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Exact Relations between Four Definitions of Productivity Indices and Indicators

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Abstract

Generalizing earlier approximation results, we establish exact relations between the Luenberger productivity indicator and the Malmquist productivity index under rather mild assumptions. Furthermore, we show that similar exact relations can be established between the Luenberger-Hicks-Moorsteen indicator and the Hicks-Moorsteen index.

Key words: Malmquist and Hicks-Moorsteen productivity indices, Luenberger and Luenberger-Hicks-Moorsteen productivity indicators, approximate relation, exact relation.

JEL: C43, D24

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1 Introduction

Caves, Christensen and Diewert (1982) introduced a technology-based, discrete-time Malmquist productivity index, defined by as a ratio of input (or output) distance functions. Luenberger (1995) generalized existing distance functions by introducing the shortage function (also known as the directional distance function), that accounts for both input contractions and output improvements simultaneously and that is dual to the profit function. Making use of the shortage function, Chambers (1996) and Chambers and Pope (1996) introduced the Luenberger productivity indicator, as a difference of directional distance functions. To maintain a total factor productivity interpretation, Bjurek (1996) proposed an alternative Malmquist TFP (or Hicks-Moorsteen) index, defined as a ratio of Malmquist output and input indices. Finally, Briec and Kerstens (2004) introduced a new difference-based variation on this Malmquist TFP index, which has been labeled as the Luenberger-Hicks-Moorsteen indicator. Thus, the renewed interest in indicators based on differences rather than ratios (Diewert (2005)) has resulted in new technology-based productivity indicators complementing earlier indexes.

These discrete-time technology-based productivity indices (especially the Malmquist index and the more recent Luenberger indicator) are rather popular and have become part of the traditional toolbox in applied production research (see, e.g., Hulten (2001)). Also a variety of extensions are available that are of relevance to agricultural and environmental economics. For instance, Jaenicke and Lengnick (1999) develop a multiplicative decomposition of the Malmquist productivity index which includes a soil-quality index. As another example, Ball et al. (2004) define a set of environmentally sensitive technology-based productivity indices including unmarketed (hence, non-priced) undesirable by-products.

The ratio-based Malmquist and Hicks-Moorsteen productivity indices have been proven to coincide under two properties: (i) inverse homotheticity of technology; and (ii) constant returns to scale (see Färe, Grosskopf and Roos (1996)). In a similar vein, Briec and Kerstens (2004) demonstrate that the difference-based Luenberger and Luenberger-Hicks-Moorsteen indicators are identical under two conditions: (i) inverse translation homotheticity of technology in the direction of a vector g; and (ii) graph translation homotheticity in the direction of a vector g (see Chambers (2002)).

Furthermore, Boussemart et al. (2003) have proven an approximate relation between the Luenberger productivity indicator and the Malmquist productivity index. In particular, this article establishes that the logarithm of the input Malmquist productivity index is twice a linear approximation of minus the Luenberger productivity indicator. Briec and Kerstens (2004) manage to establish a similar approximation result showing that the Luenberger-Hicks-Moorsteen indicator is about equal to the logarithm of the Hicks-Moorsteen index.

Therefore, in the literature all currently known productivity indices and indicators have been clearly related to one another such that applied researchers know what differences can be expected depending on their methodological choices. This contribution takes one further step by proposing exact instead of approximate relationships that hold between these productivity indices and indicators under mildly stronger assumptions.

This note unfolds as follows. Section 2 lays down the definitional groundwork. In section 3, we establish an exact relation between on the one hand the Luenberger productivity indicator and the Malmquist productivity index, and on the other hand the Luenberger-Hicks-Moorsteen indicator and the Hicks-Moorsteen index. Section 4 offers an even more general exact relation by showing that the Shephard distance function can be related to the directional distance function when computed on a logarithmic transformation of technology.

2 Technology, Distance Functions, and Productivity Indices: Definitions

2.1 Definitions of Technology and Distance Functions

Production technology transforms inputs $x \in \mathbb{R}_+^n$ into outputs $y \in \mathbb{R}_+^p$. For each time period $\tau = 0, 1$, the production possibility set T^τ summarizes the set of feasible input and output vectors and is defined as follows:

$$T^\tau = \left\{(x, y) \in \mathbb{R}_+^{n+m} : x \text{ can produce } y \text{ in period } \tau\right\}. \tag{1}$$

We define the output correspondence $P^\tau : \mathbb{R}_+^n \longrightarrow 2^{\mathbb{R}_+^p}$ as

$$P^\tau(x) = \left\{y \in \mathbb{R}_+^p : (x, y) \in T^\tau\right\}. \tag{2}$$

We assume throughout the paper that: $(P1)$ $0 \in P^\tau(x)$ for all $x \in \mathbb{R}_+^n$, i.e., the null output can always be produced; $(P2)$ $P^\tau(x)$ is a closed and bounded set of $\mathbb{R}_+^p$; i.e., it is compact; $(P3)$ For all $y \in P^\tau(x)$, $0 \leq u \leq y$ implies that $u \in P^\tau(x)$, i.e., fewer outputs can always be produced with the same inputs. These assumptions suffice for our purpose.

Efficiency is estimated relative to technologies using distance or gauge functions. The Shephard distance function in the output-orientation is the function $D_o^\tau : \mathbb{R}_+^{n+p} \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by:

$$D_o^\tau(x, y) = \inf\left\{\lambda > 0 : (x, \frac{y}{\lambda}) \in T^\tau\right\}. \tag{3}$$

The Shephard distance function in the input-orientation is the mapping $D_i^\tau : \mathbb{R}_+^{n+p} \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by:

$$D_i^\tau(x, y) = \sup\left\{\lambda > 0 : (\frac{x}{\lambda}, y) \in T^\tau\right\}. \tag{4}$$

The directional distance function $\overrightarrow{D}_T^\tau : \mathbb{R}_+^{n+m} \times \mathbb{R}_+^{n+m} \longrightarrow \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is defined by:

$$\overrightarrow{D}_T^\tau(x, y; h, k) = \sup\{\delta : (x - \delta h, y + \delta k) \in T^\tau\} \tag{5}$$

if $(x - \delta h, y + \delta k) \in T^\tau$ for some $\delta \in \mathbb{R}$ and takes the value $-\infty$ otherwise. This definition implies that if $(x, y) \in T$, then $\overrightarrow{D}_T^\tau(x, y; 0, 0) = +\infty$. However, the direction $g = (h, k)$ is fixed, and hence we suppose that $g \neq 0$. Detailed properties of the directional distance function can be found in Chambers (1996).[1] In the output-oriented case, one can write:

$$\overrightarrow{D}_T^\tau(x, y; 0, k) = \sup\{\delta : y + \delta k \in P^\tau(y)\}\ . \tag{6}$$

In the proportional output-oriented case the directional and Shephard distance functions are related as follows:

$$\overrightarrow{D}_T^\tau(x, y; 0, y) = \frac{1}{D_o^\tau(x, y)} - 1. \tag{7}$$

This relation always holds true under the assumptions $(P1)$-$(P3)$. It is rather straightforward to show that the direction vector $g = (0, y)$ is always feasible if $y \neq 0$.[2]

[1] Slightly different generalisations of the Shephard distance functions that are equally related to the profit function have been defined in, e.g., Chavas and Cox (1999) or McFadden (1978). In principle, our analysis could equally be transposed to productivity measures based on the latter distance functions (see, e.g., the definitions in Chavas and Cox (1999)).

[2] If $y = 0$, then it is also true under the convention $+\infty = \frac{1}{0^+}$.

2.2 Discrete-Time Productivity Indices and Indicators

We can now turn to the definition of the productivity indices and indicators. We start with the Malmquist productivity index and the corresponding Luenberger indicator. Thereafter, we define the Hicks-Moorsteen productivity index and the Luenberger-Hicks-Moorsteen indicator.

Following Caves et al. (1982), for the base periods $\tau = 0, 1$, the Malmquist productivity index is defined as:

$$M_o^0 = \frac{D_o^0(x^1, y^1)}{D_o^0(x^0, y^0)} \text{ and } M_o^1 = \frac{D_o^1(x^1, y^1)}{D_o^1(x^0, y^0)} \ . \tag{8}$$

The output-oriented Malmquist productivity index is defined by:

$$M_o = (M_o^0 M_o^1)^{\frac{1}{2}} \ . \tag{9}$$

Symmetrically, one can define an output-oriented Luenberger indicator for the base periods $\tau = 0, 1$ (Chambers (1996)):

$$L^0(g^0, g^1) = \overrightarrow{D}_T^0(x^0, y^0; g^0) - \overrightarrow{D}_T^0(x^1, y^1; g^1) \tag{10}$$

and

$$L^1(g^0, g^1) = \overrightarrow{D}_T^1(x^0, y^0; g^0) - \overrightarrow{D}_T^1(x^1, y^1; g^1) \ . \tag{11}$$

The output-oriented Luenberger productivity indicator is then defined by:

$$L(g^0, g^1) = \frac{1}{2}\big(L^0(g^0, g^1) + L^1(g^0, g^1)\big) \ . \tag{12}$$

A Hicks-Moorsteen productivity index with base period $\tau = 0$ is defined as the ratio of a Malmquist output quantity index at base period t over a Malmquist input quantity index at base period $\tau = 0$ (Bjurek (1996)):

$$HM^0 = \frac{MO^0}{MI^0}, \tag{13}$$

where

$$MO^0 = \frac{D_o^0(x^0, y^1)}{D_o^0(x^0, y^0)} \quad \text{and} \quad MI^0 = \frac{D_i^0(x^1,y^0)}{D_i^0(x^0,y^0)} \ . \tag{14}$$

A base period $\tau = 1$ Hicks-Moorsteen productivity index is defined as follows:

$$HM^1 = \frac{MO^1}{MI_1}, \tag{15}$$

where

$$MO^1 = \frac{D_o^1(x^1, y^1)}{D_o^1(x^1,y^0)} \quad \text{and} \quad MI^1 = \frac{D_i^1(x^1, y^1)}{D_i^1(x^0, y^1)} \ . \tag{16}$$

A geometric mean of these two Hicks-Moorsteen productivity indexes yields:

$$HM = \left[HM^0 \cdot HM^1\right]^{1/2} . \tag{17}$$

Briec and Kerstens (2004) define a Luenberger-Hicks-Moorsteen indicator with base period $\tau = 0$ as the difference between a Luenberger output quantity indicator and a Luenberger input quantity indicator:

$$LHM^0(g^0, g^1) = LO^0(k^0, k^1) - LI^0(h^0, h^1), \tag{18}$$

where

$$LO^0(k^0, k^1) = \overrightarrow{D}_T^0(x^0,y^0; 0, k^0) - \overrightarrow{D}_T^0(x^0,y^1; 0, k^1) \tag{19}$$

and

$$LI^0(h^0, h^1) = \overrightarrow{D}_T^0(x^1,y^0; h^1, 0) - \overrightarrow{D}_T^0(x^0, y^0; h^0, 0). \tag{20}$$

Also a base period $\tau = 1$ Luenberger-Hicks-Moorsteen indicator can be similarly defined as follows:

$$LHM^1(g^0, g^1) = LO^1(k^0, k^1) - LI^1(h^0, h^1), \tag{21}$$

where

$$LO^1(k^0,k^1) = \overrightarrow{D}^1(x^1,y^0;0,k^0) - \overrightarrow{D}^1(x^1,y^1;0,k^1) \tag{22}$$

and

$$LI^1(k^0,k^1) = \overrightarrow{D}^1(x^1,y^1;h^1,0) - \overrightarrow{D}^1(x^0,y^1;h^0,0). \tag{23}$$

An arithmetic mean of these two base periods Luenberger-Hicks-Moorsteen indicators is:

$$LHM(g^0,g^1) = \tfrac{1}{2}\Big[LHM^0(g^0,g^1) + LHM^1(g^0,g^1)\Big]. \tag{24}$$

3 Exact Relations between Productivity Indices and Indicators

Figure 1 summarizes the exact relations that have been established in the literature so far and delineates the contribution made here. First of all, Färe, Grosskopf and Roos (1996) established necessary and sufficient conditions to ensure the equality between the Hicks-Moorsteen and Malmquist indexes. Paralleling these results, Briec and Kerstens (2004) introduced a Luenberger-Hicks- Moorsteen indicator and established necessary and sufficient conditions to ensure its equality to the Luenberger indicator. Notice that Balk et al. (2008) have also established an exact relation between the Malmquist index and the components of the Luenberger indicator. However, our formulation is somewhat different. Moreover, these authors did not establish the converse relation linking the Luenberger indicator to the Malmquist index.

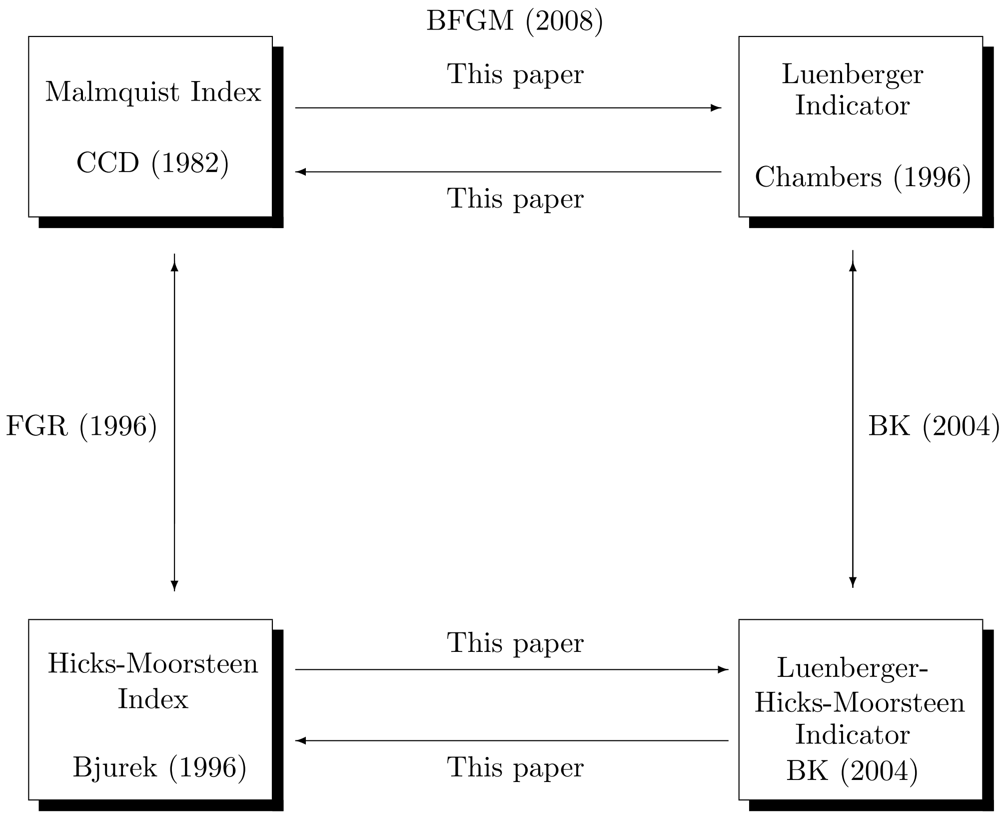

Figure 1: Exact relations between Indexes and Indicators

BK (2004) = Briec & Kerstens (2004); CCD (1982) = Caves, Christensen & Diewert (1982); FGR (1996) = Färe, Grosskopf & Roos (1996).

3.1 Exact Relation between Malmquist Index and Luenberger Indicator

Boussemart et al. (2003) define the productivity indicator as involving proportionate changes of inputs and outputs. In the output-oriented case, these authors defined their proportional Luenberger productivity indicator. For simplicity denote

$$L_o^\tau(k^0, k^1) = L^\tau(0, k^0, 0, k^1) \quad \tau = 0, 1. \tag{25}$$

Taking, $k^0 = y^0$ and $k^1 = y^1$, the output-oriented proportional Luenberger productivity indicator is then defined as:

$$L_o(y^0, y^1) = \frac{1}{2}\Big(L_o^0(y^0, y^1) + L_o^1(y^0, y^1)\Big), \tag{26}$$

where

$$L_o^0(y^0,y^1) = \frac{1}{2}\Big(\overrightarrow{D}_T^0(x^0,y^0;0,y^0) - \overrightarrow{D}_T^0(x^1,y^1;0,y^1)\Big) \tag{27}$$

and

$$L_o^1(y^0,y^1) = \frac{1}{2}\Big(\overrightarrow{D}_T^1(x^0,y^0;0,y^0) - \overrightarrow{D}_T^1(x^1,y^1;0,y^1)\Big). \tag{28}$$

Returning to the Malmquist index, we have from equation (8) and (7):

$$M_o^0 = \frac{D_o^0(x^1,y^1)}{D_o^0(x^0,y^0)} = D_o^0(x^1,y^1) = \Big[1 + \overrightarrow{D}_T^0(x^1,y^1;0,y^1)\Big]^{-1}, \tag{29}$$

where we assume that $D_o^0(x^0,y^0) = D_o^1(x^1,y^1) = 1$. From equation (7), this implies that $\overrightarrow{D}_T^0(x^0,y^0;0,y^0) = \overrightarrow{D}_T^1(x^1,y^1;0,y^1) = 0$. Consequently, we have $L_o^0(y^0,y^1) = -\overrightarrow{D}_T^0(x^1,y^1;0,y^1)$. Hence, we deduce that

$$M_o^0 = \Big[1 - L_o^0(y^0,y^1)\Big]^{-1}. \tag{30}$$

Using a similar procedure yields:

$$M_o^1 = \frac{D_o^1(x^1,y^1)}{D_o^1(x^0,y^0)} = \Big[D_o^1(x^0,y^0)\Big]^{-1} = 1 + \overrightarrow{D}_T^1(x^0,y^0;0,y^0). \tag{31}$$

Since $\overrightarrow{D}_T^1(x^1,y^1;0,y^1) = 0$, we deduce that

$$M_o^1 = 1 + L_o^1(y^0,y^1). \tag{32}$$

Finally, we obtain:

$$M_o = (M_o^0 M_o^1)^{\frac{1}{2}} = \Big(\frac{1 + L_o^1(y^0,y^1)}{1 - L_o^0(y^0,y^1)}\Big)^{1/2} = \Big(\prod_{\tau=0,1}\big(1 + (2\tau-1)L_o^\tau(y^0,y^1)\big)^{2\tau-1}\Big)^{1/2}. \tag{33}$$

Notice that if $L_o^0(y^0,y^1)$ and $L_o^1(y^0,y^1)$ are sufficiently small we retrieve the approximation result at the first order established by Boussemart et al. (2003) taking the logarithm on both sides (i.e., $\ln(M_o) \approx L_o$).

Now using equation (30) and (32) yields

$$L_o^0(y^0,y^1) = 1 - \frac{1}{M_o^0} \text{ and } L_o^1(y^0,y^1) = M_o^1 - 1, \tag{34}$$

and

$$L_o(y^0,y^1) = \frac{1}{2}\Big(M_o^1 - (M_o^0)^{-1}\Big) = \frac{1}{2}\sum_{\tau=0,1}(2\tau-1)\Big(M_o^\tau\Big)^{2\tau-1}. \tag{35}$$

This is an exact relation between the geometric mean of the Malmquist index and the arithmetic mean of the proportional Luenberger productivity indicator (33) and the reverse (35). It differs from the approximate relations established in Boussemart et al. (2003) solely in adding the assumption of efficiency with respect to the own period technology.

Notice that quite a few results in index theory require much stronger assumptions. For instance, Caves et al. (1982) derive a relation between the Malmquist and the Törnqvist productivity indices assuming, among others, that firms are cost minimisers or revenue maximisers, which subsumes our own assumption as a special case.

3.2 Exact Relation between Hicks-Moorsteen Index and Luenberger-Hicks-Moorsteen Indicator

Assuming that firms are efficient at each time period, we have, setting $g^0 = (x^0, y^0)$ and $g^1 = (x^1, y^1)$:

$$\begin{aligned} MO^0 &= D_o^0(x^0, y^1)/D_o^0(x^0, y^0) = D_o^0(x^0, y^1) \\ &= [1 + \overrightarrow{D}_T^0(x^0, y^1; 0, y^1)]^{-1} \\ &= [1 - LO^0(y^0, y^1)]^{-1} \ . \end{aligned} \tag{36}$$

Similarly, we obtain the equality:

$$\begin{aligned} MI^0 &= D_i^0(x^1, y^0)/D_i^0(x^0, y^0) = D_i^0(x^1, y^0) \\ &= [1 - \overrightarrow{D}_T^0(x^1, y^0; x^1, 0)]^{-1} \\ &= [1 - LI^0(x^0, x^1)]^{-1} \ . \end{aligned} \tag{37}$$

It follows that

$$HM^0 = \frac{1 - LI^0(x^0, x^1)}{1 - LO^0(y^0, y^1)}. \tag{38}$$

Symmetrically, we obtain:

$$HM^1 = \frac{1 - LI^1(x^0, x^1)}{1 - LO^1(y^0, y^1)}. \tag{39}$$

It follows that

$$HM = \Big[\frac{1 - LI^0(x^0, x^1)}{1 - LO^0(y^0, y^1)} \cdot \frac{1 - LI^1(x^0, x^1)}{1 - LO^1(y^0, y^1)}\Big]^{1/2} \ . \tag{40}$$

It is easy to see that one can retrieve the approximation result at the first order by Briec and Kerstens (2004) from this formulation.

Assuming that firms are efficient in each time period, and setting $g^0 = (x^0, y^0)$ and $g^1 = (x^1, y^1)$, we have:

$$\begin{aligned} LO^0(y^0, y^1) =& \vec{D}^0_T(x^0,y^0; 0, y^0) - \vec{D}^0_T(x^0,y^1; 0, y^1) \\ =& - \vec{D}^0_T(x^0,y^1; 0, y^1) \\ =& 1 - \frac{1}{D^0_o(x^0,y^1)} = 1 - \left(MO^0\right)^{-1} . \end{aligned} \tag{41}$$

Similarly, we have

$$\begin{aligned} LI^0(y^0, y^1) &= \vec{D}^0_T(x^1,y^0; x^1, 0) - \vec{D}^0_T(x^0, y^0; x^0, 0) \\ &= \vec{D}^0_T(x^1,y^0; x^1, 0) \\ &= 1 - \frac{1}{D^0_i(x^1,y^0)} = 1 - \left(MI^0\right)^{-1} . \end{aligned} \tag{42}$$

It follows that

$$LHM^0(x^0, y^0, x^1, y^1) = \left(MI^0\right)^{-1} - \left(MO^0\right)^{-1}. \tag{43}$$

Using a symmetrical procedure one can show that:

$$LHM^1(x^0, y^0, x^1, y^1) = \left(MI^1\right)^{-1} - \left(MO^1\right)^{-1}. \tag{44}$$

Hence:

$$LHM(x^0, y^0, x^1, y^1) = \frac{1}{2}\left[\left(MI^0\right)^{-1} - \left(MO^0\right)^{-1} + \left(MI^1\right)^{-1} - \left(MO^1\right)^{-1}\right] . \tag{45}$$

Again imposing the assumption that firms are efficient in each time period, we obtain an exact relation between the geometric mean Hicks-Moorsteen index and the arithmetic mean Luenberger-Hicks-Moorsteen productivity indicator (40) and the reverse (45). This assumption is again the only price to pay for moving from approximate results in Briec and Kerstens (2004) to these exact results.

4 General Exact Relations by Transposing the Algebraic Structure

4.1 Introduction

In the following we show that an exact relationship between Luenberger indicators and Malmquist index can be deduced from a suitable transposition of the algebraic structure the Euclidean space is endowed with. This is done using a special type of Maslov semimodules structures. Idempotent analysis, or the study of Maslov semimodules, has applications in optimization, optimal control, and game theory. The basic algebraic structures of semirings and of Maslov semimodules over a semiring are presented in Litvinov, Maslov and Shpitz (2001). This exact relation is more general in that we no longer impose the assumption that firms are efficient in each time period, as in the previous section. Instead, the proportional directional distance function is replaced by a directional distance function with a direction vector equal to unity.

To be more precise, let us denote by $\mathbb{1}_p$ the vector of $\mathbb{R}^p$ whose coordinates are all equal to 1. For z and z' in $(\mathbb{R} \cup \{-\infty\})^p$ let $d_{\mathrm{M}}(z, z') = \| \mathbf{e}^z - \mathbf{e}^{z'} \|_\infty$ where $\mathbf{e}^z = (e^{z_1}, \cdots, e^{z_p})$, with the convention $e^{-\infty} = 0$, and, for $y \in \mathbb{R}^p_+$, $\| y \|_\infty = \max_{i=1,\ldots,p} y_i$. The map $z \mapsto \mathbf{e}^z$ is a homeomorphism from $(\mathbb{R} \cup \{-\infty\})^p$ with the metric d_{M} to $\mathbb{R}^p_+$ endowed with the metric induced by the norm $\| \cdot \|_\infty$; its inverse is the map $\mathbf{ln}(y) = (\ln(y_1), \cdots, \ln(y_p))$ from $\mathbb{R}^p_+$ to $(\mathbb{R} \cup \{-\infty\})^p$, with the convention $\ln(0) = -\infty$.

The basic idea is to show that the Shephard distance function can be related to the directional distance function when it is computed on a logarithmic transformation of the data. At this stage we define the output directional distance function over $\mathbb{M}^p$ as:

$$\overrightarrow{D}_T^{\tau,\ln}(x, y; 0, \mathbb{1}_p) = \sup \Big\{ \delta : \mathbf{ln}(y) + \delta \mathbb{1}_p \in \mathbf{ln}\big(P^\tau(x)\big) \Big\} \tag{46}$$

if $\mathbf{ln}(y) + \delta \mathbb{1}_p \in \mathbf{ln}\big(P^\tau(x)\big)$ for some $\delta \in \mathbb{R}$ and takes the value $-\infty$ otherwise. This directional distance function with direction vector equal to the unit vector is also known as the translation function.

4.2 General Exact Relation between Luenberger Indicator and Malmquist Index

Let us define the corresponding Luenberger index as

$$L_o^{\ln}(\mathbb{1}_p, \mathbb{1}_p) = \frac{1}{2}\Big[\big(\overrightarrow{D}_T^{0,\ln}(x^0, y^0; 0, \mathbb{1}_p) - \overrightarrow{D}_T^{0,,\ln}(x^1, y^1; 0, \mathbb{1}_p)\big) + \big(\overrightarrow{D}_T^{1,\ln}(x^0, y^0; 0, \mathbb{1}_p) - \overrightarrow{D}_T^{1,\ln}(x^1, y^1; 0, \mathbb{1}_p)\big)\Big] . \tag{47}$$

Returning to the Shephard distance function and setting $\mu = \ln(\lambda)$ we obtain:

$$D_o^\tau(x, y) = \inf\big\{\lambda > 0 : \mathbf{ln}\big(\frac{y}{\lambda}\big) \in \mathbf{ln}\big(P^\tau(x)\big)\big\} \tag{48}$$

$$= \inf\big\{\lambda > 0 : \mathbf{ln}(y) - \ln(\lambda)\mathbb{1}_p \in \mathbf{ln}\big(P^\tau(x)\big)\big\} \tag{49}$$

$$= \inf\big\{\exp(\mu) : \mathbf{ln}(y) - \mu\mathbb{1}_p \in \mathbf{ln}\big(P^\tau(x)\big)\big\} \tag{50}$$

$$= \exp(\big\{\inf\big\{\mu : \mathbf{ln}(y) - \mu\mathbb{1}_p \in \mathbf{ln}\big(P^\tau(x)\big)\big\}\big) . \tag{51}$$

Hence, setting $\delta = -\mu$ yields

$$D_o^\tau(x, y) = \exp\big(-\overrightarrow{D}_T^{\tau,\ln}(x, y; 0, \mathbb{1}_p)\big) . \tag{52}$$

An elementary calculus yields now

$$M_o = \exp\Big(L_o^{\ln}(\mathbb{1}_p, \mathbb{1}_p)\Big) . \tag{53}$$

Or equivalently

$$L_o^{\ln}(\mathbb{1}_p, \mathbb{1}_p) = \ln\Big(M_o\Big) . \tag{54}$$

Thus, the output-oriented Luenberger productivity indicator with unit direction vector defined with respect to the logarithm of technology equals the logarithm of the output-oriented Malmquist productivity index (54). Reversely, the output-oriented Malmquist productivity index can be obtained from an exponential transformation of the former output-oriented Luenberger productivity indicator (53). Obviously, a similar exact relation could be defined for the input-oriented versions of these productivity indices.

4.3 General Exact Relation between Luenberger-Hicks-Moorsteen Indicator and Hicks-Moorsteen Index

Paralleling the earlier definition, we define the Luenberger-Hicks-Moorsteen indicator as

$$LHM^{\ln}(\mathbb{1}_n, \mathbb{1}_p) = \tfrac{1}{2}\Big[LHM^{0,\ln}(\mathbb{1}_p, \mathbb{1}_p) + LHM^{1,\ln}(\mathbb{1}_p, \mathbb{1}_p)\Big] \ . \tag{55}$$

At time period $t = 0$, we have:

$$LHM^{0,\ln}(\mathbb{1}_n, \mathbb{1}_p, \mathbb{1}_n, \mathbb{1}_p) = LO^{0,\ln}(\mathbb{1}_p, \mathbb{1}_p) - LI^{0,\ln}(\mathbb{1}_n, \mathbb{1}_n), \tag{56}$$

where

$$LO^{0,\ln}(\mathbb{1}_p, \mathbb{1}_p) = \overrightarrow{D}_T^{0,\ln}(x^0, y^0; 0, \mathbb{1}_p) - \overrightarrow{D}_T^{0,\ln}(x^0, y^1; 0, \mathbb{1}_p) \tag{57}$$

and

$$LI^{0,\ln}(\mathbb{1}_n, \mathbb{1}_n) = \overrightarrow{D}_T^{0,\ln}(x^1, y^0; \mathbb{1}_n, 0) - \overrightarrow{D}_T^{0,\ln}(x^0, y^0; \mathbb{1}_n, 0) \ . \tag{58}$$

At time period $t = 1$:

$$LHM^{1}(\mathbb{1}_n, \mathbb{1}_p, \mathbb{1}_n, \mathbb{1}_p) = LO^{1,\ln}(\mathbb{1}_p, \mathbb{1}_p) - LI^{1,\ln}(\mathbb{1}_n, \mathbb{1}_n), \tag{59}$$

where

$$LO^{1,\ln}(\mathbb{1}_p, \mathbb{1}_p) = \overrightarrow{D}^{1,\ln}(x^1, y^0; 0, \mathbb{1}_p) - \overrightarrow{D}^{1,\ln}(x^1, y^1; 0, \mathbb{1}_p) \tag{60}$$

and

$$LI^{1}(\mathbb{1}_n, \mathbb{1}_p) = \overrightarrow{D}^{1,\ln}(x^1, y^1; \mathbb{1}_n, 0) - \overrightarrow{D}^{1,\ln}(x^0, y^1; \mathbb{1}_n, 0) \ . \tag{61}$$

Now, let us denote:

$$\overrightarrow{D}_T^{\tau,\ln}(x, y; \mathbb{1}_n, 0) = \sup\Big\{\delta : \mathbf{ln}(x) + \delta \mathbb{1}_n \in \mathbf{ln}\big(L^{\tau}(y)\big)\Big\}. \tag{62}$$

Using a procedure similar to the one described in equations (48) to (51) we obtain:

$$D_i^{\tau}(x, y) = \exp\big(- \overrightarrow{D}_T^{\tau,\ln}(x, y; \mathbb{1}_n, 0)\big). \tag{63}$$

Hence, from (63) we have:

$$LHM^{\ln}(\mathbb{1}_n, \mathbb{1}_p) = \ln\Big(HM\Big) \ . \tag{64}$$

Or equivalently

$$HM = \exp\left(LHM^{\ln}(\mathbb{1}_n, \mathbb{1}_p)\right). \tag{65}$$

Concluding, the Luenberger-Hicks-Moorsteen productivity indicator with unit direction vectors defined with respect to the logarithm of technology equals the logarithm of the Hicks-Moorsteen index (64). Again, the Hicks-Moorsteen index is linked via an exponential transformation Luenberger-Hicks-Moorsteen indicator (65).

5 Conclusions

Existing approximate relations between primal discrete-time indices and indicators developed in Boussemart et al. (2003) and Briec and Kerstens (2004) are strengthened towards exact relations at a minimal cost in terms of additional assumptions. The first exact relations are based upon choosing a proportional distance function and assuming efficiency within each time period. The exact relations based upon a logarithmic transformation of the data just require choosing a direction vector equal to the unit vector. This result could shed some light on the problem of the choice of direction vector in the directional distance function, probably making the unit vector an interesting alternative compared to the widespread use of the observation itself (the latter leading to the proportional distance function).

These results can be useful for applied researchers to interpret similarities and differences in magnitudes of empirical results that follow from opting for different primal productivity indices and indicators.

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