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Tangency Capacity Notions Based upon the Profit and Cost Functions: A Non-Parametric Approach and a Comparison

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Abstract

This contribution provides a way to define and compute a tangency notion of economic capacity based upon the relation between the various directional distance functions and the profit and cost functions using non-parametric technologies. A new result relating profit and cost function-based tangency capacity notions is established.

Keywords: economic capacity, profit function, cost function, directional distance function, tangency.

JEL: C61, D24.

1 Introduction

Analyzing efficiency and productivity using frontier specifications of technology has become a standard empirical tool serving both regulatory and managerial purposes. However, this literature has almost ignored integrating the important notion of capacity utilization. As a consequence, part of what may be attributed to inefficiency may in fact be due to the short run fixity of certain inputs, depending on the exact definition of capacity utilization.

Capacity utilization of fixed inputs is of managerial and policy relevance at various levels of aggregation and in all economic sectors (agriculture, industry, and both private and public

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services). Traditionally, capacity utilization is employed as a leading macro-economic indicator to forecast inflationary pressures (e.g., Christiano (1981)). At the industry level, it is important to account for variable capacity utilization when measuring productivity growth (Luh and Stefanou (1991)). Capacity management has recently become an important issue in fisheries due to declining stocks of this common pool resource. For instance, various capacity measures have been used to evaluate vessel decommissioning schemes (e.g., Walden, Kirkley and Kitts (2003)). Governments worldwide must try to determine sustainable capacity levels and implement a variety of policy measures (e.g., licenses, fishing day restrictions, etc.) to curb overfishing. This has led to the development of short-run industry models based on vessel capacity estimates to plan the industry (e.g., Lindebo (2005)).

At the firm level, fluctuations in capacity utilization of investments are determined by relative factor prices, demand fluctuations, the lumpiness of certain investments, leadtimes, strategic issues, etc. A rather well-documented strategic use of excess capacities is entry deterrence in imperfectly competitive markets. However, the relative importance of precautionary and strategic uses of excess capacity vary significantly across industries (e.g., Driver (2000)). The literature on strategic capacity management develops pragmatic models (e.g., drawing upon, amongst others, the aggregate planning, the inventory and supply chain management, and the economics literatures) to determine the size, type and timing of capacity adjustment under uncertainty (see the survey in Van Mieghem (2003)). An example of a recent question of great managerial interest is the impact of IT capital goods on capacity utilization in complex supply, production and distribution systems coordinated by control systems (Nightingale et al. (2003)).

In brief, the measurement of capacity and its utilization is important for both managers and policy makers in all sectors of the economy and ideally one would like to have methods of capacity measurement available that are sufficiently general to be applicable to agriculture, manufacturing and services. However, different notions of capacity exist in the economic and managerial literature (see Christiano (1981) or Johansen (1968)). Specifically, it is customary to distinguish between technical (engineering) and economic (mostly defined in terms of cost) capacity concepts. Johansen (1968) pursued a technical approach focusing on a plant capacity notion, defined as the maximal amount that can be produced per unit of time with existing plant and equipment without restrictions on the availability of variable production factors. This definition is clearly in line with the engineering perspective and has been translated into a production frontier framework using output-based efficiency measures by Färe, Grosskopf and

Kokkelenberg (1989).

Traditionally, there are three basic ways of defining a cost-based notion of capacity (see, e.g., Nelson (1989)). The purpose of each of these notions is to isolate the short run excessive or inadequate utilization of fixed inputs. The first notion of potential output is defined in terms of the output produced at short run minimum average total cost, given existing plant and factor prices (e.g., Hickman (1964)), and stresses the need to exploit short run scale economies. The second definition of potential output is defined in terms of the output produced at long run minimum average total cost (see Cassels (1937), among others). However, it has been little used, probably because it is clearly heavily intertwined with the notion of scale economies. The third definition follows, among others, Klein (1960) and Segerson and Squires (1990) and corresponds to the output at which short and long run average total costs curves are tangent. Since this point is also the intersection of short and long run expansion paths, this notion has a strong theoretical appeal.

To implement these cost-based notions of capacity utilization on non-parametric, deterministic frontier technologies, we summarize the possibilities.[1] First, estimation of the short run minimum average total cost amounts to solving a variable cost function relative to a constant returns to scale technology (see Prior (2003)). Second, long run minimum average total cost is easily determined by computing a total cost function relative to a constant returns to scale technology (e.g., Hackman (2008) or Ray (2004)). Third, assuming inputs are fixed and cannot be changed, but outputs are adjustable such that installed capacity is utilized ex post at a tangency level we are unaware of any non-parametric method to solve for the tangency capacity notion. Therefore, the first goal of this paper is the development of a tangency capacity notion using non-parametric frontier methodologies which allows for any eventual inefficiencies.[2]

Apart from economic capacity measures defined using the cost function, there also exist capacity notions using other economic objective functions. The case of revenue functions has been treated in Segerson and Squires (1995), while the case of profit functions has been handled in Squires (1987), among others. Furthermore, Coelli, Grifell-Tatjé and Perelman (2002) define an alternative ray economic capacity measure using non-parametric frontiers that involves short-run profit maximization whereby the output mix is held constant. While this notion may have

[1] For the options using traditional parametric specifications: see Nelson (1989). For a parametric frontier application, see, e.g., Rodríguez-Álvarez and Lovell (2004).

[2] It is important to underline the issue of extrapolating the functions implied in some of these different concepts beyond the data domain. However, this is a problem for parametric and non-parametric estimation methods alike, even when model flexibility is allowed for.

some appeal, it does normally not coincide with the tangency capacity notion defined above.[3] Therefore, to cover the most general economic objective function first, this paper defines a tangency economic capacity notion based on the profit function. The case of defining a tangency point of economic capacity for the cost function is completely analogous.[4] The second goal of this contribution is then to establish a relation between profit and cost function based tangency capacity notions. Such relation is -to the best of our knowledge- new in the production literature.

The next section defines the axioms imposed on technology, the directional distance function as a representation of technology and its dual, the profit function, as well as the basic efficiency decomposition and the ensuing new definition of a tangency economic capacity notion based on the profit function allowing for inefficiencies. A similar analysis is also developed for the case of the cost function. Section 3 develops a computational procedure to obtain the profit and cost function based tangency points using non-parametric, deterministic frontier technologies that impose either constant or variable returns to scale. The fourth section focuses on establishing a new relation between the set of profit and cost tangency points. The paper ends with some concluding comments.

2 Technology and Distance Functions: Definitions, Efficiency Decomposition, and Tangency Point

This section introduces the necessary definitions of the production possibility set, the distance functions and the profit functions. To be more specific, first we define the axioms underlying technology. Then, we define both the long and short run directional distance functions characterizing technology and their corresponding long and short run profit functions. Thereafter, we develop the corresponding efficiency decompositions. This allows us to end up with a novel way of characterizing profit tangency points in terms of the short and long run allocative efficiency components. This generalizes existing approaches defining profit capacity that ignore the possibility of inefficiencies. The case for the cost function is developed in a similar way.

[3] This proposal can be attractive in case the output mix should remain (more or less) fixed, which may be relevant in certain (mainly industrial) production processes. A more profound analysis as to the relative merits of this new proposal compared to more traditional definitions is being called for.

[4] Of course, the analytical generality of this analysis must be distinguished from the differences in economic interpretation implied by adopting a profit or a cost perspective.

2.1 Technology

Production technology transforms inputs $x = (x_1, ..., x_n) \in \mathbb{R}^n_+$ into outputs $y = (y_1, ..., y_m) \in \mathbb{R}^m_+$. The set of all feasible input and output vectors is called the production possibility set T and is defined as follows:

$$T = \left\{(x, y) \in \mathbb{R}^{n+m}_+ : x \text{ can produce } y\right\}. \tag{2.1}$$

It satisfies the following standard assumptions (e.g, Hackman (2008), Shephard (1970)): (T.1) $(0, 0) \in T,\ (0, y) \in T \Rightarrow\ y = 0$, i.e., no free lunch; (T.2) the set $A(x) = \{(u, y) \in T :\ u \leqslant\ x\}$ of dominating observations is bounded $\forall x \in \mathbb{R}^n_+$, i.e., infinite outputs cannot be obtained from a finite input vector; (T.3) T is a closed set; and (T.4) $\forall(x, y) \in T$, $(u, v) \in \mathbb{R}^{n+m}_+$ and $(x, -y) \leqslant (u, -v) \Rightarrow (u, v) \in\ T$, i.e., strong input and output disposability; and (T.5) T is convex.

The estimation of efficiency relative to production frontiers relies on the theory of distance or gauge functions. In economics, Shephard (1970) distance functions are related to the efficiency measures introduced by Farrell (1957). The input distance function is dual to the cost function, while the output distance function is dual to the revenue function (e.g., Shephard (1970)).

2.2 Short Run Profit Function and Duality

We now discuss the recently introduced directional distance function $D : \mathbb{R}^{n+m}_+ \times \mathbb{R}^{n+m}_+ \to \mathbb{R} \cup \{-\infty, \infty\}$ that involves simultaneous input and output variations:

$$D(x, y; h, k) = \sup_{\delta \in \mathbb{R}} \{\delta :\ (x - \delta h, y + \delta k) \in T\}. \tag{2.2}$$

If there is no δ such that $(x - \delta h, y + \delta k) \in T$ then, by definition, $D(x, y; h, k) = -\infty$. The scalar δ attempts to contract the input vector x and to expand the output vector y in the direction of the vector h respectively k. It is a special case of the shortage function (Luenberger (1995)) and is closely related to the Farrell proportional distance (Briec (1997)), a generalization of the Debreu-Farrell measure. Input and output distance functions also appear as special cases (see Chambers, Chung and Färe (1998)).[5] We denote the general directional vector $g = (h, k)$.

[5]Slightly different generalizations of the Shephard distance functions that are equally related to the profit function have been defined in, e.g., Chavas and Cox (1999) or McFadden (1978). In principle, our analysis could equally be transposed to capacity utilization measures based on the latter distance functions.

This directional distance function is dual to the profit function (Luenberger (1995)) and therefore offers a general framework for economic analysis. This function has proven to be a useful tool in micro-economic theory (for instance, it allows Chavas and Kim (2007) to shed new light on economies of scope from a primal viewpoint).

We also need a short-run version of this directional distance function that involves simultaneous proportional variable input and output variations for a given sub-vector of fixed inputs. Therefore, we assume that the input set can be partitioned into two subsets $I_v = \{1, ..., n_v\}$ and $I_f = \{n_v + 1, ..., n\}$. I_v stands for the set of the variable inputs and I_f represents the set of fixed inputs. Obviously, we have $\{1, ..., n\} = I_v \cup I_f$. We adopt a similar decomposition of the output set. Therefore, we assume that $J_v = \{1, ..., m_v\}$ and $J_f = \{m_v + 1, ..., m\}$. J_v stands for the set of the variable outputs and J_f represents the set of fixed outputs and thus we have $\{1, ..., m\} = J_v \cup J_f$. Fixed outputs, that in the short run prevent adjusting the output mix to its profit maximizing level, may occur in case of an exclusive contract to deliver a certain amount of a sub-vector of outputs. Moreover, we assume that the inputs and the outputs are ranged such that each input-output vector is denoted $(x, y) = (x^v, x^f, y^v, y^f)$. Similarly, the short run direction vector is denoted $g = (h^v, h^f, k^v, k^f)$. The short-run directional distance function is then defined as:

$$SRD(x, y, g) = \sup_{\delta \in \mathbb{R}} \left\{ \delta : \ (x^v - \delta h^v, x^f, y^v + \delta k^v, y^f) \in T \right\} \tag{2.3}$$

$$= D(x^v, x^f, y^v, y^f; h^v, 0, k^v, 0). \tag{2.4}$$

The next element needed for our analysis is the standard long-run profit function, which can be defined as follows:

$$\Pi(w, p) = \sup_{x,y} \{p.y - w.x : (x, y) \in T\} \tag{2.5}$$

$$= \sup_{x,y} \{p.y - w.x : D(x, y; h, k) \geq 0\}. \tag{2.6}$$

Luenberger (1995) and Chambers, Chung and Färe (1998) show duality between the directional distance function and the standard long-run profit function. We have:

$$D(x, y; h, k) = \inf_{w,p} \left\{ \frac{\Pi(w, p) - (p.y - w.x)}{pk + wh} : pk + wh \neq 0 \right\}. \tag{2.7}$$

Chambers, Chung and Färe (1998) first define the overall efficiency (OE) index as the quantity:

$$OE\left(x,y,p,w\right)=\frac{\Pi\left(p,w\right)-\left(p.y-w.x\right)}{p.k+w.h}. \tag{2.8}$$

Then, they continue by characterizing a technical efficiency (TE) index as the quantity:

$$TE(x,y)=D(x,y;g). \tag{2.9}$$

Finally, the allocative efficiency (AE) index is defined as the quantity:

$$AE\left(x,y,p,w\right)=OE\left(x,y,p,w\right)-TE(x,y). \tag{2.10}$$

Obviously, the following additive decomposition identity holds:

$$OE(x,y,w,p)=AE(x,y,w,p)+TE(x,y). \tag{2.11}$$

Notice that all three components are semi-positive, with zero indicating efficiency. This implies that increases in efficiency are reflected in decreasing scores.[6]

Thus, OE is simply the ratio between maximum profit minus observed profits for the observation evaluated over the normalized value of the direction vector $g=(-h,k)$ for given input and output prices (w,p). Chambers, Chung and Färe (1998) call this Nerlovian efficiency. Technical efficiency only guarantees reaching a point on the production frontier, not necessarily a point on the frontier maximising the profit function. Allocative efficiency, by contrast, measures the adjustments in input and output mixes along the production frontier needed to achieve the maximum of the profit function given relative prices. Overall efficiency ensures that both these ideals of technical and allocative efficiency are realized simultaneously.

To define short-run or restricted profit functions, it is necessary to distinguish between input prices of variable and fixed inputs $w=(w^v,w^f)$. A similar distinction is needed for output prices: $p=(p^v,p^f)$. The short-run total profit function is then:

$$SR\Pi(w,p,\bar{x}^f,\bar{y}^f)=\sup_{(x^v,\bar{x}^f,y^v,\bar{y}^f)\in T}\left\{p^v.y^v+p^f.\bar{y}^f-w^v.x^v-w^f.\bar{x}^f\right\}, \tag{2.12}$$

[6] It is also possible to define all three components such that they are semi-negative, with zero again indicating efficiency and increasing efficiency scores now reflecting increases in efficiency.

where the bar indicates the exogenous parameters. The short-run variable profit function is defined as:

$$SRV\Pi(w^v, p^v, \bar{x}^f, \bar{y}^f) = SR\Pi(w, p, \bar{x}^f, \bar{y}^f) - p^f.\bar{y}^f + w^f.\bar{x}^f \tag{2.13}$$

$$= \sup_{x^v, y^v} \left\{ p^v.y^v - w^v.x^v : (x^v, \bar{x}^f, y^v, \bar{y}^f) \in T \right\}. \tag{2.14}$$

Adapting Blancard et al. (2006) for the case of fixed and variable outputs, one can establish duality between the short-run directional distance function and the short-run variable profit function:

$$SRD(x^v, \bar{x}^f, y^v, \bar{y}^f; g) = \inf_{\substack{w,p \geq 0 \\ p^v.k^v + w^v.h^v \neq 0}} \left\{ \frac{SRV\Pi(w^v, p^v, \bar{x}^f, \bar{y}^f) - (p^v.y^v - w^v.x^v)}{p^v.k^v + w^v.h^v} \right\}. \tag{2.15}$$

Since the OE definition of Chambers, Chung and Färe (1998) employs a long-run profit function, we first need to define similar components based upon the short-run variable profit function. Short-run overall efficiency ($SROE$) is defined as the quantity:

$$SROE(w, p, x^v, \bar{x}^f, y^v, \bar{y}^f) = \frac{SRV\Pi(w^v, p^v, \bar{x}^f, \bar{y}^f) - (p^v.y^v - w^v.x^v)}{p^v.k^v + w^v.h^v}. \tag{2.16}$$

A short-run technical efficiency ($SRTE$) index can now be defined as the quantity:

$$SRTE(x^v, \bar{x}^f, y^v, \bar{y}^f) = SRD(x^v, \bar{x}^f, y^v, \bar{y}^f; g). \tag{2.17}$$

Obviously, in analogy to expression (2.10), a short-run allocative efficiency ($SRAE$) index bridges the gap between $SROE$ and $SRTE$.

Now we provide a definition of a profit tangency point:

Definition 2.1 *Let $(w^f, \bar{w}^v, p^f, \bar{p}^v) \in \mathbb{R}_+^{n+m}$ be an input-output price vector, where the bar indicates the exogenous parameters of the problem. Under the assumptions (T.1) to (T.5) above, an input-output vector $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point if and only if*

$$\bar{p}^v.y^v + p^f.\bar{y}^f - \bar{w}^v.x^v - w^f.\bar{x}^f = SR\Pi(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f) = \Pi(w^f, \bar{w}^v, p^f, \bar{p}^v).$$

We denote $\Im(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$ the set of all the $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency points. It is clear that if a production vector $(x, y) \in T$ is such that (i) $x^f = \bar{x}^f$ and $y^f = \bar{y}^f$ and (ii) it maximizes simultaneously short- and long run profits, then it is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point.

Proposition 2.2 *Under the assumptions above, if* $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T$ *and*

$$SRAE(w, p, x^v, \bar{x}^f, y^v, \bar{y}^f) = AE(w, p, x, y) = 0,$$

if and only if

$$(\tilde{x}, \tilde{y}) = (\ x - SRD(x, y; g)\,(h^v, 0)\ ,\ y + SRD(x, y; g)\,(k^v, 0)\ )$$

is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$*-profit tangency point.*

Proof: We have $SRAE(w, p, x^v, \bar{x}^f, y^v, \bar{y}^f) = 0$ if and only if $(\tilde{x}, \tilde{y})$ maximizes the short run profit function. Moreover, $AE(w, p, x^v, \bar{x}^f, y^v, \bar{y}^f) = 0$ if and only if $(\tilde{x}, \tilde{y})$ maximizes the long run profit function. This ends the proof. □

Thus, a point that is both allocatively efficient in the short- and long-run is a profit tangency point for its short-run technically efficient projection. Notice that this definition is novel in that it allows for technical inefficiencies when defining tangency points on the short- and long-run profit function. A possibly technically inefficient point can then be projected onto the technology and satisfy both allocative efficiency conditions simultaneously.

2.3 Short Run Cost Function and Duality

We first define the input directional distance function which is derived from the directional distance function by taking a direction vector $g = (h, 0)$. We have:

$$D(x, y, h, 0) = \sup_{\delta \in \mathbb{R}} \{\delta :\ (x - \delta h, y) \in T\}\,. \tag{2.18}$$

Then we define the short run input directional distance function, taking a direction denoted $g = (h^v, 0, 0, 0)$. We have:

$$SRD(x, y; h, 0) = D(x^v, x^f, y, h^v, 0, 0) \tag{2.19}$$

$$= \sup_{\delta \in \mathbb{R}} \left\{\delta :\ (x^v - \delta h^v, x^f, y) \in T\right\}. \tag{2.20}$$

The next element needed for our analysis is the standard long-run cost function, which can be defined as follows:

$$C(w,y) = \inf_{x} \{w.x : (x,y) \in T\} \tag{2.21}$$

$$= \inf_{x} \{w.x : D(x,y,h,0) \geq 0\} .) \tag{2.22}$$

Chambers, Chung and Färe (1996) show duality between the input directional distance function (2.18) and the standard long-run cost function (2.22):

$$D(x,y;h,0) = \inf_{w,p} \left\{ \frac{w.x - C(w,y)}{w.h} : w.h \neq 0 \right\} \tag{2.23}$$

Following Chambers, Chung and Färe (1998), we first define the input overall efficiency (OE^i) index as the quantity:

$$OE^i(x,y,w) = \frac{w.x - C(w,y)}{w.h} \tag{2.24}$$

Then, they continue by characterising a technical efficiency (TE^i) index as the quantity:

$$TE^i(x,y) = D(x,y;h,0) \tag{2.25}$$

Finally, the allocative efficiency (AE^i) index is defined as the quantity:

$$AE^i(x,y,w) = OE^i(x,y,w) - TE^i(x,y) \tag{2.26}$$

Overall input efficiency ensures that both these ideals of technical and allocative efficiency are realised simultaneously. Obviously, the following additive decomposition identity holds:

$$OE^i(x,y,w) = AE^i(x,y,w) + TE^i(x,y) \tag{2.27}$$

The short-run total cost function is then:

$$SRC(w,\bar{x}^f,y) = \inf_{x^v} \left\{ w^v.x^v + w^f.\bar{x}^f : (x^v,\bar{x}^f,y) \in T \right\}, \tag{2.28}$$

while the short-run variable cost function is:

$$SRVC(w^v, \bar{x}^f, y) = SRC(w, \bar{x}^f, y) - w^f.\bar{x}^f \tag{2.29}$$

$$= \inf_{x^v} \left\{ w^v.x^v : (x^v, \bar{x}^f, y) \in T \right\}. \tag{2.30}$$

Short-run input overall efficiency ($SROE^i$) is defined as the quantity:

$$SROE^i(w, x^v, \bar{x}^f, y) = \frac{w^v.x^v - SRC(w, \bar{x}^f, y)}{w^v.h^v} \tag{2.31}$$

A short-run technical efficiency ($SRTE^i$) index can now be defined as the quantity:

$$SRTE^i(x^v, \bar{x}^f, y) = SRD(x, y; h^v, 0, 0). \tag{2.32}$$

Obviously, in analogy to the profit function, a short-run allocative efficiency ($SRAE^i$) index bridges the gap between $SROE^i$ and $SRTE^i$.

It is rather straightforward to establish duality between the short-run input directional distance function (2.20) and the short-run variable cost function (2.29). Under the assumptions above, we have:

a)

$$SRVC(w^v, \bar{x}^f, y) = \inf_{x^v} \left\{ w^v.x^v : SRD(x^v, \bar{x}^f, y; h^v, 0) \geq 0 \right\}$$

b)

$$SRD(x^v, \bar{x}^f, y; h^v, 0) = \inf_{w \geq 0} \left\{ \frac{w^v.x^v - SRVC(w^v, \bar{x}^f, y)}{w^v.h^v} : w^v.h^v \neq 0 \right\}$$

The proof is similar to the proof in Blancard et al. (2006) and is omitted.

Now we provide a definition of a cost tangency point:

Definition 2.3 *Let $(\bar{w}^v, w^f, \bar{p}) \in \mathbb{R}_+^{n+m}$ be an input-output price vector. Under the assumptions (T.1) to (T.5), an input-output vector $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point if*

$$\bar{w}^v.x^v + w^f.\bar{x}^f = SRC(\bar{w}^v, \bar{x}^f, y) = C(\bar{w}^v, w^f, y).$$

We denote $\aleph(\bar{w}^v, \bar{p}, \bar{x}^f)$ the set of all the $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency points. It is clear that if a production vector $(x, y) \in T$ is such that $x^f = \bar{x}^f$ and if x minimises the short- and long-run cost, then it is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point.

Proposition 2.4 *Under the assumptions above, if* $(x^v, \bar{x}^f, y) \in T$ *and*

$$SRAE^i(\bar{w}^v, w^f, x^v, \bar{x}^f, y) = AE^i(\bar{w}^v, w^f, x, y) = 0,$$

if and only if

$$(\tilde{x}, \tilde{y}) = (\, x - SRD(x, y; h^v, 0, 0)\,(h^v, 0)\,,\, y\,)$$

is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$*-cost tangency point.*

Proof: $SRAE^i(\bar{w}^v, w^f, x^v, \bar{x}^f, y) = 0$ if and only if $(\tilde{x}, \tilde{y})$ minimizes the short run cost function. Moreover, $AE^i(\bar{w}^v, w^f, x, y) = 0$ if and only if $(\tilde{x}, \tilde{y})$ minimizes the long run cost function. This ends the proof. □

3 Tangency Points and Non-parametric Technology

3.1 Profit Tangency Points

Now, we focus on non-parametric production models and prove how a tangency point can be computed by finding the solution to a system of linear inequalities.

Let us consider a set of r firms $A = \{(x_1, y_1), ..., (x_r, y_r)\} \in \mathbb{R}_+^{n+m}$. Production technology can be estimated by enveloping observed firms while respecting some basic economic production axioms (see Hackman (2008) or Ray (2004)). First, we focus on the constant and variable returns to scale cases and provide mathematical programs for determining solutions for the profit tangency points. This is basically just the dual solution of the mathematical program for computing the directional distance functions. When returns to scale are constant, then the non-parametric technology is the conical hull of the observed data plus the non-negative orthant. When returns to scale are variable, then the non-parametric technology is the convex hull of the observations plus the non-negative Euclidean orthant. The formulation adopted here is of particular importance to provide a system of linear inequalities.

Under constant returns to scale the production set is defined as:

$$T_{CRS} = \left\{ (x, y) : x \geq \sum_{j=1}^{r} \theta_j x_j, y \leq \sum_{j=1}^{r} \theta_j y_j, \theta \geq 0 \right\}. \tag{3.1}$$

Let $C_c(A)$ be the conical hull of A . We have $T_{CRS} = (C_c(A) + K) \cap \mathbb{R}_+^{n+m}$ where $K = \mathbb{R}_+^n \times (-\mathbb{R}_+^m)$. Let us calculate the profit function: $\forall (w,p) \in \mathbb{R}_+^{n+m}$, we have:

$$\Pi(w,p) = \begin{cases} +\infty & \text{if} \quad p.y_j - w.x_j > 0 \quad \text{for some} \quad j \\ 0 & \text{if} \quad p.y_j - w.x_j \leq 0 \quad \text{for all} \quad j, \end{cases} \tag{3.2}$$

since profits are either infinite or non-positive in the case of constant returns to scale. This yields a dual program for finding the shadow price solution $(\tilde{w}, \tilde{p})$ that is also the adjusted price function (see Luenberger (1995)):

$$D_{CRS}(x,y;g) = \min_{(w,p)\geq 0} \{\Pi(w,p) - p.y + w.x \ : \ p.k + w.h = 1\} \tag{3.3}$$

$$= \min_{\substack{(w,p)\geq 0 \\ p.k+w.h=1}} \{-p.y + w.x : p.y_j - w.x_j \leq 0 \, , j = 1,...,r\} \, , \tag{3.4}$$

where $(\tilde{w}, \tilde{p})$ is the solution of the program. We deduce the following result:

Proposition 3.1 *Assume that $T = T_{CRS}$. Let $g \neq 0$ be an arbitrary direction. Under the assumptions above, a vector $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point if and only if $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ satisfies the following system of linear inequalities:*

$$(\star) \quad \begin{cases} -\bar{p}^v.y^v - p^f.\bar{y}^f + \bar{w}^v.x^v + w^f.\bar{x}^f & = 0, \\ \bar{p}^v.y_j^v + p^f.\bar{y}_j^f - \bar{w}^v.x_j^v - w^f.\bar{x}_j^f & \leq 0, \quad j = 1,...,r \\ \bar{p}^v.k^v + p^f.k^f + \bar{w}^v.h^v + w^f.h^f & = 1, \\ (x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j, & \\ (y^v, \bar{y}^f) \leq \sum_{j=1}^{r} \theta_j y_j, & \\ \theta \geq 0, y^v \geq 0, x^v \geq 0, p^f \geq 0, w^f \geq 0. & \end{cases}$$

Proof: If (x^v, y^v, θ) satisfies $(\star)$, then $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T_{CRS}$ and moreover it maximizes profits. Since $x^f = \bar{x}^f$ and $y^f = \bar{y}^f$, we deduce that $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-tangency point. Conversely, we deduce that if $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-tangency point, then $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T^{CRS}$, thus there exists $\theta \geq 0$ such that $(x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j$ and $(y^v, \bar{y}^f) \leq \sum_{j=1}^{r} \theta_j y_j$. Moreover, from the weak version of the separation theorem, it maximizes profits, therefore: $-p^v.y^v - p^f.\bar{y}^f + w^v.x^v + w^f.\bar{x}^f = 0$ and $\bar{p}^v.y_j^v + p^f.y_j^f - \bar{w}.x_j^v - w^f.x_j^f \leq 0, \ j = 1,...,r$

and adding the normalization constraint $\bar{p}^v.k^v + p^f.k^f + \bar{w}.h^v + w^f.h^f = 1$, (x^v, y^v, θ) satisfies system $(\star)$. $\square$

The first equality imposes that profits must be zero at the tangency point (otherwise the invested capacity would be economic obsolete). The next set of j inequalities guarantees that all profits are smaller than or equal to zero (which provides an incentive to exploit the initially build capacity). The next equality imposes a normalization on the prices. All other inequalities simply impose the constant returns to scale non-parametric technology.

Thus, assuming competitive conditions, the basic logic is that for all fixed quantities, i.e., fixed inputs and outputs, we look for the corresponding prices, i.e., fixed input and output prices, that make the observed configurations of fixed and variable inputs and outputs ex post economically viable. The results are useful to assess whether certain optimal fixed quantities are technically feasible and whether the prices to make them economically viable can be supported by the market. Notice that we suppose a competitive product market, since the firm can sell any output at the optimal price.

Taking a ratio between profits at the $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point and observed profits (or eventually profits at the projection point $(\tilde{x}, \tilde{y})$) allows defining a dual measure of capacity utilization.[7] Shadow prices of fixed inputs can be compared to their observed prices: when the shadow price is larger (smaller) than the observed price, then the fixed input could be expanded (reduced). It is well-known in the literature that multiple fixed inputs can lead to ambiguous situations whereby the equality between profits at the $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point and observed profits is maintained despite the fact that shadow and actual prices of fixed inputs diverge for each fixed input due to offsetting effects (see Berndt and Fuss (1986)).

In the variable returns to scale case the production technology is:

$$T_{VRS} = \left\{ (x, y) : x \geq \sum_{j=1}^{r} \theta_j x_j, y \leq \sum_{j=1}^{r} \theta_j y_j, \sum_{j=1}^{r} \theta_j = 1, \theta \geq 0 \right\}. \tag{3.5}$$

Let $Co(A)$ be the conical hull of A. We have $T_{VRS} = (Co(A) + K) \cap R_+^{n+m}$ and we have $T_{VRS} + K = Co(A) + K$. The profit function is $\forall\, (w, p) \in R_+^{n+m}$:

$$\Pi(w, p) = \max \left\{ p.y_j - w.x_j \, : \, j = 1, ..., r \right\}. \tag{3.6}$$

[7]Alternatively, one can define a difference-based dual capacity measure.

Assume that $(x, y) \in T_{VRS} + K$. Since $T_{VRS} + K = Co(A) + K$, we obtain:

$$D_{VRS}(x,y) = \min_{\substack{(w,p)\geq 0 \\ p.k+w.h=1}} \max \{p.(y_j - y) - w.(x_j - x) : j = 1, ..., r\}. \qquad (3.7)$$

Proposition 3.2 *Assume that $T = T_{VRS}$. Let $g \neq 0$ be an arbitrary direction. Under the assumptions above, a vector $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point if and only if $(x^v, y^v, w^f, p^f, \theta)$ satisfies the following system of linear inequalities:*

$$(\star\star) \quad \begin{cases} p.y_j - w.x_j - (\bar{p}^v.y^v + p^f.\bar{y}^f - \bar{w}^v.x^v - w^f.\bar{x}^f) \leq 0 \quad , j = 1, ..., r \\ \bar{p}^v.k^v + p^f.k^f + \bar{w}^v.h^v + w^f.h^f = 1 \\ (x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j, \\ (y^v, \bar{y}^f) \leq \sum_{j=1}^{r} \theta_j y_j, \\ \sum_{j=1}^{r} \theta_j = 1, \\ \theta \geq 0, y^v \geq 0, x^v \geq 0, p^f \geq 0, w^f \geq 0. \end{cases}$$

Proof: If (x^v, y^v, θ) satisfies $(\star\star)$, then $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T^{VRS}$. Moreover, since

$$p.y_j - w.x_j \leq p^v.y^v + p^f.\bar{y}^f - w^v.x^v - w^f.\bar{x}^f \text{ for } j = 1, ..., r$$

it maximizes the profit. Since $x^f = \bar{x}^f$ and $y^f = \bar{y}^f$, we deduce that $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-tangency point. Conversely, if $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-tangency point, then $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T_{VRS}$. Thus, there exists $\theta \geq 0$ such that $\sum_{j=1}^{r} \theta_j = 1$ and $(x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j$. Moreover, from the weak version of the separation theorem, it maximizes profits, therefore:

$$p.y_j - p^v.y^v - p^f.\bar{y}^f - w.x_j + w^v.x^v + w^f.\bar{x}^f \leq 0 \text{ for } j = 1, ..., r$$

and (x^v, y^v) satisfies $(\star\star)$. $\square$

The first set of j inequalities guarantees that all observed profits are smaller than the profits at the tangency point. The next equality imposes a normalization. All other inequalities simply impose the variable returns to scale non-parametric technology. Note that the same remarks developed for the constant returns to scale case apply.

3.2 Cost Tangency Points

The following condition is not completely similar to the condition concerning the profit function because the actual output price is unknown. In particular, the system $(\star)$ is non-linear.

Proposition 3.3 *Assume that $T = T_{CRS}$. Let $h \neq 0$ be an arbitrary input direction. Under the assumptions above:*

a) If there is some $p \geq 0$ such that (x^v, y, p, w^f, θ) satisfies the following system of inequalities:

$$(\star) \quad \begin{cases} -p.y + \bar{w}^v.x^v + w^f.\bar{x}^f & = \quad 0, \\ p.y_j - \bar{w}^v.x^v_j - w^f.\bar{x}^f_j & \leq \quad 0, \ j = 1, ..., r \\ \bar{w}^v.h^v + w^f.h^f & = \quad 1, \\ (x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j, & \\ y \leq \sum_{j=1}^{r} \theta_j y_j, & \\ \theta \geq 0, y \geq 0, x^v \geq 0, p \geq 0, w^f \geq 0, & \end{cases}$$

then the vector $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point.

b) If the vector $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, then there is some $p \in \mathbb{R}^m_+$ such that (x^v, y, p, w^f, θ) is a solution of $(\star)$.

Proof: a) If (x^v, y, p, θ) satisfies $(\star)$, then $(x^v, \bar{x}^f, y) \in T_{CRS}$ and moreover it maximises profits. Thus, $(x^v, \bar{x}^f)$ minimizes costs. Since $x^f = \bar{x}^f$ we deduce that $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point. b) Conversely, if $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, then $(x^v, \bar{x}^f, y) \in T_{CRS}$. Thus, there exists $\theta \geq 0$ such that $(x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j$ and $y \leq \sum_{j=1}^{r} \theta_j y_j$. Moreover, from the weak version of the separation theorem, there exists $p \in \mathbb{R}^m_+$ such that $(x^v, \bar{x}^f, y)$ maximises profits. Therefore: $-p.y + w^v.x^v + w^f.\bar{x}^f = 0$ and $p.y_j - w.x_j \leq 0, \ j = 1, ..., r$. Thus, normalizing, (x^v, y, p, θ) satisfies $(\star)$. $\Box$

The first equality imposes that profits must be zero at the cost tangency point. The next set of j inequalities guarantees that all profits are smaller than or equal to zero. The next equality imposes a normalisation. All other inequalities simply impose the constant returns to scale non-parametric technology. Notice that the above system is non-linear since the first equation looks for output prices and outputs and in addition variable inputs that make given fixed inputs generate a zero profit.

But, if one fixes an output price vector, then we return to the linear case developed for profit maximisation under a similar returns to scale assumption in the main contribution. However, by fixing the output prices, some cost tangency points corresponding to other output price vector are omitted. Therefore, it is clear that a profit-tangency point is a cost tangency point, but the converse is not true.

As developed below, a symmetrical approach applies under a variable returns to scale assumption.

Proposition 3.4 *Assume that $T = T_{VRS}$. Let $h \neq 0$ be an arbitrary input direction. Under the assumptions above:*

a) If there is some $p \geq 0$ such that (x^v, y, p, w^f, θ) satisfies the following system of inequalities:

$$(\star\star)\quad \begin{cases} p.y_j - \bar{w}^v.x_j^v - w^f.\bar{x}_j^f - (p.y - \bar{w}^v.x^v - w^f.\bar{x}^f) & \leq \quad 0,\ j = 1, ..., r \\ \bar{w}^v.h^v + w^f.h^f & = \quad 1\ , \\ (x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j, & \\ y \leq \sum_{j=1}^{r} \theta_j y_j, & \\ \sum_{j=1}^{r} \theta_j = 1, & \\ \theta \geq 0, y \geq 0, x^v \geq 0, p \geq 0, w^f \geq 0, & \end{cases}$$

then the vector $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point.

b) If the vector $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, then there is some $p \in \mathbb{R}^m_+$ such that (x^v, y, p, w^f, θ) is a solution of $(\star\star)$.

Proof: a) If (x^v, y, p, θ) satisfies $(\star\star)$, then $(x^v, \bar{x}^f, y) \in T_{VRS}$ and moreover since

$$p.y_j - w.x_j \leq p.y - w^v.x^v - w^f.\bar{x}^f \text{ for } j = 1, ..., r$$

it maximises profits. Thus, $(x^v, \bar{x}^f)$ minimizes costs. Since $x^f = \bar{x}^f$ we deduce that $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point. b) Conversely, if $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, then $(x^v, \bar{x}^f, y) \in T_{VRS}$. Thus, there exists $\theta \geq 0$ such that $\sum_{j=1}^{r} \theta_j = 1$, $(x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j$ and $y \leq \sum_{j=1}^{r} \theta_j y_j$. Moreover, from the weak version of the separation theorem, there exists $p \in \mathbb{R}^m_+$

such that it maximises profits. Therefore:

$$p.y_j - w.x_j \leq p.y - w^v.x^v - w^f.\bar{x}^f;\ j = 1, ..., r$$

and normalizing, (x^v, y, p, θ) satisfies $(\star\star)$.$\square$

The first set of j inequalities guarantees that all profits at optimal output prices are smaller than the profits at the tangency point. The next equality imposes a normalisation. All other inequalities simply impose the variable returns to scale non-parametric technology.

4 Comparing Profit and Cost Function Tangency Points: A New Result

Assume that the outlined procedure yields on the one hand $\Im(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$, the set of all $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency points, and on the other hand $\aleph(w, \bar{p}, \bar{x}^f)$, the set of all $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency points. It is now possible to establish a relation between the characterization of both these profit and cost tangency points. This result is -to the best of our knowledge- new to the literature.

Proposition 4.1 *Let $w \in \mathbb{R}^n_+$, be an input price vector. Under the assumptions above:*
a) For all output price vector $p \in \mathbb{R}^m_+$, if an input-output vector $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point, then it is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, i.e.:

$$\Im(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f) \subset \aleph(w, \bar{p}, \bar{x}^f).$$

b) If $(x^v, \bar{x}^f, y) \in T$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, then there exists an output price vector $p \in \mathbb{R}^m_+$, such that it is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point.

Proof: a) Assume that $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point. For all $u \in L(y)$, $p.y - w.x \geq p.y - w.u \Leftrightarrow w.x \leq w.u$. Thus, $x = (x^v, \bar{x}^f)$ minimizes costs and a) is proven. b) Let us consider the subset $M(y, w) = \{(u, y) \in \mathbb{R}^{n+m} : w.u \leq C(p, y)\}$. Since there are no interior points in T lying in $M(y, w)$, there exists $(w', p) \in \mathbb{R}^{n+m}_+$, such that $M(x, p) \subset$

$\{(u,y) \in \mathbb{R}^{n+m} : p.y - w'.u \leq \Pi(w',p)\}$. Consequently, since $\Pi(w',p) < +\infty$, we deduce that $\inf\{w'.u; (u,y) \in M(y,w)\} > -\infty$. Consequently, from Farkas Lemma, we deduce that $w' = w$. Now, if $(x^v, \bar{x}^f, y) \in T$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, then $(x^v, \bar{x}^f, y) \in M(y,w) \cap T$. Thus, $p.y - w.u = \Pi(w,p)$, and $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point. □

Thus, the set of profit tangency points is a proper subset of the set of cost tangency points. Furthermore, any cost tangency point can be transformed into a profit tangency point for a particular choice of the vector of output prices.

To develop the intuition behind this result, assume for simplicity that the set of fixed outputs is empty ($J_f = \emptyset$). Then, for a given vector of fixed inputs, to find a cost tangency point one may adjust both the output quantity and price vectors such that the systems of inequalities in Proposition 3.3 (or 3.4) are satisfied. In case of the profit function, however, for a given vector of fixed inputs, to find a profit tangency point involves adjusting outputs such that profits remain maximal for the given vector of output prices (i.e., the systems of inequalities in Proposition 3.1 (or 3.2) are satisfied). Obviously, the latter exercise is much more difficult, whence the relationship.

Notice that this result is perfectly general. While the set of all $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency points $\Im(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$ and the set of all $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency points $\aleph(w, \bar{p}, \bar{x}^f)$ are currently based on non-parametric specifications of technology, this is by no means necessary. Therefore, this result holds true for any specification of technology.

5 Concluding Comments

To conclude, it is worthwhile mentioning that the analysis could eventually be extended into several directions. First, it is obvious to determine tangency points for the case of a revenue function. Another issue could be to derive tangency notions of capacity in the case of indirect technologies where output maximizing production is, e.g., subject to a budget constraint (see Ray, Mukherjee and Wu (2006) for another capacity notion proposal in this context). Also, we have assumed the presence of competitive product markets for homogeneous goods. Other cases (e.g., a monopolist offering a non-storable homogeneous good with periodic demand (i.e., a peak-load problem)) would require some modifications worthwhile pursuing. Furthermore, we have left aside the precise definition of capacity utilization measures that could be based either on ratios or on differences of optimal and observed quantities (e.g., outputs) or economic values (e.g., profit levels) and the ensuing problems (e.g., how to define a primal scalar measure in the multiple output case). Finally, we have ignored any problems of statistical inference related to the estimation of tangency points at the intersection of short- and long-run profit frontiers (see, e.g., Holland and Lee (2002) on capacity estimation or Simar and Wilson (2000) for frontier estimation in general).

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