $k^* > 0$ such that $\forall k_0 < k^*, k_0 > 0$, $\forall \underline{k}$ optimal from k_0 then $k_t = k_0, \forall t$.

Proof. Choose k^* such that $g\left(\frac{F\left(k^*\left(1+g\left(\frac{F(k^*)}{k^*}\right)\right)\right)}{k^*\left(1+g\left(\frac{F(k^*)}{k^*}\right)\right)}\right) \leq \bar{g}$ and $\frac{F'(k^*)}{\lambda} + (1+\alpha) < \frac{1}{\beta}$. Since $\frac{F(k)}{k}$ is increasing by Lemma 1 and $\lim_{k \to 0} \frac{F(k)}{k} = 0$, such a k^* exists. Let for $k_0 < k^*$, there exists a strictly increasing optimal path. We have Euler equation given as:

$$u'(c_t)\gamma'\left(\frac{k_{t+1}}{k_t} - 1\right) = \beta u'(c_{t+1})\left[F'(k_{t+1}) - \gamma\left(\frac{k_{t+2}}{k_{t+1}} - 1\right) + \gamma'\left(\frac{k_{t+2}}{k_{t+1}} - 1\right)\left(\frac{k_{t+2}}{k_{t+1}}\right)\right].$$

Our claim is that there exists t such that $k_{t+1} < k^*$ and $k_{t+2} \geq k^*$. If not, we have allways $k_t < k^*, \forall t$. Then $k_t \to k_s \geq k_0 > 0$ and $c_t \to F(k_s) > 0$ with $F'(k_s) = \frac{\lambda(1-\beta)}{\beta}$ due to Euler equation. Note that we have $\gamma'\left(\frac{k_{t+1}}{k_t} - 1\right) = \lambda, \forall t$ as $\left(\frac{k_{t+1}}{k_t} - 1\right) < g\left(\frac{F(k_t)}{k_t}\right) < g\left(\frac{F(k^*)}{k^*}\right) < g\left(\frac{F\left(k^*\left(1+g\left(\frac{F(k^*)}{k^*}\right)\right)\right)}{k^*\left(1+g\left(\frac{F(k^*)}{k^*}\right)\right)}\right) \leq \bar{g}$ and $\gamma(x) = \lambda x$ for $x \in [0, \bar{g}]$ according to Assumption (P2).

However, since F' is increasing in $[0, \bar{k}[$ and $]\bar{k}, +\infty[$ by Lemma 1 and $k_s \leq k^*$, $F'(k_s) \leq F'(k^*) < \frac{\lambda(1-\beta)}{\beta}$ leads to a contradiction.

Hence there exists t such that $k_{t+1} < k^*$ and $k_{t+2} \geq k^*$. In what follows, consider a path $\{k_0^n\}$ converging to zero and let $T_n(k_0^n)$ be the point such that

$$k_{T_n(k_0^n)+1} < k^* \text{ and } k_{T_n(k_0^n)+2} \geq k^*.$$

From $t=0$ to $T_n(k_0^n)+1$ we have:

$$u'(c_t^n)\gamma'\left(\frac{k_{t+1}^n}{k_t^n}-1\right) = \beta u'(c_{t+1}^n)\left[F'(k_{t+1}^n) - \gamma\left(\frac{k_{t+2}^n}{k_{t+1}^n}-1\right) + \gamma'\left(\frac{k_{t+2}^n}{k_{t+1}^n}-1\right)\left(\frac{k_{t+2}^n}{k_{t+1}^n}\right)\right]$$

which reduces to

$$u'(c_t^n) = \beta u'(c_{t+1}^n)\left[\frac{F'\left(k_{t+1}^n\right)}{\lambda} + (1+\alpha)\right];$$

as $\left(\frac{k_{t+2}}{k_{t+1}}-1\right) \le g\left(\frac{F(k_{t+1})}{k_{t+1}}\right) \le \frac{F(k_{t+1})}{k_{t+1}} \le \frac{F\left(k^*\left(1+g\left(\frac{F(k^*)}{k^*}\right)\right)\right)}{k^*\left(1+g\left(\frac{F(k^*)}{k^*}\right)\right)} \le \bar{g}$. Hence, $u'(c_t^n) < \beta u'(c_{t+1}^n)\left[\frac{F'(k^*)}{\lambda}+(1+\alpha)\right] < u'(c_{t+1}^n)$ so that $c_{t+1}^n < c_t^n$; $\forall t = 0,\ldots,T_n+1$. Now letting $n \to \infty$, $k_{T_n(k_0^n)+1}$ converges to $\hat{k} \le k^*$ and $k_{T_n(k_0^n)+2}$ converges to $\hat{k}' \ge k^*$. By the upper semi-continuity of φ, $\hat{k}' \in \varphi\left(\hat{k}\right)$. Since $\hat{k} > 0$, from proposition 1, $\hat{k}' > 0$. Then $c^n_{T(k_0^n)+1} \to F(\hat{k}) - \hat{k}\gamma\left(\frac{\hat{k}'-\hat{k}}{\hat{k}}\right)$. Following from Inada condition we have that $F(\hat{k}) - \hat{k}\gamma\left(\frac{\hat{k}'-\hat{k}}{\hat{k}}\right) > 0$ on the one hand. But $c^n_{T(k_0^n)+1} < c_0^n < F(k_0^n)$. Then $c^n_{T(k_0^n)+1}$ must converge to zero on the other hand: a contradiction.

Hence for k_0 small enough, $(k_0, k_0, \ldots k_0 \ldots)$ is the unique optimal path from k_0. ■

Proposition 3 *Let $\lambda > 0$ be given. There exists $k^{**} > 0$ such that $\forall k_0 > k^{**}$, $\forall \mathbf{k}$ optimal from k_0 then $k_t < k_{t+1}, \forall t$ and $k_t \to +\infty$.*

Proof. Choose k^{**} such that $F'(k^{**})\frac{\beta}{1-\beta} = \lambda$. Our claim is that if $k_0 > k^{**}$, then any optimal path is strictly increasing and grows without bound. In order to prove this, we will first show that for any $k_0 > k^{**}$, the path $\mathbf{k}_0 = (k_0, k_0, \ldots, k_0, \ldots)$ is not optimal. Then we will show that any optimal path is strictly increasing and no optimal path from $k_0 > k^{**}$ converges to a steady state.

i) Consider the path $\mathbf{k}_1 = (k_0, k_0+\varepsilon, k_0+\varepsilon, \ldots, k_0+\varepsilon, \ldots)$ that is feasible from k_0: $k_0 \le k_0 < k_0 g\left(\frac{F(k_0)}{k_0}\right) + k_0$. Since $k_0 g\left(\frac{F(k_0)}{k_0}\right) > 0$, there exists $\varepsilon > 0$ such that $k_0 + \varepsilon \le k_0 g\left(\frac{F(k_0)}{k_0}\right) + k_0$. Then for $k_1 := k_0 + \varepsilon$ and $k_t := k_1$;

$\forall t \geq 1$; we have $k_0 \leq k_1 \leq k_0 g\left(\frac{F(k_0)}{k_0}\right) + k_0$ and $k_1 \leq k_1 \leq k_1 g\left(\frac{F(k_1)}{k_1}\right) + k_1$ that is $k_t \leq k_{t+1} \leq k_t g\left(\frac{F(k_t)}{k_t}\right) + k_t$. Thus there exists $\varepsilon > 0$ such that $\tilde{k}_1$ is feasible from k_0. Now we will show that such a path increases the value of U:

$$\begin{aligned}
U(\tilde{k}_1) &= u\left(F(k_0) - k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right) + \sum_{t=1}^{\infty}\beta^t u(F(k_0+\varepsilon)) \\
&= u\left(F(k_0) - k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right) + \frac{\beta}{1-\beta}u(F(k_0+\varepsilon)). \\
U(\tilde{k}_0) &= u(F(k_0)) + \frac{\beta}{1-\beta}u(F(k_0)).
\end{aligned}$$

Then,

$$\begin{aligned}
U(\tilde{k}_1) - U(\tilde{k}_0) = {} & \left[u\left(F(k_0) - k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right) - u(F(k_0))\right] \\
& + \frac{\beta}{1-\beta}\left[u(F(k_0+\varepsilon)) - u(F(k_0))\right]
\end{aligned}$$

$$\begin{aligned}
& \geq u'\left(F(k_0) - k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right)\left(-k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right) \\
& \qquad + \frac{\beta}{1-\beta}u'(F(k_0+\varepsilon))(F(k_0+\varepsilon) - F(k_0)) \\
& \geq \varepsilon\left[u'\left(F(k_0) - k_0\gamma\left(\frac{\varepsilon}{k_0}\right)\right)\left(-\frac{k_0}{\varepsilon}\gamma\left(\frac{\varepsilon}{k_0}\right)\right) + \frac{\beta}{1-\beta}u'(F(k_0+\varepsilon))F'(k_0)\right].
\end{aligned}$$

Since $\lim_{\varepsilon\to 0}\frac{k_0}{\varepsilon}\gamma\left(\frac{\varepsilon}{k_0}\right) = \gamma'(0) = \lambda$, F' is increasing and $k_0 > k^{**}$, we have $F'(k_0) > \lambda\frac{(1-\beta)}{\beta}$. Thus, there exists $\varepsilon > 0$ such that $U(\tilde{k}_1) - U(\tilde{k}_0) > 0$ concluding that $\tilde{k}_0$ is not optimal.

ii) If there were an optimal path such that $k_1 = k_0$ then as the value function verifies the Bellman equation:

$$k_0 = k_1 \in \underset{k_0 \leq y \leq k_0 g\left(\frac{F(k_0)}{k_0}\right) + k_0}{\arg\max}\left\{u\left(F(k_0) - k_0\gamma\left(\frac{y-k_0}{k_0}\right)\right) + \beta V(y)\right\}$$

the path $\tilde{k}_0 = (k_0; k_0; ...; k_0; ...)$ would then be optimal which is impossible. So necessarily, $k_1 > k_0$. Similarly what follows is that $\forall t$, $k_{t+1} > k_t$.

iii) Writing the Euler equation :

$$
\begin{aligned}
&-\gamma'\left(\frac{k_{t+1}-k_t}{k_t}\right)u'\left(F(k_t)-k_t\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)\right)+\\
&\beta\left[F'(k_{t+1})-\gamma\left(\frac{k_{t+2}-k_{t+1}}{k_{t+1}}\right)+\gamma'\left(\frac{k_{t+2}-k_{t+1}}{k_{t+1}}\right)\left(\frac{k_{t+2}}{k_{t+1}}\right)\right]\\
&\qquad u'\left(F(k_{t+1})-k_{t+1}\gamma\left(\frac{k_{t+2}-k_{t+1}}{k_{t+1}}\right)\right)=0.
\end{aligned}
$$

Since k_t is strictly increasing, $\lim_{t\to\infty} k_t$ exists. Suppose on the contrary that k_t converges to $k_s < +\infty$. Then we would have

$$F'(k_s)\frac{\beta}{1-\beta}=\lambda.$$

But $F'_+(k^{**})\frac{\beta}{1-\beta}=\lambda$ and $k_s > k^{**}$: a contradiction since F' is increasing. So k_t diverges to $+\infty$ inducing that per-capita output grows without bound.
■

Proposition 4 Let k^{**} be defined by $F'(k^{**})\frac{\beta}{1-\beta}=\lambda$. Let $k_0 > 0$. No optimal path from $k_0 \neq k^{**}$ converges to k^{**}.

Proof. If $k_0 > k^{**}$, from proposition 3, any optimal path from k_0 will converge to $+\infty$. Consider the case where $k_0 < k^{**}$. Let $\{k_t\}$ be an optimal path from k_0 and assume it converges to k^{**}. This optimal path must be increasing and bounded above by k^{**}. There exists T_0 such that $\forall t \geq T_0$, $\frac{k_{t+1}-k_t}{k_t} \leq \bar{g}$ and hence $\gamma\left(\frac{k_{t+1}-k_t}{k_t}\right)=\lambda\frac{k_{t+1}-k_t}{k_t}$. The optimal consumption, for $t \geq T_0$ will be $c_t = F(k_t)+\lambda k_t-\lambda k_{t+1}$. We also have, $\forall t \geq T_0$, $F(k_t)-\frac{\lambda(1-\beta)}{\beta}k_t > F(k_{t+1})-\frac{\lambda(1-\beta)}{\beta}k_{t+1}$, because the function $F(k)-\frac{\lambda(1-\beta)}{\beta}k$ is decreasing when k is close to k^{**} and $k < k^{**}$. The following inequality holds:

$$\sum_{t=T_0}^{\infty}\beta^{t-T_0}\left(F(k_t)-\frac{\lambda(1-\beta)}{\beta}k_t\right)<\sum_{t=T_0}^{\infty}\beta^{t-T_0}\left(F(k_{T_0})-\frac{\lambda(1-\beta)}{\beta}k_{T_0}\right).$$

But,

$$\sum_{t=T_0}^{\infty}\beta^{t-T_0}\left(F(k_t)-\frac{\lambda(1-\beta)}{\beta}k_t\right)=\sum_{t=T_0}^{\infty}\beta^{t-T_0}\left(F(k_t)+\lambda k_t-\lambda k_{t+1}\right)-\frac{\lambda}{\beta}k_{T_0}$$

and
$$\sum_{t=T_0}^{\infty} \beta^{t-T_0} \left(F(k_{T_0}) - \frac{\delta(1-\beta)}{\beta} k_{T_0} \right) = \frac{F(k_{T_0})}{(1-\beta)} - \frac{\delta}{\beta} k_{T_0}.$$
Thus,
$$\sum_{t=T_0}^{\infty} \beta^{t-T_0} (F(k_t) + \delta k_t - \delta k_{t+1}) < \frac{F(k_{T_0})}{(1-\beta)}$$
i.e.
$$\sum_{t=T_0}^{\infty} \beta^{t-T_0} c_t < \frac{F(k_{T_0})}{(1-\beta)}.$$
By Jensen's inequality, we have
$$\sum_{t=T_0}^{\infty} (1-\beta)\beta^{t-T_0} u(c_t) \leq u\left((1-\beta) \sum_{t=T_0}^{\infty} \beta^{t-T_0} c_t \right) < u(F(k_{T_0}))$$
or
$$\sum_{t=T_0}^{\infty} \beta^{t-T_0} u(c_t) < \sum_{t=T_0}^{\infty} \beta^{t-T_0} u(F(k_{T_0})).$$
This means that the utility of the stationary path $k_t = k_{T_0}$ for $t \geq T_0$ is better than the utility from T_0 of the optimal path: a contradiction. ■

Theorem 1 *Let $\delta > 0$ be given. Then there exists k_c such that $\forall k_0 < k_c$, any optimal path $\mathbf{k}$ from k_0 will satisfy $k_t = k_0$, $\forall t$ and $\forall k_0 > k_c$, any optimal path $\mathbf{k}$ from k_0 will satisfy $k_t < k_{t+1}$, $\forall t$, and $k_t \to +\infty$.*

Proof. Let k^M be the supremum of the k_* such that if $k_0 < k_*$ then any optimal path $\mathbf{k}$ from k_0 will satisfy $k_t = k_0, \forall t$. Let k^m be the infimum of the k_{**} such that if $k_0 > k_{**}$ then any optimal path $\mathbf{k}$ from k_0 will converge to $+\infty$. Our claim is that $k^M = k^m$. It is obvious that $k^M \leq k^m$. Suppose $k^M < k^m$. Take k_0 and k_0' such that $k^M < k_0 < k_0' < k^m$. From the very definition of k^M and k^m, there exists an optimal path $\{k_t\}$ from k_0 which is strictly increasing and an optimal path $\{k_t'\}$ from k_0' which is stationary, i.e. $k_t' = k_0', \forall t$. By the increasingness property of the optimal policy, we have $k_t \leq k_0', \forall t$. Hence $k_t \to \bar{k}$. It is easy to check that $F'(\bar{k}) = \frac{\delta(1-\beta)}{\beta}$ and this contradicts proposition 4. Therefore $k^M = k^m$. Posing $k_c = k^M = k^m$ ends the proof. ■

3 Concluding Remarks

i) Let $k_0 > 0$ be given. By proposition 3, if λ is small enough then any optimal path from k_0 will converge to $+\infty$. In particular, the Romer model formalized in discrete time horizon by Le Van, Morhaim and Dimaria (2001) is the limit case of our model by letting $\lambda \to 0$. Moreover, consider two countries with the same production technology and the same level of initial capital stock. By proposition 3, when one faces a poverty-trap, the other could take-off if it is endowed with a higher quality of knowledge technology. That is to mention that the differences in the quality of knowledge technology between countries can provide an explanation for the non-convergence of the countries.

ii) Consider the equality $F'(k^{**})\frac{\beta}{1-\beta} = \lambda$ and assume that $k^{**} > \bar{k}$. We have

$$A(\delta)\rho k^{**(\rho-1)} + (\rho+\mu)k^{**(\rho+\mu-1)} = \frac{\lambda(1-\beta)}{\beta}$$

where $A(\delta) = \delta\bar{k} - \bar{k}^{\mu}$. By differentiating with respect to δ, we obtain that

$$\frac{dk^{**}}{d\delta} = -\frac{\bar{k}\rho k^{**(\rho-1)}}{A(\delta)\rho(\rho-1)k^{**(\rho-2)} + (\rho+\mu)(\rho+\mu-1)k^{**(\rho+\mu-2)}} < 0$$

because, since $k^{**} > \bar{k}$, the denominator is positive. One can conclude that, given k_0, if the influence of the fixed costs diminishes (i.e. δ increases), the economy can take-off.

Furthermore, by diffentiating the given equality $F'(k^{**}) = \frac{\lambda(1-\beta)}{\beta}$ with respect to $\bar{k}$, under the assumption of $k^{**} > \bar{k}$, we obtain that $\frac{dk^{**}}{d\bar{k}} > 0$. That is to conclude that, given k_0, if the amount of fixed costs diminishes (i.e. $\bar{k}$ decreases), the economy can take-off.

iii) To sum up, an economy which wants to take-off by using knowledge technology requires three factors:

- Large amount of initial knowledge,
- Small fixed costs,
- Good quality of knowledge technology.

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