How to Manage Inflation Risk in an Asset Allocation Problem: An Algebraic Approximated Solution*

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Abstract

This paper analyses the portfolio problem of an investor who wants to maximize the expected utility of his terminal real wealth in an incomplete financial market. The investor must cope with a set of stochastic investment opportunities and inflation risk following a jump-diffusion process. We investigate how the inflation risk affects the optimal portfolio composition and, at this aim, we present an approximated analytical solution to the portfolio choice problem based on the Feynman-Kac representation theorem. Finally, we compare our approximated solution with some exact solutions available in the literature and we find that the main qualitative results are maintained.

JEL classification: G11, C61.

Key words: asset allocation, inflation risk, Feynman-Kac theorem, stochastic investment opportunities

*We have received helpful comments from R. Anderson, G. Deelstra, O. Scaillet and D. Sciutti. We gratefully acknowledge financial support from the Belgian French Community's program "Action de Recherche Concentrée" 99/04-235.

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1 Introduction

This work analyses the issue of optimal portfolio policy in a multi-period model where investors maximize expected utility of their terminal real wealth facing, in particular, an inflation risk outside the financial market. Furthermore, our work offers a contribution to the investment problem in the rather general case where the value of assets depends on the stochastic behaviour of a set of state variables.

The vector of state variables contains all the stochastic variables directly affecting the asset prices but indirectly affecting the investors' wealth. For a review of all variables which can affect the asset prices readers are referred to Campbell (2000) who offers a survey of the most important contributions in this field.

Beside these state variables, we consider a kind of "background risk" represented by inflation. In the literature there exist some examples where the background risk is given by the investors' wages (see for instance Franke, Peterson and Stapleton, 2001).

This framework makes our model quite general because it can be applied, for example, to pension funds (see for instance Blake, 1998, Blake, Cairns and Dowd, 1998, and Boulier, Huang and Taillard, 2001) or to insurance companies (see Young and Zariphopoulou, 2000). The inflation is described by a jump-diffusion process where the jump component accounts for sudden changes in the inflation rate.

In this paper we follow the traditional route to use the stochastic dynamic programming technique (Merton, 1969,1971) leading to the Hamilton-Jacobi-Bellman (HJB) equation.[1] For the method called "martingale approach" the reader is referred to Cox and Huang (1989,1991), and Lioui and Poncet (2000). We find that the optimal portfolio is formed by three components: (i) a preference free component minimizing the investor's wealth volatility and immunizing the investor's portfolio against the inflation risk, (ii) a speculative part proportional to both the Sharpe ratio of investor's portfolio and the inverse of the Arrow-Pratt relative risk aversion index, and (iii) a component depending on the derivatives of the value function (indirect utility function) with respect to the state and inflation variables. The last component is the only one depending on the investor's time horizon.

We outline the simple solution to the maximization problem if the investor has a log-utility function and we show that, if the investor has a power utility function, then the only optimal portfolio component depending on the form of the value function is the third one. The same kind of property is found in Merton (1990, Chapter 5.9) but under the hypotheses that: (i) the riskless interest rate is constant, (ii) the only state variables are the asset prices, and (iii) there are no background risks.

In order to find the explicit solution to this value function, it is necessary to solve the HJB equation. Unfortunately, solving this highly non-linear PDE is

[1] Øksendal (2000) and Björk (1998) offer a complete derivation of the HJB equation.

the most difficult task of the stochastic optimal control approach. In fact, some algebraic solutions can only be obtained in very special cases. In particular, we refer to the works of Kim and Omberg (1996), Wachter (1998), Boulier, Huang and Taillard (2001), and Deelstra, Grasselli and Koehl (2001).

In the present work we propose a general approximated solution to the HJB equation. Even if our solution is exact under particular conditions that must hold on the value function, we find that it stays valid as an approximated solution under conditions which are not very restrictive. We compare our result both with the above-mentioned exact solutions and with another approximated solution offered by Kogan and Uppal (1999) and based on the work of Chacko and Viceira (1999). We find that the gain in computational simplicity does not generate a great error in our approximated solutions with respect to the exact ones. Furthermore, with respect to the approximation of Kogan and Uppal, our model offers a richer solution in describing the behaviour of optimal portfolio as function of the problem parameters. Moreover, all the qualitative results are maintained.

We underline that the exact solutions presented in Kim and Omberg (1996), Boulier, Huang and Taillard (2001), and Deelstra, Grasselli and Koehl (2001), consider only one state variable and do not take into account any background risk. Instead, our model is able to determine an approximated solution when there exists a set of generic state variables and the background risk is given by the inflation rate. Thus, our framework seems to be very general and able to be applied to many particular cases.

Through this work we consider agents trading continuously in a frictionless, arbitrage-free but incomplete market until time H which is the horizon of the economy.

The paper is structured as follows. Section 2 details the general economic framework and exposes the stochastic differential equations describing the behaviour of asset prices, state variables and the inflation process. In Section 3 the optimal portfolio composition is computed. This section presents our main results: the behaviour of optimal portfolio with respect to the inflation risk and the approximated algebraic solution of the HJB equation. Section 4 presents the comparisons between our solution and other exact and approximated solutions of the HJB equation in different frameworks. Section 5 concludes. All the computations relative to Section 4 can be found in the Appendix.

2 The market structure

We suppose that asset prices are affected by a set of state variables representing all the risk sources asset prices are linked to. For a review of all variables which can affect the asset prices readers are referred to Campbell (2000) who offers a survey of the most important contributions in this field.

In this paper we suppose that these risk sources follow the stochastic differential equation:

$$\begin{aligned} \underset{s\times 1}{dX} &= \underset{s\times 1}{f(X,t)}dt + \underset{s\times k}{g(X,t)'}\underset{k\times 1}{dW}, \\ X(t_0) &= X_0, \end{aligned} \tag{1}$$

where s is the number of state variables and dW is the differential of a $k-$dimensional Wiener process whose components are independent.[2]

Given these variables we can write the process describing the behaviour of asset returns like the following stochastic differential equation:

$$\begin{aligned} \underset{n\times 1}{dS} &= \underset{n\times n}{I_S}\left[\underset{n\times 1}{\mu(t,X,S)}dt + \underset{n\times k}{\Sigma(t,X,S)'}\underset{k\times 1}{dW}\right], \\ S(t_0) &= S_0, \end{aligned} \tag{2}$$

where I_S is a diagonal matrix containing asset prices (in nominal terms).

The set of risk sources is the same for the state variables and for the asset prices. This hypothesis is not restrictive because thanks to the matrices g and Σ we can model a lot of different situations. For instance, if we consider $dW = \begin{bmatrix} dW_1 & dW_2 \end{bmatrix}$, $g' = \begin{bmatrix} g_1 & 0 \end{bmatrix}$ and $\Sigma' = \begin{bmatrix} 0 & \sigma_2 \end{bmatrix}$ then the processes of X and S are not correlated.

Finally, we add the assumption that on financial market there exists a riskless asset whose price (G) follows the differential equation:

$$\begin{aligned} dG &= r(X,t)\,G dt, \\ G(t_0) &= G_0, \end{aligned}$$

where $r(X,t)$ is the nominal risk-free interest rate which is supposed to depend on the state variables X.

If we define as $\{S(t,X)\}_{t\in[t_0,H]}$ the market where there are n risky assets and one riskless asset (G) we say that the market $\{S(t,X)\}_{t\in[t_0,H]}$ is normalized if $G \equiv 1$. This hypothesis means that the riskless asset is the numeraire of the economy. Any market can always be normalized by putting $\overline{S}(t,X) = G(t,X)^{-1} S(t,X)$.

We present the main results concerning completeness and arbitrage in this kind of market (for the proofs of the two following theorems see Øksendal, 2000).

Theorem 1 *A market $\{S(t,X)\}_{t\in[t_0,H]}$ is arbitrage free if and only if there exists a $k-$dimensional vector $u(t,X)$ such that:*

$$\Sigma(t,X)'\,u(t,X) = \mu(t,X) - r(t,X)\,S(t,X)\,,$$

and such that:

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_0}^{H}\|u(t,X)\|^2 dt}\right] < \infty.$$

[2] This condition can be imposed without loss of generality because a set of independent Wiener processes can always be transformed into a set of correlated Wiener processes thanks to the Cholesky decomposition. For an application see Appendices A.3 and A.4.

Theorem 2 *A market* $\{S(t,X)\}_{t\in[t_0,H]}$ *is complete if and only if there exists a unique* $k-$*dimensional vector* $u(t,X)$ *such that:*

$$\Sigma(t,X)' u(t,X) = \mu(t,X) - r(t,X) S(t,X),$$

and such that:

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_0}^{H}\|u(t,X)\|^2 dt}\right] < \infty.$$

If on the market there are less assets than risk sources ($n < k$), then the market cannot be complete even if it is arbitrage free. In this work we assume that $n < k$ and that the rank of matrix Σ is maximum (i.e. it equals n). Thus, the results we obtain in this work are valid for a financial market which is incomplete and stay valid for a complete market ($n = k$).

2.1 The inflation risk

We suppose that the investor is subject to the inflation risk and he wants to maximize the expected value of his real welath. In particular, we suppose that the stochastic part of the inflation process can be described by two components: a Wiener process and a Poisson process. The first one is able to describe the continuous changes in the level of prices while the second one can explain the sudden changes occurring at certain times.

Thus, the inflation risk process can be represented in the following way:

$$\begin{aligned} dL &= L\left[\alpha_L(t,L,S,X)\,dt + \underset{1\times k}{\Lambda(t,L,S,X)'}\underset{k\times 1}{dW} + \underset{1\times p}{\eta(t,L,S,X)'}\underset{p\times 1}{dP}\right], \quad (3)\\ L(t_0) &= L_0, \end{aligned}$$

where L is the level of prices, dW is the same set of risk sources we have for the asset prices and the state variables, and $P(t+\tau) - P(t)$ is a $p-$dimensional Poisson process whose elements are 0 when the inflation follows its "normal" behaviour, while they are 1 if there is a jump in its value. Define formally the differential dP to be the limit of $P(t+\tau) - P(t)$ as $\tau \to dt$. The parameter η is a vector of random variables measuring the magnitude of reactions of variable L to the jumps. The drift term α_L indicates the deterministic component of inflation while the variability around this trend is measured by the matrix of diffusion terms Λ.

We suppose that dP is independent of the other stochastic differentials dW and of the random variables contained in η. Furthermore, we suppose:

$$\begin{aligned} \mathbb{E}[dP] &= \phi(t,X)\,dt,\\ Cov[dP] &= I_\phi dt, \end{aligned}$$

where $I_\phi \in \mathbb{R}^{p\times p}$ is a diagonal matrix containing the elements of vector $\phi \in \mathbb{R}^{p\times 1}$.

2.2 The investor's wealth

After what we have presented in the previous subsections, the market structure can be represented in the following way:

$$\begin{cases} \underset{s\times 1}{dX} = \underset{s\times 1}{f}(t,X)dt + \underset{s\times k}{g}(t,X)'\underset{k\times 1}{dW}, \\ \underset{n\times 1}{dS} = \underset{n\times n}{I_S}\left[\underset{n\times 1}{\mu}(t,X,S)dt + \underset{n\times k}{\Sigma}(t,X,S)'\underset{k\times 1}{dW}\right], \\ dG = Gr(t,X)\,dt, \end{cases}$$
$$dL = L\left[\alpha_L(t,L,S,X)\,dt + \underset{1\times k}{\Lambda}(t,L,S,X)'\underset{k\times 1}{dW} + \underset{1\times p}{\eta}(t,L,S,X)'\underset{p\times 1}{dP}\right]. \tag{4}$$

If we indicate with $w \in \mathbb{R}^{n\times 1}$ the vector containing the percentages of wealth invested in each asset, then the growth rate of investor's (real) wealth can be represented as:

$$\frac{dR}{R} = w' I_S^{-1} dS + (1 - w'\mathbf{1})\frac{dG}{G} - \frac{dL}{L},$$

where $\mathbf{1}$ is a vector of 1s (of suitable dimension). Actually, the growth rate of real wealth can be approximated by the difference between the growth rate of nominal wealth and the growth rate of prices.

By substituting for the differentials from system (4) into the wealth differential equation, we have:

$$dR = R\left[(r-\alpha_L) + w'(\mu - r\mathbf{1})\right]dt + R\left(w'\Sigma' - \Lambda'\right)dW - R\eta' dP. \tag{5}$$

3 The optimal portfolio

Under the market structure (4) and the evolution of investor's wealth given in equation (5), the optimization problem can be written as follows:

$$\begin{cases} \max_w \mathbb{E}_{t_0}\left[K(R(H))\right] \\ d\begin{bmatrix} z \\ R \end{bmatrix} = \begin{bmatrix} \mu_z \\ R\left[(r-\alpha_L) + w'M\right] \end{bmatrix} dt + \begin{bmatrix} \Omega' \\ R(w'\Sigma' - \Lambda') \end{bmatrix} dW + \begin{bmatrix} E' \\ -R\eta' \end{bmatrix} dP, \\ z(t_0) = z_0, R(t_0) = R_0, \quad \forall t_0 \le t \le H, \end{cases} \tag{6}$$

where:

$$\begin{aligned} \underset{(s+n+1)\times 1}{z} &\equiv \begin{bmatrix} X' & S' & L \end{bmatrix}', \\ \underset{(s+n+1)\times 1}{\mu_z} &\equiv \begin{bmatrix} f' & \mu' & \alpha_L \end{bmatrix}', \\ \underset{k\times(s+n+1)}{\Omega} &\equiv \begin{bmatrix} g & \Sigma & \Lambda \end{bmatrix}, \\ \underset{p\times(s+n+1)}{E} &\equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \eta \end{bmatrix}, \\ \underset{n\times 1}{M} &\equiv (\mu - r\mathbf{1}), \end{aligned}$$

and the function $K(R)$ is increasing and concave. The vector z contains all the state and background variables but the wealth. Hereafter, we indicate with $\mathbf{0}$ a vector of zeros of suitable dimension.

From problem (6) we have the following Hamiltonian:

$$\begin{aligned}\mathcal{H} &= J_z'\mu_z + J_R R\left[(r-\alpha_L) + w'M\right] + \frac{1}{2}tr\left(\Omega'\Omega J_{zz}\right) + \\ &\quad + R\left(w'\Sigma' - \Lambda'\right)\Omega J_{zR} + \frac{1}{2}R^2 J_{RR}\left(w'\Sigma'\Sigma w - 2w'\Sigma'\Lambda + \Lambda'\Lambda\right),\end{aligned} \tag{7}$$

where $J(R,z,t)$ is the value function solving the Hamilton-Jacobi-Bellman partial differential equation (see Section 3.2), and verifying:

$$J(R,z,t) = \sup_w \mathbb{E}_t\left[K\left(R\left(H\right)\right)\right],$$

here the subscripts indicate the partial derivative. The system of first order conditions on $\mathcal{H}$ is:[3]

$$\frac{\partial\mathcal{H}}{\partial w} = J_R RM + R\Sigma'\Omega J_{zR} + R^2 J_{RR}\left(\Sigma'\Sigma w - \Sigma'\Lambda\right) = 0,$$

from which we obtain the optimal portfolio composition:

$$w^* = \underbrace{\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Lambda}_{w^*_{(1)}} \underbrace{- \frac{J_R}{J_{RR}R}\left(\Sigma'\Sigma\right)^{-1}M}_{w^*_{(2)}} \underbrace{- \frac{1}{J_{RR}R}\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Omega J_{zR}}_{w^*_{(3)}}. \tag{8}$$

We recall that in this framework the matrix $\Sigma'\Sigma$ is invertible. In fact, $\Sigma'\Sigma$ is an $n \times n$ matrix and we suppose that $\Sigma' \in \mathbb{R}^{n\times k}$ has maximum rank. Because we put $n \leq k$, then Σ has rank n and, thus, $\Sigma'\Sigma$ is invertible. This means that, even in an incomplete market, there exists a *unique* optimal portfolio.

Thus, we can state the following result:

Proposition 1 *Under market structure (4), the portfolio composition maximizing the investor's terminal wealth (thus solving problem (6)) is formed by three components: (i) a preference free part ($w^*_{(1)}$) depending only on the diffusion terms of assets and inflation process, (ii) a part ($w^*_{(2)}$) proportional to both the portfolio Sharpe ratio and the inverse of Arrow-Pratt relative risk aversion index, and (iii) a part ($w^*_{(3)}$) depending on the state variable parameters.*

[3] The second order conditions hold if the Hessian matrix of $\mathcal{H}$:

$$\frac{\partial\mathcal{H}}{\partial w'\partial w} = R^2 J_{RR}\Sigma'\Sigma,$$

is negative definite. Because $R^2\Sigma'\Sigma$ is a quadratic form it is always positive definite and so the second order conditions are satisfied if and only if $J_{RR} < 0$ that is if the value function is concave in R. The reader is referred to Stockey and Lucas (1989) for the assumptions that must hold on the function $K(R)$ for having a strictly concave value function.

In the following subsection we analyze the role of the first preference free portfolio component ($w^*_{(1)}$). For the second part, we just outline that $w^*_{(2)}$ increases if the risk premium increases and decreases if the risk aversion or the asset variance increase. From this point of view, we can argue that this component of the optimal portfolio has just a speculative role. The third part $w^*_{(3)}$ is the only optimal portfolio component explicitly depending on the diffusion terms of the state variables. Thus, while $w^*_{(1)}$ covers the investor from the inflation risk, $w^*_{(3)}$ also covers the investor from the risk "inside" the financial market (given by variables X). We will investigate the precise role of this component after computing the functional form of the value function.

3.1 The role of the preference free portfolio component

In the previous subsection we have derived the optimal portfolio composition when the investor must cope with an inflation risk. One component of the optimal portfolio is preference free. This means that this part (hereafter $w^*_{(1)}$) does not depend on the value function $J(R, z, t)$.

Furthermore, this component hedges the investor's portfolio only against the "diffusion part" of the inflation risk, while the "jump-part" is hedged by the other two components and, in particular, by the third one (see Equation (8)).

It is quite intuitive that $w^*_{(1)}$ can hedge the optimal portfolio only against the diffusion part of the inflation risk which is linked to the asset risks. For showing this property, let us divide the matrices Σ and Λ into two sub-matrices in the following way:

$$\begin{aligned} \Sigma' &= \begin{bmatrix} \Sigma'_S & \Sigma'_L \end{bmatrix}, \\ \Lambda' &= \begin{bmatrix} \Lambda'_S & \Lambda'_L \end{bmatrix}. \end{aligned}$$

Thus, the asset prices and the inflation process can be represented as:

$$\begin{aligned} dS &= I_S \mu dt + I_S \begin{bmatrix} \Sigma'_S & \Sigma'_L \end{bmatrix} \begin{bmatrix} dW_S \\ dW_L \end{bmatrix}, \\ dL &= I_L \alpha_L dt + I_L \begin{bmatrix} \Lambda'_S & \Lambda'_L \end{bmatrix} \begin{bmatrix} dW_S \\ dW_L \end{bmatrix} + I_L \eta' dP, \end{aligned}$$

in this way the vector of Wiener differentials has been divided into two sets: the asset set and the inflation set. Thus, the matrix Σ_L contains the coefficients linking the asset prices to the inflation risk set, while the elements in the matrix Λ_S measure the correlation between inflation and the risk sources of asset prices.

Accordingly, the preference-free component of optimal portfolio can be written as follows:

$$w^*_{(1)} = \left(\Sigma'_S \Sigma_S + \Sigma'_L \Sigma_L\right)^{-1} \left(\Sigma'_S \Lambda_S + \Sigma'_L \Lambda_L\right).$$

If the risk sets of asset prices and inflation are not correlated (i.e. $\Sigma_L = \Lambda_S = \mathbf{0}$), then $w^*_{(1)}$ vanishes. So, we can state:

Proposition 2 *The preference-free component ($w^*_{(1)}$) of optimal portfolio (solving problem (6)) hedges the investor's portfolio against the diffusion part of infaltion process correlated with asset price risk sources.*

The sign of this portfolio component depends on the elements of matrices Σ_L and Λ_S because we suppose that both matrices Σ_S and Λ_L contain only positive elements. In particular, if we consider the case in which the ifnlation risk can affect the asset prices but the opposite relation is not true, then we have $\Lambda_S = \mathbf{0}$ and we can write:

$$sign\left(w^*_{(1)}\right) = sign\left(\Sigma'_L\right),$$

because the matrix $\left(\Sigma'_S\Sigma_S + \Sigma'_L\Sigma_L\right)$ is positive definite. The hypothesis of having $\Lambda_S = \mathbf{0}$ is not very restrictive because the movements in the inflation risk generally affect the asset prices on the stock exchange while the opposite relation is less likely.

Accordingly, we can state:

Proposition 3 *If the price level L does not contain any asset price risk source ($\Lambda_S = \mathbf{0}$), then, under structure (4), the preference-free component ($w^*_{(1)}$) of optimal portfolio (solving problem (6)) is directly correlated with the elements of matrix Σ_L.*

Furthermore, if we consider the analogous case in which $\Sigma_L = \mathbf{0}$, then we obtain the following condition:

$$sign\left(w^*_{(1)}\right) = sign\left(\Lambda'_S\right).$$

In this case the asset prices are not affected by the risk sources of the inflation process and we can write:

Proposition 4 *If variables S do not contain any inflation risk source ($\Sigma_L = \mathbf{0}$), then, under structure (4), the preference-free component ($w^*_{(1)}$) of optimal portfolio (solving problem (6)) is directly correlated with the elements of matrix Λ_S.*

The preference free portfolio component has another important characteristic: it minimizes the instantaneous variance of investor's wealth. In fact, from Equation (5) we can see that the wealth variance depending on the control vector w is given by:

$$R^2\left(w'\Sigma'\Sigma w - 2w'\Sigma'\Lambda + \Lambda'\Lambda\right),$$

from which we immediately see that[4]:

Proposition 5 *The preference-free component ($w^*_{(1)}$) of optimal portfolio (solving problem (6)) minimizes the instantaneous variance of investor's wealth.*

3.2 The value function

For studying the role of the portfolio components we have called $w^*_{(2)}$ and $w^*_{(3)}$ (see Equation (8)), we need to compute the value function $J(R, z, t)$. By substituting the optimal value of w into the Hamiltonian (7) we have:

$$\begin{aligned}
\mathcal{H}^* \quad = \quad & J_z' \mu_z + J_R R \left[(r - \alpha_L) + M' (\Sigma'\Sigma)^{-1} \Sigma' \Lambda \right] + \\
& - R \Lambda' \left[I - \Sigma (\Sigma'\Sigma)^{-1} \Sigma' \right] \Omega J_{zR} + \\
& + \frac{1}{2} R^2 J_{RR} \Lambda' \left[I - \Sigma (\Sigma'\Sigma)^{-1} \Sigma' \right] \Lambda + \\
& + \frac{1}{2} tr (\Omega' \Omega J_{zz}) - \frac{J_R}{J_{RR}} M' (\Sigma'\Sigma)^{-1} \Sigma' \Omega J_{zR} + \\
& - \frac{1}{2} \frac{J_R^2}{J_{RR}} M' (\Sigma'\Sigma)^{-1} M - \frac{1}{2} \frac{1}{J_{RR}} J_{zR}' \left[\Omega' \Sigma (\Sigma'\Sigma)^{-1} \Sigma' \Omega \right] J_{zR},
\end{aligned}$$

from which we can formulate the PDE whose solution is the value function. This PDE is called the Hamilton-Jacobi-Bellman equation (hereafter HJB) and can be written as follows:

$$\begin{cases} J_t + \mathcal{H}^* + \sum_{i=1}^{p} \phi_i \mathbb{E} \left[J \left(t, z + E_i', R \left(1 - \{\eta'\}_i \right) \right) - J \right] = 0, \\ \qquad\qquad\qquad\qquad J(H, R, z) = K(R(H)), \end{cases} \tag{9}$$

where $\{\eta'\}_i$ is the i^{th} element of the vector η' and $E_i' \in \mathbb{R}^{(s+n+1)\times 1}$ is the i^{th} column of the matrix $E' \in \mathbb{R}^{(s+n+1)\times p}$.

Solving this PDE is the most difficult task of the stochastic optimal control approach. There are no general analytic methods available for solving the HJB equation, so the number of optimal control problems with an analytic solution is very small indeed. In the following subsection we propose a system for solving analytically the equation (9). Our method is based on a particular specification of both the investor's utility function and the value function.

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[4] The first derivative of this term with respect to w is:

$$2R^2 \left(\Sigma'\Sigma w + \Sigma' \Lambda u \right),$$

while the second derivative is:

$$2R^2 \Sigma'\Sigma,$$

which is always positive definite because $\Sigma'\Sigma$ is a variance-covariance matrix.

3.2.1 The case of separability by sum

Here, we study which form of the utility function allows us to obtain a value function separable by sum in wealth and in the other state variables: $J(z,R,t) = U(R) + F(z,t)$.

If we substitute this functional form into the HJB equation (9), we obtain:

$$\begin{array}{l} F_t + F_z'\mu_z + U_R R\left[(r-\alpha_L) + M'(\Sigma'\Sigma)^{-1}\Sigma'\Lambda\right] + \\ +\frac{1}{2}R^2 U_{RR}\Lambda'\left[I - \Sigma(\Sigma'\Sigma)^{-1}\Sigma'\right]\Lambda + \\ +\frac{1}{2}tr(\Omega'\Omega F_{zz}) - \frac{1}{2}\frac{U_R^2}{U_{RR}}M'(\Sigma'\Sigma)^{-1}M + \\ +\sum\limits_{i=1}^{p}\phi_i\mathbb{E}\left[U(R(1-\{\eta'\}_i)) - U(R) + F(z+E_i',t) - F(z,t)\right] = 0. \end{array}$$

The value function is separable by sum if and only if the terms $U_R R$, $R^2 U_{RR}$, $\frac{U_R^2}{U_{RR}}$, and $U(R(1-\{\eta'\}_i)) - U(R)$ are constant with respect to R. The only function satisfying these conditions is the log function: $U(R) = \alpha \ln R$ under the hypothesis that the vector parameter η, as we have supposed, is independent of wealth: $\frac{\partial \eta}{\partial R} = \mathbf{0}$. Without loss of generality we can put $\alpha = 1$ because this parameter does not affect the maximization problem.

After substituting for $U(R) = \ln R$ we can write (9) as:

$$\begin{cases} F_t + F_z'\mu_z + \frac{1}{2}tr(\Omega'\Omega F_{zz}) + c(z,t) + \sum\limits_{i=1}^{p}\phi_i\mathbb{E}\left[F(z+E_i',t) - F\right] = 0, \\ \ln R + F(z,H) = K(R(H)), \end{cases} \tag{10}$$

where:

$$\begin{aligned} c(z,t) \equiv\ & (r-\alpha_L) + M'(\Sigma'\Sigma)^{-1}\Sigma'\Lambda + \\ & -\frac{1}{2}\Lambda'\left[I - \Sigma(\Sigma'\Sigma)^{-1}\Sigma'\right]\Lambda + \frac{1}{2}M'(\Sigma'\Sigma)^{-1}M + \\ & +\sum_{i=1}^{p}\phi_i\mathbb{E}\left[\ln(1-\{\eta'\}_i)\right]. \end{aligned}$$

The boundary condition in system (10) holds if and only if the utility function K has the form $K(R(H)) = \ln R$. So, we can write the boundary condition as: $F(z,H) = 0$, and we can argue that:

Proposition 6 *The value function solving the HJB equation (9) is separable by sum in wealth and in the other state variables if and only if the investor has a log-utility function.*

The same result is derived in Merton (1990, Chapter 5.9) but without considering any background risk, thus, this proposition can be considered as a generalization of Merton's result.

In this case, because the cross derivative of the value function with respect to investor's wealth and to the other state variables is zero, then the optimal portfolio is just given by its first two parts: the preference free component and the speculative component. Any hedging part vanishes because of the log utility function. In fact, the log-investor is said to be "myopic" because he does not care about hedging his portfolio against the state variable risks.

Thus, we can write:

$$w^* = \left(\Sigma'\Sigma\right)^{-1}\Sigma'\Lambda + \left(\Sigma'\Sigma\right)^{-1}M. \tag{11}$$

This solution allows us to state:

Proposition 7 *The optimal portfolio composition for an investor with log-utility function is preference-free and depends on the inflation process only through the coefficient of its diffusion component.*

Thanks to this result the optimal composition (11) can be used as the base for whatever kind of investor, this base needing to be adjusted for investors with different degree of risk aversion.

3.2.2 The case of separability by product

Now, we study which form must have the value function $J(z,R,t)$ for obtaining the following separability result: $J(z,R,t) = U(R)F(z,t)$. After substituting functions U and F into the HJB equation (9) we obtain:

$$\begin{aligned}
&F_t + \mu_z' F_z + \tfrac{U_R R}{U} F\left[(r-\alpha_L) + M'\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Lambda\right] + \\
&-\tfrac{U_R R}{U}\Lambda'\left[I + \Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\right]\Omega F_z + \\
&+\tfrac{1}{2}\tfrac{U_{RR}R^2}{U}F\Lambda'\left[I - \Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\right]\Lambda + \\
&+\tfrac{1}{2}tr\left(\Omega'\Omega F_{zz}\right) - \tfrac{U_R^2}{U_{RR}U}M'\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Omega F_z + \\
&-\tfrac{1}{2}\tfrac{U_R^2}{U_{RR}U}FM'\left(\Sigma'\Sigma\right)^{-1}M - \tfrac{1}{2}\tfrac{U_R^2}{U_{RR}UF}F_z'\left[\Omega'\Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Omega\right]F_z + \\
&+\sum_{i=1}^{p}\phi_i\mathbb{E}\left[\tfrac{U\left(R\left(1-\{\eta'\}_i\right)\right)}{U}F\left(z+E_i',t\right) - F\right] = 0.
\end{aligned}$$

Because the model is consistent the ratios: $U_R R/U$, $U_{RR}R^2/U$, $U_R^2/\left(U_{RR}U\right)$, and $U\left(R\left(1-\{\eta'\}_i\right)\right)/U$ must be constant with respect to R. The only function satisfying these properties has the form: $U(R) = \alpha R^\beta$ (α and β different from zero) under the hypothesis that the vector parameter η, as we have supposed, is independent of wealth: $\frac{\partial\eta}{\partial R} = \mathbf{0}$.

Because we want that $U(R)$ is an increasing and concave function, then the coefficients α and β must be such that: $\alpha\beta > 0$, and $\beta < 1$.

If we substitute for the function $U(R) = \alpha R^{\beta}$ into the HJB equation, we have:

$$\begin{cases} F_t + \overline{a}(z,t)' F_z + \overline{b}(z,t) F + \frac{1}{2} tr\left(\Omega' \Omega F_{zz}\right) - \frac{1}{2}\frac{\beta}{\beta-1}\frac{1}{F} F_z' C(z,t) F_z + \\ \qquad + \sum_{i=1}^{p} \phi_i \mathbb{E}\left[\left(1 - \{\eta'\}_i\right)^{\beta} F\left(z + E_i', t\right) - F\right] = 0, \\ \qquad \alpha R^{\beta} F(z,H) = K(R(H)), \end{cases} \tag{12}$$

where:

$$\begin{aligned} \overline{a}(z,t)' &\equiv \left[\mu_z' - \beta \Lambda' \left(I - \Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\right)\Omega - \frac{\beta}{\beta-1} M' \left(\Sigma'\Sigma\right)^{-1}\Sigma'\Omega\right], \\ \overline{b}(z,t) &\equiv \left[\beta\left(r - \alpha_L\right) + \beta M'\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Lambda - \frac{1}{2}\frac{\beta}{\beta-1} M'\left(\Sigma'\Sigma\right)^{-1} M \right. \\ &\quad \left. + \frac{1}{2}\beta(\beta-1)\Lambda'\left(I - \Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\right)\Lambda\right], \\ C(z,t) &\equiv \left[\Omega'\Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Omega\right]. \end{aligned}$$

The boundary condition in the system (12) holds if and only if the investor's utility function has the form $K(R(H)) = \alpha R^{\beta}$. Thus, we can state:

Proposition 8 *The value function solving the HJB equation (9) is separable by product in wealth and in the other state variables if and only if the investor has a power-utility function.*

A similar result is derived in Merton (1990) but under the hypotheses that (i) the riskless interest rate is constant and (ii) the only state variables are the asset prices. The author claims that when the utility function is a member of the HARA family,[5] then the value function is separable into a product of two functions, the first one depending on R and t and the second one on t and the other state variables.[6]

Thus, the choice of a power utility function implies that the optimal portfolio has the following composition:

$$w^* = \left(\Sigma'\Sigma\right)^{-1}\Sigma'\Lambda + \frac{1}{1-\beta}\left(\Sigma'\Sigma\right)^{-1} M + \frac{1}{1-\beta}\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Omega\frac{1}{F}F_z.$$

[5] All members of the Hyperbolic Absolute Risk Aversion (HARA) family can be expressed as:

$$K(C) = V(t)\frac{1-\gamma}{\gamma}\left(\frac{\beta C}{1-\gamma} + \delta\right)^{\gamma}.$$

[6] Under the hypotheses of Merton (1990) the value function can be written as follows:

$$J(R,z,t) = \frac{1-\gamma}{\gamma} F(z,t) V(t)\left(\frac{R}{1-\gamma} + \frac{\delta}{\beta r}\left(1 - e^{-r(H-t)}\right)\right)^{\gamma}.$$

Furthermore, for simplifying the computations, we can consider the following equivalence:

$$F(z,t) = e^{h(z,t)}.$$

We underline that this transformation is just a tool for rewriting the system (12) and eliminating the highly non linear component $\frac{1}{F}F_z'CF_z$. In fact, after this transformation, we can write (12) as:

$$\begin{cases} h_t + \overline{a}(z,t)' h_z + \overline{b}(z,t) + \frac{1}{2} tr\left[\left(\Omega'\Omega - \frac{\beta}{\beta-1}C(z,t)\right) h_z h_z' + \Omega'\Omega h_{zz}\right] + \\ \qquad + \sum_{i=1}^{p} \phi_i \mathbb{E}\left[\left(1 - \{\eta'\}_i\right)^{\beta} e^{h\left(z+E_i',t\right) - h(z,t)} - 1\right] = 0, \\ h(z,H) = 0, \end{cases} \tag{13}$$

and, accordingly, the optimal portfolio composition can be written as:

$$w^* = \left(\Sigma'\Sigma\right)^{-1} \Sigma'\Lambda + \frac{1}{1-\beta}\left(\Sigma'\Sigma\right)^{-1} M + \frac{1}{1-\beta}\left(\Sigma'\Sigma\right)^{-1} \Sigma'\Omega h_z.$$

In order to find a closed form solution we have to specify the functional form of $h(z,t)$ for computing the derivative with respect to the state variables in vector z. In the following subsections we will show how to find such a solution thanks to an approximation but now we study how to simplify the jump component in system (13).

3.2.3 The jump component

The jump component of the inflation risk makes the solution of system (13) very hard to compute in a closed form. Nevertheless, we can simplify the computations if we consider a Taylor series for the jump coefficients η tending to zero. This approximation can be justified because even a jump close to zero can have a great economic effect (we can imagine an inflation rate jumping from a value of 0.03 to a value of 0.13!). Furthermore inflation is supposed to be affected by more than one jump component ($p > 1$) and so we can consider that two or more jumps occur at the same time.

We consider the following approximation around the value $E_i \to 0$:

$$e^{h\left(z+E_i',t\right) - h(z,t)} = 1 + h_z' E_i' + \frac{1}{2} E_i \left(h_z h_z' + h_{zz}\right) E_i' + O\left(\left\|E_i'\right\|^3\right),$$

and, after substituting the first three terms in system (13) we obtain:

$$\begin{cases} h_t + a(z,t)' h_z + b(z,t) + \frac{1}{2} tr\left[\left(\Omega'\Omega - \frac{\beta}{\beta-1}C(z,t)\right) h_z h_z' + \Omega'\Omega h_{zz}\right] + \\ \qquad + \frac{1}{2} tr\left[\sum_{i=1}^{p} \phi_i \mathbb{E}\left[\left(1 - \{\eta'\}_i\right)^{\beta} E_i' E_i\right] \left(h_z h_z' + h_{zz}\right)\right] = 0, \\ h(z,H) = 0, \end{cases} \tag{14}$$

where:

$$\begin{aligned} a(z,t)' &\equiv \overline{a}(z,t)' + \sum_{i=1}^{p} \phi_i \mathbb{E}\left[\left(1 - \{\eta'\}_i\right)^{\beta} E_i \right], \\ b(z,t) &\equiv \overline{b}(z,t) + \sum_{i=1}^{p} \phi_i \mathbb{E}\left[\left(1 - \{\eta'\}_i\right)^{\beta} - 1 \right]. \end{aligned}$$

After this simplification, we show in the next subsection how to find an approximated algebraic solution for the system (14).

3.3 An analytic solution: the exact and the approximated cases

In the previous subsection we have shown that if the investor has a log-utility function, then we are able to compute the optimal portfolio composition in closed form. Instead, when the investor has a power utility function then the HJB equation does not depend on the investor's wealth but, nevertheless, it still depends on all the other state variables (in vector z).

Here, we consider system (14). Unfortunately, we cannot apply the Feynman-Kac theorem[7] because of the term $h_z h_z'$. In order to apply the theorem we should have only the term h_{zz} inside the trace operator. If we impose h_z to be zero, then also h_{zz} must be zero and we have a trivial solution. Instead, we can search for a function satisfying $h_{zz} = h_z h_z'$. After solving this differential equation, we find that $h(z,t)$ must have the following form:

$$h(z,t) = A(t) - \ln\left(B(t)' z + D(t)\right), \tag{15}$$

where $A(t), D(t) \in \mathbb{R}$, and $B(t) \in \mathbb{R}^{(s+n+1)\times 1}$ such that $B(t)' z + D(t) > 0$. In this case, in fact, we have:

$$h_{zz} = \frac{B(t) B(t)'}{\left[B(t)' z + D(t)\right]^2} = h_z h_z'.$$

Thus, if the function $h(z,t)$ has the form (15), then the HJB equation can be simplified as follows:

$$\begin{cases} h_t + a(z,t)' h_z + b(z,t) + \frac{1}{2} tr\left[\Omega' \left(2I - \frac{\beta}{\beta-1}\Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\right)\Omega h_{zz}\right] + \\ \qquad + \frac{1}{2} tr\left[2 \sum_{i=1}^{p} \phi_i \mathbb{E}\left[\left(1 - \{\eta'\}_i\right)^{\beta} E_i' E_i\right] h_{zz}\right] = 0, \\ h(H,z) = 0. \end{cases}$$

[7] For a complete exposition of the Feynman-Kac theorem the reader is referred to Duffie (1996), Björk (1998) and Øksendal (2000).

Now, it is possible to use the Feynman-Kac representation theorem. Because $I-\Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'$ is a symmetric, idempotent matrix, then, for applying the Feynman-Kac theorem, we have to find two real numbers x_1 and x_2 such that:

$$\left(x_1 I - x_2 \Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\right)^2 = 2I - \frac{\beta}{\beta-1}\Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma',$$

from which we have $x_1=\sqrt{2}$ and $x_2=\sqrt{2}\pm\sqrt{\frac{2-\beta}{1-\beta}}$. Thus, by putting:

$$\widetilde{\Omega}' \equiv \Omega'\left[\sqrt{2}I - \left(\sqrt{2}\pm\sqrt{\frac{2-\beta}{1-\beta}}\right)\Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\right]',$$

we can write the HJB equation in the following way:

$$\begin{cases} h_t + a(z,t)' h_z + b(z,t) + \frac{1}{2}tr\left(\widetilde{\Omega}'\widetilde{\Omega} h_{zz}\right) + \frac{1}{2}tr\left(N'N h_{zz}\right) = 0, \\ h(H,z) = 0, \end{cases}$$

where, after defining $N_i' \in \mathbb{R}^{(s+n+1)\times(s+n+1)}\quad \forall i=1,2,...,p$ such that:[8]

$$N_i'N_i = 2\phi_i \mathbb{E}\left[\left(1-\{\eta'\}_i\right)^\beta E_i'E_i\right], \quad \forall i=1,2,...,p$$

we can define:

$$\underset{(s+n+1)\times(s+n+1)p}{N'} \equiv \left[\begin{array}{cccc} N_1' & N_2' & ... & N_p' \end{array}\right].$$

Now, we are able to apply the Feynman-Kac theorem, and the solution of the HJB equation is given by:

$$h(z,t) = \int_t^H \mathbb{E}_t\left[b(Z_s,s)\right]ds,$$

where the variables Z_s follow:

$$\begin{aligned} dZ_s &= a(Z_s,t)\,ds + \widetilde{\Omega}(Z_s,s)'\,dW + N(Z_s,s)'\,dW_p, \\ Z_t &= z, \end{aligned}$$

and dW_p is the differential of a $(s+n+1)-$dimensional Wiener process independent of dW.

Finally, the optimal portfolio can be written as:

$$\begin{aligned} w^* &= \left(\Sigma'\Sigma\right)^{-1}\Sigma'\Lambda + \frac{1}{1-\beta}\left(\Sigma'\Sigma\right)^{-1}M + \\ &\quad + \frac{1}{1-\beta}\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Omega\int_t^H \frac{\partial}{\partial z}\mathbb{E}_t\left[b(Z_s,s)\right]ds. \end{aligned} \tag{16}$$

[8] The following decomposition is based on the hypothesis that the matrices:

$$\phi_i \mathbb{E}\left[\left(1+\{u'\eta'\}_i\right)^\beta E_i'E_i\right], \quad \forall i=1,2,...,p$$

are positive semi-definite.

We can see that the only component of optimal portfolio explicitly depending on the investor's horizon H is the third one which hedges the portfolio against the state variable risks and the background risks.

Thus, our result, can be summarized as follows:

Proposition 9 *Under market structure (4), the portfolio composition maximizing the investor's terminal power utility function ($K(R) = \alpha R^{\beta}$) is as follows:*

$$\begin{aligned} w^* &= (\Sigma'\Sigma)^{-1}\Sigma'\Lambda + \frac{1}{1-\beta}(\Sigma'\Sigma)^{-1}M + \\ &+\frac{1}{1-\beta}(\Sigma'\Sigma)^{-1}\Sigma'\Omega\int_t^H \frac{\partial}{\partial z}\mathbb{E}_t\left[b(Z_s,s)\right]ds, \end{aligned}$$

if and only if there exist functions $A(t), D(t) \in \mathbb{R}$, and $B(t) \in \mathbb{R}^{(s+n+1)\times 1}$ such that:

$$\int_t^H \mathbb{E}_t\left[b(Z_s,s)\right]ds = A(t) - \ln\left(B(t)'z + D(t)\right),$$

where:

$$\begin{aligned} dZ_s &= a(Z_s,t)\,ds + \widetilde{\Omega}(Z_s,s)'\,dW + N(Z_s,s)'\,dW_p, \\ Z_t &= z, \\ a(z,t)' &\equiv \left[\mu_z' - \beta\Lambda'\left(I - \Sigma(\Sigma'\Sigma)^{-1}\Sigma'\right)\Omega - \frac{\beta}{\beta-1}M'(\Sigma'\Sigma)^{-1}\Sigma'\Omega\right] + \\ &\quad + \sum_{i=1}^{p}\phi_i\mathbb{E}\left[\left(1-\{\eta'\}_i\right)^{\beta}E_i\right], \\ b(z,t) &\equiv \Big[\beta(r-\alpha_L) + \beta M'(\Sigma'\Sigma)^{-1}\Sigma'\Lambda + \\ &\quad +\frac{1}{2}\beta(\beta-1)\Lambda'\left(I-\Sigma(\Sigma'\Sigma)^{-1}\Sigma'\right)\Lambda + \\ &\quad -\frac{1}{2}\frac{\beta}{\beta-1}M'(\Sigma'\Sigma)^{-1}M + \sum_{i=1}^{p}\phi_i\mathbb{E}\left[\left(1-\{\eta'\}_i\right)^{\beta}-1\right]\Big], \\ \widetilde{\Omega}' &\equiv \Omega'\left[\sqrt{2}I - \left(\sqrt{2}\pm\sqrt{\frac{2-\beta}{1-\beta}}\right)\Sigma(\Sigma'\Sigma)^{-1}\Sigma'\right]', \\ N'N &\equiv 2\sum_{i=1}^{p}\phi_i\mathbb{E}\left[\left(1-\{\eta'\}_i\right)^{\beta}E_i'E_i\right]. \end{aligned}$$

We underline that when we take the limit for β tending to zero we obtain the result of the log-utility case analyzed in the previous subsections.

We have shown that, if the investor has a power utility function, then the value function can be written as: $J(R,z,t) = \alpha R^{\beta}e^{h(z,t)}$. The closed form solution (16) is valid if and only if the function $h(z,t)$ has the form: $h(z,t) = A(t) - \ln\left(B(t)' z + D(t)\right)$. If the function h cannot be written in this way, then our previous result is incorrect.

Nevertheless, we can state that the result (16) is still valid as an approximation of the true result. If we develop in Taylor series the function $h(z,t)$ around a given value of z (called z_0), then we obtain:

$$\begin{aligned} h(z,t) &= A(t) - \ln\left(B(t)' z_0 + D(t)\right) + \\ &\quad -\frac{1}{B(t)' z_0 + D(t)} B(t)' (z - z_0) + \\ &\quad +\frac{1}{2\left(B(t)' z_0 + D(t)\right)^2} (z - z_0)' B(t)' B(t) (z - z_0) + O\left(\|z - z_0\|^3\right). \end{aligned} \tag{17}$$

Thus, if the function $h(z,t)$ can be expressed in the form (17), that is as a polynomial in z, then our result can approximate the real solution.

We will show in the next section that the exact solutions available in the literature find that the function $h(z,t)$ is a polynomial in z of degree one or two and, accordingly, our solution stays valid as an approximation.

3.4 The third component of optimal portfolio and the effect of crises

As we have already underlined, the Poisson component of the background risk is able to describe the economic crises. This component appears only in the third part of optimal portfolio ($w^*_{(3)}$). From Proposition 9 it can be seen that if the parameters Σ, Λ, M, and η do not depend on state variables but depend only on time, then the derivative term $\frac{\partial}{\partial z}\mathbb{E}_t\left[b(Z_s, s)\right]$ vanishes (because $\frac{\partial}{\partial Z_s} b(Z_s, s) = 0$).

Thus, we can state:

Proposition 10 *If the coefficients of the growth rate of the asset prices and price level depend only on time, then the third component of the optimal portfolio (16) vanishes.*

It is evident that this proposition contains the case of geometric Brownian motion as a special case. Furthermore, as a corollary, we can state:

Corollary 3 *If the coefficients of the Poisson component of the growth rate of price level depend only on time, then this jump part does not affect the optimal portfolio composition.*

This means that the investor does not need to hedge against jumps in the growth rate of price level if these jumps do not depend at least on one of the other state variables.

This hypothesis is very restrictive indeed because, from an empirical point of view, it is quite difficult to suppose that there are no links between the state variables contained in the vector X and the jump component of the inflation rate.

Here, we outline that the result stated in Corollary 3 is valid when the investor has a power utility function in both exact and approximated solutions of optimal stochastic control problem. In fact, we recall that the third component of optimal portfolio can be obtained as follows:

$$w_{(3)}^{*} = \frac{1}{1-\beta} \left(\Sigma'\Sigma\right)^{-1} \Sigma'\Omega h_{z}\left(z,t\right),$$

from which we can see that if the coefficients of growth rates in the state variables depend only on time, then these variables do not enter the optimal problem. In fact, their growth rate can be substituted into the wealth growth equation and we can forget about them in the passages which follow. Thus, in this case, the function $h\left(z,t\right)$ does not depend on z but only on t and $h_{z} = \mathbf{0}$.

In the work by Lioui and Poncet (2000) it is shown that the third component of the optimal portfolio is formed only by two parts, even though the number of state variables is arbitrarily large. In particular, the first part is associated with interest rate risk and the second one with the market price of risk. Even if Lioui and Poncet use the martingale approach, here we underline that we obtain the same result. Because the authors do not introduce any inflation risk,[9] then we have to put in our framework $\alpha_{L} = 0$, $\Lambda = \mathbf{0}$ and $\eta = \mathbf{0}$. Under this hypothesis we can see from Proposition 9 that the function $h\left(z,t\right)$ is formed only by two terms and, more precisely, we have:

$$b\left(z,t\right) \equiv \beta r - \frac{1}{2}\frac{\beta}{\beta-1} M'\left(\Sigma'\Sigma\right)^{-1} M,$$

from which we can see that, independently of the number of state variables, if there are no inflation risk, then, as in Lioui and Poncet, the third component of optimal portfolio is formed by two parts. The first one is associated with interest rate risk and the second one with the market price of risk. In fact, the matrix $M'\left(\Sigma'\Sigma\right)^{-1} M$ is the square of the Sharpe ratio.

4 Some special cases

In this section we compare our approximated result with some exact solutions which are available in the literature. In particular, we use the models of: (i)

[9] We outline that they define an investor who is endowed with a portfolio of discount bonds that he chooses not to trade until his investment horizon (H). This hypothesis allows the authors to have a non-zero first portfolio component $w_{(1)}^{*}$.

Boulier, Huang and Taillard (2001) who find the optimal portfolio composition for an investor with power utility function under the Vasicek structure of interest rates, (ii) Deelstra, Grasselli and Koehl (2001) in which, instead, the interest rate structure has the Cox, Ingersoll and Ross (1985) form, and (iii) Kim and Omberg (1996) who use a mean reversion process for describing the price of risk.

Finally, we investigate how our approximated solution performs with respect to another approximated solution computed by Kogan and Uppal (1999) on the model presented in Chacko and Viceira (1999).

In the following subsections we analyze each of these models.

4.1 The comparison with Boulier, Huang and Taillard (2001)

Boulier, Huang and Taillard (2001) consider a market structure in which there are one stock and one bond. The only state variable is the interest rate following a Vasicek structure (Vasicek, 1977). The bond value depends only on the interest rate risk while the stock value depends on both its own risk and the interest rate risk. There are no background variables, thus in this model $w^*_{(1)} = \mathbf{0}$ because we have $\alpha_L = 0$, $\Lambda = \mathbf{0}$ and $\eta = \mathbf{0}$.

The market structure can be represented in the following way:

$$\begin{cases} dr = a_r\left(b_r - r\right)dt - \sigma_r dW_r, \\ \frac{dS}{S} = \left(r + \sigma_1\lambda_1 + \sigma_2\lambda_r\right)dt + \sigma_1 dW_S + \sigma_2 dW_r, \\ \frac{dB}{B} = \left(r + \lambda_r g\left(H-t\right)\sigma_r\right)dt + g\left(H-t\right)\sigma_r dW_r, \\ \frac{dG}{G} = rdt, \end{cases} \tag{18}$$

where:

$$g\left(\tau\right) = \frac{1 - e^{a_r\tau}}{a_r}.$$

The authors solve the problem for an investor having a power utility function of the form $K\left(R\right) = \frac{1}{\beta}R^{\beta}$. By applying our solution we obtain the following optimal portfolio:

$$w^* = \frac{1}{1-\beta}\begin{bmatrix} \frac{\lambda_1}{\sigma_1} \\ \frac{\lambda_r\sigma_1 - \sigma_2\lambda_1}{g(H-t)\sigma_r\sigma_1} \end{bmatrix} - \frac{1}{1-\beta}\begin{bmatrix} 0 \\ \beta \end{bmatrix}.$$

This solution, detailed in Appendix A.1, is exactly the solution obtained by the authors.

We underline that in this case the function $h\left(z,t\right)$ is linear in z and thus, our solution should be valid only as an approximation. Actually, our result is identical to the result of Boulier, Huang and Taillard because their solution does not involve any state variable but only the preference parameter β.

Thus, in this case, we have lost nothing with respect to the exact solution.

4.2 The comparison with Deelstra, Grasselli and Koehl (2001)

Deelstra, Grasselli and Koehl (2001) consider a market structure which is the same as that one after Boulier, Huang and Taillard (2001) but in which the interest rate (the only state variable) follows a Cox, Ingersoll and Ross (1985) structure (so-called CIR). Their market structure is as follows:

$$\begin{cases} dr = a_r\left(b_r - r\right)dt - \sigma_r\sqrt{r}dW_r, \\ \frac{dS}{S} = \left(r + \sigma_1\lambda_1 + \sigma_2\lambda_r r\right)dt + \sigma_1 dW_S + \sigma_2\sqrt{r}dW_r, \\ \frac{dB}{B} = \left(r + r\lambda_r g\left(H-t\right)\sigma_r\right)dt + g\left(H-t\right)\sigma_r\sqrt{r}dW_r, \\ \frac{dG}{G} = rdt, \end{cases} \tag{19}$$

where:

$$\begin{aligned} g\left(\tau\right) &= \frac{2\left(e^{\delta\tau}-1\right)}{2\delta + \left(e^{\delta\tau}-1\right)\left(\delta + b_r - \sigma_r\lambda_r\right)}, \\ \delta &= \sqrt{\left(b_r - \sigma_r\lambda_r\right)^2 + 2\sigma_r^2}. \end{aligned}$$

The authors solve the problem for an investor having a power utility function of the form $K\left(R\right) = \frac{1}{\beta}R^\beta$. As we show in Appendix A.2, the optimal portfolio composition given by our approximated solution is:

$$w^* = \frac{1}{1-\beta}\begin{bmatrix} \frac{\lambda_1}{\sigma_1} \\ \frac{\lambda_r\sigma_1 - \sigma_2\lambda_1}{g(H-t)\sigma_r\sigma_1} \end{bmatrix} + \frac{1}{1-\beta}\begin{bmatrix} 0 \\ -\left(\beta + \frac{1}{2}\frac{\beta}{1-\beta}\lambda_r^2\right)\frac{1-e^{-\widetilde{b_r}(H-t)}}{\widetilde{b_r}g(H-t)} \end{bmatrix},$$

where $\widetilde{b_r} = b_r - \frac{\beta}{\beta-1}\lambda_r\sigma_r$, while the exact result obtained by Deelstra, Grasselli and Koehl is the following one:

$$w^*_{DGK} = \frac{1}{1-\beta}\begin{bmatrix} \frac{\lambda_1}{\sigma_1} \\ \frac{\lambda_r\sigma_1 - \sigma_2\lambda_1}{g(H-t)\sigma_r\sigma_1} \end{bmatrix} + \frac{1}{1-\beta}\begin{bmatrix} 0 \\ \frac{k(H-t)}{g(H-t)} \end{bmatrix},$$

where:

$$\begin{aligned} k\left(H-t\right) &= -\frac{2\left(\beta + \frac{1}{2}\frac{\beta}{1-\beta}\lambda_r^2\right)\left(e^{\alpha(H-t)}-1\right)}{2\alpha + \left(e^{\alpha(H-t)}-1\right)\left(\alpha + b_r + \frac{\beta}{1-\beta}\lambda_r\sigma_r\right)}, \\ \alpha &= \sqrt{b_r^2 - 2\sigma_r^2\frac{\beta}{1-\beta}\left(1 + \frac{1}{2}\lambda_r^2 - b_r\lambda_r\frac{1}{\sigma_r}\right)}. \end{aligned}$$

Now, our aim is to study the difference between our solution and the correct one for seeing when the difference can be considered negligible. Here, we do not matter about the initial time t and we care only about the distance between the present date and the time horizon: $H_t \equiv H - t$. In particular, we study the behaviour of the ratio between the approximated and the exact value of the portfolio hedging component (previously called $w^*_{(3)}$) because the two first components are identical for the two solutions. If we call this ratio ϕ we obtain:

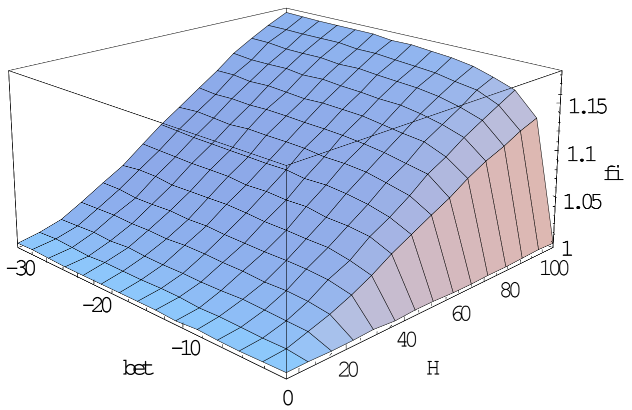
$$\phi_{DGK}\left(\beta,\lambda_r,\sigma_r,b_r,H_t\right)=\frac{1}{2}\left(1-\frac{\alpha}{\widetilde{b_r}}\frac{1+e^{\alpha H_t}}{1-e^{\alpha H_t}}\right)\left(1-e^{-\widetilde{b_r}H_t}\right).$$

The simulation of index ϕ_{DGK} is based on the following starting values and ranges for parameters:

Variables	β	λ_r	σ_r	b_r	H_t
Starting values	-9	0.0017	0.0189	0.0226	10
Simulation ranges	[-30,0]	[0,0.5]	[0,0.5]	[0,0.5]	[0,100]

The starting values are consistently chosen with Campbell and Viceira (1999). The simulations are shown in Figures 1, 2 and 3.

Figure 1: Approximation index ϕ_{DGK} as function of β and H_t

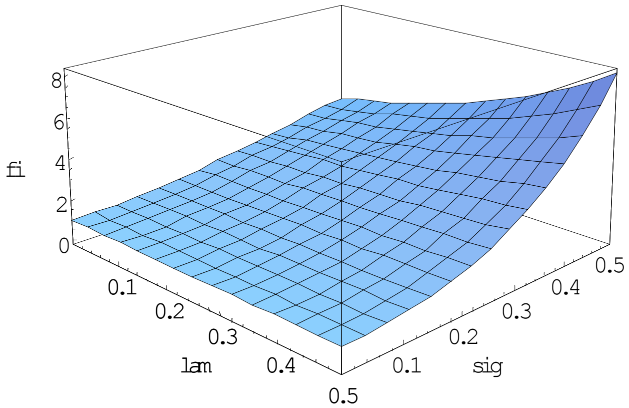


In Figure 1 the behaviour of index ϕ_{DGK} with respect to both the preference parameter β and the time horizon H_t is shown. We can see that the approximation error increases when the time horizon increases even if this error seems to be negligible for values of H_t lower than 40. Because a period length between 30 and 40 years corresponds to the work life of a worker, then this model should be able to be applied to the case of pension funds or life insurances.

If we exclude the lower values of the preference parameter β, we can see that β does not affect the approximation in a determinant way. Thus, for instance, the approximation made by Kogan and Uppal (1999) and based on a Taylor expansion around the value $\beta=0$, in this case could not be able to capture some particular behaviours of the optimal solution that our model considers.

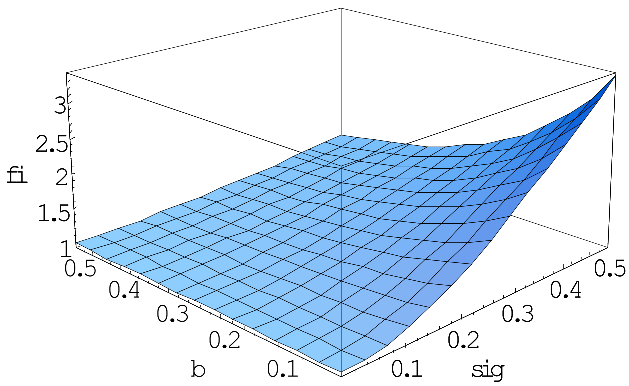
In Figure 2 we analyze the behaviour of index ϕ_{DGK} with respect to λ_r and σ_r which are respectively the constant part of the bond market price of risk and

Figure 2: Approximation index ϕ_{DGK} as function of λ_r and σ_r



the constant part of the riskless interest rate volatility. We can see that these parameters deeply affect the approximation but it is sufficient that only one of them is very small in order to have a value of ϕ_{DGK} close to 1.

Figure 3: Approximation index ϕ_{DGK} as function of b and σ_r



In Figure 3, the behaviour of index ϕ_{DGK} with respect to parameters b and σ_r is represented. Another time σ_r seems to have a great importance in determining the goodness of the approximated result. We recall that b is the coefficient measuring the strength of the mean reversion effect in the differential equation describing the behaviour of interest rate. From Figure 3 we see that it is sufficient that σ_r has a very low value or b has a very high value for having a good approximation. In fact, if b is high enough, the mean reversion effect is dominant and the interest rate follows its deterministic path very closely. The same behaviour is reached when σ_r approaches zero.

All the results we have shown confirm the idea that closer the state variables to their deterministic path, better the approximation.

4.3 The comparison with Kim and Omberg (1996)

The case analyzed by Kim and Omberg (1996) presents only one stock and one state variable representing the risk premium. Thus, the market structure is as follows:

$$\begin{cases} dx = -\lambda\left(x-\overline{x}\right)dt+\sigma_x dW_x, \\ \frac{dS}{S} = \left(r+\sigma_S x\right)dt+\sigma_S dW_S, \\ \frac{dG}{G} = rdt, \end{cases} \tag{20}$$

where the Wiener differentials of stock and risk premium are correlated:

$$\mathbb{E}\left[dW_x dW_S\right] = \rho_{xS}.$$

In the Appendix A.3 we show how to transform this setting into a framework where the Wiener processes are independent and we compute the following optimal portfolio:

$$w^* = \frac{1}{1-\beta}\frac{x}{\sigma_S}+\frac{1}{1-\beta}\frac{\rho_{xS}\sigma_x}{\sigma_S}h_x,$$

where:

$$\begin{aligned} h_x &= -\frac{1}{2}\frac{\beta}{1-\beta}\left(\left(\frac{1-e^{-(H-t)\alpha}}{\alpha}\right)^2\lambda\overline{x}+\frac{1-e^{-2(H-t)\alpha}}{\alpha}x\right), \\ \alpha &= \lambda+\frac{\beta}{\beta-1}\rho_{xS}\sigma_x. \end{aligned} \tag{21}$$

In their model, Kim and Omberg, find the same solution w^* but the function h_x has the following exact form:[10]

$$\begin{aligned} h_{x(KO)} &= -2\frac{\beta}{\beta-1}\frac{2\left(1-e^{-\frac{1}{2}\pi(H-t)}\right)^2}{\pi\left[2\pi-(\pi-2\alpha)\left(1-e^{-\pi(H-t)}\right)\right]}\lambda\overline{x}+ \\ &\quad -2\frac{\beta}{\beta-1}\frac{1-e^{-\pi(H-t)}}{2\pi-(\pi-2\alpha)\left(1-e^{-\pi(H-t)}\right)}x, \end{aligned}$$

where:

$$\pi = 2\sqrt{(-\alpha)^2+\frac{\beta}{\beta-1}\sigma_x^2\left(1-\frac{\beta}{\beta-1}\rho_{xS}^2\right)}.$$

[10] Actually, they use a HARA utility function of the form $U(R) = (R-R^*)^{\frac{\gamma-1}{\gamma}}$. Thus, here, we put $R^*=0$ and $\beta=\frac{\gamma-1}{\gamma}$. The quality of result is unaffected.

We underline that, while in the other cases our function $h(z,t)$ was an approximation of the exact function around the value $z_0 = 0$, in this case the approximation is made around a different value of z_0. In particular, in this case, it is necessary to solve the following system, in order to find the functions $A(t)$, $B(t)$ and $D(t)$ equating h_x in (21) to the derivative of (17):

$$\begin{cases} -\frac{1}{2}\frac{\beta}{\beta-1}\left(\frac{1-e^{-(H-t)\alpha}}{\alpha}\right)^2 \lambda\overline{x} = -\frac{B}{Bx_0+D} - x_0\left(\frac{B}{Bx_0+D}\right)^2, \\ -\frac{1}{2}\frac{\beta}{\beta-1}\frac{1-e^{-2(H-t)\alpha}}{\alpha} = \left(\frac{B}{Bx_0+D}\right)^2. \end{cases}$$

We can see that it is not possible to put $x_0 = 0$. Instead, this system can be solved for x_0 and D/B which are uniquely determined. We obtain:

$$\begin{aligned} \frac{D}{B} &= \frac{\lambda\overline{x}}{\alpha}\frac{1-e^{-\alpha(H-t)}}{1+e^{-\alpha(H-t)}} \pm 2\sqrt{-\frac{\beta-1}{\beta}\frac{2\alpha}{1-e^{-2\alpha(H-t)}}}, \\ x_0 &= -\frac{\lambda\overline{x}}{\alpha}\frac{1-e^{-\alpha(H-t)}}{1+e^{-\alpha(H-t)}} \pm \sqrt{-\frac{\beta-1}{\beta}\frac{2\alpha}{1-e^{-2\alpha(H-t)}}}. \end{aligned}$$

It is easy to see that our solution converges to the exact one when $\pi \to -2\alpha$. In this case, in fact we have:

$$h_{x(KO)} = -2\frac{\beta}{\beta-1}\left(\frac{\left(1-e^{\alpha(H-t)}\right)^2}{4\alpha^2 e^{2\alpha(H-t)}}\lambda\overline{x} - \frac{1-e^{2\alpha(H-t)}}{4\alpha e^{2\alpha(H-t)}}x\right).$$

The condition under which our function converges to the exact one can be written as follows:

$$\frac{\beta}{\beta-1}\sigma_x^2\left(1-\frac{\beta}{\beta-1}\rho_{xS}^2\right) \to 0,$$

and this condition holds in three cases:

1. $\beta \to 0$, actually this case is not interesting because if β tends to zero, then the investor can be described with a log-utility function and the third component $w^*_{(3)}$ of the optimal portfolio vanishes. Accordingly, in this case, the function $h(z,t)$ does not matter;

2. $\sigma_x^2 \to 0$, in this case the state variable x follows a deterministic trajectory and the component $w^*_{(3)}$ of optimal portfolio vanishes another time;

3. $\rho_{xS}^2 \to \frac{\beta-1}{\beta}$, this case is the most interesting one because we see that, under this hypothesis, the third component $w^*_{(3)}$ of optimal portfolio does not vanish. This condition is equivalent to create a relation between the investor's preference parameter β and the asset correlation with the state variable. We outline that higher the value of β, closer the value of ρ_{xS} to 1. This means that our solution converges to the exact one if we consider a highly risk averse investor and a high correlation between stock price and risk premium.

As in the previous section we analyze the behaviour of our solution with respect to the exact one through the approximation index:

$$\phi_{KO}\left(\beta,\lambda,\sigma_x,\rho_{xS},H_t,x,\overline{x}\right)=\frac{h_x}{h_{x(KO)}},$$

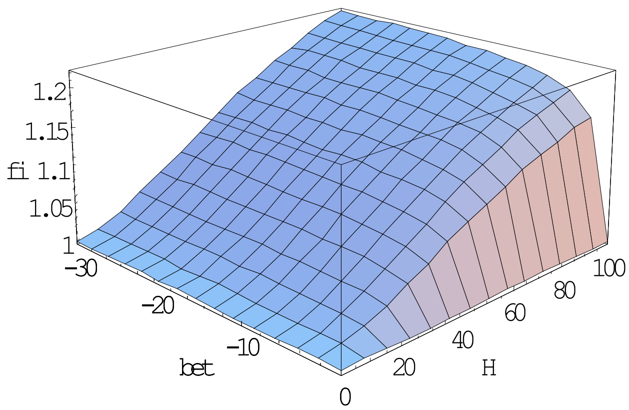
where, as before, $H_t \equiv H-t$. The numerical simulations show that the values of x and $\overline{x}$ affect the result in a negligible way. Thus, the simulation we present here only concerns the five other parameters.

The simulation of index ϕ_{KO} is based on the following starting values and ranges for parameters:

Variables	β	λ	σ_x	ρ_{xS}	H_t
Starting values	-9	0.0017	0.0189	-0.000203	10
Simulation ranges	[-30,0]	[0,0.5]	[0,0.5]	[-1,1]	[0,100]

The starting values are consistently chosen with Campbell and Viceira (1999). The simulations are shown in Figures 4, 5 and 6.

Figure 4: Approximation index ϕ_{KO} as function of β and H_t



In Figure 4 we can observe the same behaviour shown, for the same parameters, in the case analyzed by Deelstra, Grasselli and Koehl. Thus, we refer to the previous subsection for the comments.

In Figure 5 the values of approximation index ϕ_{KO} with respect to λ and σ_x are shown. These two parameters measure, respectively, the strength of the mean reversion effect and the volatility of the state variable. As in the previous section, stronger the mean reversion effect better the approximation because the state variable tends to become closer to its deterministic path. Furthermore, for the same reason, higher the volatility worst the approximation.

Finally, Figure 6 shows the approximation with respect to the volatility of the state variable (σ_x) and the correlation between the state variable risk and

Figure 5: Approximation index ϕ_{KO} as function of λ and σ_x

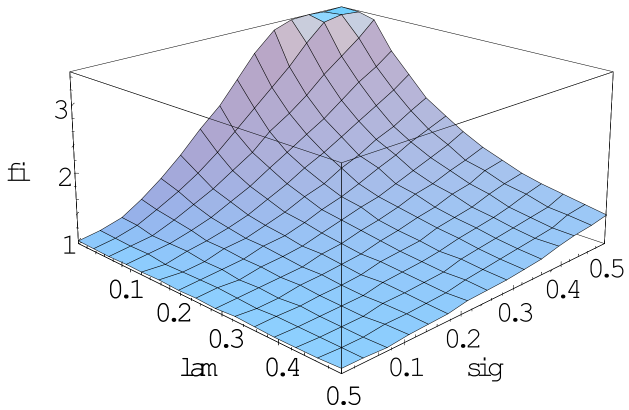
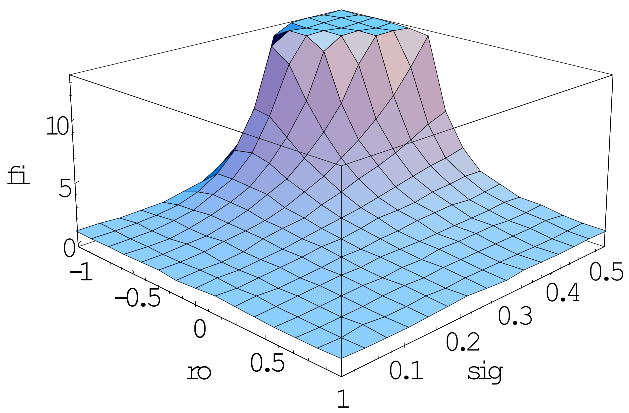


Figure 6: Approximation index ϕ_{KO} as a function of ρ_{xS} and σ_x



the asset price risk (ρ_{xS}). We can see that the most important problems arise when the volatility σ_x is high and the correlation index ρ_{xS} is very low (close to -1). A lot of empirical investigations (see for instance Barberis, 2000, and Campbell and Viceira, 1999) show that this correlation index is negative and, thus, our solution stays valid as a good approximation only if the volatility of the risk premium is low enough.

4.4 The comparison with Kogan and Uppal (1999)

Here, we consider the case analyzed by Kogan and Uppal (1999) who follow our same approach of finding an approximated solution with respect to the log-linearization technique developed in Chacko and Viceira (1999). Nevertheless, the authors linearize the function $h(z,t)$ with respect to the preference parameter β. We outline that they take the Taylor series of $h(z,t)$ around the value $\beta = 0$. Nevertheless, in this case the third component of the optimal portfolio composition ($w^*_{(3)}$) tends to vanish and thus, their analysis stays valid only for very few cases.

As in Kim and Omberg (1996) there are one stock and one state variable whose volatility is not constant but proportional to the square root of the state variable. The model studied by these two authors is as follows:

$$\begin{cases} dx = -\lambda\left(x - \overline{x}\right)dt + \sigma\sqrt{x}dW_x, \\ \frac{dS}{S} = \mu dt + \frac{1}{\sqrt{x}}dW_S, \\ dG = Grdt, \end{cases} \tag{22}$$

where the riskless interest rate is a positive constant. Because Kogan and Uppal consider the case of correlated Wiener processes, in Appendix A.4 we show how to transform their framework in a framework with two independent Wiener processes.

The authors solve the problem for an investor having a power utility function of the form $K(R) = \frac{1}{\beta}R^{\beta}$. Our approximated solution is as follows (see Appendix A.4):

$$w^* = \frac{1}{1-\beta}\left(\mu - r\right)x + \frac{1}{1-\beta}\sigma_{Sx}xh_x,$$

where:

$$h_x = \frac{1}{2}\frac{\beta}{1-\beta}\left(\mu - r\right)^2 \frac{1 - e^{-\left(\lambda - \frac{\beta}{1-\beta}(\mu - r)\sigma_{Sx}\right)(H-t)}}{\lambda - \frac{\beta}{1-\beta}\left(\mu - r\right)\sigma_{Sx}}.$$

Kogan and Uppal consider an investment horizon tending to infinity, thus the exponential term vanishes under the hypothesis that σ_{Sx} is positive, or, if negative, with an absolute value low enough for having $\lambda + \frac{\beta}{\beta-1}\left(\mu - r\right)\sigma_{Sx} > 0$. Finally, we can write the following optimal portfolio composition:

$$w^* = \frac{1}{1-\beta}\left(\mu - r\right)x + \frac{\beta}{1-\beta}\left(\mu - r\right)^2 \frac{\sigma_{Sx}x}{2}\frac{1}{\left(1-\beta\right)\lambda - \beta\left(\mu - r\right)\sigma_{Sx}},$$

while Kogan and Uppal obtain:

$$w_{KU}^* = \frac{1}{1-\beta}(\mu - r)\,x + \frac{\beta}{1-\beta}(\mu - r)^2 \frac{\sigma_{Sx} x}{2}\frac{1}{\lambda}.$$

This result is consistent with the hypothesis of β small enough for being able to approximate the function h through a Taylor polynomial around the value $\beta = 0$. Here, we want to outline that this kind of procedure would imply that the whole third portfolio component vanishes while our model is able to describe the optimal portfolio composition for each degree of risk aversion, under the hypothesis that the state variables stay around a given value.

The behaviour of the optimal portfolio with respect to the parameters σ_{Sx} and λ is qualitatively the same between w^* and w_{KU}^*. In fact, in both models it is true that $\frac{\partial w^*}{\partial \sigma_{Sx}} > 0$ and $\frac{\partial w^*}{\partial \lambda} < 0$. With respect to the risk premium $(\mu - r)$, Kogan and Uppal's model presents a derivative of w_{KU}^* with respect to $(\mu - r)$ whose sign corresponds to the sign of σ_{Sx} independently of $(\mu - r)$. Our model, instead, presents a richer range of possibilities because we have the following result (with $\beta < 0$):

$$\frac{\partial w_{(3)}^*}{\partial (\mu - r)} \gtreqless 0 \Longleftrightarrow (\mu - r) \gtreqless -2\frac{\beta - 1}{\beta}\frac{\lambda}{\sigma_{Sx}}.$$

If the asset price and the state variable are positively correlated ($\sigma_{Sx} > 0$) then, because μ must be greater than r for trivial arbitrage considerations, the third optimal portfolio component increases when the risk premium increases (as in Kogan and Uppal). Instead, when $\sigma_{Sx} < 0$ we can distinguish two different cases:

1. if the absolute value of σ_{Sx} is very high, then the optimal portfolio hedging component $(w_{(3)}^*)$ is positively correlated with the risk premium. In fact, if the correlation between the stock price and the state variable is high, then the investor needs a stronger hedging. We underline that this case must be rejected because if σ_{Sx} is negative and its absolute value is high, then the value of $w_{(3)}^*$ diverges when H tends to infinity;

2. if the absolute value of σ_{Sx} is very low, then the optimal portfolio hedging component $(w_{(3)}^*)$ is negatively correlated with the risk premium. In fact, when the state variable is not strongly correlated with the stock price, then, in order to hedge the portfolio, it is better to invest more money in the riskless asset rather than in $w_{(3)}^*$ which is not able to cover the non-correlated risk.

In Kogan and Uppal the derivative of w_{KU}^* with respect to β has the following sign:

$$\left.\frac{\partial w_{KU}^*}{\partial \beta}\right|_{\beta=0} \gtreqless 0 \Longleftrightarrow \sigma_{Sx} \gtreqless -2\frac{\lambda}{\mu - r}.$$

In our framework, we obtain an identical result and, nevertheless, our model is richer in describing the behaviour of optimal portfolio hedging component $w^*_{(3)}$ with respect to the preference parameter β because we obtain, with $\beta < -1$:

$$\frac{\partial w^*_{(3)}}{\partial \beta} \gtreqless 0 \Longleftrightarrow \begin{cases} \sigma_{Sx} \in \left] -\frac{\beta^2-1}{\beta^2}\frac{\lambda}{\mu-r}, 0 \right[, \\ \sigma_{Sx} = -\frac{\beta^2-1}{\beta^2}\frac{\lambda}{\mu-r}, \quad \sigma_{Sx} = 0, \\ \sigma_{Sx} \notin \left[-\frac{\beta^2-1}{\beta^2}\frac{\lambda}{\mu-r}, 0 \right]. \end{cases}$$

This result means that when the absolute value of σ_{Sx} is high, then the hedging component of optimal portfolio increases if the risk aversion $(1-\beta)$ increases. In fact, the portfolio part $w^*_{(3)}$ can effectively hedge the investor from the risk represented by the state variable x only if the asset price is strongly correlated with this state variable. If not, it is better to decrease the hedging portfolio component in order to increase the percentage of wealth invested in the riskless asset.

5 Conclusion

In this paper we have analyzed the asset allocation problem for an investor maximizing the expected value of his terminal power utility function. The investor faces an economic environment with stochastic investment opportunities and incomplete financial markets. Furthermore, he must cope with an inflation risk following a jump-diffusion proces.

The optimal portfolio is formed by three components: (i) a preference free part depending only on the diffusion terms of assets and inlfation process, (ii) a part proportional to both the portfolio Sharpe ratio and the inverse of Arrow-Pratt relative risk aversion index, and (iii) a part depending on the state variable parameters.

We show that the preference-free component hedges the portfolio against the diffusion part of infaltion process correlated with asset prices risk sources. Furthermore, this preference-free component minimizes the instantaneous variance of investor's wealth.

The third component of optimal portfolio vanishes when the investor has a log-utility function or when, investor having a power utility function, the drift and diffusion components of state variables and inflation depend only on time. In particular, we find that the jump component of the price process affects the optimal portfolio composition if and only if the coefficient of this jump component depends at least on one of the state variables.

For understanding the role of the third component it is necessary to explicitly compute the value function. This computation is the most difficult part of the stochastic dynamic programming technique. In this work we propose an approximated method for solving the PDE giving the value function. Our method is based on the Feynman-Kac representation theorem. We compare our approximated solution with some exact solutions available in the literature. We are able to find that all the qualitative results are maintained and the computations are simplified.

A Derivation of the approximated solutions

A.1 The comparison with Boulier, Huang and Taillard (2001)

Boulier, Huang and Taillard (2001) take the market structure (18). Thus, the only state variable is the interest rate (r), and there are two assets: a stock (S) and a bond (B). In their model there are no inflation risk and so we can put $\alpha_L = 0$, $\Lambda = \mathbf{0}$ and $\eta = \mathbf{0}$.

We underline that the changes in prices of bond and stock are Itô processes whose evolution does not depend on their own values. Thus, we can forget about the state variables S and B. Accordingly, we can transform this problem in a form suitable to compute the optimal portfolio composition in our framework:

$$
\begin{aligned}
w' &= \begin{bmatrix} w_S & w_B \end{bmatrix}', \\
z &= r, \\
\mu_z &= a_r (b_r - r), \\
\Omega &= \begin{bmatrix} -\sigma_r \\ 0 \end{bmatrix}, \\
M &= \begin{bmatrix} \sigma_1 \lambda_1 + \sigma_2 \lambda_r & g(H-t)\lambda_r \sigma_r \end{bmatrix}', \\
\Sigma' &= \begin{bmatrix} \sigma_2 & \sigma_1 \\ g(H-t)\sigma_r & 0 \end{bmatrix}, \\
dW &= \begin{bmatrix} dW_r & dW_S \end{bmatrix}'.
\end{aligned}
$$

The authors solve the problem for an investor having power utility function of the form $K(R) = \frac{1}{\beta} R^{\beta}$. By applying Formula (16) we have:

$$
w^* = \frac{1}{1-\beta} \begin{bmatrix} \frac{\lambda_1}{\sigma_1} \\ \frac{\lambda_r \sigma_1 - \sigma_2 \lambda_1}{g(H-t)\sigma_r \sigma_1} \end{bmatrix} + \frac{1}{1-\beta} \begin{bmatrix} 0 \\ -\frac{1}{g(H-t)} h_r \end{bmatrix}.
$$

Now, we have to compute the function h that in our case is:

$$
\begin{aligned}
h(z,t) &= \int_t^H \mathbb{E}\left[b(Z_s, s) \right] ds, \\
dZ_s &= a(Z_s, s)\, ds + \sqrt{\frac{2-\beta}{1-\beta}} \Omega' dW.
\end{aligned}
$$

In this special case we have:

$$
\begin{aligned}
a(z,t) &= \mu_z - \frac{\beta}{\beta-1}\Omega'\Sigma'^{-1}M = a_r(b_r - r) + \frac{\beta}{\beta-1}\lambda_r\sigma_r, \\
b(z,t) &= \beta r - \frac{1}{2}\frac{\beta}{\beta-1}M'(\Sigma'\Sigma)^{-1}M = \beta r - \frac{1}{2}\frac{\beta}{\beta-1}\left(\lambda_1^2 + \lambda_r^2\right),
\end{aligned}
$$

and, finally, we can write:

$$
\begin{aligned}
h(r,t) &= \int_t^H \mathbb{E}\left[\beta r_s - \frac{1}{2}\frac{\beta}{\beta-1}\left(\lambda^2 + \lambda_r^2\right)\right] ds, \\
dr_s &= a_r\left(\widetilde{b_r} - r_s\right) ds + \widetilde{\sigma_r} dW_r, \\
r_t &= r,
\end{aligned}
$$

where:

$$
\begin{aligned}
\widetilde{b_r} &= b_r + \frac{1}{a_r}\frac{\beta}{\beta-1}\lambda_r\sigma_r, \\
\widetilde{\sigma_r} &= \sigma_r\sqrt{\frac{2-\beta}{1-\beta}}.
\end{aligned}
$$

We can see that a component of $h(r,t)$ does not depend on time and so we can write:

$$
h(r,t) = \beta\int_t^H \mathbb{E}[r_s]\, ds - \frac{1}{2}\frac{\beta}{\beta-1}\left(\lambda_1^2 + \lambda_r^2\right)(H-t).
$$

If we compute the solution for the differential stochastic equation of r_s we obtain:[11]

$$
r_s = \widetilde{b_r} - e^{-a_r(s-t)}\left(\widetilde{b_r} - r_t\right) - \widetilde{\sigma_r}\int_s^t e^{-a_r(s-\tau)} dW_r(\tau),
$$

and, given $r_t = r$, we have $\mathbb{E}[r_s] = \widetilde{b_r} - e^{-a_r(s-t)}\left(\widetilde{b_r} - r\right)$. Thus, we can find the function $h(r,t)$:

$$
h(r,t) = \beta\int_t^H \left[\widetilde{b_r} - e^{-a_r(s-t)}\left(\widetilde{b_r} - r\right)\right] ds - \frac{1}{2}\frac{\beta}{\beta-1}\left(\lambda_1^2 + \lambda_r^2\right)(H-t),
$$

[11] Given the equation:

$$
dr_s = a_r\left(\widetilde{b_r} - r_s\right) ds - \widetilde{\sigma_r} dW_r,
$$

we apply the Itô's lemma to $Y(s) = e^{a_r s}\left(\widetilde{b_r} - r_s\right)$ and we have:

$$
\begin{aligned}
dY(s) &= \left[a_r e^{a_r s}\left(\widetilde{b_r} - r_s\right) - a_r e^{a_r s}\left(\widetilde{b_r} - r_s\right)\right] dt + e^{a_r s}\widetilde{\sigma_r} dW_r, \\
dY(s) &= e^{a_r s}\widetilde{\sigma_r} dW_r,
\end{aligned}
$$

from which, by integrating between t and s we have:

$$
\begin{gathered}
e^{a_r s}\left(\widetilde{b_r} - r_s\right) - e^{a_r t}\left(\widetilde{b_r} - r_t\right) = \widetilde{\sigma_r}\int_t^s e^{a_r \tau} dW_r(\tau), \\
r_s = \widetilde{b_r} - e^{a_r(t-s)}\left(\widetilde{b_r} - r_t\right) - \widetilde{\sigma_r}\int_s^t e^{a_r(\tau - s)} dW_r(\tau).
\end{gathered}
$$

from which we obtain:

$$h_r = \beta \int_t^H e^{-a_r(s-t)} ds = \beta \frac{1-e^{-a_r(H-t)}}{a_r}.$$

We recall that $g(H-t) = \frac{1-e^{-a_r(H-t)}}{a_r}$ and so:

$$w^* = \frac{1}{1-\beta} \begin{bmatrix} \frac{\lambda_1}{\sigma_1} \\ \frac{\lambda_r\sigma_1 - \sigma_2\lambda_1}{g(H-t)\sigma_r\sigma_1} \end{bmatrix} - \frac{1}{1-\beta} \begin{bmatrix} 0 \\ \beta \end{bmatrix},$$

which is exactly the solution obtained by Boulier, Huang and Taillard (2001).

A.2 The comparison with Deelstra, Grasselli and Koehl (2001)

Deelstra, Grasselli and Koehl (2001) consider the market structure (19) where the changes in prices of bond and stock are Itô processes whose evolution does not depend on the values themselves. Thus, we can forget about the state variables S and B. Furthermore, there are no liabilities and so we have $\alpha_L = 0$, $\Lambda = \mathbf{0}$ and $\eta = \mathbf{0}$.

Accordingly, we can transform this problem in a form suitable to compute the optimal portfolio composition in our framework:

$$\begin{aligned} w' &= \begin{bmatrix} w_S & w_B \end{bmatrix}', \\ z &= r, \\ \mu_z &= (a_r - b_r r), \\ \Omega &= \begin{bmatrix} -\sigma_r\sqrt{r} \\ 0 \end{bmatrix}, \\ M &= \begin{bmatrix} \sigma_1\lambda_1 + \sigma_2\lambda_r r & g(H-t)\lambda_r\sigma_r r \end{bmatrix}', \\ \Sigma' &= \begin{bmatrix} \sigma_2\sqrt{r} & \sigma_1 \\ g(H-t)\sigma_r\sqrt{r} & 0 \end{bmatrix}, \\ dW &= \begin{bmatrix} dW_r & dW_S \end{bmatrix}'. \end{aligned}$$

The authors solve the problem for an investor having power utility function of the form $K(R) = \frac{1}{\beta}R^\beta$. By applying Formula (16) we have:

$$w^* = \frac{1}{1-\beta} \begin{bmatrix} \frac{\lambda_1}{\sigma_1} \\ \frac{\lambda_r\sigma_1 - \sigma_2\lambda_1}{g(H-t)\sigma_r\sigma_1} \end{bmatrix} + \frac{1}{1-\beta} \begin{bmatrix} 0 \\ -\frac{1}{g(H-t)}h_r \end{bmatrix}.$$

Now, we have to compute the function h that in our case is:

$$\begin{aligned} h(z,t) &= \int_t^H \mathbb{E}\left[b(Z_s,s)\right] ds, \\ dZ_s &= a(Z_s,s)\,ds + \sqrt{\frac{2-\beta}{1-\beta}}\Omega' dW. \end{aligned}$$

In this special case we have:

$$
\begin{aligned}
a(z,t) &= \mu_z - \frac{\beta}{\beta-1}\Omega'\Sigma'^{-1}M = (a_r - b_r r) + \frac{\beta}{\beta-1}\lambda_r\sigma_r r, \\
b(z,t) &= \beta r - \frac{1}{2}\frac{\beta}{\beta-1}M'\left(\Sigma'\Sigma\right)^{-1}M = \beta r - \frac{1}{2}\frac{\beta}{\beta-1}\left(\lambda_1^2 + \lambda_r^2 r\right),
\end{aligned}
$$

and, finally, we can write:

$$
\begin{aligned}
h(r,t) &= \int_t^H \mathbb{E}\left[\left(\beta - \frac{1}{2}\frac{\beta}{\beta-1}\lambda_r^2\right) r_s - \frac{1}{2}\frac{\beta}{\beta-1}\lambda_1^2\right] ds, \\
dr_s &= \left(a_r - \widetilde{b_r} r_s\right) ds + \widetilde{\sigma_r}\sqrt{r_s}\, dW_r, \\
r_t &= r,
\end{aligned}
$$

where:

$$
\begin{aligned}
\widetilde{b_r} &= b_r - \frac{\beta}{\beta-1}\lambda_r\sigma_r, \\
\widetilde{\sigma_r} &= \sigma_r\sqrt{\frac{2-\beta}{1-\beta}}.
\end{aligned}
$$

We can see that a component of $h(r,t)$ does not depend on time and so we can write:

$$
h(r,t) = \left(\beta - \frac{1}{2}\frac{\beta}{\beta-1}\lambda_r^2\right)\int_t^H \mathbb{E}[r_s]\, ds - \frac{1}{2}\frac{\beta}{\beta-1}\lambda_1^2 (H-t).
$$

From Cox, Ingersoll and Ross (1985) we obtain the following solution for the expected value of the stochastic differential equation:

$$
\mathbb{E}[r_s] = \frac{a_r}{\widetilde{b_r}} - e^{-\widetilde{b_r}(s-t)}\left(\frac{a_r}{\widetilde{b_r}} - r\right).
$$

By substituting this expected value into $h(r,t)$ we have:

$$
\begin{aligned}
h_r &= \left(\beta - \frac{1}{2}\frac{\beta}{\beta-1}\lambda_r^2\right)\int_t^H \frac{\partial}{\partial r}\mathbb{E}[r_s] = \\
&= \left(\beta - \frac{1}{2}\frac{\beta}{\beta-1}\lambda_r^2\right)\frac{1-e^{-\widetilde{b_r}(H-t)}}{\widetilde{b_r}}.
\end{aligned}
$$

Thus, the optimal portfolio composition is given by the following approximated formula:

$$
w^* = \frac{1}{1-\beta}\left[\begin{array}{c} \frac{\lambda_1}{\sigma_1} \\ \frac{\lambda_r\sigma_1 - \sigma_2\lambda_1}{g(H-t)\sigma_r\sigma_1} \end{array}\right] + \frac{1}{1-\beta}\left[\begin{array}{c} 0 \\ -\left(\beta - \frac{1}{2}\frac{\beta}{\beta-1}\lambda_r^2\right)\frac{1-e^{-\widetilde{b_r}(H-t)}}{\widetilde{b_r}g(H-t)} \end{array}\right].
$$

A.3 The comparison with Kim and Omberg (1996)

Kim and Omberg (1996) consider the market structure (20) where there are two correlated Wiener processes: $\mathbb{E}\left[dW_x dW_S\right] = \rho_{xS}$. Thus, we can write:

$$Cov\left(dx, \frac{dS}{S}\right) = \begin{bmatrix} \sigma_x^2 & \sigma_S\sigma_x\rho_{xS} \\ \sigma_S\sigma_x\rho_{xS} & \sigma_S^2 \end{bmatrix}.$$

We can lead this case back to our approach by using the Cholesky decomposition. Because the variance and covariance matrix is always positive semi-definite, we can write:

$$\begin{bmatrix} \sigma_x & 0 \\ \sigma_S\rho_{xS} & \sigma_S\sqrt{1-\rho_{xS}^2} \end{bmatrix} \begin{bmatrix} \sigma_x & \sigma_S\rho_{xS} \\ 0 & \sigma_S\sqrt{1-\rho_{xS}^2} \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \sigma_S\sigma_x\rho_{xS} \\ \sigma_S\sigma_x\rho_{xS} & \sigma_S^2 \end{bmatrix}.$$

Thus, the market structure (20) can be written in the following way:

$$\begin{cases} dx = -\lambda\left(x - \overline{x}\right)dt + \sigma_x d\widetilde{W_x}, \\ \frac{dS}{S} = \left(r + \sigma_S x\right)dt + \sigma_S\rho_{xS}d\widetilde{W_x} + \sigma_S\sqrt{1-\rho_{xS}^2}\,d\widetilde{W_S}, \\ \frac{dG}{G} = rdt, \end{cases}$$

where $\widetilde{W_S}$ and $\widetilde{W_x}$ are two independent Wiener processes.[12] For leading this kind of problem back to our approach we put:

$$\begin{aligned} w &= w_S, \\ z &= x, \\ \mu_z &= -\lambda\left(x - \overline{x}\right), \\ \Omega &= \begin{bmatrix} \sigma_x \\ 0 \end{bmatrix}, \\ M &= x\sigma_S, \\ \Sigma' &= \begin{bmatrix} \sigma_S\rho_{xS} & \sigma_S\sqrt{1-\rho_{xS}^2} \end{bmatrix}, \\ dW &= \begin{bmatrix} d\widetilde{W_x} & d\widetilde{W_S} \end{bmatrix}'. \end{aligned}$$

[12] We underline that this transformation is equivalent to the following one:

$$\begin{bmatrix} \sigma_x\sqrt{1-\rho_{xS}^2} & \sigma_x\rho_{xS} \\ 0 & \sigma_S \end{bmatrix} \begin{bmatrix} \sigma_x\sqrt{1-\rho_{xS}^2} & 0 \\ \sigma_x\rho_{xS} & \sigma_S \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \sigma_S\sigma_x\rho_{xS} \\ \sigma_S\sigma_x\rho_{xS} & \sigma_S^2 \end{bmatrix},$$

from which we obtain the system:

$$\begin{cases} dx = -\lambda\left(x - \overline{x}\right)dt + \sigma_x\sqrt{1-\rho_{xS}^2}\,d\widetilde{W_x} + \sigma_x\rho_{xS}d\widetilde{W_S}, \\ \frac{dS}{S} = \left(r + \sigma_S x\right)dt + \sigma_S d\widetilde{W_S}, \\ \frac{dG}{G} = rdt. \end{cases}$$

Here, we have decided to maintain a single risk source for the state variable x because this representation is more intuitive from an economic point of view. In fact, the state variable affects the stock price which also has its own risk source.

However, we outline another time that the final result is identical for both transformations.

The authors use a HARA utility function of the form $U(R) = (R - R^*)^{\frac{\gamma-1}{\gamma}}$. Thus, here, we put $R^* = 0$ and $\beta = \frac{\gamma-1}{\gamma}$ for having our form $K(R) = R^\beta$. From Formula (16), the optimal portfolio composition is:

$$w^* = \frac{1}{1-\beta}\frac{x}{\sigma_S} + \frac{1}{1-\beta}\frac{\rho_{xS}\sigma_x}{\sigma_S}h_x.$$

After computing the following functions:

$$\begin{aligned} a(x,t) &= -\lambda(x - \overline{x}) - \frac{\beta}{1-\beta}\rho_{xS}\sigma_x x, \\ b(x,t) &= \beta r - \frac{1}{2}\frac{\beta}{1-\beta}x^2, \end{aligned}$$

the problem to solve is:

$$\begin{aligned} h(x,t) &= \int_t^H \mathbb{E}\left[\beta r - \frac{1}{2}\frac{\beta}{1-\beta}X_s^2\right] ds = \beta r(H-t) - \frac{1}{2}\frac{\beta}{1-\beta}\int_t^H \mathbb{E}\left[X_s^2\right] ds, \\ dX_s &= \left[\lambda\overline{x} - \left(\lambda + \frac{\beta}{1-\beta}\rho_{xS}\sigma_x\right)X\right] ds + \sigma_x\sqrt{2 - \frac{\beta}{1-\beta}\rho_{xS}^2 d\widetilde{W_x}}, \\ X_t &= x. \end{aligned}$$

From the stochastic differential equation we can compute the behaviour of X_s as in the previous appendix (see also Cox, Ingersoll and Ross, 1985) and we obtain:

$$\mathbb{E}\left[X_s^2\right] = \left[\frac{\lambda\overline{x}}{\alpha} - e^{\alpha(t-s)}\left(\frac{\lambda\overline{x}}{\alpha} - x\right)\right]^2 + \sigma_x^2\left(2 - \frac{\beta}{1-\beta}\rho_{xS}^2\right)\int_s^t e^{2\alpha(\tau-t)}d\tau,$$

where:

$$\alpha = \lambda + \frac{\beta}{1-\beta}\rho_{xS}\sigma_x.$$

Thus, we can complete the solution:

$$\begin{aligned} h_x &= -\frac{1}{2}\frac{\beta}{1-\beta}\int_t^H \frac{\partial}{\partial x}\mathbb{E}\left[X_s^2\right] ds = \\ &= -\frac{\beta}{1-\beta}\int_t^H \left[\frac{\lambda\overline{x}}{\alpha}e^{\alpha(t-s)} - e^{2\alpha(t-s)}\left(\frac{\lambda\overline{x}}{\alpha} - x\right)\right] ds = \\ &= -\frac{1}{2}\frac{\beta}{1-\beta}\left(\left(\frac{1-e^{-\alpha(H-t)}}{\alpha}\right)^2\lambda\overline{x} + \frac{1-e^{-2\alpha(H-t)}}{\alpha}x\right). \end{aligned}$$

A.4 The comparison with Kogan and Uppal (1999)

Kogan and Uppal (1999) consider the market structure (22) of Chacko and Viceira (1999) where there are two correlated Wiener processes such that:

$$Cov\left(dx, \frac{dS}{S}\right) = \begin{bmatrix} \sigma^2 x & \sigma_{Sx} \\ \sigma_{Sx} & \frac{1}{x} \end{bmatrix}.$$

We can lead this case back to our approach by using the Cholesky decomposition. Because the variance and covariance matrix is always positive semi-definite, we can write:

$$\begin{bmatrix} \sigma\sqrt{x} & 0 \\ \frac{\sigma_{Sx}}{\sigma\sqrt{x}} & \frac{1}{\sqrt{x}}\sqrt{1-\frac{\sigma_{Sx}^2}{\sigma^2}} \end{bmatrix} \begin{bmatrix} \sigma\sqrt{x} & \frac{\sigma_{Sx}}{\sigma\sqrt{x}} \\ 0 & \frac{1}{\sqrt{x}}\sqrt{1-\frac{\sigma_{Sx}^2}{\sigma^2}} \end{bmatrix} = \begin{bmatrix} \sigma^2 x & \sigma_{Sx} \\ \sigma_{Sx} & \frac{1}{x} \end{bmatrix}.$$

Thus, the previous problem can be written in the following way:

$$\begin{cases} dx = -\lambda\left(x-\overline{x}\right)dt + \sigma\sqrt{x}d\widetilde{W_x}, \\ \frac{dS}{S} = \mu dt + \frac{\sigma_{Sx}}{\sigma\sqrt{x}}d\widetilde{W_x} + \frac{1}{\sqrt{x}}\sqrt{1-\frac{\sigma_{xS}^2}{\sigma^2}}d\widetilde{W_S}, \\ \frac{dG}{G} = rdt, \end{cases}$$

where $\widetilde{W_S}$ and $\widetilde{W_x}$ are two independent Wiener processes.[13]

Because the change in the stock price is an Itô process whose evolution does not depend on the value itself, then we can forget about the state variable S. Furthermore, there are no inflation risk and so we have $\alpha_L = 0$, $\Lambda = \mathbf{0}$ and $\eta = \mathbf{0}$.

Accordingly, we can transform this problem in a form suitable to compute

[13] We underline that this transformation is equivalent to the following one:

$$\begin{bmatrix} \sqrt{x}\sqrt{\sigma^2-\sigma_{Sx}^2} & \sigma_{Sx}\sqrt{x} \\ 0 & \frac{1}{\sqrt{x}} \end{bmatrix} \begin{bmatrix} \sqrt{x}\sqrt{\sigma^2-\sigma_{Sx}^2} & 0 \\ \sigma_{Sx}\sqrt{x} & \frac{1}{\sqrt{x}} \end{bmatrix} = \begin{bmatrix} \sigma^2 x & \sigma_{Sx} \\ \sigma_{Sx} & \frac{1}{x} \end{bmatrix},$$

from which we obtain the system:

$$\begin{cases} dx = -\lambda\left(x-\overline{x}\right)dt + \sqrt{x}\sqrt{\sigma^2-\sigma_{Sx}^2}d\widetilde{W_x} + \sigma_{Sx}\sqrt{x}d\widetilde{W_S}, \\ \frac{dS}{S} = \mu dt + \frac{1}{\sqrt{x}}d\widetilde{W_S}, \\ \frac{dG}{G} = rdt. \end{cases}$$

Here, we have decided to maintain a single risk source for the state variable x because this representation is more intuitive from an economic point of view. In fact, the state variable affects the stock price which has also its own risk source.

However, we outline another time that the final result is identical for both transformations.

the optimal portfolio composition in our framework:

$$
\begin{aligned}
w &= w_S, \\
z &= x, \\
\mu_z &= -\lambda\left(x-\overline{x}\right), \\
\Omega &= \begin{bmatrix} \sigma\sqrt{x} \\ 0 \end{bmatrix}, \\
M &= \mu - r, \\
\Sigma' &= \begin{bmatrix} \frac{\sigma_{Sx}}{\sigma\sqrt{x}} & \frac{1}{\sqrt{x}}\sqrt{1-\frac{\sigma_{Sx}^2}{\sigma^2}} \end{bmatrix}, \\
dW &= \begin{bmatrix} d\widetilde{W_x} & d\widetilde{W_S} \end{bmatrix}'.
\end{aligned}
$$

The authors solve the problem for an investor having power utility function of the form $K\left(R\right)=\frac{1}{\beta}R^{\beta}$. By applying Formula (16) we have:

$$
w^* = \frac{1}{1-\beta}\left(\mu-r\right)x + \frac{1}{1-\beta}\sigma_{Sx}xh_x.
$$

Given the following values of functions $a\left(x,t\right)$, $b\left(x,t\right)$, and $C\left(x,t\right)$:

$$
\begin{aligned}
a\left(x,t\right) &= -\lambda\left(x-\overline{x}\right) + \frac{\beta}{1-\beta}\left(\mu-r\right)\sigma_{Sx}x, \\
b\left(x,t\right) &= \beta r + \frac{1}{2}\frac{\beta}{1-\beta}\left(\mu-r\right)^2 x,
\end{aligned}
$$

we have to solve the problem:

$$
\begin{aligned}
dX_s &= \left[\lambda\overline{x} - \left(\lambda - \frac{\beta}{1-\beta}\left(\mu-r\right)\sigma_{Sx}\right)X_s\right]ds + \sqrt{2\sigma^2 + \frac{\beta}{1-\beta}\sigma_{Sx}^2}\sqrt{X_s}d\widetilde{W_x}, \\
X_t &= x.
\end{aligned}
$$

By using the solution already exposed for the CIR model (see Cox, Ingersoll and Ross, 1985), we have:

$$
\begin{aligned}
\mathbb{E}\left[X_s\right] &= \frac{\lambda\overline{x}}{\lambda - \frac{\beta}{1-\beta}\left(\mu-r\right)\sigma_{Sx}} + \\
&\quad - e^{-\left(\lambda-\frac{\beta}{1-\beta}\left(\mu-r\right)\sigma_{Sx}\right)\left(s-t\right)}\left(\frac{\lambda\overline{x}}{\lambda-\frac{\beta}{1-\beta}\left(\mu-r\right)\sigma_{Sx}} - x\right),
\end{aligned}
$$

and, accordingly, we can write:

$$
\begin{aligned}
h\left(x,t\right) &= \int_t^H \mathbb{E}\left[\beta r + \frac{1}{2}\frac{\beta}{1-\beta}\left(\mu-r\right)^2 X_s\right]ds = \\
&= \beta r\left(H-t\right) + \frac{1}{2}\frac{\beta}{1-\beta}\left(\mu-r\right)^2\int_t^H \mathbb{E}\left[X_s\right]ds.
\end{aligned}
$$

Because we are interested in the derivative of h with respect to x, then:

$$\begin{aligned} h_x &= \frac{1}{2}\frac{\beta}{1-\beta}(\mu-r)^2 \int_t^H \frac{\partial}{\partial x}\mathbb{E}[X_s] = \\ &= \frac{1}{2}\frac{\beta}{1-\beta}(\mu-r)^2 \frac{1-e^{-\left(\lambda-\frac{\beta}{1-\beta}(\mu-r)\sigma_{Sx}\right)(H-t)}}{\lambda-\frac{\beta}{1-\beta}(\mu-r)\sigma_{Sx}}. \end{aligned}$$

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