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Abstract

This note generalizes analytical relationships among activity variables of DEA models previously derived by Boussemart, Briec and Leleu (2007). We relax the asumption of constant returns to scale by showing that the key results hold under a weaker asumption of homogeneity. We use the notion of α-returns to scale to extend the analysis to strictly increasing and decreasing returns, covering now the whole range of returns to scale for multi-output homogenous technologies.

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Introduction

In this technical note, we generalize analytical relationships among scores and activity variables of DEA models previously derived by Boussemart, Briec and Leleu (2007) [BBL (2007) in the remainder of the text]. The results were established under a constant returns to scale (CRS) assumption that we relax here. More formally, we show that our previous results do not necessitate the CRS assumption but a weaker assumption of homogeneity. We can therefore consider other kinds of returns to scale. Following Boussemart et al. (2008) we refer to the notion of α -returns to scale to deal with strictly increasing and decreasing returns for multi-output homogeneous technologies. As in BBL (2007) we analyze the relationships among optimal solutions of the input-based, the output-based, the hyperbolic, and the proportional distance functions.

1. Background

1.1. Production Technology: Definition and Assumptions

The production technology transforms inputs $x=(x_1,\cdots,x_n)\in R_+^n$ into outputs $y=(y_1,\cdots,y_p)\in R_+^p$ under the technology T:

$$T=\left\{(x,y)\in R_+^{n+p} : x \text{ can produce } y\right\} \tag{1}$$

We suppose that the technology obeys the following axioms:

- T1: $(0,0)\in T$, $(0,y)\in T \Rightarrow y=0$ i.e., no free lunch;
- T2: the set $A(x)=\left\{(u,y)\in T : u\leq x\right\}$ of dominating observations is bounded $\forall x\in R_+^n$, i.e., infinite outputs cannot be obtained from a finite input vector;
- T3: T is closed;
- T4: For all $(x,y)\in T$, and all $(u,v)\in R_+^{n+p}$, we have $(x,-y)\leq(u,-v)\Rightarrow(u,v)\in T$ (free disposability of inputs and outputs).

1.2. α-Returns to Scale Technologies and Distance Functions

We first define the distance functions. The input and output Farrell measures are defined respectively by $E^I(x,y) = \inf_{\theta^I}\left\{\theta^I \geq 0 : (\theta^I x, y) \in T\right\}$ and by $E^O(x,y) = \sup_{\theta^O}\left\{\theta^O \geq 0 : (x, \theta^O y) \in T\right\}$. It is also possible to define graph technical efficiency measures. The hyperbolic measure is defined by $E^H(x,y) = \inf_{\theta^H}\left\{\theta^H \geq 0 : (\theta^H x, \frac{1}{\theta^H} y) \in T\right\}$ and the proportional measure is given by: $E^P(x,y) = \sup_{\theta^P}\left\{\theta^P \geq 0 : \left((1-\theta^P)x, (1+\theta^P)y\right) \in T\right\}$.

In BBL (2007) these distance functions were considered under a constant returns to scale technology. We extend the class of technologies to homogeneous multi-ouput technologies. A production technology T is said to be *homogeneous of degree* α if for all $\beta > 0$:

$$(x,y) \in T \Rightarrow (\beta x, \beta^\alpha y) \in T. \tag{2}$$

Lau (1978) termed these technologies "almost homogeneous technologies of degree 1 and α" for all $\beta > 0$. A complete characterization is given by Färe and Mitchell (1993). Obviously, CRS corresponds to $\alpha = 1$ while strictly increasing returns corresponds to $\alpha > 1$ and strictly decreasing returns corresponds to $\alpha < 1$. Boussemart et. al. (2008) termed this property of the technology α*-returns to scale*. Under such an assumption they established the following equalities linking both input, output and graph measures:

$$E^O(x,y) = \left[E^I(x,y)\right]^{-\alpha}, \tag{3}$$

$$E^H(x,y) = \left[E^I(x,y)\right]^{\frac{\alpha}{\alpha+1}} \quad \text{and} \quad E^H(x,y) = \left[E^O(x,y)\right]^{-\frac{1}{\alpha+1}} \tag{4}$$

We first extend these relationships to the proportional distance function.

Lemma 1.1. Suppose the technology T satisfies T1-T4. Under an assumption of α-returns to scale:

$$E^I(x,y) = \frac{1 - E^P(x,y)}{\left[1 + E^P(x,y)\right]^{1/\alpha}} \tag{5}$$

Proof: Since α-returns to scale hold, $\forall (x,y) \in T$, $\beta \geq 0 \Rightarrow (\beta x, \beta^\alpha y) \in T$. This implies that

$(\beta E^I(x,y)x, \beta^\alpha y) \in T$. Since the projected vector $(E^I(x,y).x, y)$ is a frontier point in T that achieves the Debreu-Farrell efficiency measure and since:

$$\left([1 - E^P(x,y)]x, [1 + E^P(x,y)]y\right) \tag{6}$$

achieves the proportional distance function, we need to find some $\beta > 0$ that satisfies the relationship:

$$(\beta E^I(x,y)x, \beta^\alpha y) = \left([1 - E^P(x,y)]x, [1 + E^P(x,y)]y\right) \tag{7}$$

Then, we deduce both the following equalities:

$$\beta E^I(x,y) = (1 - E^P(x,y)) \tag{8}$$

and

$$\beta^\alpha = (1 + E^P(x,y)) \tag{9}$$

Dividing equation 8 by equation 9 yields: $E^I(x,y) = \frac{1 - E^P(x,y)}{\left[1 + E^P(x,y)\right]^{1/\alpha}}$.

All these relationships are important because they show that, under an assumption of α-returns to scale, most of the existing measures can be expressed in term of the Farrell input measure of technical efficiency.

2. α-Returns to Scale Technologies and DEA Models

We further propose a nonparametric model of production technologies for which the four distance functions can be calculated by solving DEA models. Let us consider a set of J firms $A = \{(x_1, y_1), ..., (x_J, y_J)\} \in R_+^{n+p}$. We denote $J = \{1, \cdots, J\}$. The production technology can be estimated by enveloping observed firms. Under this DEA framework, the production set for constant returns to scale is defined as:

$$T_{CRS} = \left\{ (x,y) \in R_+^{n+p} : x \geq \sum_{j \in J} \lambda_j x_j, y \leq \sum_{j \in J} \lambda_j y_j, \lambda \geq 0 \right\} \tag{10}$$

We also use a more general $CES - CET$ model introduced by Färe, Grosskopf and Njinkeu (1988) and adapted by Boussemart et. al. (2008) to α-returns to scale. It consists in two parts: the output part which is characterized by a *Constant Elasticity of Transformation* formula and the input part which is characterized by a *Constant Elasticity of Substitution* formula. Formally, we consider the map $\Phi_r : R_+^m \to R_+^m$ defined as $\Phi_r(z) = (z_1^r, \cdots, z_m^r)$. For all $r > 0$, this function is an isomorphism from R_+^m to itself and its reciprocal is defined on R_+^m as: $\Phi_r^{-1}(z) = (z_1^{1/r}, \cdots, z_m^{1/r})$. To simplify the technical exposition we will denote in the remainder of the paper:

$$\Phi_r(z) = z^r, \tag{11}$$

for all $r > 0$. Let us consider the following set:

$$T_{\gamma,\delta} = \{(x, y) : x \geq (\sum_{j\in J} \lambda_j x_j^{\gamma})^{1/\gamma}, y \leq (\sum_{j\in J} \lambda_j y_j^{\delta})^{1/\delta}, \lambda \geq 0\} \tag{12}$$

$T_{\gamma,\delta}$ satisfies T1-T4. It is obvious to see that $T_{CRS} = T_{1,1}$.

We have by definition of the distance functions considered here:

$$E^I(x_k, y_k) = \min_{\theta^I, \lambda\geq 0} \{\theta^I : \theta^I x_k \geq (\sum_{j\in J} \lambda_j x_j^{\gamma})^{\frac{1}{\gamma}}, y_k \leq (\sum_{j\in J} \lambda_j y_j^{\delta})^{\frac{1}{\delta}}\} \tag{13}$$

$$E^O(x_k, y_k) = \max_{\theta^O, \lambda\geq 0} \{\theta^O : x_k \geq (\sum_{j\in J} \lambda_j x_j^{\gamma})^{\frac{1}{\gamma}}, \theta^O y_k \leq (\sum_{j\in J} \lambda_j y_j^{\delta})^{\frac{1}{\delta}}\} \tag{14}$$

$$E^H(x_k, y_k) = \min_{\theta^H, \lambda\geq 0} \{\theta^H : \theta^H x_k \geq (\sum_{j\in J} \lambda_j x_j^{\gamma})^{\frac{1}{\gamma}}, \frac{1}{\theta^H} y_k \leq (\sum_{j\in J} \lambda_j y_j^{\delta})^{\frac{1}{\delta}}\} \tag{15}$$

$$\begin{aligned} E^P(x_k, y_k) = \max_{\theta^P, \lambda\geq 0} \{\theta^P : (1-\theta^P) x_k \geq (\sum_{j\in J} \lambda_j x_j^{\gamma})^{\frac{1}{\gamma}}, \\ (1+\theta^P) y_k \leq (\sum_{j\in J} \lambda_j y_j^{\delta})^{\frac{1}{\delta}}\} \end{aligned} \tag{16}$$

3. Main Results

Let us denote by, P^I, P^O, P^H and P^P the mathematical programs defined in equations (13), (14), (15), and (16) respectively. Let $\Lambda_k^I, \Lambda_k^O, \Lambda_k^H$ and Λ_k^P the optimal values of λ in programs. These sets may not be singletons when there exist multiple solutions, i.e. in case of degeneracy of the system of linear inequalities. From P^I, we have for all observed production vector k and all $\lambda_*^I \in \Lambda_k^I$:

$$([E^I(x_k, y_k)] \cdot x_k, y_k) = ((\sum_{j\in J} \lambda_{*,j}^I x_j^{\gamma})^{\frac{1}{\gamma}}, (\sum_{j\in J} \lambda_{*,j}^I y_j^{\delta})^{\frac{1}{\delta}}) + (s, -t) \tag{17}$$

where $(s,t)\in R_{+}^{n+p}$ is a nonnegative input-output slack vector.

Let us consider $\alpha=\gamma/\delta$. First we establish a relationship between the optimal solutions of the programs computing the Farrell input and output measures respectively. Since $\left(E^{I}(x_k,y_k)\cdot x_k,y_k\right)\in T_{\gamma,\delta}$ and since $T_{\gamma,\delta}$ satisfies α-returns to scale, it follows from 2 that:

$$([E^{O}(x_k,y_k)]^{1/\alpha}\cdot E^{I}(x_k,y_k)\cdot x_k,[E^{O}(x_k,y_k)]\cdot y_k)\in T_{\gamma,\delta} \tag{18}$$

Now from the relationship on distance functions under α-returns to scale: $E^{O}(x,y)=\left[E^{I}(x,y)\right]^{-\alpha}$, it follows that:

$$\left(x_k,[E^{O}(x_k,y_k)]\cdot y_k\right)\in T_{\gamma,\delta} \tag{19}$$

Combining equations 17 and 19 we obtain:

$$\begin{aligned}(x_k,[E^{O}(x_k,y_k)]\cdot y_k)&=((\sum_{j\in J}[E^{O}(x_k,y_k)]^{\delta}\cdot\lambda_{*,j}^{I}x_j^{\gamma})^{\frac{1}{\gamma}},\\ &(\sum_{j\in J}[E^{O}(x_k,y_k)]^{\delta}\cdot\lambda_{*,j}^{I}y_j^{\delta})^{\frac{1}{\delta}})+(s',-t').\end{aligned} \tag{20}$$

Where $(s',-t')=([E^{O}(x_k,y_k)]^{1/\alpha}\cdot s,-[E^{O}(x_k,y_k))]\cdot t)\geq 0$. Consequently, $[E^{O}(x_k,y_k)]^{\delta}\cdot\lambda_{*}^{I}\in\Lambda_k^{O}$ and we can now deduce that:

$$[E^{O}(x_k,y_k)]^{\delta}\cdot\Lambda_k^{I}\subset\Lambda_k^{O} \tag{21}$$

Using a similar procedure if $\lambda_{*}^{O}\in\Lambda_k^{O}$ then there is an input-output slack $(s,-t)\geq 0$ such that:

$$(E^{I}(x_k,y_k)\cdot x_k,y_k)=(E^{I}(x_k,y_k)\cdot x_k,[E^{I}(x_k,y_k)]^{\alpha}[E^{O}(x_k,y_k)]\cdot y_k) \tag{22}$$

$$=((\sum_{j\in J}[E^{I}(x_k,y_k)]^{\gamma}\cdot\lambda_{*,j}^{O}x_j^{\gamma})^{\frac{1}{\gamma}},(\sum_{j\in J}[E^{I}(x_k,y_k)]^{\gamma}\cdot\lambda_{*,j}^{O}y_j^{\delta})^{\frac{1}{\delta}})+(s,-t). \tag{23}$$

It follows from 23 that:

$$[E^{I}(x_k,y_k)]^{\gamma}\cdot\Lambda_k^{O}\subset\Lambda_k^{I} \tag{24}$$

Since $[E^{I}(x_k,y_k)]^{-\gamma}=[E^{O}(x_k,y_k)]^{\delta}$, we deduce that $\Lambda_k^{O}\subset[E^{O}(x_k,y_k)]^{\delta}\cdot\Lambda_k^{I}$ which yields the following result.

Lemma 3.1. For all $k\in J$, we have $[E^{O}(x_k,y_k)]^{\delta}\cdot\Lambda_k^{I}=\Lambda_k^{O}$, with $\delta=\gamma/\alpha$.

Second let us establish a relationship between the optimal solutions of the programs computing the hyperbolic measure and the Farrell input measure. Since $\left(E^{I}(x_k,y_k)\cdot x_k,y_k\right)\in T_{\gamma,\delta}$ and since $T_{\gamma,\delta}$ satisfies α-returns to scale, it follows from 2 that:

$$([E^H(x_k,y_k)]^{-\frac{1}{\alpha}} E^I(x_k,\cdot y_k)\cdot x_k,[E^H(x_k,y_k)]^{-1}\cdot y_k)\in T_{\gamma,\delta} \tag{25}$$

Now from the relationship on distance functions under α-returns to scale: $E^H(x,y)=\left[E^I(x,y)\right]^{\frac{\alpha}{\alpha+1}}$ and it follows that:

$$([E^H(x_k,y_k)]\cdot x_k,[E^H(x_k,y_k)]^{-1}\cdot y_k)\in T_{\gamma,\delta} \tag{26}$$

Combining equations 17 and 26 we obtain:

$$([E^H(x_k,y_k)]\cdot x_k,[E^H(x_k,y_k)]^{-1}\cdot y_k) \tag{27}$$

$$=((\sum_{j\in J}[E^H(x_k,y_k)]^{-\delta}\cdot\lambda^I_{*,j}x_j^{\gamma})^{\frac{1}{\gamma}},(\sum_{j\in J}[E^H(x_k,y_k)]^{-\delta}\cdot\lambda^I_{*,j}y_j^{\delta})^{\frac{1}{\delta}})+(s',-t').$$

where $(s',-t')=([E^H(x_k,y_k)]^{-\frac{1}{\alpha}}\cdot s,[E^H(x_k,y_k)]^{-1}\cdot t)\geq 0$. Consequently, we obtain $[E^H(x_k,y_k)]^{-\delta}\cdot\lambda^I_*\in\Lambda^H_k$ and we can now deduce that:

$$[E^H(x_k,y_k)]^{-\delta}\cdot\Lambda^I_k\subset\Lambda^H_k \tag{28}$$

Using a similar procedure if $\lambda^H_*\in\Lambda^H_k$ then there is an input-output slack $(s,-t)\geq 0$ such that:

$$(E^I(x_k,y_k)\cdot x_k,y_k) \tag{29}$$

$$=([E^H(x_k,y_k)]^{\frac{1}{\alpha}}\cdot E^H(x_k,y_k)\cdot x_k,E^H(x_k,y_k)\cdot[E^H(x_k,y_k)]^{-1}\cdot y_k) \tag{30}$$

$$=((\sum_{j\in J}[E^H(x_k,y_k)]^{\delta}\cdot\lambda^H_{*,j}x_j^{\gamma})^{\frac{1}{\gamma}},(\sum_{j\in J}[E^H(x_k,y_k)]^{\delta}\cdot\lambda^H_{*,j}y_j^{\delta})^{\frac{1}{\delta}})+(s,-t). \tag{31}$$

It follows that:

$$[E^H(x_k,y_k)]^{\delta}\cdot\Lambda^H_k\subset\Lambda^I_k. \tag{32}$$

This implies that $\Lambda^H_k\subset[E^H(x_k,y_k)]^{-\delta}\cdot\Lambda^I_k$ yields the following result.

Lemma 3.2. For all $k\in J$, we have $\Lambda^H_k=[E^H(x_k,y_k)]^{-\delta}\cdot\Lambda^I_k$, with $\delta=\gamma/\alpha$.

Finally, we establish the link between the solutions of the programs computing Farrell and proportional distance functions. Since $\left(E^I(x_k,y_k)\cdot x_k,y_k\right)\in T_{\gamma,\delta}$ and since $T_{\gamma,\delta}$ satisfies α-returns to scale, it follows from 2 that:

$$(\left[1+E^P(x,y)\right]^{1/\alpha}E^I(x_k,\cdot y_k)\cdot x_k,\left[1+E^P(x,y)\right]\cdot y_k)\in T_{\gamma,\delta} \tag{33}$$

Now from Lemma 1: $E^I(x,y)=\frac{1-E^P(x,y)}{\left[1+E^P(x,y)\right]^{1/\alpha}}$ and it follows that:

$$(\left[1-E^P(x,y)\right]\cdot x_k,\left[1+E^P(x,y)\right]\cdot y_k)\in T_{\gamma,\delta} \tag{34}$$

Combining equations 17 and 34 we obtain:

$$
\begin{aligned}
&(\left[1-E^{P}(x,y)\right]\cdot x_k,\left[1+E^{P}(x,y)\right]\cdot y_k)\\
&=((\sum_{j\in J}\left[1+E^{P}(x,y)\right]^{\delta}\cdot\lambda^{I}_{*,j}x_j^{\gamma})^{\frac{1}{\gamma}},(\sum_{j\in J}\left[1+E^{P}(x,y)\right]^{\delta}\cdot\lambda^{I}_{*,j}y_j^{\delta})^{\frac{1}{\delta}})+(s',-t').
\end{aligned}
\tag{35}
$$

where $(s',-t')=(\left[1+E^{P}(x,y)\right]^{1/\alpha}\cdot s,\left[1+E^{P}(x,y)\right]\cdot t)\geq 0$. Consequently, we obtain $[1+E^{P}(x_k,y_k)]^{\delta}\cdot\lambda^{I}_{*}\in\Lambda^{P}_{k}$ and we can now deduce that:

$$\left[1+E^{P}(x,y)\right]^{\delta}\cdot\Lambda^{I}_{k}\subset\Lambda^{P}_{k} \tag{36}$$

Using a similar procedure if $\lambda^{P}_{*}\in\Lambda^{P}_{k}$ then there is an input-output slack $(s,-t)\geq 0$ such that:

$$(E^{I}(x_k,y_k)\cdot x_k,y_k) \tag{37}$$

$$=([1+E^{P}(x_k,y_k)]^{-\frac{1}{\alpha}}\cdot[1+E^{P}(x_k,y_k)]\cdot x_k,[1+E^{P}(x_k,y_k)]^{-1}\cdot[1+E^{P}(x_k,y_k)]\cdot y_k) \tag{38}$$

$$=((\sum_{j\in J}[1+E^{P}(x_k,y_k)]^{-\delta}\cdot\lambda^{P}_{*,j}x_j^{\gamma})^{\frac{1}{\gamma}},(\sum_{j\in J}[1+E^{P}(x_k,y_k)]^{-\delta}\cdot\lambda^{P}_{*,j}y_j^{\delta})^{\frac{1}{\delta}})+(s,-t). \tag{39}$$

It follows that:

$$[1+E^{P}(x_k,y_k)]^{-\delta}\cdot\Lambda^{P}_{k}\subset\Lambda^{I}_{k}. \tag{40}$$

This implies that $\Lambda^{P}_{k}\subset[1+E^{P}(x_k,y_k)]^{\delta}\cdot\Lambda^{I}_{k}$ yields the following result.

Lemma 3.3. For all $k\in J$, we have $\Lambda^{P}_{k}=[1+E^{P}(x_k,y_k)]^{\delta}\cdot\Lambda^{I}_{k}$, with $\delta=\gamma/\alpha$.

By letting $\delta=\gamma=1$, Lemmas 3.1 to 3.3 reduce to the results found in BBL (2007) for constant returns to scale technologies.

Conclusion

This note formally establishes the relationships among the input-based, the output-based, the hyperbolic, and the proportional distance functions as well as their relative DEA optimal solutions for multi-output homogeneous technologies. In BBL (2007), the assumption of constant returns to scale was considered as the key assumption to allow some links among the various DEA models. For practitioners, it could have been seen like a weakness since CRS is often judged as a restrictive assumption for empirical works. The main contribution of this generalization is to show that previous results were not specific to constant returns but rely on a weaker assumption of homogeneity.

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