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Conflict Networks

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Abstract
Conflict parties are frequently involved into more than one conflict at a given time. In this paper the interrelated structure of conflictive relations is modeled as a conflict network where opponents are embedded in a local structure of bilateral conflicts. Conflict parties invest in specific conflict technology to attack their respective rivals and defend their own resources. We show that there exists a unique equilibrium for this conflict game and examine the relation between aggregated equilibrium investment (interpreted as conflict intensity) and underlying network characteristics. The derived results have implications for peaceful resolutions of conflicts because neglecting the fact that opponents are embedded into an interrelated conflict structure might have adverse consequences for conflict intensity.

JEL Classification: C72, D74, D85

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1 Introduction

Violent conflicts and wars are frequently observed phenomena in human history and have always been in the research focus of social scientists. Recently, the economists profession became interested in establishing general, and therefore abstract models of conflicts based on a game-theoretical setup. Most of this growing literature is based on stylized models of unique and isolated conflicts between two or more conflict parties. The nature of the respective conflict is then analyzed depending on specific idiosyncratic characteristics of the involved conflict parties, e.g., Esteban and Ray (2008), Esteban and Ray (1999), Caselli and Coleman (2006), Basu (2005), Skaperdas (1992), and Beviá and Corchón (2008). We depart from this literature by assuming that conflictive relations between conflict parties are interrelated, i.e., a conflict party may be involved into two or more different conflicts involving different opponents at the same time. This interrelated structure results in local externalities when behavior by agents is affected by changes in behavior of direct (and also indirect) opponents. Hence, our objective is to clarify the relation between the structure of those interrelated conflicts and equilibrium behavior.

In this paper we restrict our analysis to bilateral conflicts which allows us to represent the overall conflict structure as a network, or graph, where conflict parties are linked if they are in a conflictive relation among each other. Hence, we can interpret the structure of interrelated conflicts as a simultaneously played conflict game, consisting of several distinctive bilateral conflicts, played on a fixed and given network.

Conflict parties can affect the probability of winning against a particular direct rival by investing into conflict specific technology, e.g., military equipment, mercenaries, etc. The network structure of conflictive relations implies that the investment decision of a conflict party may not only affect the decision of direct rivals but also of other parties that are not directly involved. This type of interdependencies might induce spill-over effects that are common in network models of social interaction.

We model each bilateral conflict as a transfer contest, where contested resources are transfered from the winner to the loser, see Appelbaum and Katz (1986), Hillman and Riley (1989), and Leininger (2003).[1] Hence, resources are reallocated between direct rivals depending on their relative conflict investment. This implies that conflict investment is socially inefficient because conflict investment is social waste. In principle, a conflict party has also the option not to invest anything into a bilateral conflict which would reduce social waste. However, this option can be exploited by their rival and therefore will never occur in equilibrium, i.e., investing into conflict specific technology resembles a

[1] The model is formulated in general terms such that different interpretations for the underlying conflictive environment are possible: For instance, lobbying of several firms for several distinctive issues at different authorities could also be interpreted as a network of bilateral conflicts where two firms are connected if they lobby for the same issue. Analyzing the dependence of overall lobbying activity on the underlying relations of lobby issues and firms is an important issue due to the social waste that is generated by these activities.

prisoner's-dilemma structure.

In this study we are especially interested in the relation between conflict intensity (measured as the total equilibrium investment into conflict technology by all opponents) and the underlying network structure. For reasons of tractability we restrict our analysis in the first part of the paper to the three classes of conflict structures.[2]

- Regular conflict networks are characterized by a large degree of symmetry among the opponents. In this context we briefly discuss the anthropological concept of conflictive peer polity interaction, that can be interpreted as a description of regular conflict networks.

- Star-shaped conflict networks are characterized by a large degree of asymmetry between a center and its periphery. Historically, the multitude of conflictive relations among an empire and its surrounding neighbors, e.g., the Western Roman Empire, has this kind of core-periphery structure.

- Complete bipartite conflict networks consist of two coalitions that are in conflict against each other. Ideological conflicts can be interpreted as bipartite conflict networks because all members of one coalition share the same ideology and consider each member of the hostile coalition as a potential enemy, as it could be observed in the ideological conflicts of the 20th century, e.g., World War II.

For these classes of conflict networks we find an intuitive relation between the underlying network characteristics and conflict intensity: Within each considered class, conflict intensity is increasing in the number of conflictive relations and the density of the network. Based on a prominent centrality measure, i.e., eigenvector centrality, we are also able to compare conflict intensity across the three different classes of conflict networks.

The relation between conflict intensity and network characteristics can also be stated from the perspective of peaceful conflict resolution. Peaceful resolution of conflicts is here interpreted as an exogenous ad-hoc deletion of specific conflictive links within the conflict network.[3] For each of the three considered classes the established positive relation between conflict intensity and the number of bilateral conflicts implies then that peaceful conflict resolution is beneficial because total conflict intensity is decreased if the number of links in the respective class is reduced.

However, this result does not carry over to conflict networks that are not in the considered classes. In fact, peaceful resolution of bilateral conflicts might have adverse consequences: we provide a specific example outside the considered

[2]The examples of historical conflicts mentioned below are also frequently applied in theoretical concepts from social anthropology. We discuss those connections briefly in the main text.

[3]In most cases the conflict parties that are directly affected by peaceful conflict resolution will benefit because no socially-wasteful conflict investments will be exerted for a resolved conflict. Hence, for the affected conflict parties peaceful conflict resolution can be interpreted as exogenously enforced solution of the prisoner dilemma situation of conflict investment.

classes where conflict intensity is increased as a consequence of peaceful conflict resolution.

Obtaining general results for irregular networks is a complex issue due to the fact that no closed form solution for an equilibrium exist. Nevertheless, we are able to characterize indirectly equilibrium behavior in general irregular networks. This equilibrium characterization can be used to derive the counterintuitive result that an agent spending relatively more in total conflict investment than its opponent will in expectation loose the bilateral conflict against the respective rival.

Our approach is related to the recent network literature that considers games that are played on a fixed and given network structure, for instance, Bramoullé and Kranton (2007), Goyal and Moraga-González (2001), Calvó-Armengol and Zenou (2004) and Ballester et al. (2006). As we are interested in local externalities, we depart from this literature in an important aspect: In our set-up the individual action is link-specific (and therefore multi-dimensional) because conflict investment is specific for each bilateral conflict.[4] This is in contrast with most of the network literature based on games played on fixed and given networks where an individual's strategy space is usually assumed to be unidimensional (and hence common for all neighbors).[5] Our extension provides a richer structure that also allows to analyze explicitly how a specific agent reallocates her conflict investment among its different bilateral conflicts.

Besides this difference there exists a close relation to Ballester et al. (2006) where also the consequences of the network structure for aggregated equilibrium actions are analyzed. In their model simple linear quadratic payoff functions are considered that facilitate the analysis substantially. However, we are interested in the analysis of conflict situations and therefore consider payoff functions that are based upon so called contest success functions. This type of functional form is frequently applied in the literature on conflict analysis and has a simple and intuitive interpretation in this context. The caveat is that this functional form is not linear quadratic which makes the analysis more complex.

The rest of the paper is structured as follows. In the next section we set up a general model of conflict networks and show, in section 3, that a unique equilibrium exists for our framework. We analyze three specific classes of conflict network in section 4, and reinterpret and discuss our results from the perspective of peaceful conflict resolution in section 5. In section 6 we analyze irregular conflict networks. Finally, section 7 concludes.

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[4] This might be due to the different nature of the conflict, e.g., naval forces are more suited than air forces in specific conflicts. Moreover, even if the same type of force is suitable in various conflicts, the conflict party has to decide where they should be employed. Link-specificity then simply implies that forces cannot be employed at two distinct locations at the same time.

[5] For a recent exception with multi-dimensional individual strategy space, see Goyal et al. (2008).

2 The Model

There is a set $N = \{1, \ldots, n\}$ of conflicting parties (from now on called agents) that are embedded in a fixed structure of bilateral conflicts, i.e., each agent i is engaged into bilateral conflicts with some agents called her rivals. The set of rivals of agent i is denoted by $N_i \subseteq N \setminus \{i\}$ which implies that agent i is involved in $n_i = |N_i|$ conflicts. The underlying structure of bilateral conflicts can be interpreted as a fixed network which is represented by a graph $\mathbf{g}$ consisting of nodes (agents) and links (conflicts). Hence, if agent i is in conflict with opponent j then $g_{ij} = 1$, while if there is no conflictive relation between them then $g_{ij} = 0$. It is assumed that both opponents in a bilateral conflict are affected in the same way, i.e., the network is undirected and symmetric: $g_{ij} = g_{ji}$ for all $i \neq j$. The set N_i of rivals of agent i can then be defined as $N_i = \{j \in N \setminus \{i\} : g_{ij} = 1\}$.

The outcome of each bilateral conflict is probabilistic and depends on the investment into conflict specific technology by the respective rivals. The investment of agent i into the conflict against rival $j \in N_i$ is denoted by $e_{ij} \in \Re_+$ and the n_i-dimensional vector of conflict spendings of agent i against all her rivals (her strategy) by $\mathbf{e_i} = (e_{ij})_{j \in N_i}$. The vector of conflict spending that is directed against agent i by all of her respective rivals is denoted by $\mathbf{e_{-i}} = (e_{ji})_{j \in N_i}$.

Our study is focused on the analysis of the effects of the network structure on equilibrium outcome. As the network structure by itself induces endogenously heterogeneity on the agents (depending on their location), we exclude all other sources of heterogeneity in our model to be able to concentrate exclusively on this channel.[6] Hence, it is assumed that all bilateral conflicts are symmetric in the sense that rivals have identical perceptions with respect to potential gains and losses in each bilateral conflict in which they are involved: If agent i wins the conflict against any of her rivals $j \in N_i$ she obtains an amount V of resources of rival j, if agent i looses against j an amount V of her own resources are transferred to the winning agent j, and vice versa.[7] In other words, as a result of the conflict contested resources are purely redistributed among direct rivals, i.e., the loser has to fully compensate the winner. This assumption reflects the frequently observed fact that underlying motivations for conflict are contested natural resources, or territory, and that looting is and was a frequently observed behavior of the winning conflict party. [8]

The outcome of each bilateral conflict is governed by a probability function that maps the conflict specific investments of the respective two opposing rivals into a probability to win the respective conflict, i.e., agent i wins the bilateral conflict against rival j with probability $p_{ij} = p(e_{ij}, e_{ji}) \in [0, 1]$, which is twice differentiable, increasing and strictly concave in own spendings e_{ij} for each level

[6] In section 7 we briefly discuss the consequences of an additional source of heterogeneity.

[7] This assumption implies that agents that have more rivals might potentially gain more but also loose more resources than agents with a lower number of hostile neighbors, see also the discussion in section 7.

[8] In Collier and Hoeffler (2004), for instance, it is shown that economic factors ('greed'), like primary commodities and opportunity costs for conflict activity, have more predictive power for the outbreak of civil war than political factors ('grievance'), e.g. inequality or ethnic polarization.

of spending e_{ij} by its respective opponent j. It is also assumed that p_{ij} is decreasing and strictly convex in the spending e_{ji} of its rival j. Moreover, it is symmetric in the sense that if two direct rivals i and j spend the same amount, $e = e_{ij} = e_{ji}$, then they will win the conflict with the same probability: $p_{ij} = p(e, e) = p_{ji}$.

Spending in conflict against rivals is related with a cost $c(\mathbf{e_i})$ that is a continuous, increasing and convex function with $c(0, \ldots, 0) = 0$.

The expected payoff function of agent i is additively separable in costs and expected wins and losses of all bilateral conflicts in which she is involved, and can be stated in the following way:

$$\pi_i(\mathbf{e_i}, \mathbf{e_{-i}}; \mathbf{g}) = \sum_{j \in N_i} p_{ij} V - \sum_{j \in N_i} p_{ji} V - c(\mathbf{e_i}).$$

For notational simplicity we reformulate this expression as follows:

$$\pi_i(\mathbf{e_i}, \mathbf{e_{-i}}; \mathbf{g}) = W(\mathbf{e_i}, \mathbf{e_{-i}}) - c(\mathbf{e_i}), \tag{1}$$

where $W(\mathbf{e_i}, \mathbf{e_{-i}}; \mathbf{g}) = V \sum_{j \in N_i} (p_{ij} - p_{ji})$ denotes the expected 'revenue' of conflict for agent i, i.e., the aggregated expected amount of transfered resources that agent i wins or looses in all her bilateral conflicts. Note that, due to the fact that each conflict is modeled as a transfer contest where losers have to compensate the winner, the total expected revenue of the overall conflict game (or, in other words, the aggregated value of contested and transfered resources) is zero independently of the network structure:

$$\sum_{i \in N} W(\mathbf{e_i}, \mathbf{e_{-i}}; \mathbf{g}) = 0. \tag{2}$$

This implies that equilibrium behavior does purely depend on the strategic response to the network structure and is not confounded by the fact that different network structures induce different values of aggregated resources.

Our objective is the analysis of overall conflict intensity, denoted by $E^*(\mathbf{g})$, and formally defined as the aggregated level of conflict investment in equilibrium by all agents in all bilateral conflicts:

$$E^*(\mathbf{g}) = \sum_{i \in N} E_i^*(\mathbf{g}) = \sum_{i \in N} \sum_{j \in N_i} e_{ij}^*(\mathbf{g}),$$

where $E_i = \sum_{j \in N_i} e_{ij}$ is the aggregated conflict investment of agent i against all her rivals $j \in N_i$. The variation of conflict intensity for different networks can then be determined by analyzing how $E^*(\mathbf{g})$ depends on the variables that characterize the respective network structure.

3 Equilibrium Analysis

For the conflict network game we will establish the existence of a unique equilibrium by using results from Rosen (1965) and Goodman (1980).[9] They establish existence and uniqueness for a concave n-person game based on the concept of diagonal strict concavity of the joint (and weighted) payoff function:[10]

$$\sigma(\mathbf{e},\mathbf{r}) = \sum_{i\in N} r_i\, \pi_i(\mathbf{e_i},\mathbf{e_{-i}}), \tag{3}$$

where $\mathbf{e} = (\mathbf{e_1},\dots,\mathbf{e_n})$, $\mathbf{r} = (r_1,\dots,r_n)$, and $r_i \geq 0$. Intuitively, this technical condition guarantees that an agent has more control over her payoff than the other players.

The conflict network game satisfies the requirement of a concave n-person game by assumption. Theorem 1 and 2 of Rosen (1965) imply that a unique equilibrium exists in a concave n-person game with orthogonal constraint set[11] if and only if the function $\sigma(\mathbf{e},\mathbf{r})$ is diagonally strictly concave. Using a characterization of Goodman (1980), the following conditions on the payoff functions are equivalent for diagonally strict concavity of function $\sigma(\mathbf{e},\mathbf{r})$:

(i) $\pi_i(\mathbf{e_i},\mathbf{e_{-i}})$ is strictly concave in $\mathbf{e_i}$ for all $\mathbf{e_{-i}}$,

(ii) $\pi_i(\mathbf{e_i},\mathbf{e_{-i}})$ is convex in $\mathbf{e_{-i}}$ for all $\mathbf{e_i}$,

(iii) $\sigma(\mathbf{e},\mathbf{r})$ is concave in $\mathbf{e}$ for some $\mathbf{r}$ with $r_i > 0$ for all $i \in N$.

Proposition 1 *There exists a unique equilibrium in the conflict network game.*

Proof. The conflict network game is a concave n-person game by assumption. To proof that it is also diagonally strictly concave it suffices to show that conditions (i), (ii) and (iii) are satisfied.

(i) $W(\mathbf{e_i},\mathbf{e_{-i}})$ is strictly concave in $\mathbf{e_i}$ for all $\mathbf{e_{-i}}$ and for all $i \in N$ because its Hessian is negative definite, i.e., it is a (diagonal) matrix with entries $\frac{\partial^2 W(\mathbf{e_i},\mathbf{e_{-i}})}{\partial e_{ij}^2} = \frac{\partial^2 p_{ij}}{\partial e_{ij}^2} - \frac{\partial^2 p_{ji}}{\partial e_{ij}^2} < 0$ on the diagonal (the last inequality holds by assumption) and $\frac{\partial^2 W(\mathbf{e_i},\mathbf{e_{-i}})}{\partial e_{ij}\partial e_{ik}} = \frac{\partial^2 p_{ij}}{\partial e_{ij}\partial e_{ik}} = 0$ elsewhere for all $j,k \in N_i$ with

[9]The conflict game is neither an aggregative game (the reaction functions cannot be expressed in terms of aggregated strategies of all the other players), nor a supermodular game (because the reaction functions are non-monotonic). This implies that common existence proofs that are based on those characteristics, e.g. Galeotti et al. (2009), are not applicable in this setup.

[10]For reasons of notational simplicity the dependence of the payoff functions on graph $\mathbf{g}$ is suppressed in the following paragraphs.

[11]A constraint set is orthogonal if it is uncoupled. This is the case in the conflict game because the strategy space of each individual does not depend on the strategies of her rivals. Note also that in Rosen (1965) the strategy space for each $i \in N$ is convex and compact, which is, in principle, not the case in the conflict game defined above (here it is the non-negative orthant). However, we can construct a (sufficiently high) upper limit $\bar{e}$ such that all strategies $e_{ij} > \bar{e}$ are strictly dominated (for instance by choosing $e_{ij} = 0$) due to the fact that $W(\mathbf{e_i},\mathbf{e_{-i}}) \in [-n_iV, n_iV]$ is bounded while $c(\mathbf{e_i})$ is unbounded. Hence, without loss of generality we can restrict attention to the strategy space $[0,\bar{e}]^{n_i}$ of non-dominated strategies for each individual $i \in N$ which is convex and compact.

$j \neq k$. As $c(\mathbf{e_i})$ is convex for all $i \in N$, the payoff function $\pi_i(\mathbf{e_i}, \mathbf{e_{-i}})$ is the sum of a constant, a strictly concave and a concave function in $\mathbf{e_i}$. Hence, it is strictly concave in $\mathbf{e_i}$ for all $\mathbf{e_{-i}}$ and for all $i \in N$.

(ii) Note that $W(\mathbf{e_i}, \mathbf{e_{-i}}) = V \sum_{j \in N_i} (p_{ij} - p_{ji})$. Take a generic factor of this sum, for instance $(p_{ij} - p_{ji})$. It is strictly convex in e_{ji} for $j \in N_i$ because $\frac{\partial^2 p_{ij}}{\partial e_{ji}^2} - \frac{\partial^2 p_{ji}}{\partial e_{ji}^2} > 0$ by assumption. Moreover, it is convex in $\mathbf{e_{-i}}$ for all $\mathbf{e_i}$ because its Hessian is positive semi-definite, i.e., it is a (diagonal) matrix with positive entries, $\frac{\partial^2 (p_{ij} - p_{ji})}{\partial e_{ji}^2} > 0$, or zero entries, $\frac{\partial^2 (p_{ij} - p_{ji})}{\partial e_{ki}^2} = 0$, on the diagonal and zero entries, $\frac{\partial^2 (p_{ij} - p_{ji})}{\partial e_{ji} \partial e_{ki}} = 0$, elsewhere for all $j, k \in N_i$ with $j \neq k$. This implies that $W(\mathbf{e_i}, \mathbf{e_{-i}})$ is a sum of functions that are all convex in $\mathbf{e_{-i}}$ for all $\mathbf{e_i}$. Hence, $W(\mathbf{e_i}, \mathbf{e_{-i}})$ is also convex in $\mathbf{e_{-i}}$ for all $\mathbf{e_i}$. As the cost function does not depend on conflict spending of the rivals, the function $\pi_i(\mathbf{e_i}, \mathbf{e_{-i}})$ is convex in $\mathbf{e_{-i}}$ for all $\mathbf{e_i}$.

(iii) Assume that $r_i = r > 0$ for all $i \in N$. Then Eq. (3) simplifies substantially due to the fact that the aggregated value of contested resources is zero, compare Eq. (2):

$$\sigma(\mathbf{e}, \mathbf{r}) = -r \sum_{i \in N} c(\mathbf{e_i})$$

By assumption, the cost function is convex in own conflict spending. Hence, the function $\sigma(\mathbf{e}, \mathbf{r})$ is a sum of concave functions which is also concave. ■

To derive closed form equilibrium expressions we adopt the following functional form for the pay-off function.[12] The cost function has the following quadratic form:

$$c(\mathbf{e_i}) = c(E_i) = (E_i)^2 = (\sum_{j \in N_i} e_{ij})^2. \tag{4}$$

This functional form captures the externalities of the network structure because the marginal cost of conflict technology for a specific bilateral conflict also depends on the spending in all other bilateral conflicts in which agent i is involved.[13] Intuitively, this could be attributed to the fact that agent i has access to a centralized (and convex) production process where conflict technology for all her different bilateral conflicts has to be produced. Agent i then allocates conflict technology to the different bilateral conflicts in which she is involved.

The outcome of a bilateral conflict is realized according to a contest success function in the style of Tullock (1980), which is frequently applied in models of conflict and contests.[14] Under this contest success function the winning

[12] In section 7 the relevance of the specific functional form for the derived results is discussed.

[13] An additive separable cost function, for instance $c(\mathbf{e_i}) = \sum_{j \in N_i} (e_{ij}^2)$, would not induce externalities because neither marginal benefits, nor marginal costs are affected by the network structure of conflicts. For this scenario the network structure would not have any impact. As we are interested in the externalities that are induced by the underlying network, we stick to the functional form presented above.

[14] Recent surveys that review the literature that is based on this functional form are Corchon (2007) and Konrad (2007, 2009) for models of contests, as well as Garfinkel and Skaperdas (2006) for conflict models.

probability of agent i in the bilateral conflict against rival j is simply determined as the relation between individual conflict investments:

$$p_{ij} = \begin{cases} \frac{e_{ij}}{e_{ij}+e_{ji}} & \text{if } e_{ij}+e_{ji} > 0, \\ 1/2 & \text{if } e_{ij}+e_{ji} = 0. \end{cases} \tag{5}$$

This functional form does not fit exactly the setup as introduced before because of its discontinuity at point $(0, 0)$, i.e., in the case that two rivals do not spend anything in the respective bilateral conflict. However, properties (i), (ii) and (iii) of the existence proof in Proposition 1 still hold for this specific functional form. In the appendix it is shown how the existence result from Proposition 1 can still be applied to this framework.

Corollary 10 in the appendix states that the unique equilibrium in the conflict network game is interior. This implies that equilibrium investment turns out to be socially inefficient: Resources are merely reallocated while all agents invest positive amounts into conflict technology and therefore face real costs. Hence, aggregated expected equilibrium payoff will be negative. This captures the idea that conflicts are generally highly destructive and socially undesirable. In fact, the socially efficient outcome in this kind of conflict game would be not to invest in conflict spending at all. In principle, all agents could voluntarily decide to refrain from conflict investment. However, as in the prisoner's dilemma, the strategy of not investing into a specific bilateral conflict is exploitable by the respective rival. To avoid exploitation all agents invest in equilibrium strictly positive amounts in each of their bilateral conflicts.

Interiority also implies that the equilibrium solves the following system of first order conditions:

$$\frac{e^*_{ki}(\mathbf{g})}{(e^*_{ik}(\mathbf{g}) + e^*_{ki}(\mathbf{g}))^2} V = E^*_i(\mathbf{g}), \qquad \text{for all } k \in N_i \text{ and all } i \in N. \tag{6}$$

This is a non-linear system with $\sum_{i \in N} n_i$ equations that does not allow a closed form solution for general conflict structures. Therefore we will concentrate our analysis at first on three distinct classes of more structured conflict networks that allow closed form solutions of the above system. An analysis for general irregular networks follows in section 6.

4 Characteristic Classes of Conflict Networks

The three considered classes of conflict networks are distinct with respect to their grade of symmetry. In our framework asymmetry is induced through the underlying network structure in the sense that agents with a high number of conflictive relations can potentially gain and also loose more resources than agents with less conflicts. Hence, we consider on one side highly symmetric conflict structures where each agent has the same number of conflicts, and, on the other side, highly asymmetric conflict networks, e.g., conflicts among center and periphery. An intermediate class are complete bipartite conflict networks consisting of two hostile coalitions where members of one coalition are in conflict with each member of the opposed coalition.

The considered classes of conflict networks share some characteristics that are emphasized in anthropological theories of (hostile) interaction among societies. Although the conflict game is highly stylized we try to relate and discuss the derived results in light of this literature.

4.1 Regular Conflict Networks

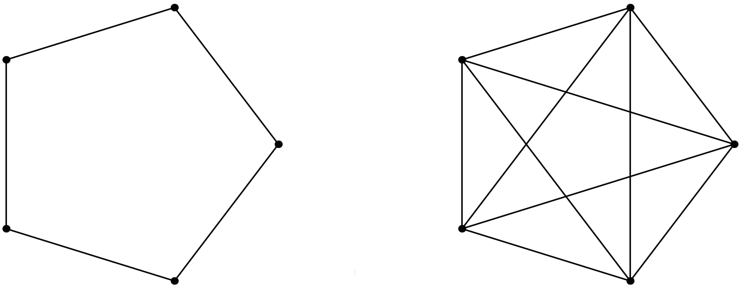
Regular conflict networks are characterized by their high degree of symmetry among rivals. The symmetry property among social entities in a local environment is also the crucial element in the concept of 'peer polity interaction'. This concept was introduced in Renfrew and Cherry (1986) to describe the historical fact that complex societies often developed through interaction of autonomous and homogeneous social units that were not related to each other in forms of dominance and subordination. Peer polity interaction also included warfare and conflict. Historical examples that could be subsumed under this concept are:

> The Mycenaean states, the later small city-states of the Aegean and the Cyclades, or the centers of the Maya Lowlands, that interact on an approximately equal level. [...] The evolution of such clusters of peer polities is conditioned not by some dominant neighbor, but more usually by their own mutual interaction, which may include both exchange and conflict. Tainter (1988, p. 201)

Our focus is on hostile interaction among peer polities and we associate the symmetric nature of peer polity interaction with a regular conflict network.

Formally, a graph $\mathbf{g}^R$ is called *regular of degree d* if each agent $i \in N$ has the same number d of opponents: $n_i = d$ for all $i \in N$. Hence, a regular graph $\mathbf{g}^R$ can be characterized by its degree d and the total number n of agents. The corresponding class of regular networks is denoted by R and incorporates cases such as the fully connected network, where $d = n - 1$, and a ring structure, where $d = 2$, compare figure 1.

Figure 1: Regular Conflict Structures: Ring (left) and Complete Network (right)



The following proposition describes the relation between those characteristics and conflict intensity for the class R of regular networks.

Proposition 2 *In conflict networks of class R,*

(i) Conflict intensity is increasing in its degree d and in the total number n of agents in the network.

(ii) Conflict intensity in a regular network $\mathbf{g}^{R1}$ is higher than in $\mathbf{g}^{R2}$ if and only if

$$n_1\sqrt{d_1} > n_2\sqrt{d_2}.$$

(iii) Individual conflict investment and expected payoff in equilibrium is decreasing in d and does not depend on n. Moreover, expected equilibrium payoff is negative for all agents.

Proof. By Corollary 10 there exists a unique and interior equilibrium of the conflict game. The following (symmetric) conflict investment $e^* \equiv e^*_{ij}$ for all $i \neq j$ solves the system of first order conditions; hence it must be the unique equilibrium:

$$e^*(\mathbf{g}^R) = \frac{1}{2}\sqrt{\frac{V}{d}}, \qquad \text{for all } i \in N.$$

Total conflict intensity is defined as aggregated equilibrium spending:

$$E^*(\mathbf{g}^R) = \sum_{i\in N}\sum_{j\in N_i} e^*(\mathbf{g}^R) = n\,d\,e^*(\mathbf{g}^R) = \frac{n}{2}\sqrt{d\,V}.$$

The last expression is increasing in the total number of agents and also in its degree d. Simplifying the inequality $E^*(\mathbf{g}^{R1}) > E^*(\mathbf{g}^{R2})$ yields the condition presented in (ii).

As the equilibrium is symmetric, the probability to win (or loose) each bilateral conflict is identical for all agents, i.e., $p^*_{ij} = \frac{1}{2}$ for all $i \neq j$. Hence, expected equilibrium payoff is:

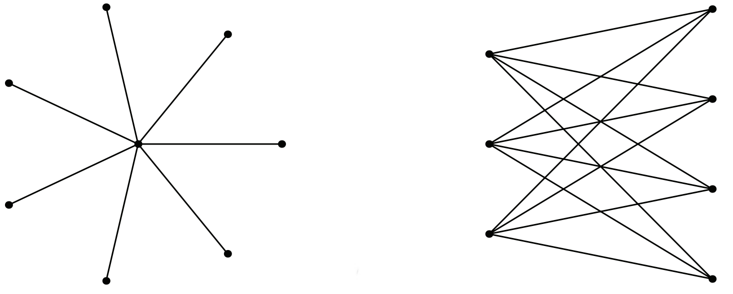
$$\pi(\mathbf{e}^*_\mathbf{i}, \mathbf{e}^*_{-\mathbf{i}}; \mathbf{g}^R) = -d\frac{V}{4}, \qquad \text{for all } i \in N.$$

Clearly, $e^*(\mathbf{g}^R)$ and $\pi(\mathbf{e}^*_\mathbf{i}, \mathbf{e}^*_{-\mathbf{i}}; \mathbf{g}^R)$ are decreasing in d and independent of n which establishes the statements of the proposition. ■

In equilibrium all agents choose the same level of conflict investment which implies that they win each bilateral conflict with the same probability. By definition the sum of expected transfered resources is equal to zero for each agent. Hence, equilibrium payoff is negative because an agent also faces the cost of conflict spending. This situation is socially (and also Pareto) inefficient because universal peace would result in zero expected payoff. The fact that such a conflict structure induces socially inefficient results is also acknowledged in the historical analysis of the above mentioned examples:

Successful competition by any Mycenaean polity would yield little real return. The result was probably constant investment in defense, military administration, and petty warfare, with any single polity rarely experiencing a significant return on that investment. (ibid., p. 204).

Figure 2: A Star-Shaped (left) and a Bipartite Conflict Network (right)



4.2 Star-Shaped Conflict Networks

We now focus our attention on asymmetric conflict structures that are star-shaped, i.e., where one agent is in conflict with all other remaining rivals while none of the rivals is in conflict with each other, see the left part of figure 2. This class of conflict networks has a center-periphery structure which is reminiscent of historical empires that were frequently in permanent conflict with rivals at their periphery, (e.g., the Western Roman Empire at the point of its largest expansion).

Formally, a star-shaped conflict network consists of a center agent c who is in conflict with all other agents such that $g_{ci} = 1$ for all $i \in N_c$ and $N_c = N \setminus \{c\}$. All agents of set N_c at the periphery are only in conflict with the center but not with each other, $g_{ij} = 0$ for all $i, j \neq c$ and thus $n_i = 1$ for all $i \in N_c$. This implies that there are in total $n - 1$ bilateral conflicts in the star network. Hence, the class of star networks, from now on denoted by S, is completely characterized by $n_c = n - 1$, the number of agents in the periphery.

The payoff of the center agent c can be written as

$$\pi_c(\mathbf{e_c}, \mathbf{e_i}; \mathbf{g}^S) = \sum_{i \in N_c} \frac{e_{ci}}{e_{ci} + e_{ic}} 2V - (E_c)^2 - (n-1)V, \tag{7}$$

and the corresponding payoff by an agent p in the periphery is

$$\pi_p(e_{pc}, e_{cp}; \mathbf{g}^S) = \frac{e_{pc}}{e_{cp} + e_{pc}} 2V - (e_{pc})^2 - V. \tag{8}$$

The following proposition summarizes the equilibrium in this class of star-shaped conflict networks.

Proposition 3 *In conflict networks of class S*

(i) Conflict intensity is increasing in the number n_c of agents in the periphery.

(ii) For the center agent individual (aggregated) conflict investment is decreasing (increasing) in n_c, while equilibrium probability and expected payoff is decreasing in n_c. For the periphery agent the same relation holds with respect to individual conflict investment, while the relation is inversed for equilibrium probability and payoff.

Proof. By Corollary 10 the equilibrium is interior and unique. Inspection of the first order conditions reveals that the center agent invests the same amount in each of her conflicts, i.e., $e_c^*(\mathbf{g}^S) \equiv e_{ci}^*(\mathbf{g}^S)$ for all $i \in N_c$. This also holds for each agent p in the periphery: $e_p^*(\mathbf{g}^S) \equiv e_{jc}^*(\mathbf{g}^S)$ for all $j \in N_c$. Calculating those expressions yields:

$$e_i^*(\mathbf{g}^S) = p_i^*(\mathbf{g}^S)\sqrt{\frac{V}{\sqrt{n_c}}} \qquad \text{for } i \in \{c, p\}\ , \tag{9}$$

where $p_c^*(\mathbf{g}^S) = \frac{1}{1+\sqrt{n_c}}$ and $p_p^*(\mathbf{g}^S) = 1 - p_c^*(\mathbf{g}^S)$ are the equilibrium probabilities for the center and the periphery agent to win a bilateral conflict. Note also, that $\frac{\partial p_c^*(\mathbf{g}^S)}{\partial n_c} < 0$ and that $\frac{\partial p_p^*(\mathbf{g}^S)}{\partial n_c} > 0$.

Individual conflict investment as expressed in Eq. (9) is decreasing in n_c for the center agent c and the periphery agent p. Aggregated conflict investment of the center agent is $E_c^*(\mathbf{g}^S) = n_c\ e_c^*(\mathbf{g}^S) = \frac{n_c^{3/4}}{1+\sqrt{n_c}}\sqrt{V}$, which is increasing in n_c. Plugging the obtained expressions into the payoff functions in Eq. (7) and (8) yields the following expected equilibrium payoff:

$$\begin{aligned}\pi_c(\mathbf{e_c^*}, \mathbf{e_p^*}; \mathbf{g}^S) &= -\frac{n_c(\sqrt{n_c} + n_c - 1)}{(1+\sqrt{n_c})^2}V, \\ \pi_p(e_p^*, e_c^*; \mathbf{g}^S) &= \frac{n_c - 1 - \sqrt{n_c}}{(1+\sqrt{n_c})^2}V.\end{aligned}$$

Note that $\frac{\partial \pi_c(\mathbf{e_c}, \mathbf{e_p}; \mathbf{g}^S)}{\partial n_c} < 0$ and that $\frac{\partial \pi_p(e_p, e_c; \mathbf{g}^S)}{\partial n_c} > 0$. Hence, equilibrium payoff for the center agent is decreasing in n_c, while it is increasing for an agent at the periphery.

Finally, conflict intensity in a star conflict network can be expressed as:

$$E^*(\mathbf{g}^S) = n_c[e_c^*(\mathbf{g}^S) + e_i^*(\mathbf{g}^S)] = n_c^{3/4}\sqrt{V}, \tag{10}$$

which is increasing in n_c. ■

The results stated in proposition 4 imply that the center agent is worse off if she faces more bilateral conflicts with the periphery. This is intuitive because additional opponents of the center agent will also invest in the bilateral conflict which forces the center agent to invest more into total conflict spendings

($E_c^*(\mathbf{g}^S)$ is increasing in the number of agents in the periphery). However, this is not sufficient to induce equal or higher probability to win in each of her conflicts. The marginal cost of the center is relatively higher compared with a star-shaped network with less agents which explains why $p_{ci}^*(\mathbf{g}^S)$ is decreasing in n_c. In addition, for $n > 2$ we have that $p_{ci}^*(\mathbf{g}^S) < \frac{1}{2}$. Hence, in conflict networks with more rivals the center will more frequently loose conflicts in expectation.

As a consequence, expected equilibrium payoff $\pi_c^*(\mathbf{g}^S)$ is strictly decreasing in the number of rivals. This result bears some similarities to the historically observed tendency of expanding empires to collapse at some point in time because expansion requires more total investment for an increasing number of conflicts. However, this types of investment are related with diminishing marginal returns, as is argued in Tainter (1981). The following quotation clarifies his argumentation:

> The economics of territorial expansion dictate, as a simple matter of mathematical probability, that an expanding power will ultimately encounter a frontier beyond which conquest and garrisoning are unprofitable. [...] The combined factors of increased costliness of conquest, and increased difficulty of administration with distance from the capital, effectively require that at some point a policy of expansion must end. (ibid, p. 148 f.)

Although based on a specific historical case, i.e., the decline of the western roman empire, this quotation reflects the importance of marginal increasing costs (or equivalently declining marginal returns on investment) which is also the driving force in the results for star-shaped conflict networks.

4.3 Complete Bipartite Conflict Networks

An intermediate case with respect to the symmetry of the underlying conflict structure are complete bipartite conflict networks where the members of two hostile coalitions are in conflict among each other. Hence, each agent of a coalition is in conflictive relations with all the members of the hostile coalition, as represented in the right part of figure 2. This type of conflict structure resembles an ideological bipolar conflict because members of the two hostile coalitions perceive each other as enemies.[15]

Moreover, the common ideology among coalition members implies that there are no conflictive relations among agents of the same ideology. Historical examples of conflicts that fit to this description are the massive ideological conflicts in the 20th century, especially the second world war where each country of the Axis Powers were (at least at some point in time) in conflict with nearly each member of the Allies.

[15] As mentioned earlier our basic assumption is that the underlying conflict network is exogenously given. For a recent contribution that shows how a bipolar coalition structure can be the stable equilibrium outcome in a coalition formation game embedded in a conflict framework, see Jackson and Morelli (2007).

A complete bipartite network, denoted by B, consists of two sets (coalitions) of agents, X and Y, that each have $x = |X|$ and $y = |Y|$ members. All members of set X are in conflict with each member of set Y and vice versa, such that $g_{ij} = 1$ for all $i \in X$ and all $j \in Y$. Agents of the same coalition are not in conflict among each other: $g_{ij} = 0$ for all $i, j \in X$ or $i, j \in Y$ which also implies that $X = N_j$ for all $j \in Y$ and vice versa.

The payoff function of an agent $i \in X$ in a complete bipartite network B can be stated as follows:

$$\pi_i(\mathbf{e_i}, \mathbf{e_{-i}}; \mathbf{g}^B) = \sum_{j \in Y} \frac{e_{ij}}{e_{ij} + e_{ji}} 2V - (E_i)^2 - yV, \tag{11}$$

and vice versa for an agent that is member of coalition Y. Note also, that the star-shaped network is a special case of a bipartite network where one coalition only consists of one (center) agent, i.e., $S \subseteq B$. Therefore, the following proposition is a generalized version of Proposition 3.

Proposition 4 *In conflict networks of class B,*

(i) Conflict intensity is increasing in the number x and y of each coalition.

(ii) Conflict intensity in a complete bipartite network $\mathbf{g}^{B1}$ is higher than in $\mathbf{g}^{B2}$ if and only if:

$$x_1\, y_1 > x_2\, y_2.$$

(iii) A larger coalition is beneficial for its members, i.e., each member of the more numerous coalition invests less in total conflict investment, wins each bilateral conflict with higher probability and has higher equilibrium payoff.

Proof. By Corollary 10 the equilibrium is interior and unique. Inspection of the first order conditions reveals that each member of the same coalition invests the same amount in each of her conflicts, e.g., for $i \in X$: $e^*_x(\mathbf{g}^B) = e^*_{ij}(\mathbf{g}^B)$ for all $j \in N_i$ and all $i \in X$. A symmetric observation holds for all agents $j \in Y$. Hence, there exist only two levels of individual equilibrium conflict investment in a bipartite conflict network:

$$e^*_i(\mathbf{g}^B) = p^*_i(\mathbf{g}^B) \sqrt{\frac{V}{\sqrt{x\, y}}} \qquad \text{for } i \in \{X, Y\}\ , \tag{12}$$

where $p^*_x(\mathbf{g}^B) = \frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}}$ for members of coalition X and $p^*_y(\mathbf{g}^B) = 1 - p^*_x(\mathbf{g}^B)$ for members of coalition Y. Note also, that $\frac{\partial p^*_x(\mathbf{g}^B)}{\partial x} > 0$, $\frac{\partial p^*_x(\mathbf{g}^B)}{\partial y} < 0$, and that $p^*_x(\mathbf{g}^B) > p^*_y(\mathbf{g}^B)$ if and only if $x > y$. Total conflict investment of an agent $i \in X$ can be calculated as:

$$E^*_i(\mathbf{g}^B) = \frac{\sqrt{V \sqrt{x\, y}}}{1 + \sqrt{\frac{x}{y}}}.$$

This expressions is strictly increasing in x as long as $x < y$ and becomes strictly decreasing for $x > y$. Using the derived solutions to calculate expected equilibrium payoff for an agent $i \in X$ as specified in Eq. (11) yields:

$$\pi_i(\mathbf{e_i^*}, \mathbf{e_{-i}^*}; \mathbf{g}^B) = \frac{y(x - y - \sqrt{x\,y})}{(\sqrt{x} + \sqrt{y})^2} V,$$

which is strictly increasing in x. Note that this relation also implies that, for $i \in X$ and $j \in Y$, we have that $\pi_i(\mathbf{e_i^*}, \mathbf{e_{-i}^*}; \mathbf{g}^B) > \pi_j(\mathbf{e_j^*}, \mathbf{e_{-j}^*}; \mathbf{g}^B)$ if and only if $x > y$.

Conflict intensity in a complete bipartite network can be expressed as:

$$E^*(\mathbf{g}^B) = \sqrt{(x\,y)^{\frac{3}{2}} V}. \tag{13}$$

This expression is obviously increasing in x, as well as in y, the number of agents in each coalition, which proves (i). Solving the inequality $E^*(\mathbf{g}^{B1}) > E^*(\mathbf{g}^{B2})$ yields the condition stated in (ii). ■

In a complete bipartite conflict network a coalition becomes more powerful if it has more members. The intuition for this result is similar to the star-shaped network: Assume for a moment that $x > y$, then Proposition 5 implies that members of coalition X will in expectation win each bilateral conflict with higher probability, i.e., for $i \in X$ and $j \in Y$, we have that $p^*_{ij}(\mathbf{g}^B) > p^*_{ji}(\mathbf{g}^B) = 1 - p^*_{ij}(\mathbf{g}^B)$, and therefore $p^*_{ij}(\mathbf{g}^B) > \frac{1}{2}$. Moreover, aggregated conflict investment $E^*_j(\mathbf{g}^B)$ of an agent j of coalition Y is increasing in x but this increase is not sufficient to keep the win probabilities constant among all bilateral conflicts in which she is involved. This relation also holds with respect to equilibrium payoff that is increasing in the number of members of the respective own coalition. If the lead in coalition membership is sufficiently large (out of the perspective of coalition X: if $x > y(3+\sqrt{5})/2 \approx 2.62\,y$) then it is possible that the conflict game results in positive equilibrium payoff for the members of the more numerous coalition.[16]

4.4 Conflict Intensity and Centrality

In the previous sections equilibrium outcomes were analyzed separately for each of the considered conflict classes. In this section we present a result that allows to compare conflict networks that belong to different classes within the considered ones. This cross-comparison is facilitated by establishing a relation between conflict intensity and network centrality, here eigenvector centrality, across the considered classes of conflict networks R, S, and B. Its union is denoted by $C \equiv R \cup S \cup B$.[17]

[16] As a star-shaped network is a special case of a complete bipartite network, this result also holds for star-shaped conflict network, i.e., if the number of players in the periphery is 3 or larger, then agents at the periphery have positive equilibrium payoff.

[17] A star-shaped network is a special case of a complete bipartite network, i.e. $S \subseteq B$. We included it in the definition for completeness.

The following additional notation is used: The symmetric adjacency matrix[18] G represents graph $\mathbf{g}$ and has elements g_{ij} where $g_{ii} = 0$ for all $i \in N$ (because no agent is in a conflictive relation with herself). The largest eigenvalue of G, denoted by $\mu(G)$, is real-valued and positive because G is symmetric. By the Perron-Frobenius theorem the components $(\mu_1(G), \ldots, \mu_n(G))$ of the eigenvector that corresponds to the largest eigenvalue $\mu(G)$ are all positive and frequently interpreted as a centrality measure of the respective nodes of graph $\mathbf{g}$. Solving the characteristic equation for the considered classes of conflict networks implies that:

- for regular networks: $\mu(\mathbf{g}^R) = d$ and $\mu_i(\mathbf{g}^R) = 1$ for all $i \in N$.
- for star-shaped networks: $\mu(\mathbf{g}^S) = \sqrt{n-1}$, and

$$\begin{aligned} \mu_i(\mathbf{g}^S) &= 1 \text{ for all } i \in N_c, \\ \mu_c(\mathbf{g}^S) &= \sqrt{n-1}. \end{aligned} \tag{14}$$

- for complete bipartite networks:[19] $\mu(\mathbf{g}^B) = \sqrt{x\,y}$, and, assuming without loss of generality that $x > y$:

$$\begin{aligned} \mu_i(\mathbf{g}^B) = 1 \qquad & \text{for all } i \in X, \\ \mu_j(\mathbf{g}^B) = \sqrt{\frac{x}{y}} \qquad & \text{for all } j \in Y. \end{aligned}$$

Based on this notation the following relation holds:

Corollary 5 *For the class C of conflict networks total conflict intensity is positively related to the number of conflicts in the respective network and negatively to the largest eigenvalue of its adjacency matrix. Individual conflict spending is negatively related to individual eigenvector centrality and the largest eigenvalue.*

$$E^*(\mathbf{g^C}) = \sum_{i \neq j} \frac{g_{ij}}{2} \sqrt{\frac{V}{\mu(G)}}, \qquad \textit{and} \tag{15}$$

$$e^*_{ij}(\mathbf{g^C}) = \frac{\mu_j(G)}{\mu_i(G) + \mu_j(G)} \sqrt{\frac{V}{\mu(G)}}. \tag{16}$$

Proof. Applying the derived eigenvector results for the different network classes to the equilibrium solutions for each type of conflict network yields the statement in the corollary. ■

The comparison of equilibrium outcomes between the considered classes R, S, or B based on this corollary is straight forward. The following example, that

[18] To save on notation we identify a network class with its adjacency matrix.

[19] It was already mentioned that a star-shaped network is a special case of a complete bipartite network. This fact is also reflected by the expressions for eigenvalues and -vectors, i.e., the eigenvalues and -vector of a star-shaped network can be obtained from those of a bipartite network by simply setting $x = c$ and $y = n - 1$.

compares ring- and star-shaped conflict networks, provides the insight that the local externality that is generated by the center agent plays an important role for conflict intensity.

Two types of network structures are considered: $\mathbf{g}^1$ is a ring-shaped network, i.e., a regular network of class R with degree $d = 2$, $\mathbf{g}^2$ is a star-shaped network of class S, and both conflict networks have the same number of bilateral conflicts:

$$\sum_{i \neq j} \frac{\mathbf{g}_{ij}^1}{2} = \sum_{i \neq j} \frac{\mathbf{g}_{ij}^2}{2}. \tag{17}$$

The last condition implies, in combination with Eq. (15), that the difference in conflict intensity between $\mathbf{g}^1$ and $\mathbf{g}^2$ is solely determined by the relation of their largest eigenvalues: $E^*(\mathbf{g}^1) > E^*(\mathbf{g}^2)$ iff $\mu(G^1) < \mu(G^2)$. It also should be mentioned that a ring-shaped network with n agents is compared with a star-shaped network that involves $n+1$ agents, i.e., there is always one agent more involved in $\mathbf{g^2}$ than in $\mathbf{g^1}$.

The largest eigenvalue of the ring-shaped network $\mathbf{g^1}$ is equal to its degree, $\mu(G^1) = 2$, and therefore independent of the number of involved agents. However, the largest eigenvalue of the star-shaped network $\mathbf{g^2}$ is equal to the centrality of its center, $\mu(G^2) = \mu_c(G^2) = \sqrt{n}$, which is increasing in the number of involved agents. Hence, the comparison of conflict intensity between $\mathbf{g^1}$ and $\mathbf{g^2}$ does crucially depend on the centrality of the center in $\mathbf{g^2}$.

The direct comparison between $\mathbf{g^1}$ and $\mathbf{g^2}$ reveals that for $n > 4$ the relation $E^*(\mathbf{g}^1) < E^*(\mathbf{g}^2)$ holds. In other words, although the number of conflicts is identical among $\mathbf{g}^1$ and $\mathbf{g}^2$ the star-shaped network induces higher conflict intensity for a low number of bilateral conflicts which is related to the fact that an additional agent is involved in the star-shaped network $\mathbf{g}^2$. For higher number of conflicts the centrality of the center agent in the star-shaped network is increased. Being more central implies higher externalities which tend to reduce conflict investments by affected agents. There exists a cut-off value for the number of bilateral conflicts ($n = 4$) from which on the relation of conflict intensity between $\mathbf{g}^1$ and $\mathbf{g}^2$ is reversed: for $n > 4$ we can observe that $E^*(\mathbf{g}^1) > E^*(\mathbf{g}^2)$. Hence, for a sufficiently high number of bilateral conflicts the local externality that is induced through the center agent on each agent on the periphery becomes so dominant that conflict intensity is lower in the star-shaped network.

5 Peaceful Conflict Resolution

In our framework agents are linked if there is a conflictive relation among them and then they decide how much to invest in each of their bilateral conflicts. Lemma 2 revealed that in equilibrium all agents invest positive amounts into each of their bilateral conflicts. This also implies that conflict parties cannot by themselves induce a peaceful outcome with zero conflict investment for the respective bilateral conflict.[20]

[20] This result holds although in most cases both direct rivals would profit from zero investment. As already mentioned, the prisoner-dilemma like structure implies that they cannot

Hence, peaceful conflict resolution is interpreted as the ad-hoc deletion of conflictive relations in a given conflict network. Formally, the graph $\mathbf{g}$ that remains after peaceful conflict resolution is a subset of the original graph $\mathbf{g}'$. The crucial question is how conflict intensity is affected by changing exogenously the network structure, or, in other words, if peaceful conflict resolution is beneficial in the sense that conflict intensity is reduced.

The following proposition answers this question under the restriction that the original and the resulting conflict network belongs to class C of regular, star-shaped, or bipartite conflict networks.

Proposition 6 *For class C peaceful conflict resolution is beneficial with respect to conflict intensity:*

$$\textit{If } \mathbf{g} \subset \mathbf{g}', \textit{ where } \mathbf{g} \in C \textit{ and } \mathbf{g}' \in C, \textit{ then } E^*(\mathbf{g}) < E^*(\mathbf{g}'). \tag{18}$$

Proof. From proposition 2, 3, and 4 peaceful conflict resolution implies reduced conflict intensity if the resulting conflict network after peaceful conflict resolution remains within the same class. It remains to check whether this result also hold across the considered classes. For peaceful conflict resolution of star-shaped conflict networks the proof is trivial because the resulting conflict network is always star-shaped. Also, peaceful conflict resolution of bipartite networks is clearly beneficial if the resulting network is star-shaped because star-shaped are subclasses of bipartite networks (Proposition 4 can be applied directly). The remaining two cases are therefore:

1. Case: $\mathbf{g} \subset \mathbf{g}'$, with $\mathbf{g} \in R$ and $\mathbf{g}' \in B$

   Peaceful conflict resolution is beneficial if $E^*(\mathbf{g}) < E^*(\mathbf{g}')$. We calculate the largest possible conflict intensity $\bar{E}^*(\mathbf{g}) = \max_{\mathbf{g} \subset \mathbf{g}'} E^*(\mathbf{g})$ for a regular network $\mathbf{g}$ (with maximal degree $\bar{d}$ and maximal number $\bar{n}$ of agents) that results from a bipartite network with x (y) members of coalition X (Y) through deleting links. Assume without loss of generality that $x < y$. Then $\bar{d} = \min\{x, y\} = x$, and similarly for $\bar{n} = \min\{x, y\} = x$. Hence, maximal conflict intensity for a regular network that results from a bipartite network is $\bar{E}^*(\mathbf{g}) = \sqrt{x^3 V}$. This is clearly less than $E^*(\mathbf{g}') = \sqrt{(xy)^{\frac{3}{2}} V}$, which proves the statement.

2. Case: $\mathbf{g} \subset \mathbf{g}'$, with $\mathbf{g} \in B$ and $\mathbf{g}' \in R$

   We derive the largest possible conflict intensity $\bar{E}^*(\mathbf{g}) = \max_{\mathbf{g} \subset \mathbf{g}'} E^*(\mathbf{g})$ of the resulting bipartite network with x (y) members of coalition X (Y) that stems from a regular network $\mathbf{g}$ of degree d with n agents. We then show that $\bar{E}^*(\mathbf{g}) < E^*(\mathbf{g}')$ which proves the statement.

   The bipartite network must satisfy the following inequalities: $x + y \leq n$,

mutually commit to the efficient strategy of zero investment because it will be exploited by the rival.

$x \leq d$, and $y \leq d$, where one of this equation must be strict.[21] Conflict intensity for a bipartite network, $E^*(\mathbf{g}) = \sqrt{(xy)^{\frac{3}{2}} V}$ will be maximal if $x = y$ because $(xy)^{\frac{3}{2}}$ is a concave and symmetric function. Hence, there are two cases to check for the inequality $\bar{E}^*(\mathbf{g}) < E^*(\mathbf{g'})$ to be satisfied:

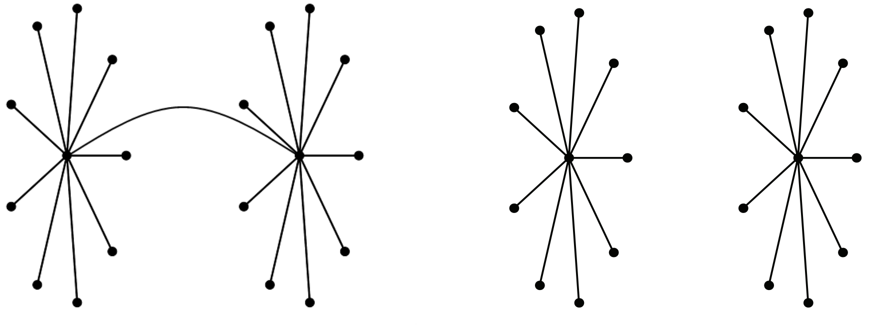
- $x = y < d$ and $x + y = n$ which implies that $x = y = \frac{n}{2}$ and $n < 2d$: Based on this information the inequality $\bar{E}^*(\mathbf{g}) < \bar{E}^*(\mathbf{g'})$ can be reduced to: $(\frac{n}{2})^3 < \frac{n^2 d}{4}$, which is satisfied because $n < 2d$.
- $x = y = d$ and $x + y < n$ which implies that $n > 2d$: Based on this information the inequality $\bar{E}^*(\mathbf{g}) < E^*(\mathbf{g'})$ can be reduced to: $d^3 < \frac{n^2 d}{4}$, which is satisfied because $n > 2d$.

As the inequality $\bar{E}^*(\mathbf{g}) < E^*(\mathbf{g'})$ is satisfied for both cases, this relation also holds for $E^*(\mathbf{g}) < E^*(\mathbf{g'})$, which proves the statement.

■

Proposition 6 is restrictive in the sense that it only covers cases where the process of peaceful conflict resolution starts and ends with a conflict network in class C. For irregular conflict networks that are not member of class C the proposition makes no statement.

Figure 3: Example S_2 before (left) and after (right) peaceful conflict resolution



In fact, peaceful conflict resolution can have highly adverse consequences. In the following example, for instance, peaceful resolution of bilateral conflicts does lead to an increase in total conflict intensity. Here, two centers of two identical star networks (that each have $n_c = n - 1$ agents in their respective periphery such that there are in total $2n$ agents in this conflict network) are in

[21] Without the last restriction the following case could occur: $x = y = d$ where $n = d/2$. Here, the regular network with $n = 2d$ describes a complete bipartite network with $x = y = d$. As we consider a resulting regular network after the deletion of links, this case can be excluded.

conflictive relation with each other. We are interested in the consequences for overall conflict intensity induced through resolving the central conflict between the center agents. The resulting graph of two stars with linked centers is denoted by S_2 and is represented, together with the situation after conflict resolution, i.e., two isolated stars, in figure 3.

The payoff function of a center agent that is linked with the other center is:

$$\pi_c(\mathbf{e_c}, \mathbf{e_{-c}}; \mathbf{g}^{S_2}) = \sum_{i \in N_c} \frac{e_{ci}}{e_{ci} + e_{ic}} 2V + \frac{e_{cc}}{e_{cc} + e_{cc}} 2V - (\sum_{i \in N_c} e_{ci} + e_{cc})^2 - nV,$$

where e_{cc} denotes the conflict spending of one center against the other.[22]

Based on numerical solution techniques it is possible to calculate the conflict intensity $E^*(\mathbf{g}^{S_2})$ for this network constellation and to compare it with $2E^*(\mathbf{g}^S)$, i.e., conflict intensity in a network with two isolated star-shaped conflict networks based on Eq. (10). Results are presented in table 1.[23]

$\mathbf{n_c}$	$\mathbf{E}^*(\mathbf{g}^{\mathbf{S_2}})$	$\mathbf{2E}^*(\mathbf{g}^{\mathbf{S}})$
8	8.639	8.607
9	9.529	9.514
10	10.392	10.392
11	11.234	11.247
12	12.055	12.080
13	12.856	12.895

Table 1: Conflict intensity for $\mathbf{g}^{\mathbf{S_2}}$ and $\mathbf{g}^{\mathbf{S}}$

Surprisingly, resolving the central conflict may actually imply an increase in conflict intensity. More precisely, if $n_c > 10$ then $E^*(\mathbf{g}^{S_2}) < 2E^*(\mathbf{g}^S)$, i.e., peaceful conflict resolution induces higher conflict intensity if each of the two stars has more than ten agents in the periphery. If $n_c < 10$ then conflict intensity decreases.

This result can be related to the negative externality that the center agent exerts on the respective rivals in the periphery. This externality is induced through the decrease in the number of conflictive relations of the center agent: The direct effect is that the two center agents spend less in aggregated conflict investment because they face less direct conflicts. However, they also shift part of the conflict investment from the resolved central conflict to the periphery. Agents in the periphery react to this increase in conflict spending of their rival (the respective center agent) by also increasing investment into this conflict.

[22]Note that both star networks have the same number of agents in their periphery. It can be shown that the only interior equilibrium is symmetric in the sense that both center agent invest the same amount into the conflict against each other. Hence, it is not necessary to discriminate between the two center agents.

[23]To apply those techniques it is assumed that $V = 1$. Results are similar for different numerical values of V.

Hence, if the number of periphery agents is sufficiently large, those indirect effects (the externality induced by resolving the central conflict) by the periphery dominate the direct effect of decreased aggregated spending by the two center agents.

This example shows, that paying attention to the underlying structure of conflicts is crucial for the success of peaceful conflict resolution. It also suggests that the resolution of bilateral conflicts should be targeted with respect to the underlying conflict structure to guarantee a reduction of conflict intensity. However, finding the bilateral conflict that would (by peacefully resolving it) induce the maximal decrease in conflict intensity requires an analytical solution of Eq. (19) which in general does not exist for conflict networks outside of class C. In the next section we derive partial results that indirectly characterize the bilateral conflict that induces the highest aggregated conflict spending. This bilateral conflict might constitute a presumably valuable target for peaceful conflict resolution.

6 General Irregular Conflict Structures

The results derived in the previous section are based on the assumption that the conflict network belongs to class C. This assumption is now relaxed by extending the analysis to general irregular networks. As already mentioned, individual conflict spending can be characterized as the solution to the following system of $\sum_{i \in N} n_i$ first order equations:[24]

$$\frac{e_{ki}^*}{(e_{ik}^* + e_{ki}^*)^2} V = E_i^*, \qquad \text{for all } k \in N_i \text{ and all } i \in N. \tag{19}$$

Analytical solutions for this system of non-linear equations do in general not exist. However, the following reformulation allows to derive some additional indirect results.

Combining the two first order conditions for two direct rivals i and j implies that in equilibrium:

$$\frac{e_{ij}^*}{e_{ji}^*} = \frac{E_j^*}{E_i^*}, \text{ and} \tag{20}$$

$$e_{ij}^* + e_{ji}^* = \frac{V}{E_i^* + E_j^*}. \tag{21}$$

The probability to win the bilateral conflict can be expressed in terms of aggregated conflict spending of the two rivals:

$$p(e_{ij}^*, e_{ji}^*) = \frac{E_j^*}{E_i^* + E_j^*} \tag{22}$$

[24] The dependence of equilibrium conflict spendings on graph $\mathbf{g}$ is suppressed in the following section for notational convenience.

It is assumed that the agents are ordered according to their level of aggregated equilibrium investment, i.e., $E_1^* \leq E_2^* \leq \ldots \leq E_n^*$.[25] Then the following relation holds in equilibrium:

Proposition 7 *If agent i is in conflict with agent j and $i < j$ then agent i will win this conflict with higher probability.*

Proof. If $i < j$ then $E_i^* \leq E_j^*$ because the agents are ordered. By Eq. (20) this implies that $e_{ij}^* \geq e_{ji}^*$. By the definition of the probability $p(e_{ij}^*, e_{ji}^*) \geq p(e_{ji}^*, e_{ij}^*)$ and hence $p_{ij}^* \geq \frac{1}{2}$. ■

This result is counter-intuitive at first sight because it states that agent i will win the conflict against j although she invests in total less in all her conflicts than her rival j. However, the reason for j investing relatively more in total conflict spending in equilibrium is due to the fact that she either faces very aggressive rivals or because she has a lot of them. Both situations favor her direct rival i who can guarantee a high winning probability because agent j will not invest too much into the conflict against i in comparison to her other conflicts. Note, that the same intuition has been discussed in the section on star-shaped networks where the center agents is the one with highest total conflict investments but still looses in expectation in each bilateral conflict with the periphery.

By combining Eq. (20) and (21) it becomes obvious that individual conflict spending into a singular bilateral conflict is totally determined by the aggregated conflict spending of the two directly affected rivals:

$$e_{ij}^* = \frac{E_j^*}{(E_i^* + E_j^*)^2} V.$$

This allows the simplification of the system of $\sum_{i \in N} n_i$ first order conditions as specified in Eq. (19) to a system of only n equations:

$$E_i^* = \sum_{j \in N_i} \frac{E_j^*}{(E_i^* + E_j^*)^2} V \text{ for all } i \in N.$$

However, this system of equations is still non-linear and can not be solved analytically. Overall conflict intensity can be indirectly expressed in the following way:

$$E^* = \sum_{i \in N} E_i^* = \sum_{i \neq j} \frac{g_{ij}}{2} \frac{V}{E_i^* + E_j^*}. \tag{23}$$

Note that the same caveat as in footnote 25 applies here because E^*, as well as E_i^*, depend on the whole network structure $\mathbf{g}$. This implies that changes in the network structure $\mathbf{g}$, for instance ad-hoc deletion of conflictive links, affect

[25] Ordering agents according to their aggregated individual equilibrium spending obviously depends on the network structure. In this sense Proposition 7 is an indirect result because this order is based on equilibrium outcomes.

E^* in two ways: First, there is a direct effect because at least one g_{ij} takes on a value of zero which tends to reduce conflict intensity (at least one term in the sum of Eq. (23) is eliminated. Second, there are also indirect effects because all agents in the network will react to the change in aggregated conflict investment that is induced by the direct effect. This implies that all remaining terms in the sum are altered. In general, it is not clear which effect dominates the other because the indirects effect depend in a complex way on the network structure $\mathbf{g}$.[26]

Nevertheless, those results can be used to determine the bilateral conflict that induces the highest aggregate conflict investment $\max_{j \in N_i} E^*_{ij}$ for all $i \in N$ where $E^*_{ij} = e^*_{ij} + e^*_{ji}$. By Eq. (21) E^*_{ij} is maximal for the bilateral conflict between those agents that have the lowest aggregated conflict investment. This seems to suggest that isolated bilateral conflicts where affected agents do not have any additional rivals induce the highest levels of conflict spending. Intuitively, for agents in isolated conflicts there is no local externality from other conflicts that tends to reduce conflict investment. The question whether such an isolated conflict is the optimal target for peaceful conflict resolution must remain open because the feedback effects that are induced by resolving alternative (and more embedded) bilateral conflicts cannot be compared based on the indirect results derived here.

7 Discussion

The model as presented in section 4 ff. is based on specific functional forms. Here, we argue that our results are (at least qualitatively) robust to more general specifications as long as they are in the framework of section 2. For instance, we additionally considered a convex (but not necessarily quadratic) cost function of the type $c(\mathbf{e_i}) = E_i^r$ with $r > 1$ that did neither alter our results for conflict networks of the combined class C, nor the indirect results derived in section 6.[27] Moreover, a more general contest success function of the type $p_{ij} = \frac{e^s_{ij}}{e^s_{ij} + e^s_{ji}}$ with $s \in (0, 1]$ should not alter our results substantially.

In our set up heterogeneity among agents is implicitly induced by the relevant position in the potentially irregular network structure. Hence, adding another source of heterogeneity, for instance different valuations or different cost functions, might influence our results. If this second source of heterogeneity is sufficiently important then it will dominate differences in behavior induced

[26] In the example presented in section 5 the indirect effects dominated the direct effects, while this is never the case for peaceful conflict resolution of conflict networks within class C.

[27] The numerical values that we derived in section 4.3 and section 5 would obviously be different without affecting the argumentation in these sections. Also the relation to eigenvector centrality is sensitive with respect to quadratic cost functions, compare Ballester et al. (2006) for a brief discussion of this issue. It should also be mentioned that for $r \to 1$ the externality that is induced through the network structure vanishes because for $r = 1$ the marginal costs of investing into a specific bilateral conflict is independent of the other conflict investments of an agent.

by different locations in the network. The following extension clarifies this intuition for conflict networks of class C. We relax the assumption that each agent can win (or loose) in principle the same amount of resources in each conflict. Instead, the value of contested resources depends now on the number of rivals that an agent faces: $V_i = \frac{V}{n_i}$. Equilibrium investment is now identical for each agent in a given network of class C: $e^*_{ij}(\mathbf{g^c}) = k(\mathbf{g^c})$ for all $j \in N_i$ and all $i \in N$, where $k(\mathbf{g^c})$ is a constant that depends on the underlying network structure.[28] Hence, in this specification the heterogeneity of the network structure is exactly balanced out by the heterogeneity in valuations.

In our framework we assume that the underlying conflict network is given ex-ante without specifying how it came into existence. Hence, the formation process of the conflict network could be an interesting extension. However, the usual stability conditions, for instance pairwise stability, see Jackson and Wollinsky (1996), are not applicable in our context because at least one rival in each bilateral conflict has negative payoff in equilibrium. This would imply that at least one agent would sever the respective link in each bilateral conflict such that all conflict networks would be unstable. Still, our model has some elements of network formation because investing zero into a specific bilateral conflict is part of the individual strategy space of an agent. Two direct rivals could therefore leave a link inactive if they mutually decided not to invest into the respective conflict. We already argued that this mutually beneficial strategy cannot occur in equilibrium because it is exploitable by the involved agents. Hence, the reason why agents in our set up do not decide to endogenously dissolve links is not related with network stability but with the lack of commitment devices that would allow them to coordinate on peaceful behavior.

8 Concluding Remarks

Analyzing conflict situations that are embedded in a structure of conflictive relations yields constructive results with respect to equilibrium conflict intensity and network characteristics. While we confirm the intuitive statement that more conflictive relations imply higher conflict intensity for an important class of conflict structures, i.e., regular, star-shaped, and complete bipartite conflict networks, we also provide an example under which this statement is not true. Nevertheless, a general relation between conflict intensity and a prominent centrality measure is established for the three mentioned classes of conflict networks.

Extending the analysis to more general irregular networks is a complex issue due to the fact that no closed form equilibrium solutions exist. Indirect results allow us to characterize the bilateral conflict that induces maximal conflict investment. Moreover an inverse relation between individual aggregated conflict spending and the individual win probability for each bilateral conflict is established.

[28] For regular networks individual conflict investment is now also independent of the respective degree, i.e., all regular conflict networks induce the same individual conflict investment.

An advantage of our simple model is that, contrary to most of the literature that considers games played on fixed networks, the resulting equilibrium is interior and unique. This feature in combination with the very intuitive contest rule should make our model especially attractive for experimental approaches which is part of our ongoing research.

Appendix

Due to the discontinuity of the applied contest success function (CSF) with $p_{ij} = \frac{e_{ij}}{e_{ij}+e_{ji}}$ at the point $(0,0)$ the existence result established in Proposition 1 cannot be applied directly.[29] Therefore, an extended existence result for the conflict network game as specified in Eq. (1), (4) and (5) is provided in this appendix based on the following line of arguments. In Lemma 8 we show that an equilibrium (if it exists) in the conflict network game as specified in Eq. (1), (4) and (5) must be interior. We then use a suggestion by Myerson and Wärneryd (2006), where it is observed that the CSF $p_{ij} = \frac{e_{ij}}{e_{ij}+e_{ji}}$ can be obtained as the limit of the function $\bar{p}_{ij} = \frac{e_{ij}+a}{e_{ij}+e_{ji}+2a}$ as $a \to 0$ for $a > 0$. This alternative CSF[30] is continuous everywhere and also satisfies all the conditions of Proposition 1. Therefore, Lemma 9 states that there exists a unique equilibrium in a conflict network game based on this alternative CSF $\bar{p}_{ij}$ which is also interior. As p_{ij} can be obtained as the limit of $\bar{p}_{ij}$ where the same relation also holds for its first order conditions, Corollary 10 then states that there must exist a unique and interior equilibrium for the original conflict network game based on Eq. (1), (4) and (5).

Lemma 8 *If an equilibrium exists in the conflict network game as specified in Eq. (1), (4) and (5) then it is interior.*

Proof. The proof consists of two parts:

1. *Claim: Two direct rivals cannot exert zero conflict investment in equilibrium in their respective bilateral conflict.*

   Consider an arbitrary strategy profile $(\mathbf{e_i}, \mathbf{e_{-i}})$, where $e_{ij} = e_{ji} = 0$ and $j \in N_i$. Consider now the following strategy $\mathbf{e'_i} = (e_{i1}, \ldots, e'_{ij}, \ldots, e_{in_i})$ where $e'_{ij} = \epsilon$ for ϵ sufficiently small. This is a profitable deviation because $\pi_i(\mathbf{e'_i}, \mathbf{e_{-i}}; \mathbf{g}) > \pi_i(\mathbf{e_i}, \mathbf{e_{-i}}; \mathbf{g})$ as $p(e'_{ij}, e_{ji}) = 1 > p(e_{ij}, e_{ji}) = 1/2$ due to the discontinuity at $(0,0)$ while $lim_{\epsilon \to 0} c(\mathbf{e'_i}) = c(\mathbf{e_i})$ because the cost function is continuous.

2. *Claim: An agent cannot exert zero conflict investment in equilibrium against a rival with positive conflict investment.*

[29] There also exist alternative approaches that provide existence proofs for discontinuous payoff functions which might be applicable in our framework, for instance, Baye et al. (1993), and Reny (1999). However, in this literature the issue of uniqueness is usually not addressed which is crucial for comparative static analysis.

[30] A similar functional form is also used, for instance, in Nti (1997).

Assume by contradiction that there exists an equilibrium strategy profile $\mathbf{e}^* = (\mathbf{e}_1^*, \ldots, \mathbf{e}_i^*, \ldots, \mathbf{e}_j^*, \ldots, \mathbf{e}_n^*)$ with $j \in N_i$ where agent i invests $e_{ij}^* = 0$ and its rival j invests $e_{ji}^* > 0$ into the respective bilateral conflict. The following strategy is a profitable deviation: $\mathbf{e}_j' = (e_{ji}', \mathbf{e}_{j-i}^*)$ where $e_{ji}' \in (0, e_{ji}^*)$ and $\mathbf{e}_{j-i}^* = \{e_{jk}^*\}_{k \in N_j/i}$, i.e., agent j only reduces conflict spending against rival i without altering conflict investment in all other conflicts. Note that $p(e_{ji}', e_{ij}^*) = p(e_{ji}^*, e_{ij}^*) = 1$ while $c(\mathbf{e}_j') < c(\mathbf{e}_j^*)$ and therefore $\pi_j(\mathbf{e}_j', \mathbf{e}_{-j}^*; \mathbf{g}) > \pi_j(\mathbf{e}_j^*, \mathbf{e}_{-j}^*; \mathbf{g})$. Hence, $\mathbf{e}^*$ cannot be an equilibrium strategy profile.

■

Interiority of equilibrium would imply that an equilibrium can be characterized by first order conditions. Note also, that the strategy that is related with this discontinuity cannot be part of an equilibrium strategy.

Based on the mentioned suggestion by Myerson and Wärneryd (2006), we now substitute the CSF in the original conflict network game by the following expression:

$$\bar{p}_{ij} = \frac{e_{ij} + a}{e_{ij} + e_{ji} + 2a} \text{ with } \alpha > 0. \tag{24}$$

Note first that the CSF p_{ij} can be obtained as the limit from the alternative CSF $\bar{p}_{ij}$ for $a \to 0$: $p_{ij} = \lim_{\alpha \to 0} \bar{p}_{ij}$, because $\frac{e_{ij}}{e_{ij}+e_{ji}} = \lim_{\alpha \to 0} \frac{e_{ij}+a}{e_{ij}+e_{ji}+2a}$. The same relation holds for a generic equation from the system of first order conditions for the two CSFs: $\frac{\partial p_{ij}}{\partial e_{ij}} = \lim_{\alpha \to 0} \frac{\partial \bar{p}_{ij}}{\partial e_{ij}}$, because $\frac{e_{ji}}{(e_{ij}+e_{ji})^2} = \lim_{\alpha \to 0} \frac{e_{ji}+a}{(e_{ij}+e_{ji}+2a)^2}$. Note also that Eq. 24 is continuous everywhere and that a conflict network game based on Eq. (1), (4) and (24) satisfies all the conditions mentioned in Proposition 1. Hence, by the same proposition a unique equilibrium exists for this setup. The following Lemma summarizes this result and additionally shows that this equilibrium is interior.

Lemma 9 *There exists a unique and interior equilibrium for the alternative conflict network game based on Eq. (24).*

Proof. The alternative CSF $\bar{p}_{ij}$ is everywhere continuous and a conflict network game based on this alternative CSF is a concave game which is diagonally strictly concave. Hence, Proposition 1 can be applied to proof that there exists a unique equilibrium. This equilibrium is also interior for a sufficiently small, which is proved in a similar way as in Lemma 8:

- *Claim 1: For a sufficiently low two direct rivals cannot exert zero conflict investment in equilibrium in their respective bilateral conflict .*

  Assume by contradiction that there exists an equilibrium strategy profile $\mathbf{e}^* = (\mathbf{e}_1^*, \ldots, \mathbf{e}_i^*, \ldots, \mathbf{e}_j^*, \ldots, \mathbf{e}_n^*)$ with $j \in N_i$ where $e_{ij}^* = e_{ji}^* = 0$. Evaluation of the respective first order derivative at the equilibrium point reveals that $\frac{\partial \pi_i(\mathbf{e}_i^*, \mathbf{e}_{-i}^*)}{\partial e_{ij}} = \frac{1}{4a} - 2\sum_{j \in N_i} e_{ij}^*$, which should be non-positive to sustain an equilibrium $\mathbf{e}^*$. However, the last expression will be positive for small a which is a contradiction to the statement that $\mathbf{e}^*$ is an equilibrium.

- *Claim 2: For a sufficiently low an agent cannot exert zero conflict investment in equilibrium against a rival with positive conflict investment.*

  Assume by contradiction that there exists an equilibrium strategy profile $\mathbf{e}^* = (\mathbf{e}^*_\mathbf{1}, \ldots, \mathbf{e}^*_\mathbf{i}, \ldots, \mathbf{e}^*_\mathbf{j}, \ldots, \mathbf{e}^*_\mathbf{n})$ with $j \in N_i$ where agent i invests $e^*_{ij} > 0$ and its rival j invests $e^*_{ji} = 0$ into the respective bilateral conflict. This implies that in equilibrium the respective first order condition for player i is satisfied by equality: $\frac{\partial \pi_i(\mathbf{e}^*_\mathbf{i}, \mathbf{e}^*_{-\mathbf{i}})}{\partial e_{ij}} = \frac{a}{(e^*_{ij}+2a)^2} - \sum_{j \in N_i} e^*_{ij} = 0$. However, in the limit for $a \to 0$ the last equality cannot hold which contradicts the statement that $\mathbf{e}^*$ is an equilibrium.

■

The results obtained so far imply that, by Lemma 8, an equilibrium in the original conflict network game based on CSF p_{ij} is interior (if it exists) and therefore characterized by first order conditions. By Lemma 9 the alternative conflict network game based on the continuous CSF $\bar{p}_{ij}$ has a unique and interior equilibrium which is therefore also characterized by first order conditions. We also showed that p_{ij} as well as a generic equation from its system of first order conditions can be obtained in the limit for $a \to 0$ from the alternative CSF $\bar{p}_{ij}$. Hence, for the limit $a \to 0$ the equilibrium result carries through which is summarized in the following corollary:

Corollary 10 *There exists a unique and interior equilibrium for the conflict network game as specified in Eq. (1), (4) and (5).*

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