Closed-Form Solution for a Two-Sector Endogenous Growth Model with two Controols*

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Abstract

In this paper we show a method for solving in closed form a particular family of four-dimension non-linear modified Hamiltonian dynamic systems, with two states and two co-states, which arises from a two-sector endogenous growth model where the physical capital stock is combined with a renewable natural capital stock as essential inputs for production.

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1 Introduction

Since non-linear dynamic systems cannot be usually solved in closed form, stationarity becomes particularly suitable when we look for analytical solutions. If the dynamic system have a unique, or multiple but isolated, structurally stable rest point we can proceed by solving the linear approximation, which retains the main dynamic properties from the original non-linear system. However, the new growth theory based on endogenous growth models generates dynamic systems for which it does not exist a well-defined hyperbolic steady state. In such cases, linearization is not allowed for solving in closed form, and then we suffer from the lack a general resolution method.

In this paper we show a method for solving in closed form a particular family of non-linear dynamic systems. They are four-dimension non-linear modified Hamiltonian dynamic systems with two states and two co-states. These systems arise in a natural way from two-sector endogenous growth models like the one studied in Lucas (1988), which considers the consequences of human capital accumulation in addition to physical capital, or the more recent one studied in Aznar-Márquez and Ruiz-Tamarit (2002), which combines both the traditional physical capital stock and a renewable natural capital stock as essential inputs for production. Looking at these two endogenous growth models, the first one has been largely analyzed in Benhabib and Perli (1994), Xie (1994), and others. Whereas Benhabib and Perli proceed by reducing dimension and solving the linear approximation around the so artificially generated steady state, Xie takes directly the original system and solves it under additional parameter constraints. The results from the latter have been recently revised in Ruiz-Tamarit (2002). In this paper we solve the original non-linear modified Hamiltonian dynamic system arising from the second endogenous growth model. Moreover, in line with Xie's procedure we impose an additional parameter constraint which allows us to solve in closed form for the original variables of the model. In Section 2 we shortly describe the model economy, the optimization problem for a competitive economy, and the dynamic system that governs the state of this economy over time. In Section 3 we provide the complete analytical solution.

2 The dynamic system

Consider the intertemporal optimization problem corresponding to the economic model studied in Aznar-Márquez and Ruiz-Tamarit (2002). There are two state variables, K and Q, representing the levels of physical capital and natural capital, respectively. Each of these states has an accompanying accumulation equation that determines its evolution over time. The value for each state at any moment in time depends on the parameters connected with technology and nature, and also on the values of two control variables, c and z, representing the flows of consumption and harvesting that denote the extraction from the stocks decided in each period. There is also a return function that takes the form of a CIES ($\sigma^{-1} > 0$) instantaneous utility function, which only depends on consumption. Behind the natural resource there is a biotic law of motion suggested by the intrinsic rate of growth, $\delta > 0$, which should imply exponential growth at a constant rate out of any human economically-based intervention. In the final good sector technology is represented by $Y = AK^{\beta} (zQ)^{1-\beta} Q_a^{\gamma}$, where Q_a plays the role of a production externality, $A > 0$ is an efficiency parameter, $1 > \beta > 0$ is the elasticity of output with respect to physical capital, and $\gamma > 0$ captures the weight of the external effect. This production function exhibits constant returns to scale over private internal factors. Moreover, for fixed Q_a, there are diminishing returns to K and Q. If, however, Q_a rises along with K diminishing returns will not arise because of the increasing returns to scale with respect to all the accumulable factors taken together. The absence of diminishing returns to the factors that can be accumulated is at the origin of the endogenous growth result, and the presence of an externality opens the possibility for multiple equilibrium paths. Under the previous technological assumptions, individual agent still face to a concave optimization problem, but the external effect introduces a distortion between private and social marginal productivity of natural capital. That is, the competitive solution to this intertemporal optimization problem does not correspond to a social optimum.

The problem facing up this economy, for a given constant intertemporal discount rate $\rho > 0$, consists in choosing the controls $c(t)$ and $z(t)$ $\forall t \geq 0$, which solve the following optimization problem:

$$Max \int_0^{\infty} \frac{c(t)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} \, dt \tag{P}$$

subject to:

$$\overset{\bullet}{Q}(t) = \delta\left(1 - z(t)\right)Q(t) - z(t)\,Q(t) \tag{1}$$

$$\overset{\bullet}{K}(t) = AK^{\beta}(t)\left(z(t)\,Q(t)\right)^{1-\beta}Q_a(t)^{\gamma} - c(t) \tag{2}$$

for $K_0 > 0$ and $Q_0 > 0$ given.

In order to calculate the competitive sub-optimal equilibrium, the representative agent takes $Q_a(t)$ as given when he optimizes, but we will impose hereafter the additional ex-post equilibrium condition $Q_a = Q$ to make individual decisions compatible at the aggregate level. If we introduce θ_1 and θ_2 as the co-state variables (shadow prices) associated with K and Q, respectively, then the set of equations arising from the Pontryagin's Maximum Principle as first order necessary conditions are:

$$c^{-\sigma} = \theta_1 \tag{3}$$

$$\theta_1\left(1-\beta\right)AK^{\beta}z^{-\beta}Q^{1-\beta+\gamma} = \theta_2\left(1+\delta\right)Q \tag{4}$$

$$\overset{\bullet}{\theta}_1 = \rho\theta_1 - \theta_1\beta AK^{\beta-1}z^{1-\beta}Q^{1-\beta+\gamma} \tag{5}$$

$$\overset{\bullet}{K} = AK^{\beta}z^{1-\beta}Q^{1-\beta+\gamma} - c \tag{6}$$

$$\overset{\bullet}{\theta}_2 = \rho\theta_2 - \theta_1(1-\beta)AK^{\beta}z^{1-\beta}Q^{-\beta+\gamma} - \theta_2\delta\left(1 - \left(\frac{1+\delta}{\delta}\right)z\right) \tag{7}$$

$$\overset{\bullet}{Q} = \delta\left(1 - \left(\frac{1+\delta}{\delta}\right)z\right)Q \tag{8}$$

The boundary conditions include the two initial conditions K_0 and Q_0, as well as the transversality conditions:

$$\lim_{t\to\infty}\theta_1 K\exp\left\{-\rho t\right\} = 0 \tag{9}$$

$$\lim_{t\to\infty} \theta_2 Q \exp\{-\rho t\} = 0 \tag{10}$$

From (3) and (4) we get the two control functions:

$$c = \theta_1^{-\frac{1}{\sigma}} \tag{11}$$

$$z = \left(\frac{(1-\beta)\,A}{(1+\delta)}\right)^{\frac{1}{\beta}} \left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{\beta}} Q^{\frac{\gamma}{\beta}-1} K \tag{12}$$

After substituting in (5)-(8), we obtain the following dynamic system:

$$\overset{\bullet}{\theta}_1 = \rho\theta_1 - \xi\theta_1^{\frac{1}{\beta}}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)} Q^{\frac{\gamma}{\beta}} \tag{13}$$

$$\overset{\bullet}{K} = \frac{\xi}{\beta}\theta_1^{\frac{1-\beta}{\beta}}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)} K Q^{\frac{\gamma}{\beta}} - \theta_1^{-\frac{1}{\sigma}} \tag{14}$$

$$\overset{\bullet}{\theta}_2 = -\left(\delta - \rho\right)\theta_2 \tag{15}$$

$$\overset{\bullet}{Q} = \delta Q - \left(\frac{1-\beta}{\beta}\right)\xi\theta_1^{\frac{1}{\beta}}\theta_2^{-\frac{1}{\beta}} K Q^{\frac{\gamma}{\beta}} \tag{16}$$

where $\xi \equiv \frac{\beta(1+\delta)}{(1-\beta)}\left(\frac{(1-\beta)A}{(1+\delta)}\right)^{\frac{1}{\beta}} > 0$. These equations, together with the initial conditions K_0 and Q_0 and the transversality conditions (9) and (10), determine the equilibrium dynamics over time.

3 The complete analytical solution

In this section we resolve the non-linear dynamic system (13)-(16) together with the initial conditions K_0 and Q_0 and the transversality conditions (9) and (10). We characterize the whole parameter space identifying the main parameter constraints that delimitate the different sub-spaces for which the states, co-states and controls show, respectively, a unique equilibrium trajectory, a multiplicity of equilibrium trajectories, or even no equilibrium trajectory at all. Once trajectories have been completely obtained, we conclude about the short-run and the long-run dynamics for each variable. We also analyze features like positiveness and the sign of the long-run rates of growth. We will proceed sequentially and using instrumental variables when necessary. The main competitive equilibrium results will be provided under the form of Propositions with their corresponding proofs. Let us start with the co-state variable θ_2.

Proposition 1 *: Along any equilibrium path, θ_2 grows permanently at a constant rate, $-(\delta-\rho) \lesseqgtr 0$. Each of these paths, in turn, represents a balanced growth path for θ_2.*

Proof. From (15) we obtain $\overset{\bullet}{\theta}_2 / \theta_2$ constant. Hence,

$$\theta_2 = \theta_2(0) \exp\{-(\delta-\rho)t\} \tag{17}$$

where $\theta_2(0)$ has still to be determined. ■

Consider now the instrumental variable X defined as:

$$X \equiv \theta_1^{\frac{1}{\sigma}} K = \frac{K}{c} \tag{18}$$

By totally differentiating and substituting from (13) and (14) we get:

$$\overset{\bullet}{X} = \frac{1}{\sigma}\frac{\overset{\bullet}{\theta}_1}{\theta_1}X + \frac{\overset{\bullet}{K}}{K}X = \frac{\rho}{\sigma}X - \frac{\xi}{\sigma}\theta_1^{\frac{1}{\beta}-1}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)}Q^{\frac{\gamma}{\beta}}X + \frac{\xi}{\beta}\theta_1^{\frac{1}{\beta}-1}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)}Q^{\frac{\gamma}{\beta}}X - \frac{X}{\theta_1^{\frac{1}{\sigma}}K}$$

which under the assumption that $\sigma=\beta$, transforms into the following non homogeneous first-order first-degree linear differential equation with constant coefficients:

$$\overset{\bullet}{X} = \frac{\rho}{\sigma}X - 1 \tag{19}$$

Now, given the initial condition K_0 and a certain initial value $\theta_1(0)$, although for the moment unknown, we can generate an initial condition for X, namely $X(0) = \theta_1^{\frac{1}{\sigma}}(0)K_0$. Then, a particular solution to (19) will be given by the expression:

$$X = \frac{\sigma}{\rho} + \left[X(0) - \frac{\sigma}{\rho}\right]\exp\left\{\frac{\rho}{\sigma}t\right\} \tag{20}$$

The transversality condition (9) allows us to establish and prove the following:

Proposition 2 *: Along any equilibrium path, X remains constant at the stationary value $X = \frac{\sigma}{\rho}$.*

Proof. From (18) and (20), under the assumption $\sigma = \beta$, we get:

$$\theta_1 K = X\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} = \frac{\sigma}{\rho}\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} + \left[X(0) - \frac{\sigma}{\rho}\right]\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\left\{\frac{\rho}{\sigma}t\right\}$$

Then, the transversality condition (9) may be written as:

$$\lim_{t\to\infty} \theta_1 K \exp\{-\rho t\} = \lim_{t\to\infty} \frac{\sigma\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\{-\rho t\}}{\rho}$$

$$+ \lim_{t\to\infty} \left[X(0) - \frac{\sigma}{\rho}\right]\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\left\{\rho\left(\frac{1-\beta}{\beta}\right)t\right\} = 0 \qquad (21)$$

Given that in the long-run X is always different from zero, the transversality condition imposes as necessary but not sufficient condition:

$$\lim_{t\to\infty} \theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\{-\rho t\} = 0 \qquad (22)$$

Consequently, looking at the second right-hand term of (21), we realize that the transversality condition also imposes the constraint $X(0) = \frac{\sigma}{\rho}$, from which we deduce the stationarity of X simply by substituting in (20). This is the unique non-explosive solution trajectory for X, which implies a constant value given by the initial condition. Moreover, this result also implies a particular and well-defined initial value for θ_1:

$$\theta_1(0) = \left(\frac{\sigma}{\rho}\frac{1}{K_0}\right)^{\sigma} \qquad (23)$$

where σ, the inverse of the intertemporal elasticity of substitution, is equal to the elasticity of goods production with respect to physical capital stock, β. ■

Proposition 3 *: Under the equilibrium conditions,*

i) if $\gamma > \beta$ and $\delta(1+\gamma-\beta) - \rho > 0$ then there exist a continuum of equilibrium paths for Q starting from Q_0. These paths may be characterized by the multiplicity of initial values $\theta_2(0) = (1+\epsilon)\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} Q_0^{\gamma-\beta}$, where $\epsilon \gtrless 0$ is indeterminate.

ii) if $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho \leqslant 0$ then it does not exist any equilibrium path for Q starting from Q_0.

iii) if $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho \geqslant 0$ then it does not exist any equilibrium path for Q starting from Q_0.

iv) if $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$ then there exist a unique equilibrium path for Q starting from Q_0. This unique path may be characterized by the initial value $\theta_2(0) = \left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} Q_0^{-(\beta-\gamma)}$.

Proof. Making use of the previous result about the instrumental variable X, we can reconsider the non-linear dynamic system (13)-(16), which may be sequentially solved in closed form. We do not need to transform this original modified Hamiltonian dynamic system by reducing its dimension. Instead, we can substitute the results from Propositions 1 and 2 in (16) getting:

$$\overset{\bullet}{Q} = \delta Q - \psi_1 Q^{\frac{\gamma}{\beta}} \tag{24}$$

where $\psi_1 = \left(\frac{1-\beta}{\beta}\right)\xi\theta_2^{-\frac{1}{\beta}}(0)\frac{\sigma}{\rho}\exp\left\{\frac{\delta-\rho}{\beta}t\right\}$. Equation (24) may be solved in two steps using Bernoulli's method, which leads to the general solution:

$$Q = \left\{\left[Q_0^{\frac{\beta-\gamma}{\beta}} + W_1\right]\exp\left\{\frac{\delta(\beta-\gamma)}{\beta}t\right\} - W_1\exp\left\{\frac{\delta-\rho}{\beta}t\right\}\right\}^{\frac{\beta}{\beta-\gamma}} \tag{25}$$

where:

$$W_1 = -\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\theta_2^{-\frac{1}{\beta}}(0)\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}$$

The transversality condition (10), in turn, may be written as:

$$0 = \lim_{t\to\infty}\left[(\theta_2(0)Q_0)^{\frac{\beta-\gamma}{\beta}} - \frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\theta_2^{-\frac{1+\gamma-\beta}{\beta}}(0)\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right.$$

$$\left.+\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\theta_2^{-\frac{1+\gamma-\beta}{\beta}}(0)\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\exp\left\{\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t\right\}\right]^{\frac{\beta}{\beta-\gamma}} \tag{26}$$

and the different cases in Proposition 3 arise almost automatically in a natural way. ■

As we have seen in Proposition 3, the initial value for the shadow price of natural capital admits the following general specification:

$$\theta_2(0) = (1+\epsilon)\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} Q_0^{\gamma-\beta} \tag{27}$$

This expression will correspond to case *i)* under the additional constraints: $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$ for any $\epsilon \gtrless 0$. Moreover, it will correspond to case *iv)* under the alternative set of constraints: $\epsilon = 0$, $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$. On the other hand, the coefficient W_1 appearing in (25) may be simplified by defining $W_1 = -(1+\Delta)\,Q_0^{\frac{\beta-\gamma}{\beta}}$, where $1+\Delta \equiv (1+\epsilon)^{-\frac{1}{\beta}}$ and $\Delta \gtrless 0$ depending on whether $\epsilon \lessgtr 0$. Now, we can use this definition to derive a general expression for Q, which encompasses the two cases *i)* and *iv)* from Proposition 3:

$$Q = \frac{Q_0}{\left[1+\Delta-\Delta\exp\left\{-\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t\right\}\right]^{\frac{\beta}{\gamma-\beta}}}\exp\left\{\frac{\rho-\delta}{\gamma-\beta}t\right\} \tag{28}$$

This expression will correspond to case *i)* under the constraints: $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$, for any $\Delta \gtrless 0$. It shows a multiplicity of solution trajectories for Q because of the indeterminate value of the parameter Δ. Moreover, it will correspond to case *iv)* under the constraints: $\epsilon = \Delta = 0$, $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$, showing a unique solution trajectory for Q because in this case the parameter Δ takes a definite value.

Proposition 4 *: Under the equilibrium conditions,*

a) if $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$ then there exist a continuum of equilibrium paths for θ_2. These paths may be characterized by the multiplicity of initial values $\theta_2(0) = (1+\Delta)^{-\beta}\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} Q_0^{\gamma-\beta}$, where $\Delta \gtrless 0$ is indeterminate.

b) if $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$ then there exist a unique equilibrium path for θ_2. This unique path may be characterized by the initial value $\theta_2(0) = \left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} Q_0^{-(\beta-\gamma)}$.

Otherwise it does not exist any equilibrium path for θ_2.

Proof. We only need to take (17) and substitute the value for $\theta_2(0)$ just determined in (27). Then, we get:

$$\theta_2 = (1+\Delta)^{-\beta} \left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\,\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho} \right)^{\beta} Q_0^{\gamma-\beta} \exp\{-(\delta-\rho)\,t\} \qquad (29)$$

Multiplicity appears associated with the indeterminate value of Δ, while in case *b)*, where $\Delta = 0$, the indetermination disappears and we find a unique trajectory. ■

Lemma 1 *: The equilibrium paths for θ_2 and Q take only positive values if and only if $\Delta > -1$.*

Proof. From (29), given the correlation among the signs of the parameter constraints as indicated in Proposition 4, we conclude that the positiveness of θ_2 depends on the constraint $\Delta > -1$ alone. From (28), the positiveness of Q also depends on the constraint $\Delta > -1$, given the sign of the parameter constraints. ■

Proposition 5 *: If $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$ then any of the multiple equilibrium trajectories for Q starting from Q_0, while describing transitional dynamics, approaches asymptotically to an undetermined positive balanced growth path where the natural capital stock grows permanently at a constant rate $\bar{g}_Q^I = \frac{\rho-\delta}{\gamma-\beta} \gtreqless 0$, depending on whether $\rho \gtreqless \delta$.*

Proof. Under the above parameter constraints, looking at (28) we find that in the long-run any of the multiple equilibrium trajectories for Q evolves transitionally approaching to its associated positive balanced growth path:

$$\bar{Q}_I = \frac{Q_0}{[1+\Delta]^{\frac{\beta}{\gamma-\beta}}} \exp\left\{ \frac{\rho-\delta}{\gamma-\beta} t \right\} \qquad (30)$$

for any $\Delta > -1$. Along these asymptotic paths Q grows at a constant rate, which is positive or negative depending on whether $\rho \gtreqless \delta$. ■

Corollary 1 *: Under the parameter constraints assumed in the previous Proposition, any of the long-run equilibrium trajectories or balanced growth paths, which implies permanent and positive (negative) growth for Q, also implies permanent and positive (negative) growth for its associated shadow price θ_2. Nevertheless, along any of such trajectories we find non-explosivity because the transversality condition is satisfied.*

Proposition 6 *: If $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$ then, associated with the unique equilibrium trajectory for Q starting from Q_0, it does not exist transitional dynamics at all, and the natural capital stock grows forever along such a balanced growth path at a constant rate $\bar{g}_Q^{II} = \frac{\delta-\rho}{\beta-\gamma} \gtreqless 0$, depending on whether $\delta \gtreqless \rho$.*

Proof. Under the constraint of a weak externality that means $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$ then, according to Proposition 3, the constraint $\Delta = 0$ applies too. Thus, substituting the latter in (28), we get the following positive balanced growth path:

$$Q = \bar{Q}_{II} = Q_0 \exp\left\{\frac{\delta-\rho}{\beta-\gamma}t\right\} \tag{31}$$

Consequently, Q grows at a constant rate, which is positive or negative depending on whether $\delta \gtreqless \rho$. ■

Corollary 2 *: Under the parameter constraints assumed in the previous Proposition, the unique equilibrium trajectory and balanced growth path which implies permanent and positive (negative) growth for Q, also implies a continuous decreasing (increasing) movement for its associated shadow price θ_2. Along this trajectory the transversality condition is satisfied.*

Proposition 7 *: Under the equilibrium conditions,*

I) if $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$ then there exist a continuum of equilibrium paths for θ_1 starting from $\theta_1(0)$. These paths may be characterized by the indeterminate value of the parameter Δ.

II) if $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$, hence $\Delta = 0$, then there exist a unique equilibrium path for θ_1 starting from $\theta_1(0)$.

Otherwise it does not exist any equilibrium path for θ_1 starting from $\theta_1(0)$.

Proof. Using (28) for Q and (29) for θ_2, we can substitute in (13), getting the non-linear differential equation:

$$\overset{\bullet}{\theta}_1 = \rho\theta_1 - \psi_2\theta_1^{\frac{1}{\beta}} \tag{32}$$

where $\psi_2 = \xi\left(\frac{1}{1+\Delta}\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta-1}\left[1+\Delta-\Delta\exp\left\{-\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t\right\}\right]^{\frac{-\gamma}{\gamma-\beta}}$ $Q_0^{1+\gamma-\beta}\exp\left\{\frac{\delta-\rho}{\beta-\gamma}(1+\gamma-\beta)\,t\right\}$. Equation (32) may be solved as before applying Bernoulli's method, which leads to the solution:

$$\theta_1 = \left[\left(\frac{\rho}{\beta}K_0\right)^{1-\beta} + C_\Delta^0 Q_0^{1+\gamma-\beta} I_\Delta(t)\right]^{\frac{-\beta}{1-\beta}} \exp\{\rho t\} \tag{33}$$

where $C_\Delta^0 = \frac{\left(\frac{1-\beta}{\beta}\right)\xi}{(1+\Delta)^{\frac{\beta(1+\gamma-\beta)}{\gamma-\beta}}\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{1-\beta}}$ is an indeterminate constant, which depends on the value of the parameter Δ, and $I_\Delta(t)$ represents the following definite integral, which also depends on the parameter Δ:

$$I_\Delta(t) = \int_0^t \frac{\exp\left\{-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta}s\right\}}{\left[1-\frac{\Delta}{1+\Delta}\exp\left\{-\frac{\delta(1+\gamma-\beta)-\rho}{\beta}s\right\}\right]^{\frac{\gamma}{\gamma-\beta}}}ds \tag{34}$$

Equation (33) gives a continuum of solution trajectories for θ_1 depending on the indeterminate value of Δ as well as on the value of the remaining structural parameters. Hence, we will study this shadow price under two sets of parameter constraints. First, consider $\gamma > \beta$, $\delta(1+\gamma-\beta)-\rho > 0$ and $\Delta > -1$. In this case, the necessary transversality condition (21) which imposes the non-explosivity constraint $\lim_{t\to\infty} \theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\{-\rho t\} = 0$, given (33) may be simplified to the following necessary condition: $\lim_{t\to\infty} I_\Delta(t)\exp\left\{-\frac{\rho}{\beta}t\right\} = 0$. Moreover, given the above parameter constraints, we can see the integrand function in (34) as a function converging in the long-run to the pure exponential function $\exp\left\{-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta}s\right\}$. Therefore, this suggest a bound to the function $I_\Delta(t)$ as in the following integral function:

$$I_b(t) = \int_0^t \exp\left\{-\frac{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\,\beta}s\right\}ds$$

$$= \frac{(\gamma-\beta)\,\beta\left(1-\exp\left\{-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta}t\right\}\right)}{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}$$

Then, given the applicability of the transversality condition in the limit as t tends to infinity, we can reconsider the previous necessary condition in terms of the bounding function just introduced, which allows us to write: $\lim_{t\to\infty} I_b(t)\exp\left\{-\frac{\rho}{\beta}t\right\} = 0$. It is easy to see that, under the prevailing set of parameter constraints, this condition always holds and no other parameter constraint is needed.

Second, consider $\gamma < \beta$, $\delta\,(1+\gamma-\beta)-\rho < 0$ and $\Delta = 0$. In this case, (33) simplifies to:

$$\theta_1 = \left[\left(\frac{\rho}{\beta}K_0\right)^{1-\beta} + C_0^0 Q_0^{1+\gamma-\beta} I_0(t)\right]^{\frac{-\beta}{1-\beta}} \exp\{\rho t\}$$

where $C_0^0 = \frac{\left(\frac{1-\beta}{\beta}\right)\xi}{\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{1-\beta}} > 0$ is the value for the constant C_Δ^0 when $\Delta = 0$, and $I_0(t) = \frac{(\gamma-\beta)\beta\left(1-\exp\left\{-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta}t\right\}\right)}{\delta\beta(1+\gamma-\beta)-\gamma\rho}$ represents the solution to the integral function $I_\Delta(t)$ under $\Delta = 0$. After some substitutions and rearranging terms we find the following expression for the shadow price of physical capital:

$$\theta_1 = [\left\{\left(\frac{\rho}{\beta}K_0\right)^{1-\beta} - \frac{(\beta-\gamma)\,\beta C_0^0 Q_0^{1+\gamma-\beta}}{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}\right\}\exp\left\{-\frac{(1-\beta)\,\rho}{\beta}t\right\}$$

$$+\frac{(\beta-\gamma)\,\beta C_0^0 Q_0^{1+\gamma-\beta}}{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}\exp\left\{\frac{(\delta-\rho)\,(1+\gamma-\beta)}{(\beta-\gamma)}t\right\}]^{\frac{-\beta}{1-\beta}} \tag{35}$$

Thus, given the solution for θ_1 and the prevailing set of parameter constraints, the transversality condition (21) which imposes the non-explosivity constraint $\lim_{t\to\infty} \theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\{-\rho t\} = 0$ will be always met with no additional constraint on the parameter values. In this case, there exist a unique equilibrium path for θ_1 starting from $\theta_1(0)$. The initial value for θ_1 depends only on K_0, as shown in (23). However, subsequent values also depend on the initial natural capital stock Q_0. ■

Finally, using the previous results for the variables θ_1, Q and θ_2 we can substitute in (14) in such a way that we get:

$$\dot{K} = \psi_3 K - \psi_4 \tag{36}$$

where:

$$\psi_3 = \frac{\xi}{\beta}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)}\theta_1^{\frac{1}{\beta}-1}Q^{\frac{\gamma}{\beta}} = \frac{1}{\beta}\psi_2\theta_1^{\frac{1-\beta}{\beta}}$$

$$= \frac{\frac{1}{\beta}\xi\left(\frac{1}{1+\Delta}\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta-1} Q_0^{1+\gamma-\beta}\exp\left\{\left(\frac{\delta-\rho}{\beta-\gamma}(1+\gamma-\beta)+\frac{\rho}{\beta}-\rho\right)t\right\}}{\left[1+\Delta-\Delta\exp\left\{-\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t\right\}\right]^{\frac{\gamma}{\gamma-\beta}}\left[\left(\frac{\rho}{\beta}K_0\right)^{1-\beta}+C_\Delta^0 Q_0^{1+\gamma-\beta}I_\Delta(t)\right]}$$

and

$$\psi_4 = \theta_1^{-\frac{1}{\sigma}} = \left[\left(\frac{\rho}{\beta}K_0\right)^{1-\beta}+C_\Delta^0 Q_0^{1+\gamma-\beta}I_\Delta(t)\right]^{\frac{1}{1-\beta}}\exp\left\{-\frac{\rho}{\beta}t\right\}$$

The general solution to (36) is:

$$K = K_0\exp\left\{\int_0^t \psi_3(s)\ ds\right\} - \int_0^t \psi_4(r)\exp\left\{\int_r^t \psi_3(z)\ dz\right\}\ dr \tag{37}$$

This is an exact solution for K which depends only on the parameters and the initial conditions. Nevertheless, the above expression is quite complex and we would like to find an alternative way for getting the trajectory solution for physical capital stock. We can do that by using some previous results like the one established in Proposition 2.

Proposition 8 *: Under the equilibrium conditions,*

I) if $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$ then there exist a continuum of equilibrium paths for K starting from K_0. These paths may be characterized by the indeterminate value of the parameter Δ.

II) if $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$, hence $\Delta = 0$ and then there exist a unique equilibrium path for K starting from K_0.

Otherwise it does not exist any equilibrium path for K starting from K_0.

Proof. Given definition (18) as well as the constant value for X under the assumption $\sigma = \beta$, and the general solution for θ_1 given in (33), we can write:

$$K = \frac{\beta}{\rho}\theta_1^{-\frac{1}{\beta}} = \frac{\beta}{\rho}\left[\left(\frac{\rho}{\beta}K_0\right)^{1-\beta} + C_\Delta^0 Q_0^{1+\gamma-\beta} I_\Delta(t)\right]^{\frac{1}{1-\beta}} \exp\left\{-\frac{\rho}{\beta}t\right\} \tag{38}$$

Equation (38) shows a continuum of solution trajectories for K depending on the indeterminate value of the parameter Δ, which correspond to the following set of parameter constraints: $\gamma > \beta$, $\delta(1+\gamma-\beta)-\rho > 0$ and $\Delta > -1$. Instead, when the prevailing set of parameter constraints is: $\gamma < \beta$, $\delta(1+\gamma-\beta)-\rho < 0$ and $\Delta = 0$, the expression for physical capital stock simplifies to:

$$K = \frac{\beta}{\rho}[\left\{\left(\frac{\rho}{\beta}K_0\right)^{1-\beta} - \frac{(\beta-\gamma)\beta C_0^0 Q_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right\}\exp\left\{-\frac{(1-\beta)\rho}{\beta}t\right\}$$

$$+\frac{(\beta-\gamma)\beta C_0^0 Q_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\exp\left\{\frac{(\delta-\rho)(1+\gamma-\beta)}{(\beta-\gamma)}t\right\}]^{\frac{1}{1-\beta}} \tag{39}$$

In this case, there exist a unique equilibrium path for K starting from K_0. Subsequent values of K also depend on the initial natural capital stock Q_0.

On the other hand, given the direct dependence of K with respect to θ_1, as established by the constancy of variable X, which arises from the transversality condition, the different cases considered in Proposition 7 necessarily have to reflect the corresponding ones in Proposition 8. ■

Lemma 2 *: In the case where $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$, if $\Delta > -1$ and $\delta\beta(1+\gamma-\beta)-\gamma\rho < 0$ the multiple equilibrium paths for θ_1 and K take only positive values.*

Proof. Looking at (33) and (38), if C_Δ^0 and $I_\Delta(t)$ are always positive then we get always positive values for θ_1 and K. Given the signs of the parameter constraints, both C_Δ^0 and $I(t)$ are always positive if $\Delta > -1$ and $\delta\beta(1+\gamma-\beta)-\gamma\rho<0$. ■

Lemma 3 *: In the case where $\gamma<\beta$ and $\delta(1+\gamma-\beta)-\rho<0$, and hence $\Delta=0$, if $\delta\beta(1+\gamma-\beta)-\gamma\rho>0$ the unique equilibrium paths for θ_1 and K take only positive values.*

Proof. This is a result which arises immediately from (35) and (39). ■

Proposition 9 *: If $\gamma>\beta$ and $\delta(1+\gamma-\beta)-\rho>0$ then any of the multiple equilibrium trajectories for K starting from K_0, while describing transitional dynamics, approaches asymptotically to an undetermined positive balanced growth path where the physical capital stock grows permanently at a constant rate $\bar{g}_K^I=\frac{1+\gamma-\beta}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)\gtrless 0$, depending on whether $\rho\gtrless\delta$.*

Proof. Under the above parameter constraints, looking at (33) we find that in the long-run any of the multiple equilibrium trajectories for θ_1 evolves transitionally approaching to its associated positive balanced growth path:

$$\bar{\theta}_{1I}=\left(-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta C_\Delta^0 Q_0^{1+\gamma-\beta}}\right)^{\frac{\beta}{1-\beta}}\exp\left\{\frac{-\beta(1+\gamma-\beta)}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)t\right\} \tag{40}$$

for any $\Delta>-1$ and $\delta\beta(1+\gamma-\beta)-\gamma\rho<0$. Consequently, given definition (18) and the constant value for X under the assumption $\sigma=\beta$, in the long-run any of the multiple equilibrium trajectories for K evolves transitionally approaching to its associated positive balanced growth path:

$$\bar{K}_I=\frac{\beta}{\rho}\left(-\frac{(\gamma-\beta)\beta C_\Delta^0 Q_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}}\exp\left\{\frac{1+\gamma-\beta}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)t\right\} \tag{41}$$

for any $\Delta>-1$ and $\delta\beta(1+\gamma-\beta)-\gamma\rho<0$. Along these asymptotic paths K and θ_1 grow at a constant rate. It is easy to see that these variables evolve in opposite directions: K increase (decrease) while θ_1 decrease (increase) depending on whether $\rho\gtrless\delta$. Moreover, these trajectories show a direct dependence on Q_0 as well as on the parameter Δ but, instead, they are absolutely independent of K_0. ■

Proposition 10 *: If* $\gamma < \beta$ *and* $\delta(1+\gamma-\beta)-\rho < 0$ *then the unique equilibrium trajectory for* K *starting from* K_0*, while describing transitional dynamics, approaches asymptotically to the unique positive balanced growth path where the physical capital stock grows permanently at a constant rate* $\bar{g}_K^{II} = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right) \gtrless 0$*, depending on whether* $\delta \gtrless \rho$*.*

Proof. Under the constraint of a weak externality that means $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$ then, according to Proposition 7, the constraint $\Delta = 0$ applies too. Therefore, substituting in (38) we find an expression for K, which is unique and in the long-run approaches to the unique positive balanced growth path:

$$\bar{K}_{II} = \frac{\beta}{\rho}\left(\frac{(\beta-\gamma)\,\beta C_0^0 Q_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}} \exp\left\{\frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\} \tag{42}$$

given $\delta\beta(1+\gamma-\beta)-\gamma\rho > 0$. Along this path K grows at a constant rate, which is positive or negative depending on whether $\delta \gtrless \rho$. Once again, we can see that this asymptotic path for K depends on Q_0 but is completely independent of K_0.

In addition, looking for a global description of the solution we use definition (18) and the constant value for X under the assumption $\sigma = \beta$. Then, we can see that in the long-run the unique trajectory for θ_1 evolves transitionally approaching to the unique positive balanced growth path:

$$\bar{\theta}_{1II} = \left(\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\beta-\gamma)\,\beta C_0^0 Q_0^{1+\gamma-\beta}}\right)^{\frac{\beta}{1-\beta}} \exp\left\{\frac{-\beta(1+\gamma-\beta)}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\} \tag{43}$$

given $\delta\beta(1+\gamma-\beta)-\gamma\rho > 0$. ■

Corollary 3 *: For any of the cases considered along the two previous Propositions, the long-run equilibrium trajectories or balanced growth paths to which asymptotically moves the physical capital stock imply permanent and positive (negative) growth for* K*, and a continuous decrease (increase) for its associated shadow price* θ_1*. In any case, these two variables always move in opposite directions and the transversality condition is satisfied.*

The three next Propositions will give us the complete solution for the two control variables of the model: the per capita consumption c, and the harvesting rate z.

Proposition 11 *: Under the equilibrium conditions,*

I) If $\gamma > \beta$, $\delta(1+\gamma-\beta)-\rho>0$, $\delta\beta(1+\gamma-\beta)-\gamma\rho<0$ and $\Delta > -1$ then there exist a continuum of equilibrium paths for c starting from $c(0) = \frac{\rho}{\beta}K_0$. Along each equilibrium path, which may be characterized by the indeterminate value of the parameter Δ, per capita consumption takes only positive values. Moreover, while describing transitional dynamics, every equilibrium trajectory approaches asymptotically to an undetermined positive balanced growth path, along which c grows permanently at a positive or negative constant rate, $\bar{g}_c^I = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right) \gtreqless 0$, depending on whether $\rho \gtreqless \delta$.

II) If $\gamma < \beta$, $\delta(1+\gamma-\beta)-\rho<0$, $\delta\beta(1+\gamma-\beta)-\gamma\rho>0$ and $\Delta = 0$ then there exist a unique equilibrium path for c starting from $c(0) = \frac{\rho}{\beta}K_0$. Along this equilibrium path per capita consumption takes only positive values. Moreover, while describing transitional dynamics, it approaches asymptotically to the unique positive balanced growth path, along which c grows permanently at a positive or negative constant rate, $\bar{g}_c^{II} = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right) \gtreqless 0$, depending on whether $\delta \gtreqless \rho$.

Proof. Given the control function (11), definition (18) as well as Proposition 2, which assigns a constant value to X under the assumption $\sigma = \beta$, we get:

$$c = \frac{\rho}{\beta}K \tag{44}$$

Consequently, the above statements become a natural extension from those which have been stated for the variable physical capital stock along the previous Propositions. ■

Proposition 12 *: Under the equilibrium conditions,*

a) if $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho>0$ then there exist a continuum of equilibrium paths for z. These paths may be characterized by the multiplicity of initial values $z(0) = (1+\Delta)\left(\frac{\delta(1+\gamma-\beta)-\rho}{(1+\delta)(\gamma-\beta)}\right)$, where $\Delta \gtreqless 0$ is indeterminate. Moreover, any of the multiple equilibrium trajectories asymptotically approaches to the same constant value, which represents the unique balanced growth path.

b) if $\gamma < \beta$ *and* $\delta(1+\gamma-\beta)-\rho < 0$ *then there exist a unique equilibrium path for* z*. This unique path for which there is no transitional dynamics, may be characterized by the initial value* $z(0) = -\frac{\delta(1+\gamma-\beta)-\rho}{(1+\delta)(\beta-\gamma)}$*, which also represents the unique balanced growth path.*

Otherwise it does not exist any equilibrium path for z*.*

Proof. Take the control function (12) which, given the constancy of $X \equiv \theta_1^{\frac{1}{\sigma}} K$ according to Proposition 2 and the general solutions for Q and θ_2 according to (28) and (29), may be reduced to the following expression:

$$z = \frac{1}{\left[1 - \frac{\Delta}{1+\Delta}\exp\left\{-\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t\right\}\right]} \frac{\delta(1+\gamma-\beta)-\rho}{(1+\delta)(\gamma-\beta)} \tag{45}$$

When $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$ the previous equation gives a continuum of solution trajectories for z because of the indeterminate value of the parameter Δ. Moreover, it is easily derived from (45) that, in the long-run, any of the multiple equilibrium trajectories for z evolves transitionally approaching to the same constant path:

$$\bar{z}_I = \frac{\delta(1+\gamma-\beta)-\rho}{(1+\delta)(\gamma-\beta)} \tag{46}$$

When $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$, hence $\Delta = 0$, the indetermination disappears and we find a unique and constant equilibrium trajectory:

$$z = \bar{z}_{II} = -\frac{\delta(1+\gamma-\beta)-\rho}{(1+\delta)(\beta-\gamma)} \tag{47}$$

In this case, the above expression means that there is no transitional dynamics for z. This variable remains always constant. ■

Proposition 13 *: Under the equilibrium conditions,*

a) In the case where $\gamma > \beta$ *and* $\delta(1+\gamma-\beta)-\rho > 0$ *the variable* z *satisfies the constraint* $1 > z > 0$*, if and only if* $\delta < \rho + (\gamma-\beta)$ *and* $\frac{(\gamma-\beta)+(\rho-\delta)}{\delta(1+\gamma-\beta)-\rho} > \Delta > -1$.

b) In the case where $\gamma < \beta$ *and* $\delta(1+\gamma-\beta)-\rho < 0$*, along with* $\Delta = 0$*, the variable* z *satisfies the constraint* $1 > z > 0$*, if and only if* $\delta > \rho + (\gamma-\beta)$.

Proof: As we have seen along the proof of the previous Proposition, in case *a)* any of the multiple equilibrium trajectories for z starting from the indeterminate value:

$$z(0) = (1+\Delta)\left(\frac{\delta\,(1+\gamma-\beta)-\rho}{(1+\delta)\,(\gamma-\beta)}\right) \tag{48}$$

approaches monotonically to $\bar{z}_I$, as given in (46). It is immediate to prove that $1 > z(0) > 0$ if and only if $\frac{(\gamma-\beta)+(\rho-\delta)}{\delta(1+\gamma-\beta)-\rho} > \Delta > -1$, but also that $1 > \bar{z}_I > 0$ if and only if $\delta < \rho + (\gamma - \beta)$.
On the other hand, in case *b)* the variable z follows a constant trajectory associated with the initial value:

$$z = \bar{z}_{II} = z(0) = -\frac{\delta\,(1+\gamma-\beta)-\rho}{(1+\delta)\,(\beta-\gamma)} \tag{49}$$

In this case, the constraint $1 > z > 0$ holds if and only if $\delta > \rho + (\gamma - \beta)$. ■

This completes the analytical closed-form solution corresponding to the competitive equilibrium. Along the previous Propositions we have shown several results, all of them derived under the simplifying assumption $\sigma = \beta$. However, given our interest in theoretical properties of the transitional dynamics and the explicit trajectories for the different variables, the above assumption does not seem too restrictive. In fact, we can identify the following shortcomings: first, consumption is proportional to physical capital stock; second, the initial physical capital stock does not contribute to determine any of the long-run balanced growth paths; and third, transitional dynamics corresponding to all the variables are partially simplified, although they still retain the main original features.

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