Endogenous Growth, Capital Utilization and Depreciation*

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December, 2001.

Abstract

We study the one sector model of growth when a linear production technology is combined with adjustment costs and a technology for capital maintenance. Agents are allowed to under-use the installed capital and to vary the depreciation rate. This economy decides endogenously how much resources devotes to the accumulation of new capital and how much to maintenance and repair activities. We find as striking results that the long-run depreciation and capital utilization rates are positively related to the population growth rate, and that both depend negatively on the initial conditions. The long-run growth rate appears positively correlated with the depreciation rate.

Keywords: Maintenance, Depreciation, Capital Utilization, Endogenous Growth.

JEL classification: O40, E22, D90.

*We thank Raouf Boucekkine for his helpful comments. J. Aznar-Márquez acknowledges the support of the Belgian research programmes "Poles d'Attraction interuniversitaires" PAI P4/01, and "Action de Recherches Concertée" 99/04-235. J. R. Ruiz-Tamarit acknowledges financial support from the Spanish CICYT, Projects SEC99-0820 and SEC2000-0260.

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1 Introduction

In the standard neoclassic growth model, the decision about how much to save is based on the comparison, in welfare terms, between the costs and the benefits of a higher consumption today rather than tomorrow. Once the amount of savings has been decided, they are automatically channelled into investment. Investment thus has an entirely passive role. This model assumes full installed capital utilization and, moreover, that the depreciation suffered by capital equipment is a constant exogenously determined proportion of capital stock. These assumptions, however, do not conform to observed facts as available data shows quite a different reality. It seems reasonable, from our point of view, to think that firms do not always use all of their installed capital and that they are able to decide and act upon the depreciation rate that capital stock experiences. This later point is made possible by devoting resources to the preservation, that is, the repair and maintenance, of capital stock that has deteriorated either through use in the production process or simply through the natural process of ageing.

Recently, McGrattan and Schmitz (1999) have highlighted the quantitative relevance of repair and maintenance activities. These authors obtain in Canada for the period 1961-1993, that up to 6% of gross national product was devoted to repair and maintenance activities, which is approximately half the expenditure made on the acquisition of new capital goods. In addition, Gylfason and Zoega (2001), using data currently published by the World Bank, have studied the relationship between depreciation and growth. Among the results they pointed out, we would like to emphasize the following ones: i) increased population growth accelerates depreciation, ii) increased efficiency increases depreciation, and iii) increased long-run growth also accelerates depreciation. Furthermore, they document an important positive correlation between the per capita income growth rate and the depreciation rate over the period 1965-1998 for a sample of 85 countries.

Despite the above cited empirical evidence, the depreciation rate has been regarded as an exogenous parameter in growth theory which, in the case of neoclassical models, negatively affects long-run variable levels and short-run growth rates, and in the case of endogenous growth models, affects also the long-run rates of growth. In this study, we explore the analytical relationship between the determinants of both depreciation and growth in the context of the one sector model of growth when a linear production technology is combined with adjustment costs and a technology for capital maintenance.

Agents are allowed to under-use the installed capital as well as to vary the depreciation rate. So, this economy decides endogenously the amount of resources to be devoted to the accumulation of new capital and also the amount to be devoted to repair and maintenance activities. This latter decision appears to be directly related to the matter of the endogeneity of both the capital utilization rate and the depreciation rate.

The maintenance of capital stock, that allows to break with the strong hypothesis of a constant depreciation rate even in the absence of obsolescence, has been left aside for many years after the seminal contributions of the seventies. Nevertheless, various attempts to reintroduce the variability of the depreciation rate have been made with the implementation of the hypothesis Depreciation-in-Use. That is, the causality line which connects biunivocally high/low rates of capital utilization, usually associated to high/low levels of economic activity, with higher/lower depreciation rates. This hypothesis has been used in microeconomic studies at the firm level [Epstein and Denny (1980), Bischoff and Kokkelenberg (1987), Motahar (1992)] as well as in macroeconomic studies concerning both the neoclassical growth theory [Rumbos and Auernheimer (1997)] and the real business cycle theory [Burnside and Eichenbaum (1996)]. Although the depreciation rate is transformed into an endogenous variable, this approach does not seem to be completely satisfactory because of the residual role assigned to capital depreciation. More recently, the above hypothesis has been extended to incorporate the maintenance activity which allows for the depreciation rate to be a decision variable analogous to the capital utilization rate. We would like to mention the effort that has been made at the firm level by Boucekkine and Ruiz-Tamarit (2001) as well as at an aggregate level in a neoclassical growth model by Licandro, Puch and Ruiz-Tamarit (2001) and in a real business cycle model by Licandro and Puch (2000) and Collard and Kollintzas (2000).

As mentioned above, the basic general equilibrium growth models do not allow for the separation of the household saving decisions from the investment decisions of firms. However, by introducing adjustment costs connected with gross investment expenditures it is possible to overcome the essentially passive role of investment in the models[1]. In this paper, we take the canonical model of Rebelo (1991) and introduce both an adjustment cost function and a

[1] The active role of investment has been studied by Abel and Blanchard (1983) in a neoclassical Ramsey-like model, and also by Barro and Sala-i-Martín (1992) in an endogenous growth model of the AK type.

maintenance cost function affecting the objective functional. It is well-known that in the one-sector models of growth, the linear technology constitutes a useful referent for easily modeling, directly or asymptotically, the endogenous growth phenomenon. At this respect we would like to know whether the introduction of those new functions into the model may break the previous association. In this context, depreciation is no longer a residual variable. Together with investment and the rate of capital utilization, it becomes one of the instruments used by economic agents in setting their optimal plans. In short, we want to elucidate whether incorporating the maintenance and repair expenditures into a model of aggregate economic activity, the answer to the question of what happens in the short-run, particularly in terms of the convergence hypothesis, as well as in the long-run, concerning the determinants of the growth rate and the new endogenous variables, may substantially change. Our technology assumptions allow us to augment the model in such a way that well-defined investment, depreciation, and utilization functions may be derived. However, there are no theoretical contributions, that we know of, aimed at pointing out all these topics. This paper is devoted to this end and, consequently, it is primarily dedicated to investigating the short-run dynamics and the balanced growth path. As an anticipation, we may assure that the big changes will concentrate upon the long-run results.

The article is organized as follows. Section 2 describes the economy and introduces the assumptions featuring the different parts of the general equilibrium model. In section 3, we solve the intertemporal optimization problem and study the resulting dynamic system which governs the economy. Section 4 is devoted to obtaining and interpreting results, connecting with the empirical literature which parallels the present work. Finally, section 5 summarizes and highlights the central aspects of the model.

2 The economy

Let us consider an economy populated with many identical infinitely-lived individuals, N_t. Population is assumed to grow at a constant and exogenously given rate $n \geq 0$. We normalize the initial population to unity and then we get $N_t = e^{nt}$. Moreover, it is assumed that people facing up to an infinite planning horizon will discount the future at a positive constant rate $\rho > n$. Individual preferences are represented by an instantaneous utility function $U(c_t)$, which is assumed increasing, twice differentiable and strictly concave. This function only depends on the per capita consumption c_t, and it is assumed that Inada conditions are satisfied, $\lim_{c\to 0^+} U'(c_t) = +\infty$ and $\lim_{c\to +\infty} U'(c_t) = 0$.

On the other hand, there are many identical firms producing a single good. For simplicity, we assume that each firm uses a linear technology[2] of the AK type, the capital stock K_t being the only relevant factor. We interpret capital in a broad sense, so that it includes physical capital as well as human capital, which usually comes embodied in workers. In this sense, human capital is considered a rival and excludable factor as physical capital is. Labour, measured as the number of workers and independently of the index of human capital that has been considered as perfect substitute for physical capital, is not necessary for production. Therefore, total current output Y_t is a function of the effectively used capital $K_t u_t$, where u_t is the variable proportion of installed capital that firm decides to use, and the efficiency parameter A represents a constant technological level. Moreover, this parameter may also be read as the marginal productivity as well as the average productivity of the effectively used capital. So, given the constant returns inherent to a linear production function, we write this function in per capita terms:

$$y_t = A k_t u_t \tag{1}$$

Because of our interest in long-run endogenous growth paths, we leave aside the hypothesis of exogenous technological progress. So, it is possible to identify more easily the consequences of the assumed constant returns to capital for the rate of growth, the rate of capital utilization and the depreciation rate.

In this economy, the produced single-good may be allocated to consumption, to the accumulation of new capital or to preserving the inherited capital. While the present consumption contributes directly to increase welfare, the other uses of resources are connected with the rise of capital stock which allows for a greater consumption in the future. In this context, accumulation of new capital has not only to do with investment purchases but also with adjustment or installation activities. Moreover, the preservation of old capital has to do with maintenance and repair activities. Consequently, we have

[2]Similar results could be derived under a more general production function with constant returns to scale if we introduce, following Romer (1986), the learning-by-investing device together with the knowledge spillovers assumption.

to introduce in our framework the two corresponding cost functions.

First, let us assume that adjustment costs, which are internal to the firm, are represented by a linearly homogeneous function $\Phi(I_t, K_t)$, increasing in gross investment, $I_t > 0$, and decreasing in the total installed capital stock, $K_t > 0$. Then $\Phi(I_t, K_t) = \Phi(I_t/K_t, 1)K_t = \phi(i_t)K_t$, where i_t is the rate of gross investment over capital, and $\phi(i_t)$ is assumed non-negative, twice differentiable, increasing and strictly convex for $i_t > 0$, with $\lim_{i\to 0^+} \phi(i) = 0$ and $\lim_{i\to +\infty} \phi'(i) = +\infty$. Consequently, per capita adjustment costs are then written as $\phi(i_t)k_t$.

Second, in order to preserve the inherited capital stock, we assume that period by period it is possible to reduce the depreciation due to the deterioration that arises from equipment ageing and use[3], by means of the corresponding maintenance and repair activities. These activities entail specific maintenance costs which are internal to the firm and, by assumption, will be represented by a linearly homogeneous function $M(D_t, K_t u_t)$, decreasing in total depreciation, $D_t > 0$, and increasing in effectively used capital. Redefining variables we get $M(D_t, K_t u_t) = M(D_t/K_t, K_t u_t/K_t)K_t = m(\delta_t, u_t)K_t$, where $\delta_t > 0$ is the endogenous rate of depreciation over capital stock and $u_t \in]0, 1[$ is the intensity of use of the installed capital stock. The function $m(\delta_t, u_t)$, the average maintenance costs, is assumed non-negative, twice differentiable, convex and linearly homogeneous. Furthermore, we assume $m_\delta(\delta_t, u_t) < 0$, $m_u(\delta_t, u_t) > 0$, $m_{\delta\delta}(\delta_t, u_t) > 0$, $m_{uu}(\delta_t, u_t) > 0$ and $m_{\delta u}(\delta_t, u_t) < 0$ for $u \in]0, 1[$ and $\delta > 0$ with $\lim_{\delta\to 1} m(\delta_t, u_t) = 0$ and $\lim_{u\to 0} m(\delta_t, u_t) = 0$. The larger the utilization of capital, the larger the costs of maintenance, and the larger the maintenance costs, the smaller the depreciation rate of capital[4]. The homogeneity assumption implies that $m_{\delta\delta}(\delta_t, u_t)m_{uu}(\delta_t, u_t) - m_{\delta u}(\delta_t, u_t)^2 = 0$. Consequently, per capita maintenance costs are then written as $m(\delta_t, u_t)k_t$.

[3] Here we are refering strictly to physical wear and tear but, contrary to the standard procedure, we take this depreciation as an economic phenomenon because firms are assumed to optimally decide how much resources have to be allocated to maintenance. In this paper we ignore obsolescence as a source of depreciation. Factors usually causing obsolescence are left out of the present analytical framework because of the assumed perfect malleability of capital.

[4] An equivalent representation of the problem would correspond to the assumption that the depreciation rate is a function of both the utilization rate and the rate of maintenance costs to capital. This alternative view has been adopted by Boucekkine and Ruiz-Tamarit (2001) in a partial equilibrium context to develop the study of firm investment and depreciation decisions.

The aggregate resource constraint is $Y_t = C_t + I_t + \Phi\left(I_t, K_t\right) + M\left(D_t, K_t u_t\right)$ where $\overset{\bullet}{K}_t = I_t - \delta_t K_t$. In per capita terms the resource constraint appears determined by the following equalities:

$$c_t + \left(i_t + \phi(i_t) + m(\delta_t, u_t)\right) k_t = A k_t u_t \tag{2}$$

$$\overset{\bullet}{k}_t \; = \left(i_t - \delta_t - n\right) \; k_t, \tag{3}$$

where $\overset{\bullet}{k}$ denotes the time derivative of per capita capital considered in its broad sense.

3 The optimization problem

In an economy without externalities and no other market failures, such as imperfections or incompleteness which could appear in conflict with the assumptions of any of the two basic welfare theorems, the competitive equilibrium solution to the intertemporal resources allocation problem will correspond to the central planner solution. Therefore, in our present setting, every optimal solution may be decentralized as a competitive equilibrium. The planner's optimization problem is to choose at each moment in time the three controls: the rate of capital utilization, the rate of investment and the rate of depreciation, which solve the following problem.

$$\underset{\{u_t, i_t, \delta_t\}}{Max} \qquad W = \int_0^\infty u(c_t) e^{-(\rho - n)t} dt \tag{P}$$

subject to the resource constraints (2) and (3), and given the initial capital stock, k_0.

The assumption of a time-preference rate larger than the population growth rate together with the properties imposed upon the instantaneous utility function, ensure that the previous integral is upper bounded[5]. Then, the current value Hamiltonian function associated to this problem, after dropping time subscripts, may be written as follow:

$$H^c = U\left(Aku - \left[i + \phi\left(i\right) + m\left(\delta, u\right)\right] k\right) + \mu[i - \delta - n]k$$

[5]Further on it will be shown how this constraint derives from the transversality condition alone.

where μ is a co-state variable. According to the Maximum Principle[6], an interior optimal solution to problem (P) must satisfy the following set of first-order conditions:

$$A = m_u(\delta, u) \tag{4}$$

$$\mu = U'(c)\ [1 + \phi'(i)] \tag{5}$$

$$\mu = -U'(c)\ m_\delta(\delta, u) \tag{6}$$

the Euler equation:

$$\dot{\mu} = -U'(c)\ [A\ u - i - \phi(i) - m(\delta, u)] + \mu\ [\rho + \delta - i] \tag{7}$$

the constraints (2) and (3), as well as the initial condition $k_0 > 0$ and the corresponding transversality condition:

$$\lim_{t \to \infty} e^{-(\rho - n)t} \mu_t k_t = 0 \tag{8}$$

The multiplier μ defines the shadow price, measured in units of utility, of an additional unit of installed capital. The term $1 + \phi'(i)$ is the marginal opportunity cost of gross investment. Then, equation (5) states that this marginal cost measured in units of utility must be equal to the shadow price of capital. On the other hand, $-m_\delta(\delta, u)$ is the marginal saving in maintenance costs associated to an increase in the depreciation rate. An increase in δ reduces capital stock and, consequently, diminishes maintenance expenditures. So, equation (6) states that this marginal saving measured in units of utility must be equal to the shadow price of the lost capital. The term $m_u(\delta, u)$ is the marginal maintenance cost associated to an increase in the utilization rate. Equation (4) states that this marginal cost must be equal to the marginal productivity of such an increase in the utilization rate, measured by the term A.

Consider now equation (7) and, after some manipulations, solve forward subject to the transversality condition (8) which avoids explosive solutions. In doing so we have to use two fundamental relationships:

[6] Under the more restrictive assumption that the Hamiltonian function is strictly concave with respect to the control variables, the solution functions are continuous and the first order conditions become necessary and sufficient for a maximum.

$$-\frac{m_{\delta\delta}(\delta,u)}{m_{\delta u}(\delta,u)}=-\frac{m_{\delta u}(\delta,u)}{m_{uu}(\delta,u)}=\frac{u}{\delta} \tag{9}$$

$$1+\phi'(i)=\frac{\mu}{U'(c)}=-m_{\delta}(\delta,u) \tag{10}$$

the first one coming from the homogeneity assumption and the second one arising from equations (5) and (6). As a result from that, the price μ must be equal to the present discounted value of the marginal products:

$$\mu_t=\int_t^{\infty}U'(c_s)[Au_s+i_s\phi'(i_s)-\phi(i_s)-m(\delta_s,u_s)]e^{-\int_t^s[\rho+\delta_z]\cdot dz}ds \tag{11}$$

In this expression, the integrand represents the total marginal product of installed capital measured in units of utility, and the discount term takes into account the fact that the depreciation rate is variable.

On the other hand, the first order conditions (4)-(6), plus the resource constraint (2) and the production function (1) implicitly define the optimal functions relating each control variable to the state and co-state variables. These control functions may be represented as $u=u(k,\mu,\Theta)$, $i=i(k,\mu,\Theta)$, $\delta=\delta(k,\mu,\Theta)$, $c=c(k,\mu,\Theta)$ and $y=y(k,\mu,\Theta)$, where Θ represents a vector of structural parameters. In appendix I we widely analyze all these functions and show, among other things, that investment rate takes a constant value depending only on the structural parameters of the model. Consequently, we study now the dynamic system which describes the evolution of state and costate variables. First, we introduce the control functions and transform the accumulation equation (3) into the following differential equation:

$$\overset{\bullet}{k}=[i-\delta(k,\mu)-n]\,k \tag{12}$$

Then, making use of the relationship (10) as well as of the linear homogeneity assumption on the maintenance cost function (9) and the first order conditions (4)-(6), which allow for further simplifications as shown in appendix II, the Euler equation (7) may be written in the following way:

$$\overset{\bullet}{\mu}=[-H(i)+\rho]\,\mu \tag{13}$$

In this equation, the constant coefficient on the right hand side involves the use of the function $H(i)=\frac{i\,\phi'(i)-\phi(i)}{1+\phi'(i)}$, which by the assumed convexity on $\phi(i)$ gives positive values for any $i > 0$. This function is a monotonous increasing function given that $H(0) = 0$ and $H'(i) = \frac{i\ \phi''(i)+\phi(i)\ \phi''(i)}{[1+\phi'(i)]^2} > 0$.

We have previously shown the constancy of the investment rate. So, the function $H(i)$ gives a constant value and hence a constant coefficient for the differential equation (13). Then, the system (12)-(13) have a structure very similar to the one characterizing the standard AK models. However, we cannot go forward as is usually done because of two reasons. First, we cannot translate the above system from the state-costate space to the state-control space because our first order conditions do not allow for an immediate substitution as in the basic model, where there is only one control variable and the transformed dynamic system becomes linear. Second, the presence of the depreciation rate control function in equation (12) is so general that we cannot identify any partial and separated linear form with respect to the state variable. Consequently, if we want to apply an analytical resolution method for this non-linear system, we have to proceed by analyzing the dynamic system under the particular forms as were assumed in appendix I for each structural function in the model. Thus, given that $H(i) = \left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}} - 1\right]^2 / \frac{2b\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}$, we can write our dynamic system as:

$$\overset{\bullet}{k} = \left[\frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}} - 1\right]^2}{\frac{2b\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}} - n\right] k - \left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{1}{\Phi}-1} \mu^{\frac{-1}{\Phi}} \tag{14}$$

$$\overset{\bullet}{\mu} = [-\frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}} - 1\right]^2}{\frac{2b\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}} + \rho]\ \mu \tag{15}$$

This system is non-linear and it does not admit a linearization because of the lack of a well defined steady state. However, its structure allows for a complete closed solution working sequentially with the equations and the boundary conditions. Instead of that, we will attach our analysis to the procedure explained in Ruiz-Tamarit and Ventura-Marco (2000), where the authors study and solve in closed form a general non-linear modified Hamiltonian dynamic system for which the previous one may be seen as a particular case.

4 Results

In appendix III the reader may find an sketch of the method applied to solve the particular version of the system which drives our economic system. The unique non-explosive particular solution trajectories for the variables of the system are:

$$k(t,\Theta) = k_0 \exp\left\{ \frac{1}{\Phi}\left[\frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}} - 1\right]^2}{\frac{2b\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}} - \rho \right] t \right\} \tag{16}$$

$$\mu(t,\Theta) = \mu(0)\ \exp\left\{ \left[\rho - \frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}} - 1\right]^2}{\frac{2b\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}} \right] t \right\} \tag{17}$$

with k_0 known and $\mu(0)$ given by the expression:

$$\mu(0) = \frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^{1-\Phi}}{\left[\left(1-\frac{1}{\Phi}\right)\frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}} - 1\right]^2}{\frac{2b\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}} + \frac{\rho}{\Phi} - n\right]^{\Phi}} \frac{1}{k_0^{\Phi}} \tag{18}$$

Then, using the results (16) and (17) we can compute the term $\mu^{\frac{-1}{\Phi}}k^{-1}$ which is needed in order to determine the complete particular trajectories for the control variables. We find $\mu^{\frac{-1}{\Phi}}k^{-1} = \frac{\mu(0)}{k_0}$, and substituting in equations (I.1)-(I.3) from appendix I we get the following trajectories for $i(t,\Theta)$, $\delta(t,\Theta)$ and $u(t,\Theta)$:

$$i(\Theta) = \frac{1}{b}\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}} - 1\right] \tag{19}$$

$$\delta(\Theta) = \frac{\frac{1}{2b}\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^2 - \frac{1}{2b}}{\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}} + \frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{1}{\Phi}-\Phi}}{k_0^{1+\Phi}\left[\left(1-\frac{1}{\Phi}\right)\frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}} - 1\right]^2}{\frac{2b\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}} + \frac{\rho}{\Phi} - n\right]^{\Phi}} \tag{20}$$

$$u(\Theta)=\frac{\frac{1}{2b}\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^{2}-\frac{1}{2b}}{\varepsilon\left(\frac{A}{1+\varepsilon}\right)}+\frac{\left(\frac{A}{1+\varepsilon}\right)^{\frac{1}{\varepsilon}}\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{1}{\Phi}-\Phi}}{k_0^{1+\Phi}d^{\frac{1}{\varepsilon}}\left[\left(1-\frac{1}{\Phi}\right)\frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}-1\right]^{2}}{\frac{2b\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}}+\frac{\rho}{\Phi}-n\right]^{\Phi}} \tag{21}$$

First, we can see that the investment rate, the depreciation rate as well as the capital utilization rate remain constant along the particular solution trajectory which, as we will see below, does correspond to the unique balanced growth path arising from the dynamic model. The result concerning the investment rate could have been anticipated because of the previously proved independence of the associated control function with respect to the endogenous variables. However, in the case of the depreciation and utilization rates, the aforesaid result is due to the compensating effects exerted by capital stock and its shadow price on each of the variables along the solution trajectory.

The previous functions, in turn, show several interesting parameter dependences that we would like to remark on. First of all, the three variables react together in the same direction face to parameter changes. However, we can particularize some of those effects in the following way: i) the greater the productivity of effectively used capital or efficiency parameter A, the higher the investment rate as well as the depreciation and capital utilization rates; ii) the bigger the rate of population growth, the greater the depreciation and capital utilization rates, but the same investment rate; iii) the higher the weight of installation and maintenance costs in gross product, represented by parameters b and d respectively, the lower the investment rate as well as the depreciation and capital utilization rates. Moreover, leaving off the technological parameters and focussing on preferences, our results show that both the capital utilization rate and the depreciation rate are negatively related with the level of impatience characterizing economic agents. That is, iv) the lower the intertemporal elasticity of substitution in consumption $\sigma=\frac{1}{\Phi}$, and/or the greater the rate of discount ρ, the lower the depreciation rate as well as the capital utilization rate, even though the investment rate appears as independent of such preference parameters[7]. Finally, as an additional and unusual feature arising from this model, we have to point out

[7] Among the signs reflecting the influence of every parameter on the control variables,

that v) whereas investment rate is independent of initial conditions, both the depreciation rate and the capital utilization rate show a negative dependence on the initial capital stock value, k_0. All these results are in strong contradiction to the ones appearing throughout the most common endogenous growth models where the depreciation rate is assumed constant and capital stock is used at full capacity. However, most of them are consistent with empirical facts as recently have been reported by Gylfason and Zoega (2001).

On the other hand, from the definitions in (1) and (2) and the above control trajectories, we can also derive the particular solution trajectories for per capita consumption and output per capita.

$$y(t,\Theta) = A\ u(\Theta)\ k_0\ \exp\left\{\frac{1}{\Phi}\left[\frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}-1\right]^2}{\frac{2b\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}}-\rho\right]t\right\} \tag{22}$$

$$c(t,\Theta) = \Gamma(\Theta)\ k_0\ \exp\left\{\frac{1}{\Phi}\left[\frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}-1\right]^2}{\frac{2b\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}}-\rho\right]t\right\} \tag{23}$$

Here the expression $\Gamma(\Theta) = A\ u(\Theta) - i(\Theta) - \phi(i(\Theta)) - m(\delta(\Theta), u(\Theta)) > 0$ is time independent and represents the ratio $\frac{c(t,\Theta)}{k(t,\Theta)}$ which will be constant along the non-explosive solution trajectory. Moreover, given the previous results it is easy to derive the saving rate corresponding to this model. By definition we have $s(t,\Theta) = 1 - \frac{c(t,\Theta)}{y(t,\Theta)}$. Consequently, we get the value:

$$s(\Theta) = \frac{A\ u(\Theta) - \Gamma(\Theta)}{A\ u(\Theta)} = \frac{i(\Theta) + \phi(i(\Theta)) + m(\delta(\Theta), u(\Theta))}{A\ u(\Theta)} \tag{24}$$

Household saving just finances the two kind of expenditures related with the capital accumulation process: gross investment expenditures, including adjustment costs, and capital maintenance expenditures. This variable, by construction, is also time independent. So, as usual in AK models, associated to a constant investment rate we get a constant saving rate and a constant

those corresponding to the impact of A, b and d on the depreciation and utilization rates could not be analytically proved when $\Phi > 1$. Nevertheless, for such a case we have numerically checked the signs of the partial derivatives for a wide range of parameter values and alternative calibrations.

consumption-capital ratio. However, given the endogeneity of both the depreciation rate and the utilization rate, their dependence on parameters become more rich and complex in this model than in others.

The previous solution trajectories allow us to identify a balanced growth path. Thus, given the complete closed solution for each of the involved variables, it is easy to conclude about the growth rates in our economy.

$$\gamma_i = \gamma_\delta = \gamma_u = 0 \tag{25}$$

$$\gamma_k = \gamma_y = \gamma_c = \gamma = \frac{1}{\Phi}\left[H(i) - \rho\right] = \frac{1}{\Phi}\left[\frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}} - 1\right]^2}{\frac{2b\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}} - \rho\right] \tag{26}$$

The latter constant growth rate depends on the structural parameters in the following way: i) the greater the efficiency level for effectively used capital A, the higher the growth rate; ii) the higher the weight of installation and maintenance costs in gross product, represented by parameters b and d respectively, the lower the rate of growth. But also, iii) the greater the patience of economic agents, that is, the higher the intertemporal elasticity of substitution in consumption $\sigma = \frac{1}{\Phi}$ and/or the lower the rate of discount ρ, the higher the rate of growth. Finally, we find that: iv) the economy's growth rate does not depend on the population growth rate.

Looking at the relationship between the economy's growth rate and the set of macroeconomic variables including the investment rate, the depreciation rate and the rate of capital utilization, we identify a strong positive correlation between the first one and each of the controls. According to the traditional AK model, the growth rate appears positively related to investment but negatively to the depreciation rate. In our general endogenous growth model, the growth rate appears positively related to the investment rate but also to the depreciation and capital utilization rates. This result is in accordance to the observed facts concerning growth and depreciation, as pointed out in the introduction.

One of the main features, inherited from standard AK models, is that there is no transitional dynamics. Variables like k, y and c, starting from k_0, $y(0) = Au(\Theta)k_0$ and $c(0) = \Gamma(\Theta)k_0$ respectively, conform a balanced growth path for which (26) always hold. The remaining variables i, δ and u satisfy (25). The growth rate associated with non-stationary variables does not depend on the initial capital stock k_0. The per capita income growth rate is not related to the initial income level nor to any other per capita income level either. The absence of convergence may be illustrated by taking two equally parameterized economies except for their initial capital stocks. In such a case, any initial per capita income difference among the two economies will always be increased over time, never reduced. Nevertheless, things are not as simple as they might seem at first sight. Considering these two economies with different levels of capitalization, the poorest one will experience higher depreciation and capital utilization rates though the same investment rate. Consequently, we cannot decide which economy will initially produce a greater per capita production and which one will initially consume more. Because of the direct influence of the utilization rate, a lower capital being used more intensively may produce more output than otherwise. It could be perfectly possible that the economy with a lower capital stock may produce and consume more than the one with the highest capital stock. Moreover, because of the compensating influence of u and δ on the maintenance costs, the greater per capita consumption will usually appear associated with the greater per capita production, and vice versa. In any case, the absence of convergence will have as consequence the amplification of any initial difference.

Finally, this model accounts for most of the growth facts that were pointed out by Parente and Prescott (1993). To see that, we define the relative per capita income levels for two equally parameterized countries that only differ in their capital endowments $y^a/y^b = k_0^a \ u(\Theta; k_0^a)/k_0^b \ u(\Theta; k_0^b)$, and then observe that this ratio remains constant over time and far from unity because of the difference between initial per capita capital stocks. Consequently, our model may explain the great disparity between rich and poor countries, as well as the constancy of that disparity over time[8]. On the other hand, associated to the nature of our theoretical results, we notice that all countries become somewhat richer, the poorest too. Even so, we must recognize that our model cannot explain the demonstrated ability of some countries to change their positions within the per capita income distribution. Putting this in terms of the absolute levels of income per capita, we can say that countries adopt different growth patterns: some grow steadily, some do not

[8]However, in a recent work, Easterly and Levine (2000) find in data a massive divergence in the absolute levels of income per capita over the last thirty years, caused because the rich grew faster than the poor.

grow for long periods and then suddenly start to grow at high rates, and others that were growing steadily stop for long periods. In short, we cannot explain miracles and disasters.

5 Conclusions

As usually occurs in standard AK models, in our model there is no transitional dynamics. Variables like k, y and c conform a unique balanced growth path from the beginning, while the remaining variables i, δ and u stand at their initial constant values forever. In addition, associated to the constant investment rate we get a constant saving rate and a constant consumption-capital ratio. This is so because the adjustment and maintenance cost functions included here are assumed linear with respect to k. The absence of convergence implies that the per capita income growth rate is not related to the initial income level nor to any other per capita income level. Even though, some parameter dependences have to be pointed out. We find that: i) the bigger the rate of population growth, the greater the depreciation and capital utilization rates, although this parameter does not affect the investment rate nor the rate of growth; ii) there is a negative dependence of both the depreciation rate and the capital utilization rate on initial conditions; iii) the higher the weight of installation and maintenance costs in gross product, the lower the investment, the utilization and the depreciation rates, as well as the rate of growth; iv) the greater the efficiency level for effectively used capital, the higher the investment, the utilization and the depreciation rates, as well as the general growth rate; and v) the greater the patience level of economic agents, the higher the rate of growth but also the greater the depreciation and utilization rates.

Notwithstanding, looking at the relationship between the economy's growth rate and the set of macroeconomic variables including the investment rate, the depreciation rate and the rate of capital utilization, we identify a strong positive correlation between the first one and each of the latter. According to the traditional AK model, the growth rate appears positively related to investment but negatively to the depreciation rate. The capital stock is assumed to be fully used. In our model, the growth rate appears positively related to the investment rate but also to the depreciation and capital utilization rates. This result is in accordance to the observed facts concerning growth and depreciation but contradicts previous theoretical results.

Furthermore, because of the direct influence of the utilization rate, a lower capital being used more intensively may produce more output than otherwise. It could be perfectly possible that one economy with a lower capital stock produces and consumes more than other with a higher capital stock. However, a greater per capita consumption will usually appear associated with a greater per capita production. In any case, the absence of convergence will have as consequence the amplification of any initial difference. Hence, our model may explain the great disparity between rich and poor countries as well as the constancy of that disparity over time, but we cannot explain the experiences of growth which are known as miracles.

6 Appendix I

From the first order conditions (4)-(6), the resource constraint (2) and the production function (1) we define implicitly the following control functions: $u = u(k, \mu, \Theta)$, $i = i(k, \mu, \Theta)$, $\delta = \delta(k, \mu, \Theta)$, $c = c(k, \mu, \Theta)$ and $y = y(k, \mu, \Theta)$, where Θ represents a vector including the structural parameters of the model. By total differentiation, the implicit function theorem allow us to identify the following partial effects:

$$
\begin{array}{rcl}
u_k = \frac{-m_{\delta u}}{m_{uu}} \frac{Au-i-\phi-m}{m_\delta \; k} < 0 & ; & u_\mu = \frac{-m_{\delta u}}{m_{uu}} \frac{1}{(m_\delta)^2 k \; U''} < 0 \\
i_k = 0 & ; & i_\mu = 0 \\
\delta_k = \frac{Au-i-\phi-m}{m_\delta \; k} < 0 & ; & \delta_\mu = \frac{1}{(m_\delta)^2 k \; U''} < 0 \\
c_k = 0 & ; & c_\mu = \frac{-1}{m_\delta \; U''} < 0 \\
y_k = \frac{A \; m_{\delta u}(i+\phi)}{m_\delta \; m_{uu}} > 0 & ; & y_\mu = \frac{-A \; m_{\delta u}}{m_{uu}(m_\delta)^2 U''} < 0
\end{array}
$$

These results show some interesting features of the model. First, assuming that capital stock and its shadow price evolve in opposite directions, it is very difficult to decide at first sight the evolution of the variables capital utilization rate and depreciation rate. Second, the investment rate over capital stock remains constant for the given parameter values, implying that gross investment share will move parallel to the capital-output ratio. Finally, per capita consumption evolves inversely proportional to the movement in the shadow price of capital stock, as well as per capita production which, in addition, moves directly proportional to capital stock.

Now, we are going to illustrate the previous statements concerning the control functions by specifying particular forms for each structural function implied in our model. We will consider a CRRA instantaneous utility function $U(c) = \frac{c^{1-\Phi}-1}{1-\Phi}$, where Φ is a non negative constant representing the inverse of intertemporal elasticity of substitution. Per capita production is obtained from a linear technology depending on the effectively used capital stock according to equation (1). Adjustment costs will be represented by a quadratic function as $\phi(i) = \frac{bi^2}{2}$, where b is a positive constant. Maintenance costs are assumed to be represented by the function $m(\delta, u) = d\delta^{-\varepsilon}u^{1+\varepsilon}$, where $\varepsilon > 0$ approaches the elasticity of such average maintenance cost with respect to the depreciation rate and the utilization rate, and d is a positive constant. These particular functions satisfy all the assumed general properties.

Solving the optimization problem and focussing on the control functions for these particular forms, we get the following expressions for investment rate, depreciation rate, and utilization rate:

$$i(\Theta) = \frac{1}{b}\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}} - 1\right] \tag{I.1}$$

$$\delta(k, \mu, \Theta) = \frac{\frac{1}{2b}\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^2 - \frac{1}{2b}}{\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}} + \frac{\mu^{\frac{-1}{\Phi}}k^{-1}}{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^{1-\frac{1}{\Phi}}} \tag{I.2}$$

$$u(k, \mu, \Theta) = \frac{\frac{1}{2b}\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^2 - \frac{1}{2b}}{\varepsilon\left(\frac{A}{1+\varepsilon}\right)} + \frac{\left(\frac{A}{1+\varepsilon}\right)^{\frac{1}{\varepsilon}}\mu^{\frac{-1}{\Phi}}k^{-1}}{d^{\frac{1}{\varepsilon}}\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^{1-\frac{1}{\Phi}}} \tag{I.3}$$

As previously set down, we can see that the investment rate only depends on structural parameters and so, it takes a constant value determined by such parameters as in equation (I.1). Equations (I.2) and (I.3) show the particular expressions for δ and u, and it is easy to see that both depend negatively on the state and costate variables. Moreover, these two variables are linearly and positively related to each other. The particular expressions for consumption and output, in turn, may be immediately derived substituting the previous control functions in the resources constraint (2) and the production function (1). Therefore, we could also check for the sign of their partial derivatives.

7 Appendix II

Consider the first order conditions (4)-(6) and the Euler equation (7) written as follows:

$$\overset{\bullet}{\mu}= \frac{\mu\left\{-f'(k\ u)\ u+\phi\left(i\right)-i\ \phi'\left(i\right)+m\left(\delta,u\right)+\left[1+\phi'\left(i\right)\right]\left(\delta+\rho\right)\right\}}{\left[1+\phi'\left(i\right)\right]} \tag{II.1}$$

where $f'(k\ u) = A$ because of the AK nature of our model. Now, define the function $H(i) = \frac{i\ \phi'(i)-\phi(i)}{1+\phi'(i)}$, which by the assumed convexity on $\phi(i)$ gives positive values for any $i > 0$. This function is a monotonous increasing function given that $H(0) = 0$ and $H'(i) = \frac{i\ \phi''(i)+\phi(i)\ \phi''(i)}{[1+\phi'(i)]^2} > 0$. Moreover, we know that $H(i) - i = -\frac{i+\phi(i)}{1+\phi'(i)} < 0$ for any $i > 0$. Then, equation (II.1) may be rewritten as:

$$\overset{\bullet}{\mu}= \mu\ \left[\rho+\delta+\frac{m(\delta,u)}{1+\phi'(i)}-H(i)-\frac{f'(k\ u)\ u}{1+\phi'(i)}\right] \tag{II.2}$$

In this model, by analogy to the standard models, the term $\frac{f'(ku)u}{1+\phi'(i)}+H(i)-\frac{m(\delta,u)}{1+\phi'(i)}$ is the total gross marginal product of capital. So, we can define the total net marginal product of capital as:

$$r = \frac{f'(k\ u)\ u}{1+\phi'(i)}+H(i)-\frac{m(\delta,u)}{1+\phi'(i)}-\delta \tag{II.3}$$

Given our linear homogeneity assumption on the maintenance cost function, the above expression reduces to $r = H(i)$. Consequently, the Euler equation may be reduced to:

$$\overset{\bullet}{\mu}= -\mu\left[r-\rho\right] = \mu\left[-H\left(i\right)+\rho\right] \tag{II.4}$$

which corresponds to equation (13) in the main text. Moreover, given the definition of $H(i)$, we know that it express in a summarized way the whole marginal effects of investment on the Hamiltonian function, that is to say, the full marginal productivity of investment. Implicitly, this function includes both the higher investment expenditures and the lower adjustment costs due to an increase in capital stock.

An alternative but complementary view of that function may be obtained from the following reorganization of terms:

$$H(i) = U'(c)\ \frac{i}{\mu}\ [\phi'(i) - \frac{\phi(i)}{i}] \tag{II.5}$$

where the term on the right hand side shows the difference between marginal and average adjustment cost, multiplied by the investment rate and divided by the shadow price of capital. This value appears converted into utility units when we multiply by the marginal utility factor.

8 Appendix III

The modified Hamiltonian dynamic system analyzed by Ruiz-Tamarit and Ventura-Marco (2000) in search of a closed form solution, and for which they study existence, uniqueness vs. multiplicity, positivity, transitional dynamics and long-run growth, takes the form:

$$\overset{\bullet}{k}\ (t) = \Delta_k\ k(t) - \Sigma_k\ k(t)^{a_{11}} \mu(t)^{a_{22}} \tag{III.1}$$

$$\overset{\bullet}{\mu}\ (t) = \Delta_\mu\ \mu(t) + \Sigma_\mu\ k(t)^{a_{11}-1} \mu(t)^{1+a_{22}} \tag{III.2}$$

$$k(t_0) = k_0 \tag{III.3}$$

$$\lim_{t \to \infty} \mu(t)\ k(t)\ \exp\{-(\rho - n)(t - t_0)\} = 0 \tag{III.4}$$

The elements $\Delta_k > 0$, $\Delta_\mu \geqslant -\Delta_k$, $\Sigma_k > 0$, $\Sigma_\mu \geqslant 0$, $a_{11} \gtreqless 0$, $a_{22} < 0$, k_0, t_0 and ρ are constant parameters, while k, μ and t are the variables. Moreover, the next general parameter constraints are assumed: $\Sigma_k > \Sigma_\mu$, $1 - a_{11} > 0$ and $1 + a_{22} \gtreqless 0$.

It is easy to show that the above dynamic system simplifies to (14)-(15) under the following specific parameter values: $\Delta_k = H(i) - n$, $\Delta_\mu = \rho - H(i)$, $\Sigma_k = \left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}} \left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{1}{\Phi}-1} > 0$, $\Sigma_\mu = 0$, $a_{11} = 0$ and $a_{22} = \frac{-1}{\Phi} < 0$. From these values we can see that $\Delta_k + \Delta_\mu = \rho - n > 0$, where the right hand term represents the effective intertemporal rate of discount. At the same time, we have $\Delta_k > \rho - n$ provided that $H(i) - \rho > 0$ or, according to what we saw in appendix II, as long as the net marginal product of capital r be always higher than the rate of time preference ρ. This also implies $\Delta_\mu < 0$ as well as $\Delta_k > 0$. Under these conditions the authors apply a method that works in three steps. First, define the instrumental variable $X(t) = k(t)\mu(t)^{\frac{1}{\Phi}}$. By totally differentiating and substituting from equations (III.1) and (III.2) we get:

$$\overset{\bullet}{X}(t) = a_x\, X(t) - b_x \tag{III.5}$$

This is an autonomous non-homogeneous linear differential equation with constant coefficients $a_x = \Delta_k + \frac{\Delta_\mu}{\Phi} = \left(1 - \frac{1}{\Phi}\right) H(i) + \frac{\rho}{\Phi} - n \gtrless 0$ and $b_x = \Sigma_k > 0$. Given the initial condition k_0 and a certain, for the moment unknown, initial value $\mu(t_0)$ which allow us to determine the initial condition $X(t_0) = k_0\mu(t_0)^{\frac{1}{\Phi}}$, any particular solution to (III.5) must be of the form:

$$X(t) = \frac{b_x}{a_x} + \left[X(t_0) - \frac{b_x}{a_x}\right] \exp\left\{a_x\ (t - t_0)\right\} \tag{III.6}$$

Once we know the fixed value of every parameter and the initial ones of the variables, the above expression determines the value for the instrumental variable $X(t)$ at any moment in time. In a second step we transform the initial non-linear system and get the following two separated, non-autonomous but homogeneous, linear differential equations for the primary variables:

$$\overset{\bullet}{k}(t) = \left(\Delta_k - \frac{\Sigma_k}{X(t)}\right) k(t) \tag{III.7}$$

$$\overset{\bullet}{\mu}(t) = \Delta_\mu\ \mu(t) \tag{III.8}$$

The expressions for the particular solutions are respectively:

$$k(t) = k_0\ \exp\left\{\int_{t_0}^{t}\left(H(i) - n - \frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{1}{\Phi}-1}}{X(s)}\right) ds\right\} \tag{III.9}$$

$$\mu(t) = \mu(t_0)\ \exp\left\{-\left(H(i) - \rho\right)\ (t - t_0)\right\} \tag{III.10}$$

The third step consists in determining the initial value of the costate variable $\mu(t)$, for which trajectories are non-explosive. Given k_0 known, this may be done by determining $X(t_0)$. All what is needed in this step can be deduced from the transversality condition. This necessary condition, for the signs of the parameters that we are considering, may be simplified to:

$$\lim_{t\to\infty} \left| \frac{b_x \ \exp\{-a_x(t-t_0)\}}{a_x \ X(t_0)} + 1 - \frac{b_x}{a_x \ X(t_0)} \right| = 0 \tag{III.11}$$

In particular, given that $b_x > 0$, this condition holds if, and only if, both $a_x = \left(1 - \frac{1}{\Phi}\right) H(i) + \frac{\rho}{\Phi} - n > 0$ and $X(t_0) = \frac{b_x}{a_x} = \frac{\left[\frac{\varepsilon}{d^{\frac{1}{\varepsilon}}}\left(\frac{A}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{1}{\Phi}-1}}{\left(1-\frac{1}{\Phi}\right)H(i)+\frac{\rho}{\Phi}-n}$. Coming back to (III.6) we find that the instrumental variable $X(t)$ will remain constant and equal to its initial stationary value $X(t_0)$, $\forall t \geq t_0$. Consequently, the non-explosive solution trajectories for the variables involved in the modified Hamiltonian dynamic system (III.1)-(III.4), are unique and may be written as in equations (16), (17) and (18). Finally, we would like to point out that the constraint $\frac{\Sigma_\mu}{-\Delta_\mu} = 0 < \frac{\Sigma_k}{\Delta_k} = \frac{\Sigma_k}{H(i)-n} < \frac{b_x}{a_x} = \frac{\Sigma_k}{\left(1-\frac{1}{\Phi}\right)H(i)+\frac{\rho}{\Phi}-n}$ also holds, therefore Proposition 4 from Ruiz-Tamarit and Ventura-Marco (2000) applies here, and we conclude that $H(i) > \rho$.

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