Investment Strategies for HARA Utility Function: A General Algebraic Approximated Solution*

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Abstract

In an incomplete financial market where an investor maximizes the expected HARA utility of his terminal real wealth, we present an algebraic approximated solution for the optimal portfolio composition. We take into account: (i) a (finite) set of assets, (ii) a (finite) set of state variables and (iii) a consumption price process, all of them described by general Itô processes. Finally, we supply an easy test for checking the goodness of the approximated result.

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Key words: Incomplete Market; Inflation Risk; Hamilton-Jacobi-Bellman equation; HARA utility function.

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1 Introduction

In this paper we consider the problem of an investor wanting to maximize the expected utility of his terminal real wealth. The utility function is supposed to belong to the HARA (Hyperbolic Absolute Risk Aversion) family. Thus, the framework we develop in this work is very general indeed. In fact, the results which can be obtained for a CRRA (Constant Relative Risk Aversion), a CARA (Constant Absolute Risk Aversion), a log, or a quadratic utility function, come out to be particular cases of our more general one.

Furthermore, the values of the financial assets are supposed to depend on a set of stochastic state variables and a stochastic inflation risk is considered.

In the literature about the optimal portfolio rules two main fields of research can be found. On the one hand some authors concentrate on establishing the existence (and uniqueness) of a viscosity solution for the Hamilton-Jacobi-Bellman equation deriving from the stochastic optimal control approach (see for instance Crandall et al., 1992; and Buckdahn and Ma, 2001a, 2001b). On the other, some authors offer an algebraic closed form solution to the optimal portfolio composition. In particular, we refer to the works of Kim and Omberg (1996), Wachter (1998), Chacko and Viceira (1999), Deelstra et al. (2000), Boulier et al. (2001), Zariphopoulou (2001) and Menoncin (2002). The two last works use a solution approach based on the Feynman-Kač theorem,[1] in an incomplete market and in a complete market with a background risk respectively.

Unfortunately, the former literature is not useful for an easy application of its results since it is not able to explicitly derive the form of the optimal portfolio, while the latter can be easily applied but lies on the assumption that the asset values and the state variables behave in a very precise way.

Our work is aimed at finding a "third way" to the investment problem by supplying, on a very general framework, an approximated closed form solution for the optimal portfolio composition.

In all the works where a closed form solution is derived the market structure is as follows: (i) there exists only one state variable (the riskless interest rate or the risk premium) following the Vasiček (1977) model or the Cox et al. (1985) model; (ii) there exists only one risky asset; and (iii) a bond may exist. Some works consider a complete financial market (Wachter, 1998; Deelstra et al., 2000; Boulier et al., 2001; and Menoncin 2002) while others deal with an incomplete market (Kim and Omberg, 1996; Chacko and Viceira, 1999; and Zariphopoulou, 2001). Furthermore, all these works consider a CRRA utility function, with the exception of Kim and Omberg (1996) who deal with a HARA utility function and of Menoncin (2002) who considers a CARA utility function.

As we aim at providing a very general analysis, our framework considers a (finite) set of assets, a (finite) set of state variables and a consumption price process, all of them following general Itô processes. Furthermore, we do not need the hypothesis of completeness for the financial market and we take into account the most general form for the utility function (HARA).

[1] For a complete exposition of the Feynman-Kač theorem the reader is referred to Duffie (1996), Björk (1998) and Øksendal (2000).

In this paper, we follow the traditional stochastic dynamic programming technique (Merton, 1969, 1971) leading to the Hamilton-Jacobi-Bellman (HJB) equation (Øksendal, 2000; and Björk, 1998 offer a complete derivation of the HJB equation). As regard the "martingale approach" the reader is referred to Cox and Huang (1989, 1991), and Lioui and Poncet (2001).

We define some fundamental matrices which the optimal portfolio composition is based on. They are given by a combination of both preference parameters, drift and diffusion terms for both assets and state variables. We approximate the value of these matrices thanks to a Taylor series. After this approximation the value function solving the HJB equation turns out to be log-linear in the state variables. Thus, the solution for the optimal portfolio becomes very easy to compute and we also present an easy way for checking the goodness of this approximated solution.

In the literature there exists another example where an approximated solution to the optimal portfolio composition is computed. We refer to Kogan and Uppal (1999) who solve the HJB equation by approximating it near a given value of the risk aversion index. Their work takes into account a CRRA utility function and it is valid for a value of the Arrow-Pratt risk aversion index close to zero. On the contrary, in our work, we allow for a more general pattern of consumer preferences since we do not take into account any restriction on the preference parameters of the HARA utility function. In fact, we compute the Taylor approximation around given values of the state variables.

Through this work we consider agents trading continuously in a frictionless, arbitrage-free, and incomplete market until time H, which is the horizon of the economy.

The paper is structured as follows. Section 2 details the general economic framework, exposes the stochastic differential equations describing the behaviour of asset prices, state variables, and consumpiton price index and derives the dynamic behaviour of the investor's real wealth. In Section 3, both the implicit form of the optimal portfolio and the HJB equation are computed. Section 4 presents our main result, that is to say an algebraic approximated solution for the optimal portfolio composition. This section ends by presenting an easy way for computing the goodness of the approximation. Section 5 concludes.

2 The market structure

The financial market is supposed to have the following structure:

$$\begin{cases} \underset{s\times 1}{dX} = \underset{s\times 1}{f(t,X)}dt + \underset{s\times k}{g(t,X)'}\underset{k\times 1}{dW}, \qquad X(t_0) = X_0, \\ \underset{n\times 1}{dS} = \underset{n\times 1}{\mu(t,X,S)}dt + \underset{n\times k}{\Sigma(t,X,S)'}\underset{k\times 1}{dW}, \qquad S(t_0) = S_0, \\ dG = Gr(t,X)\,dt, \qquad G(t_0) = G_0, \end{cases} \tag{1}$$

where X is a vector containing all the state variables affecting the asset whose values are contained in vector S. For a review of all variables which can affect the asset prices the reader is referred to Campbell (2000) who offers a survey of the most important contributions in this field. We have indicated with G the value of a riskless asset paying the instantaneous riskless interest rate r. Hereafter, the prime denotes transposition.

All the functions $f(t,X)$, $g(t,X)$, $\mu(t,X,S)$, $\Sigma(t,X,S)$, and $r(t,X)$ are supposed to be $\mathcal{F}_t-$measurable. The $\sigma-$algebra $\mathcal{F}$ is defined on a set Θ wherethrough the complete probability space $(\Theta,\mathcal{F},\mathbb{P})$ is defined. Here, $\mathbb{P}$ can be considered as the "historical" probability measure.

The stochastic equations in System (1) are driven by a set of risks represented by dW which is the differential of a $k-$dimensional Wiener process whose components are independent.[2]

The set of risk sources is the same for the state variables and for the asset prices. This hypothesis is not restrictive because thanks to the elements of matrices g and Σ we can model a lot of different frameworks. For instance, if we consider $dW = \left[\begin{array}{cc} dW_1 & dW_2 \end{array}\right]$, $g' = \left[\begin{array}{cc} g_1 & 0 \end{array}\right]$, and $\Sigma' = \left[\begin{array}{cc} 0 & \sigma_2 \end{array}\right]$ then the processes of X and S are not correlated even if they formally have the same risk sources.

We recall the main result concerning completeness and arbitrage in this kind of market (for the proof of the following theorem see Øksendal, 2000).

Theorem 1 *A market $\{S(t,X)\}_{t\in[t_0,H]}$ is arbitrage free (complete) if and only if there exists a (unique) $k-$dimensional vector $u(t,X)$ such that*

$$\Sigma(t,X)'\,u(t,X) = \mu(t,X) - r(t,X)\,S(t,X)\,,$$

and such that

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_0}^{H}\|u(t,X)\|^2 dt}\right] < \infty.$$

If on the market there are less assets than risk sources ($n<k$), then the market cannot be complete even if it is arbitrage free. In this work, we assume that $n \leq k$ and that the rank of matrix Σ is maximum (i.e. it equals n). Thus, the results we obtain in this work are valid for a financial market which is incomplete as well as for a complete market ($n=k$).

2.1 The inflation and the real wealth

We suppose that the set of the state variables X in the market structure (1) also contains the consumption price process (P) that behaves according to the following stochastic differential equation:

$$\begin{aligned} dP &= P\mu_\pi(X,P,t)\,dt + P\underset{1\times k}{\sigma_\pi(X,P,t)'}\underset{k\times 1}{dW}, \\ P(t_0) &= 1. \end{aligned}$$

[2]This condition can be imposed without loss of generality because a set of independent Wiener processes can always be transformed into a set of correlated Wiener processes thanks to the Cholesky decomposition.

The initial value of the price consumption process is conventionally put equal to 1 without loss of generality because prices can always be normalized. For the sake of generality we do not specify any particular form for the drift and the diffusion coefficients of this process. The reader is referred to Cox, Ingersoll, and Ross (1985) for two particular functional forms which can be used for modeling inflation.

If we indicate with $\theta(t) \in \mathbb{R}^{n\times 1}$ and $\theta_G(t) \in \mathbb{R}$ the number of risky assets held and the quantity of riskless asset held respectively, then the investor's nominal wealth can be written as

$$R_N = \theta(t)' S + \theta_G(t) G, \tag{2}$$

After differentiating the budget constraint (2) and considering the self-financing condition[3] we obtain

$$dR_N = \theta(t)' dS + \theta_G(t) dG,$$

and, after substituting the differentials from System (1), we finally have

$$dR_N = \left(\theta'\mu + \theta_G G r\right) dt + \theta'\Sigma' dW. \tag{3}$$

This equation describes the dynamic behaviour of investor's nominal wealth. However, we consider that an investor should be more interested in maximizing the expected utility of his terminal real wealth. The behaviour of the real wealth can be obtain from Equation (3) by recalling that the real wealth level is defined as the ratio between the nominal wealth and the price level. Accordingly, we have to differentiate the following formula:

$$R = \frac{R_N}{P}.$$

By applying the Itô's lemma we obtain[4]

$$\begin{aligned} dR &= \left(\frac{1}{P}\left(\theta'\mu - \theta'\Sigma'\sigma_\pi + \theta_G G r\right) - \frac{R_N}{P}\left(\mu_\pi - \sigma_\pi'\sigma_\pi\right)\right) dt \\ &\quad + \left(\frac{1}{P}\theta'\Sigma' - \frac{R_N}{P}\sigma_\pi'\right) dW, \end{aligned}$$

[3] The self-financing condition can be written as

$$d\theta'(S + dS) + d\theta_G G = 0.$$

[4] We recall that the Jacobian of the real wealth is:

$$\nabla_{R_N,P} R = \begin{bmatrix} \frac{1}{P} \\ -\frac{R_N}{P^2} \end{bmatrix},$$

while its Hessian is:

$$\nabla^2_{R_N,P} R = \begin{bmatrix} 0 & -\frac{1}{P^2} \\ -\frac{1}{P^2} & 2\frac{R_N}{P^3} \end{bmatrix}.$$

which can be written, after substituting for the value of R_N given in Equation (2), as

$$dR = w'Mdt + w'\Gamma'dW, \tag{4}$$

where

$$\begin{aligned}
\underset{(n+1)\times 1}{w} &\equiv \begin{bmatrix} \theta \\ \theta_G \end{bmatrix}, \\
\underset{(n+1)\times 1}{M} &\equiv \frac{1}{P}\begin{bmatrix} \mu - S\left(\mu_\pi - \sigma_\pi'\sigma_\pi\right) - \Sigma'\sigma_\pi \\ Gr - G\left(\mu_\pi - \sigma_\pi'\sigma_\pi\right) \end{bmatrix}, \\
\underset{(n+1)\times k}{\Gamma'} &\equiv \frac{1}{P}\begin{bmatrix} \Sigma' - S\sigma_\pi' \\ -G\sigma_\pi' \end{bmatrix}.
\end{aligned}$$

Thus, during our analysis we will consider Equation (4) instead of Equation (3). Thus, we will suppose that the aim of the investor is to maximize the expected utility of his terminal real wealth. With respect to the usual approach, in Equation (4) we lack the term containing the wealth level.[5] This characteristic comes from the following consideration: the investor is not interested in the level of his nominal wealth R_N, and furthermore, he cannot invest his real wealth R because this is just a fictitious index (only nominal wealth can be actually invested). We recall that inflation can be considered as the opportunity cost of investing in financial (and not real) assets. Thus, in this work, by considering inflation, we are able to take into account the investment opportunity in the real market.

From Equation (4) we can immediately see that the riskless asset looses its characteristic for becoming like a risky asset. In particular, it acquires a diffusion coefficient corresponding to the opposite of the price diffusion term. In fact, when the inflation rises, the real value of the riskless asset decreases and vice versa. Accordingly, the diffusion matrix of the "real market" (Γ) has one more column with respect to the nominal one (Σ). Thus, hereafter, the completeness will be defined on the real market, that is to say on the existence of the inverse of matrix Γ, considering $n+1$ risky assets.

Finally, we outline that the matrix M containing the risk premium, does not measure the difference between the asset returns and the riskless interest rate as in the usual "nominal" analysis. Instead, in our framework, it contains the difference between the nominal asset return and the inflation drift term. Furthermore, this difference is adjusted for the diffusion terms of assets and inflation.

[5] Without inflation and under market structure (1), the wealth differential equation should be written as follows:

$$dR = \left(Rr + \theta'\left(\mu - rS\right)\right)dt + \theta'\Sigma'dW,$$

where there exists a term proportional to the wealth level (Rr) which creates some problems for solving the partial differential equation deriving from the stochastic dynamic programming technique.

3 The optimal portfolio

Under the market structure (1) and the evolution of investor's real wealth given in Equation (4), the optimization problem for an investor maximizing the expected HARA utility of his terminal real wealth, can be written as

$$\begin{cases} \max_{w} \mathbb{E}_{t_0}\left[\left(\alpha+\gamma R\left(H\right)\right)^{1-\frac{\beta}{\gamma}}\right] \\ d\begin{bmatrix} z \\ R \end{bmatrix} = \begin{bmatrix} \mu_z \\ w'M \end{bmatrix} dt + \begin{bmatrix} \Omega' \\ w'\Gamma' \end{bmatrix} dW, \\ z\left(t_0\right)=z_0, \qquad R\left(t_0\right)=R_0, \quad \forall t_0 \leq t \leq H, \end{cases} \tag{5}$$

where

$$\begin{aligned} \underset{(s+n+1)\times 1}{z} &\equiv \begin{bmatrix} X' & S' & G \end{bmatrix}', \\ \underset{(s+n+1)\times 1}{\mu_z} &\equiv \begin{bmatrix} f' & \mu' & Gr \end{bmatrix}', \\ \underset{k\times(s+n+1)}{\Omega} &\equiv \begin{bmatrix} g & \Sigma & \mathbf{0} \end{bmatrix}, \end{aligned}$$

and H is the investor's time horizon. The vector z contains all the state variables but the investor's wealth. Hereafter, we will indicate with $\mathbf{0}$ a matrix of suitable dimension containing only zeros.

The utility function we consider belongs to the HARA family since it has an Hyperbolic Absolute Risk Aversion index

$$-\frac{\frac{\partial^2}{\partial R^2}\left(\left(\alpha+\gamma R\right)^{1-\frac{\beta}{\gamma}}\right)}{\frac{\partial}{\partial R}\left(\left(\alpha+\gamma R\right)^{1-\frac{\beta}{\gamma}}\right)} = \frac{\beta}{\alpha+\gamma R}.$$

This kind of utility function is very general indeed since it contains the most used utility functions as particular cases:

1. the CARA (Constant Absolute Risk Aversion) or exponential utility function when $\alpha = 1$ and $\gamma \to 0$ in the form

$$U\left(R\right) = \lim_{\gamma\to 0}\left(1+\gamma R\right)^{1-\frac{\beta}{\gamma}} = e^{-\beta R};$$

2. the CRRA (Constant Relative Risk Aversion) or power utility function when $\alpha = 0$;

3. the log-utility function when, after putting $\alpha = 0$, $\gamma = k^{-\frac{1}{k}}$, and $\beta = k^{-\frac{1}{k}}\left(1-k\right)$, we consider the result for k tending to zero; in this case, in fact, we have

$$U\left(R\right) = \frac{1}{k}R^k,$$

whose form is generally led back to the log utility when k tends to zero;

4. the quadratic utility function when $1-\frac{\beta}{\gamma}=2 \Rightarrow \gamma=-\beta$.

We recall that the optimization problem is well defined if the objective function is increasing and concave in its argument R. This occurs when

$$0<\frac{\beta}{\gamma}<1.$$

The only restriction on parameter α comes from the condition

$$\alpha+\gamma R>0,$$

which must hold for having a well defined power function. Thus, nothing prevents α from being negative. This means that we could write the HARA utility function also as

$$\left(\gamma R-R^{*}\right)^{1-\frac{\beta}{\gamma}},$$

where R^{*} can be considered as the lowest level of wealth the investor is willing to accept. Nevertheless, in what follows we continue with the more general framework, without considering any particular form for α.

From Problem (5) we have the Hamiltonian

$$\mathcal{H}=\mu_{z}^{\prime} J_{z}+J_{R} w^{\prime} M+\frac{1}{2} \operatorname{tr}\left(\Omega^{\prime} \Omega J_{z z}\right)+w^{\prime} \Gamma^{\prime} \Omega J_{z R}+\frac{1}{2} J_{R R} w^{\prime} \Gamma^{\prime} \Gamma w, \tag{6}$$

where $J(R, z, t)$ is the value function solving the Hamilton-Jacobi-Bellman partial differential equation (see Section 3.1), verifying

$$J(R, z, t)=\sup_{w} \mathbb{E}_{t}[K(R(H))],$$

and the subscripts on J indicate the partial derivatives.

The system of the first order conditions on $\mathcal{H}$ is[6]

$$\frac{\partial \mathcal{H}}{\partial w}=J_{R} M+\Gamma^{\prime} \Omega J_{z R}+J_{R R} \Gamma^{\prime} \Gamma w=0,$$

from which we obtain the optimal portfolio composition

$$w^{*}=\underbrace{-\frac{J_{R}}{J_{R R}}\left(\Gamma^{\prime} \Gamma\right)^{-1} M}_{w_{(1)}^{*}} \underbrace{-\frac{1}{J_{R R}}\left(\Gamma^{\prime} \Gamma\right)^{-1} \Gamma^{\prime} \Omega J_{z R}}_{w_{(2)}^{*}}. \tag{7}$$

[6] The second order conditions hold if the Hessian matrix of $\mathcal{H}$

$$\frac{\partial \mathcal{H}}{\partial w^{\prime} \partial w}=J_{R R} \Gamma^{\prime} \Gamma$$

is negative definite. Because $\Gamma^{\prime} \Gamma$ is a quadratic form it is always positive definite and so the second order conditions are satisfied if and only if $J_{R R}<0$, that is if the value function is concave in R. The reader is referred to Stockey and Lucas (1989) for the assumptions that must hold on the function $K(R)$ for having a strictly concave value function.

For having a unique solution to the optimal portfolio problem we must check that the matrix $\Gamma'\Gamma \in \mathbb{R}^{(n+1)\times(n+1)}$ is invertible. This condition is satisfied if $\Gamma' \in \mathbb{R}^{(n+1)\times k}$ has rank equal to $n+1$ and $n+1 \leq k$. Actually, due to the inflation risk, the riskless asset becomes a risky asset, acquiring the price diffusion term. Thus, the completeness must be defined on $n+1$ assets (including the riskless one) and no more on n assets. In what follows we will define a complete market as the market where the matrix Γ is invertible (that is to say it is a square $(n+1)\times(n+1)$ matrix and its rank is maximum).

We just outline that $w^*_{(1)}$ increases if the real returns on assets (M) increase and decreases if the risk aversion ($-J_{RR}/J_R$) or the asset variance ($\Gamma'\Gamma$) increase. From this point of view, we can argue that this component of the optimal portfolio has just a speculative role.

The second part $w^*_{(2)}$ is the only optimal portfolio component explicitly depending on the diffusion terms of the state variables (Ω). We will investigate the precise role of this component after computing the functional form of the value function.

We recall that Kogan and Uppal (1999) call $w^*_{(1)}$ the "myopic" component and $w^*_{(2)}$ the "hedging" component of the optimal portfolio. In fact, in the next section we will see that $w^*_{(2)}$ is the only part of w^* depending on the financial time horizon (H). From this point of view $w^*_{(1)}$ can be properly called "myopic". Instead, the hedging nature of $w^*_{(2)}$ depends on its charactersitic to contain the volatility matrix of state variables. In this way, we can say that the second portfolio component $w^*_{(2)}$ can hedge the optimal portfolio against the risk represented by the state variables.

3.1 The value function

For studying the exact role of the portfolio components we have called $w^*_{(1)}$ and $w^*_{(2)}$ (see Equation (7)), we need to compute the value function $J(R,z,t)$. By substituting the optimal value of w into the Hamiltonian (6) we have

$$\begin{aligned}\mathcal{H}^* &= \mu_z' J_z - \frac{1}{2}\frac{J_R^2}{J_{RR}} M' \left(\Gamma'\Gamma\right)^{-1} M - \frac{J_R}{J_{RR}} M' \left(\Gamma'\Gamma\right)^{-1} \Gamma'\Omega J_{zR} \\ &\quad + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega J_{zz}\right) - \frac{1}{2}\frac{1}{J_{RR}} J_{zR}'\Omega'\Gamma\left(\Gamma'\Gamma\right)^{-1}\Gamma'\Omega J_{zR}.\end{aligned}$$

From this equation we can formulate the PDE whose solution is the value function. This PDE is called the Hamilton-Jacobi-Bellman equation (hereafter HJB) and it can be written as follows:

$$\begin{cases} J_t + \mathcal{H}^* = 0, \\ J(H,R,z) = K(R(H)). \end{cases} \tag{8}$$

One of the most common way to solve this kind of PDE is to try a separability condition. In the literature (since Merton, 1969, 1971), a separability by product is generally found. Here, we suppose that the value function $J(z,R,t)$ is separable by product in wealth and in the other state variables according to the following form: $J(z,R,t) = U(R)\,e^{h(z,t)}$. After substituting this functional form into the HJB equation (8) and dividing by J we obtain

$$\begin{aligned} 0 &= h_t + \mu_z' h_z - \frac{1}{2}\frac{U_R^2}{U_{RR}U} M'(\Gamma'\Gamma)^{-1} M - \frac{U_R^2}{U_{RR}U} M'(\Gamma'\Gamma)^{-1}\Gamma'\Omega h_z \\ &\quad + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega\left(h_{zz} + h_z h_z'\right)\right) - \frac{1}{2}\frac{U_R^2}{U_{RR}U} h_z'\Omega'\Gamma(\Gamma'\Gamma)^{-1}\Gamma'\Omega h_z, \end{aligned} \tag{9}$$

where the boundary condition has become $h(z,H) = 0$. Now, since the utility function belongs to the HARA family, we have

$$\frac{U_R^2}{U_{RR}U} = 1 - \frac{\gamma}{\beta},$$

and so the Equation (9) can finally be written as

$$\begin{cases} h_t + a(z,t)' h_z + b(z,t) + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega h_{zz}\right) + \frac{1}{2} h_z' D(z,t)\, h_z = 0, \\ \hfill h(z,H) = 0, \end{cases} \tag{10}$$

where

$$\begin{aligned} a(z,t)' &\equiv \mu_z' - \left(1 - \frac{\gamma}{\beta}\right) M'(\Gamma'\Gamma)^{-1}\Gamma'\Omega, \\ b(z,t) &\equiv -\frac{1}{2}\left(1 - \frac{\gamma}{\beta}\right) M'(\Gamma'\Gamma)^{-1} M, \\ D(z,t) &\equiv \Omega'\left(I - \left(1 - \frac{\gamma}{\beta}\right)\Gamma(\Gamma'\Gamma)^{-1}\Gamma'\right)\Omega. \end{aligned}$$

We just underline that the matrix $D(z,t)$ is positive semi-definite and $b(z,t)$ is a non negative scalar. The latter result can be immediately checked because, in order to have a well defined maximization problem, we supposed γ/β to be grater than one. The former result holds because the matrix $D(z,t)$ can be written as a quadratic form in the following way:

$$\begin{aligned} D(z,t) &= \Omega'\left(I - \left(1 + \sqrt{\frac{\gamma}{\beta}}\right)\Gamma(\Gamma'\Gamma)^{-1}\Gamma'\right)' \\ &\quad \times \left(I - \left(1 + \sqrt{\frac{\gamma}{\beta}}\right)\Gamma(\Gamma'\Gamma)^{-1}\Gamma'\right)\Omega. \end{aligned}$$

By using the HARA utility function and the separability condition, we can write the optimal portfolio composition as in the following proposition.

Proposition 1 *The optimal portfolio solving Problem (5) is given by*

$$w^* = \underbrace{\frac{\alpha + \gamma R}{\beta}(\Gamma'\Gamma)^{-1} M}_{w^*_{(1)}} + \underbrace{\frac{\alpha + \gamma R}{\beta}(\Gamma'\Gamma)^{-1}\Gamma'\Omega\frac{\partial h(z,t)}{\partial z}}_{w^*_{(2)}}, \tag{11}$$

where $h(z,t)$ solves the HJB equation (10).

Thus, for finding a closed form solution, we must compute the function $h(z,t)$ solving Equation (10). In the following section we present an approximated solution.

We just underline that the optimal portfolio composition (11) is an affine transformation of wealth. This means that the percentage of wealth invested in each asset does depend on the wealth level. We can immediately see that:

1. for a CARA utility function ($\alpha = 1$, $\gamma \to 0$) the number of asset held in portfolio does not depend on the wealth level;

2. for both a CRRA and a log-utility function ($\alpha = 0$) the percentage of wealth invested in each asset ($R^{-1}w^*$) does not depend on the wealth level.

Menoncin (2002) takes into account a more general framework in which there exists also a set of background risks. He shows that if a CARA utility function is considered in a framework of a complete financial market for $n+1$ assets (i.e. the matrix Γ^{-1} does exist) then $D(z,t) = 0$ and the HJB equation (10) can be solved thanks to the Feynman-Kač theorem.[7]

In the following section we present a more general approximated solution when the utility function belongs to the HARA family and the market is not necessarily complete.

4 A general approximated solution

In order to find an approximated solution to the HJB equation (10) we propose to develop in Taylor series the fundamental matrices $a(z,t)$, $b(z,t)$, $\Omega(z,t)' \Omega(z,t)$, and $D(z,t)$. In particular, our proposal relays on the literature showing a particular closed form solution to the optimal portfolio problem. In this literature (see for instance Chacko and Viceira, 1999; Deelstra et al., 2000; and Boulier et al., 2001) all the above-mentioned matrices are linear in z. We just underline that in Kim and Omberg (1996) the scalar $b(z,t)$ is a second order polynomial in z but, on the other way, $\Omega'\Omega$ and $D(z,t)$ are both constant with respect to time and to z.

Accordingly, we propose the following simplification, based on the expansion

[7] For a complete exposition of the Feynman-Kač theorem the reader is referred to Duffie (1996), Björk (1998) and Øksendal (2000).

in Taylor series around the values z_0:

$$\begin{aligned}
a(z,t) &\approx a(z_0,t) + \left.\frac{\partial a(z,t)}{\partial z}\right|_{z=z_0}(z-z_0) \equiv a_0(t) + A_1(t)' z, \\
b(z,t) &\approx b(z_0,t) + \left.\frac{\partial b(z,t)}{\partial z'}\right|_{z=z_0}(z-z_0) \equiv b_0(t) + b_1(t)' z, \\
\Omega(z,t)'\Omega(z,t) &\approx \Omega(z_0,t)'\Omega(z_0,t) \equiv C_0(t), \\
D(z,t) &\approx D(z_0,t) \equiv D_0(t).
\end{aligned}$$

The choice of approximating the matrices $\Omega'\Omega$ and $D(z,t)$ with a constant rather than with a first order polynomial in z prevents us from having to solve a Riccati matrix equation. Since we let all the matrices depend on time, then we would not be able to solve this Riccati matrix equation without knowing a particular solution to it, and this is not the case.

It is worthnoting that Boulier et al. (2001) take into account a financial marke whose foundamental matrices $a(z,t)$, $b(z,t)$, $\Omega(z,t)'\Omega(z,t)$, and $D(z,t)$ have exactly the form we use here as an approxiation (i.e. the two first functions are linear in z while the last two are just constants).

Since, after the approximation, all the functions of z appearing in Equation (10) are linear in z, then we can suppose also the function $h(z,t)$ to be linear in z having the following form:

$$h(z,t) = h_0(t) + h_1(t)' z.$$

After substituting these approximations into Equation (10) we have

$$\frac{\partial h_0}{\partial t} + z'\left(\frac{\partial h_1}{\partial t}\right) + \left(a_0(t)' + z' A_1(t)\right) h_1 + b_0(t) + b_1(t)' z + \frac{1}{2} h_1' D_0(t) h_1 = 0.$$

This equation can be split into two equations, by putting equal to zero the constant terms (with respect to z) and the terms containing z. Thus, we can write down the following system:

$$\begin{cases} \frac{\partial h_0}{\partial t} + a_0(t)' h_1 + b_0(t) + \frac{1}{2} h_1' D_0(t) h_1 = 0, \\ \frac{\partial h_1}{\partial t} + A_1(t) h_1 + b_1(t) = 0, \end{cases} \tag{12}$$

and the original boundary condition on $h(z,t)$ is now defined on the two new functions $h_0(t)$ and $h_1(t)$ as follows:

$$\begin{cases} h_0(H) = 0, \\ h_1(H) = \mathbf{0}. \end{cases}$$

Since in the optimal portfolio component only the first derivative of $h(z,t)$ with respect to z plays a role, then we must care just about the function $h_1(t)$, whose value is completely defined by the second equation in System (12). Then, we have to solve the matrix ODE

$$\begin{cases} \frac{\partial h_1}{\partial t} + A_1(t) h_1 + b_1(t) = 0, \\ h_1(H) = \mathbf{0}, \end{cases}$$

whose solution is given by

$$h_1(t) = \int_t^H e^{\int_t^s A_1(\tau)d\tau} b_1(s)\, ds, \tag{13}$$

where the exponential matrix is computed, as usual, as[8]

$$e^{\int_t^s A_1(\tau)d\tau} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_t^s A_1(\tau)\, d\tau \right)^n .$$

Finally, the approximated value of the optimal portfolio can be formulated as in Proposition 2.

Proposition 2 *The second component* ($w^*_{(2)}$) *of the optimal portfolio solving Problem (5) can be approximated as follows:*

$$w^*_{(2)} \approx \frac{\alpha + \gamma R}{\beta} \left(\Gamma'\Gamma\right)^{-1} \Gamma'\Omega \int_t^H e^{\int_t^s A_1(\tau)d\tau} b_1(s)\, ds, \tag{14}$$

where

$$\begin{aligned} A_1(t) &\equiv \frac{\partial}{\partial z}\left(\mu_z - \left(1 - \frac{\gamma}{\beta}\right) \Omega'\Gamma\left(\Gamma'\Gamma\right)^{-1} M \right)\Bigg|_{z=z_0}, \\ b_1(t) &\equiv -\frac{1}{2}\left(1 - \frac{\gamma}{\beta}\right) \frac{\partial}{\partial z}\left(M'\left(\Gamma'\Gamma\right)^{-1} M \right)\Bigg|_{z=z_0}. \end{aligned}$$

We underline that, during our procedure, we have never used the inverted matrix Γ^{-1}. Since we do not need this matrix, then our result is valid also in the very general case of an incomplete market.

The approximated solution for $w^*_{(2)}$ given in Equation (14) is very simple to apply and to implement with a mathematical software. The weak point of this approximated solution lies on the difference $(z - z_0)$. When this difference increases the approximation becomes more and more inaccurate. Nevertheless, the integrals in Formula (14) can be easily computed numerically, and so the strategy of recomputing the optimal portfolio when z becomes farther off z_0 does not seem to be too much expensive.

[8] Thus, for instance, if

$$A_1(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

then

$$\int_t^s A_1(\tau)\, d\tau = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} (s - t),$$

and

$$e^{\int_t^s A_1(\tau)d\tau} = e^{(s-t)} \begin{bmatrix} \cosh(2(s-t)) & \sinh(2(s-t)) \\ \sinh(2(s-t)) & \cosh(2(s-t)) \end{bmatrix}.$$

Since the values $a(z,t)$ and $b(z,t)$ enter the Approximation (14), then it can be useful to underline their role in our framework. For this purpose we begins with the following definition.[9]

Definition 1 *Given Problem (5), the market price of risk is*

$$\xi \equiv \Gamma \left(\Gamma'\Gamma\right)^{-1} M.$$

If the market is complete, i.e. if the matrix Γ^{-1} does exist, then we obtain the usual result:

$$\xi \equiv \Gamma'^{-1} M,$$

according to which the market price of risk is just given by the Sharpe ratio (the ratio between the return exceeding the riskless interest rate and the volatility). We underline that in our framework where we consider the inflation risk, the excess return is not defined as usual by the difference between an asset's return and the riskless interest rate. This excess return, instead, is the difference between an asset's return and the inflation rate, corrected by the inflation volatility and the correlation between inflation and asset volatility.

Accordingly, the functions $a(z,t)$ and $b(z,t)$ can be written as follows:

$$\begin{aligned} a(z,t)' &\equiv \mu_z' - \left(1 - \frac{\gamma}{\beta}\right) \xi'\Omega, \\ b(z,t) &\equiv -\frac{1}{2}\left(1 - \frac{\gamma}{\beta}\right) \xi'\xi. \end{aligned}$$

Thus, we can conclude that:

1. $b(z,t)$ is the square of the market price of risk, weighted by a combination of preference parameters;
2. $a(z,t)$ contains the drift of all the state variables corrected by the product between the price of state variable's risk and a combination of preference parameters.

From Proposition 2 we can easily check that if the market price of risk is constant (with respect to the values of the state variables z) then the approximated optimal portfolio shrinks to contain only one component, i.e. the first one. This result is stated in the following corollary.

Corollary 1 *The approximated second component ($w^*_{(2)}$) of the optimal portfolio solving Problem (5) vanishes when the market price of risk does not depend on the values of the state variables ($\frac{\partial \xi}{\partial z} = \mathbf{0}$).*

In the following subsection we present a computation of the error implied in the approximation presented in Proposition 2.

[9] For a similar approach see Dana and Jeanblanc-Picqué (1998).

4.1 The maximum error

In the previous section we have presented an approximated solution for the optimal portfolio composition solving Problem 5. Here, we recall that the approximation in Taylor series of the matrices $a(z,t)$, $b(z,t)$, $\Omega(z,t)'\,\Omega(z,t)$, and $D(z,t)$ presents a maximum error (ε) which is respectively:[10]

$$\begin{aligned}
\varepsilon_{a_i}(t,z) &\equiv \max_{\lambda\in[0,1]}\left\{\frac{1}{2}(z-z_0)'\left(\left.\frac{\partial^2 a_i(z,t)}{\partial z'\partial z}\right|_{z=z_0+\lambda(z-z_0)}\right)(z-z_0)\right\}, && (15)\\
\varepsilon_{b}(t,z) &\equiv \max_{\lambda\in[0,1]}\left\{\frac{1}{2}(z-z_0)'\left(\left.\frac{\partial^2 b(z,t)}{\partial z'\partial z}\right|_{z=z_0+\lambda(z-z_0)}\right)(z-z_0)\right\}, && (16)\\
\varepsilon_{C}(t,z) &\equiv \max_{\lambda\in[0,1]}\left\{\left.\frac{\partial}{\partial z}\left(\Omega'(z,t)\,\Omega(z,t)\right)\right|_{z=z_0+\lambda(z-z_0)}(z-z_0)\right\}, && (17)\\
\varepsilon_{D}(t,z) &\equiv \max_{\lambda\in[0,1]}\left\{\left.\frac{\partial D(z,t)}{\partial z}\right|_{z=z_0+\lambda(z-z_0)}(z-z_0)\right\}, && (18)
\end{aligned}$$

where we have indicated with a_i the i^{th} element ($i\in[1,...,s+n+1]$) of vector a. Now, after putting

$$\varepsilon_a(z,t)=\{\varepsilon_{a_i}(z,t)\}_{i=1,\ldots,s+n+1}, \tag{19}$$

we can substitute the error values into the HJB equation and conclude what follows.

Proposition 3 *Let function $h_1(t)$ be as in (13) and the error terms $\varepsilon_a(z,t)$, $\varepsilon_b(z,t)$, and $\varepsilon_D(z,t)$ be as in (15)-(19), then the closer*

$$\varepsilon_a(z,t)'\,h_1(t)+\varepsilon_b(z,t)+\frac{1}{2}h_1(t)'\,\varepsilon_D(z,t)\,h_1(t) \tag{20}$$

to zero, the better the approximated solution for the second optimal portfolio component presented in Proposition 2.

Thus, after choosing an initial level z_0 for approximating the matrices $a(z,t)$, $b(z,t)$, $\Omega(z,t)'\,\Omega(z,t)$, and $D(z,t)$, the goodness of these approximations can be easily checked by computing the absolute value of Expression (20).

[10] We consider here the so-called Lagrange's error term.

5 Conclusion

In this paper we have considered the problem of an investor maximizing the expected HARA utility of his terminal real wealth. All the variables taken into account are supposed to follow general Itô processes. In particular, we consider: (i) a (finite) number of financial assets; (ii) a (finite) number of state variables; and (iii) a consumption price process.

We compute the Hamilton-Jacobi-Bellman (HJB) equation solving our dynamic programming problem and we propose an approximated solution to it. In particular, we define four fundamental matrices whose values are given by combinations of preference parameters, drift and diffusion terms for both assets and state variables.

We approximate the HJB equation by computing a Taylor series of the abovementioned matrices around a given level for state variables. Thus, the value function, solving the HJB equation turns out to be a log-linear function of the state variables.

Finally, we present an easy way for checking the goodness of this approximated solution.

With respect to the present literature, our model presents a higher degree of generality in terms of both financial market structure and investor's preferences it deals with. In particular, while the literature is mainly concerned with the problem of the existence of a solution without providing an actual form of it, our model supplies an approximation which can be uselful for computing the actual solution to the proposed problem.

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