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The Folk Theorem and Bertrand Competition

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Abstract: We examine if the folk theorem of perfect competition holds under Bertrand competition (when firms supply all demand), both when entry is exogenous, as well as when it is free. *Inter alia*, we also characterize the limit equilibrium sets.

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1 Introduction

This paper examines if the folk theorem of perfect competition holds under Bertrand competition, both when entry is exogenous, as well as free. *Inter alia*, it also characterizes the limit equilibrium sets.

We focus on price competition when firms supply all demand. The assumption that firms supply all demand can, in fact, be traced back to Chamberlin (1933).[1] This assumption is appropriate when the costs of turning away customers are very high (see Dixon (1990), or Vives (1999)). Such costs may arise because of either reputational reasons, or governmental regulations. Vives (1999) argues that such regulations are operative in U.S. industries like electricity and telephone.

We study the properties of the *limit equilibrium set* under Bertrand competition both when entry is exogenous, as well as when it is free. Under the exogenous entry approach, pioneered by Ruffin (1971) and Okuguchi (1973) for the case of Cournot competition, we solve for the n-firm Bertrand equilibrium where demand is given and all firms are active in equilibrium. Under the free entry approach, pioneered by Novshek (1980) for the case of Cournot competition, we consider an r-fold replication demand and then solve for the free entry Bertrand equilibrium where at least one firm is inactive. We then examine the limit equilibrium sets under both these approaches. The objective is to examine if, for the Bertrand framework, the folk theorem of perfect competition holds, in the sense that the set of limit equilibrium prices contains the perfectly competitive price(s), and no other price(s).[2]

We then briefly summarize our main results.

[1] It has also been adopted, among others, by authors like Bulow, Geanakoplos and Klemperer (1985), Dastidar (1995), Novshek and Roy Chowdhury (2003), and Vives (1990, 1999).

[2] While the folk theorem is relatively well explored in the Cournot framework, (see, among others, Novshek (1980), Okuguchi (1973) and Ruffin (1971)), it is much less so in the Bertrand framework.

First consider the case with exogenous entry. To begin with we characterize the limit-equilibrium set. We then use this characterization to show that the folk theorem fails to hold unless average cost is constant (and, for all prices greater than the average cost, the demand function achieves its maximum at the average cost).

Under the free entry case, we begin by characterizing the limit equilibrium set for average cost functions that are ultimately either increasing, or decreasing. For these class of cost functions we demonstrate that the folk theorem fails to hold. However, it does hold if the average cost function is constant (and, for all prices greater than the average cost, the demand function achieves its maximum at the average cost).

Finally, in the Appendix we show that irrespective of whether entry is exogenous, or free, our results regarding the folk theorem goes through even if we allow for multiple price equilibria.

We then relate our paper to the literature. This problem has been examined earlier by Novshek and Roy Chowdhury (2003) (NRC from now on), though for the case when the demand function is negatively sloped and the average cost function is primarily either U-shaped, or increasing.[3] The assumptions on the demand and the cost functions imposed by NRC are certainly quite reasonable. Given the importance of the issue, however, it is of interest to re-examine the problem under a minimal set of restrictions on the demand and the cost functions.

Hence in this paper we essentially only assume that the demand function is continuous and intersects both the axes, and that the cost function is continuous (except possibly at the origin). Moreover, for the free entry case the average cost function is assumed to be ultimately monotonic. In particular, we do not assume that the demand function is negatively sloped, or that the average cost function is either increasing, or U-shaped. Further,

[3]NRC, of course, also characterize the limit equilibrium set when average costs are constant, or decreasing, or have a capacity constraint.

in the paper we derive conditions under which the characterizations derived in this paper coincide with those in NRC.

The rest of the paper is organized as follows. In the next section we describe the model. The case with exogenous entry is analyzed in Section 3, whereas Section 4 considers the case with free entry. Section 5 concludes. In the Appendix we allow for multiple price equilibria.

2 The Model

The market $M(n)$ comprises the demand function $f(p)$ and n firms, all producing a single homogeneous good, and having the same cost function, $c(q)$ and the average cost function $AC(q)$.[4]

The market demand function $f(p)$ satisfies the following assumption.

Assumption 1: (a) $f : [0, \infty) \to [0, \infty)$.[5] Moreover, $f(p)$ is continuous.

(b) There exists a choke-off price $\hat{p}$ (> 0) such that, $\forall p \geq \hat{p}$, $f(p) = 0$, and, $\forall p < \hat{p}$, $f(p) > 0$.

Note that the demand function is not necessarily negatively sloped.[6]

The cost function satisfies the following assumption.

Assumption 2: (a) $c : [0, \infty) \to [0, \infty)$. Moreover, $c(0) = 0$ and $c(q) > 0$, $\forall q > 0$.

(b) The cost function is continuous, except possibly at the origin.

(c) $AC : (0, \infty) \to (0, \infty)$.[7] Moreover, there exists p such that $p >$

[4]For ease of comparison, the notations in this paper closely follow those in NRC.

[5]Note that this implies that $f(0)$ is finite.

[6]Other papers to allow for a general class of demand functions include, among others, Allen and Hellwig (1986) and Maskin (1986) (both these papers are in the Bertrand-Edgeworth framework).

[7]Given Assumptions 2(a) and 2(b), $AC(q)$ is well defined and continuous on $(0, \infty)$.

$AC(f(p))$.[8] Finally, $b = \lim_{q\to 0} AC(q)$ is well defined (allowing for infinity as a possible limit).

Note that we do not assume that the average cost function is necessarily either increasing, or U-shaped (we say that $AC(q)$ is U-shaped if there exists $q^* > 0$ such that the average cost function is strictly decreasing for all $0 < q < q^*$, and strictly increasing for all $q > q^*$).

We examine a game of Bertrand competition where the firms simultaneously announce their prices, and the Chamberlin (1933) assumption holds.

Assumption 3. The firms supply all demand.

Let $D_i(p_1, \cdots, p_i, \cdots, p_n)$ denote the residual or contingent demand facing firm i when the announced price vector is $(p_1, \cdots, p_i, \cdots, p_n)$. We assume that the residual demand is the parallel, or the efficient one.[9] Further, firms charging the same price share the residual demand equally between them. Thus

$$D_i(p_1, \cdots, p_i, \cdots, p_n) = \frac{\max\{0,\ f(p_i) - \sum_{j:p_j<p_i} D_j(p_1, \cdots, p_n)\}}{m_i}, \qquad (1)$$

where m_i denotes the number of firms charging p_i.

The profit of the i-th firm

$$\pi_i(p_1, \cdots, p_n) = (p_i - AC(D_i(p_1, \cdots, p_n)))D_i(p_1, \cdots, p_n). \qquad (2)$$

We solve for the pure strategy Nash equilibrium in prices, i.e. Bertrand equilibrium.

Definition. A *Bertrand equilibrium* for the market $M(n)$ consists of a

[8] This implies that the optimal monopoly profit is strictly positive. It is equivalent to the NRC assumption that $f(p)$ and $AC(q)$ intersect at least once in the $p-q$ plane.

[9] Our results are not dependent on the specific rationing rule being used though.

price vector $(p_1, \cdots, p_i, \cdots, p_n)$ such that, $\forall i$ and $\forall p_i'$,

$$\pi_i(p_1, \cdots, p_i, \cdots, p_n) \geq \pi_i(p_1, \cdots, p_i', \cdots, p_n). \tag{3}$$

In this paper we shall be mainly concerned with *single price* Bertrand equilibria, where all active firms (i.e. firms with positive output) charge the same price. For an important class of demand functions, i.e. negatively sloped ones, it is easy to see that all Bertrand equilibria are necessarily single priced (follows from Assumption 3).[10]

Finally, a Bertrand equilibrium is said to be a *free entry equilibrium* if some of the firms are not active, i.e. have zero demand.

3 Exogenous Entry

In this section we examine a situation where the number of active firms is taken to be exogenously given. We study the limiting equilibrium outcomes as the number of active firms goes to infinity. Following NRC, we characterize the set of all prices p such that if the number of firms n is large enough, then, for the market $M(n)$, there is some single price equilibrium where all firms are active and the equilibrium price is arbitrarily close to p.

Definition: $S = \{p$: there is a sequence $p(n)$ that converges to p such that, for each sufficiently large n, all firms setting a price $p(n)$ is an equilibrium for the market $M(n)\}$.

We need some more notations before we can characterize S.

$c^* = \inf_q \ AC(q)$.[11]
$\tilde{p} = \ \text{argmax}_{p \in [0,\hat{p}]} \ f(p)$.[12]

[10] For Cournot competition also, all equilibria are single priced. In the Appendix we briefly allow for equilibria that are not single-priced.

[11] Given that $AC : (0, \infty) \to (0, \infty)$, c^* is finite.

[12] Given that $f(p)$ is continuous, $\tilde{p}$ and $f(\tilde{p})$ are well defined.

$\tilde{d} = \inf \{p : \ p > AC(f(p))\}$.[13]

d is the minimum p such that $AC(f(p)) = p$.[14]

$F(r) = \{p : \ \forall \ p' > p$, either (a) $f(p') \leq f(p)$, or (b) $f(p') > f(p)$ and $p' \leq AC(rf(p') - rf(p))\}$.[15] For ease of exposition we write $F(1) = F$.

$\overline{p} = \text{argmax}_{p \in (c^*, \hat{p}]} \ f(p)$.

For any set A, let $\overline{A}$ denote the closure of A.

We then impose the following regularity condition.

Assumption 4. (i) If $b = \tilde{d}$, then the cost function is either linear, or there exists $t > 0$ such that $AC(q)$ is negatively sloped for all $q \in (0, t)$.

(ii) If $\overline{p}$ is well defined, then $f(\overline{p}) \neq f(c^*)$.

It may be argued that Assumption 4 is not very strong.[16] Recall that in the NRC framework, $b = \tilde{d}$ implies that the average cost function is U-shaped, so that Assumption 4 is necessarily satisfied.

Proposition 1 below characterizes the set S.

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[13]Given Assumption 2(c), the set $\{p : p > AC(f(p))\}$ is non-empty. Since $p = 0$ is a lower bound, there is a least upper bound. Hence $\tilde{d}$ is finite.

[14]Given Assumption 2(c), d is well defined.

[15]Clearly, $F(r)$ is closed. Suppose $p \notin F(r)$. Then $\exists p' > p$ such that $f(p') > f(p)$ and $p' > AC(rf(p') - rf(p))$. Clearly, for any $p + \epsilon$, $\epsilon > 0$ but sufficiently small, $p' > p + \epsilon$, $f(p') > f(p + \epsilon)$ and $p' > AC(rf(p') - rf(p + \epsilon))$.

[16]Consider 4(i). This is because in general $b \neq \tilde{d}$. Take $f(p)$ and $AC(q)$ such that $b = \tilde{d}$. Now if the two functions are perturbed slightly (in an appropriate manner), then it will no longer be the case that $b = \tilde{d}$. For example, consider the family of demand functions $\lambda f(p)$, λ different from, but close to 1. Let $\tilde{d}(\lambda)$ denote the appropriately modified version of $\tilde{d}$ for the demand function $\lambda f(p)$. Assuming that $AC(q)$ does not have horizontal segments, and $f(p)$ does not have vertical segments (in the $p-q$ plane), for λ close to 1, $\tilde{d}(\lambda) \neq \tilde{d} = b$. Next consider 4(ii). Suppose that $f(\overline{p}) = f(c^*)$. Let us perturb $AC(q)$ by considering the family of functions $\lambda AC(q)$, where λ is close to 1. Thus unless $c^* = 0$, $c^*(\lambda) \neq c^*$, for $\lambda \neq 1$ (where $c^*(\lambda)$ is the obvious extension of c^* for $\lambda f(p)$). Thus if $f(p)$ does not have a vertical section then, for λ close enough to 1, $f(\overline{p}) \neq f(c^*(\lambda))$.

Proposition 1. *Let Assumptions 1, 2, 3 and 4 hold.*

(i) If $b < \tilde{d}$*, then* $S = \overline{(b, \tilde{d}) \cap F}$*.*

(ii) If $b > \tilde{d}$*, then* S *is empty.*

(iii) If $b = \tilde{d}$*, then* $S = \{b\}$*, if either (a)* $AC(q)$ *is constant, and* $\overline{p}$ *is not well defined, or (b) there exists some sequence* $< p(n) >$ *in* F *such that,* $\forall n$*,* $p(n) < b$*, and* $< p(n) >$ *converges to* b*. Otherwise,* S *is empty.*

Proof: To begin with we argue that no price less than b, or greater than $\tilde{d}$, or not in F, can belong in S. Suppose that $p(n)$ converges to p as n increases and for each sufficiently large n, all n firms setting a price $p(n)$ is an equilibrium for $M(n)$.

Note that the output per active firm is at most $\frac{f(\tilde{p})}{n}$, which converges to zero as n goes to infinity. Thus if $p < b$, then for all sufficiently large n, $p(n) < AC(\frac{f(p(n))}{n})$, so that $p(n)$ cannot be an equilibrium price.

Next let $p > \tilde{d}$. Note that profit per active firm, $[p(n) - AC(\frac{f(p(n))}{n})]\frac{f(p(n))}{n}$, is less than $(\hat{p} - c^*)\frac{f(\tilde{p})}{n}$. Thus, for n large, profit per active firm converges to zero. Moreover, from the definition of $\tilde{d}$, there exists p' such that $\tilde{d} < p' < p(n)$ and $p' > AC(f(p'))$. Undercutting to such a price p' yields a strictly positive profit that depends on p', but not on n. Thus, for n large, undercutting is strictly profitable.

Next consider some $p \notin F$. As argued earlier, for n large, profit per active firm goes to zero. Since $p \notin F$, there exists $p' > p$ such that $f(p') > f(p)$ and $[p' - AC(f(p') - f(p))][f(p') - f(p)] > 0$. Given that F is closed, for n sufficiently large, we can find some $p(n)$ such that $p' > p(n)$, $f(p') > f(p(n))$ and $[p(n) - AC(\frac{f(p(n))}{n})]\frac{f(p(n))}{n} < [p' - AC(f(p') - f(p(n)))][f(p') - f(p(n))]$. Hence one of the firms can deviate to p', and make a strict gain.

We then argue that every price in the interval $(b, \tilde{d}) \cap F$ is in the limit set. If $p > b$, then, for any sufficiently large n, if n firms set such a price then each firm will produce an output at which p exceeds average cost, and thus obtain a positive profit. Undercutting is unprofitable since for any p

strictly less than $\tilde{d}$, an undercutting firm cannot make a positive profit as $p \leq AC(f(p))$. Finally, since $p \in F$, none of the firms can charge a higher price and gain.

Next we consider $p \in \overline{(b, \tilde{d}) \cap F} - (b, \tilde{d}) \cap F$. Suppose $b < \tilde{d}$. Since $p \in \overline{(b, \tilde{d}) \cap F}$, any such p can be obtained as the limits of appropriate sequences of equilibrium prices, $p(n)$, described above.

Finally, let $b = \tilde{d} < \hat{p}$.[17] First suppose average cost is constant, and $\overline{p}$ is not well defined. Since $\overline{p}$ is not well defined, $f(c^*) > f(p)$, for all $p > c^*$ (otherwise $\exists p > c^*$ such that $f(p) \geq f(c^*)$. But then $\overline{p} = \arg\max_{[c^*, \hat{p}]} f(p)$, which is well defined). Then $p = b = c^*$ can be sustained as a Bertrand equilibrium for all n. Further, any $p > b$ will be undercut. Whereas if $AC(q)$ is constant, and $\overline{p}$ is well defined, then the only possible equilibrium involves the firms charging $b = c^*$, when they have an incentive to deviate to $\overline{p}$. This follows since from Assumption 4(ii), $f(\overline{p}) > f(c^*)$.

Next, given Assumption 4(i), we assume that there exists $t > 0$ such that $AC(q)$ is negatively sloped for all $q \in (0, t)$. Consider some $p \in (AC(t), b)$. Let $\widetilde{q}(p)$ be the unique q, $0 < q < t$, such that $AC(\widetilde{q}(p)) = p$. Next, let $n(p)$ satisfy $\frac{f(p)}{n(p)} = \widetilde{q}(p)$, where $n(p)$ can be a non-integer. Given that $f(b) > 0$ and $\lim_{p \uparrow b} \widetilde{q}(p) = 0$, it follows that $\lim_{p \uparrow b} n(p) \to \infty$. Next, let $\tilde{n}(p)$ be the largest possible integer such that $p \geq AC(\frac{f(p)}{n})$ (this is well defined for $n(p)$ large enough). Clearly, there exists some largest interval (b', b), $AC(t) \leq b' < b$, such that for all $p \in (b', b)$, $\tilde{n}(p)$ is well defined. Given that $|n(p) - \tilde{n}(p)| < 1$ and $\lim_{p \uparrow b} n(p) \to \infty$, we have that $\lim_{p \uparrow b} \tilde{n}(p) \to \infty$. Let $\hat{n} = \min_{p \in (b', b)} \tilde{n}(p)$.

We then construct a sequence $< p(n) > \subseteq F$ such that $\forall i \in \{0, 1, 2, \ldots\}$, $p(\hat{n} + i)$ is some $p \in (b', b)$ such that $\hat{n} + i = \tilde{n}(p)$. Note that for $n \geq \hat{n}$, the pair $(n, p(n))$ belongs to the graph of $\tilde{n}(p)$. Thus $p(n) \geq AC(\frac{f(p(n))}{n})$, so that all firms earn non-negative profits. Moreover, since $p(n) < b = \tilde{d}$, no firm can

[17] Since $f(p)$ is negatively sloped at $\hat{p}$, it cannot be the case that $b = \tilde{d} > \hat{p}$. If $b = \tilde{d} = \hat{p}$, then all firms charging $\hat{p}$ and having zero demand and supply is an equilibrium for $M(n)$.

undercut profitably. Finally, we argue that the sequence $< p(n) >$ converges to b. Suppose not. Then there exists some $\epsilon > 0$ and some sub-sequence $< p(n_i) >$ such that $p(n_i) \leq b - \epsilon$, $\forall n_i$. Note that $\lim_{n_i \to \infty} \frac{f(p(n_i))}{n_i} \leq \lim_{n_i \to \infty} \frac{f(\tilde{p})}{n_i} = 0$. Hence, for n_i large enough, $p(n_i) < AC(\frac{f(n_i)}{n_i})$. This, however, is a contradiction since for all n_i, $(n_i, p(n_i))$ belongs to the graph of $\tilde{n}(p)$. ■

3.1 The Folk Theorem

We next use Proposition 1 to examine whether the folk theorem of perfect competition holds or not, i.e. if $S = \{c^*\}$.

Definition. $f(p)$ is said to be of limited variation if, $\forall p \in [c^*, \hat{p}]$, $\exists \epsilon(p) > 0$, such that $f(p)$ is monotonic over $[p, p + \epsilon(p)]$.

We need one more assumption before we can proceed further.

Assumption 5. (i) $f(p)$ is of *limited variation.*
(ii) If $b = c^*$, then $AC(q)$ is either constant, or increasing.

It can be argued that Assumption 5(ii) is not very strong.[18] We can now write down our next proposition.

Proposition 2. *Suppose Assumptions 1, 2, 3, 4 and 5 hold. Then* $S = \{c^*\}$, *if and only if* $AC(q)$ *is constant and* $\overline{p}$ *is not well defined.*

Proof. There are two cases to consider.
Case 1. $c^* < b$. Since $c^* \notin \overline{(b, \tilde{d}) \cap F}$, the folk theorem cannot hold.
Case 2. $c^* = b$ (thus b is finite). From Assumption 5(ii), $AC(q)$ is either constant, or increasing. To begin with we consider the case where $AC(q)$ is

[18]One can argue that in general $b \neq c^*$. Let us perturb $AC(q)$ by considering the family of functions $AC(q) + \alpha q$, where α is close to 1. Unless c^* is achieved at $q = 0$ (this is true if $AC(q)$ is increasing or constant), then, for α close to zero, $c^*(\alpha) \neq c^* = b$.

increasing, so that $b = c^* < \tilde{d}$. From Assumption 5(i), $\exists \epsilon(c^*) > 0$, such that $f(p)$ is monotonic over $[c^*, c^* + \epsilon(c^*)]$. We consider two sub-cases.

Case (2a). Suppose $\overline{p}$ is not well defined. Then $f(p)$ is negatively sloped over $[c^*, c^* + \epsilon(c^*)]$. (Suppose not, then $\exists p \in (c^*, c^* + \epsilon(c^*)]$ such that $f(p) \geq f(c^*)$. But then $\overline{p} = \text{argmax}_{p \in [c^*, \hat{p}]} f(p)$, which is well defined.) W.l.o.g. let $\epsilon(c^*) < \tilde{d}$. Suppose there exists some $p \in (c^*, c^* + \epsilon(c^*)]$, such that $f(p') \leq f(p)$, $\forall p' > p$, then $p \in F$, and hence $p \in S$. Since $p > c^*$, $S \neq \{c^*\}$. Thus, we next consider the case where no such p exists. Then we can find a monotone decreasing sequence $< p(n) >$ in $(c^*, c^* + \epsilon(c^*)]$ converging to c^* such that $\forall p(n)$, $\exists p'(p(n))) > p(n)$, such that $f(p'(p(n))) > f(p(n))$. Fix some $\overline{n}$ such that $f(p'(p(\overline{n})) > f(p(\overline{n}))$. Since $f(p)$ is decreasing over $[c^*, c^* + \epsilon(c^*)]$, $f(p'(p(\overline{n}))) > f(p(n))$, $\forall n > \overline{n}$. Taking limits, $f(p'(p(\overline{n}))) \geq f(c^*)$. However, in that case $\overline{p} = \text{argmax}_{p \in [c^*, \hat{p}]} f(p)$, which is well defined.

Case (2b). We next consider the case where $\overline{p}$ is well defined. Clearly, $f(\overline{p}) \geq f(c^*)$. In fact, from Assumption 4(ii), $f(\overline{p}) > f(c^*)$. Then, from the continuity of $f(p)$, $\exists p(c^*) > c^*$ such that $f(p(c^*)) = f(c^*)$ and, $\forall p'(c^*)$ strictly less than, but sufficiently close to $p(c^*)$, $f(p'(c^*)) > f(c^*)$ and $p'(c^*) > AC(f(p'(c^*)) - f(c^*))$ (this follows since $\lim_{p'(c^*) \to p(c^*)} AC(f(p'(c^*)) - f(c^*)) = b < p(c^*)$). Thus $\exists p'(c^*) > c^*$ such that $f(p'(c^*)) > f(c^*)$ and $[p'(c^*) - AC(f(p'(c^*)) - f(c^*))][f(p'(c^*)) - f(c^*)] = 2\delta$, where $\delta > 0$. Hence $\exists \epsilon' > 0$ such that $\forall p \in [c^*, c^* + \epsilon']$, $\exists p'(p) > p$, such that $f(p'(p)) > f(p)$ and $[p'(p) - AC(f(p'(p)) - f(p))][f(p'(p)) - f(p)] > \delta > 0$. Consider a sequence $< p(n) >$ converging to c^*, such that $p(n)$ is an equilibrium for $M(n)$. We then note that $\exists n'$ such that for all $n > n'$, $p(n) < c^* + \epsilon'$, and the profit of the active firms is less than δ. Hence they have an incentive to deviate to $p'(p)$.

Finally, we consider the case where $AC(q)$ is constant. Note that any single price equilibrium must involve all the firms charging c^* (any $p > c^*$ will be undercut). Suppose $\overline{p}$ is not well defined. Then, $\forall p > c^*$, $f(c^*) > f(p)$. Thus, $\forall n$, there is an equilibrium where all the firms charge c^*. Next suppose

$\overline{p}$ is well defined, so that from Assumption 4(ii), $f(\overline{p}) > f(c^*)$. But then, the firms charging c^* will have an incentive to deviate to $\overline{p}$. ■

Thus, in the class of continuous demand and cost functions (the cost function may possibly be discontinuous at the origin), the folk theorem fails unless the average cost function is constant and $\overline{p}$ is not well defined.

Finally, let us relate our results to those for Cournot competition with exogenous entry. It is well known that the folk theorem goes through if average cost is constant. Further, from Ruffin (1971), the folk theorem goes through if $AC(q)$ is increasing, but not if its U-shaped, and the cost function is continuous.

Remark. While we adopt the efficient rationing rule, it is easy to see that Proposition 1 goes through for other rationing rules, including the proportional one. This follows since the choice of the rationing rule does not affect the *definition* of F, which is what we need for Proposition 1.

3.2 Relating Proposition 1 to NRC

We then relate the characterization of S in Proposition 1 to the corresponding one in NRC (i.e. Theorem 1). Note that if the demand function is negatively sloped, then $F = [0, \infty)$. Further, if the average cost function is either increasing, or U-shaped, and $b \leq \tilde{d}$, then $\tilde{d} = d$.[19] Thus under these assumptions the characterization of S in Proposition 1 coincides with that in NRC.

We then argue that for negatively sloped demand functions, the characterization of S in NRC holds for a large class of cost functions satisfying some mild regularity condition.

[19] If $b \leq \tilde{d}$, then, under the NRC formulation, the average cost function must be positively sloped at $\tilde{d}$. Thus there does not exist any $p' < \tilde{d}$ such that $p' = AC(f(p'))$. Of course if $b > \tilde{d}$, then S is empty.

Definition. $f(p)$ is said to be *tangent* to $AC(q)$ at some $\overline{p}$, if $\overline{p} = AC(f(\overline{p}))$ and $\exists\epsilon(\overline{p}) > 0$ such that $\forall p \in (\overline{p} - \epsilon(\overline{p}), \overline{p}) \cup (\overline{p}, \overline{p} + \epsilon(\overline{p}))$, either $p \geq AC(f(p))$, or $p \leq AC(f(p))$.

Assumption 6. At any $p < \tilde{d}$ such that $p = AC(f(p))$, the demand and the average cost functions cannot be tangent to each other.

Note that assumption 6 above is not very strong.[20]

We are now in a position to prove Proposition 3.

Proposition 3. *Let Assumptions 1, 2, 3, 4 and 6 hold and let the demand function be negatively sloped. Then $S = [b, d]$ if $b \leq d$, S is empty otherwise.*

Proof. Note that $F = [0, \infty)$ since the demand function is negatively sloped. Hence given Proposition 1, it is sufficient to show that, under Assumption 6, $\tilde{d} = d$. Clearly, $\tilde{d} = AC(f(\tilde{d}))$. Since d is the minimum p such that $p = AC(f(p))$, $\tilde{d} \geq d$. Next suppose that $\tilde{d} > d$. From the definition of $\tilde{d}$, $p \leq AC(f(p))$ for all $p \in [0, \tilde{d})$. Moreover, since $d = AC(f(d))$, $f(p)$ and $AC(q)$ are tangent to each other at d, thus violating assumption 6. ■

Thus, Theorem 1 in NRC can be substantially generalized to allow for a large class of average cost functions. In contrast, the assumption that the demand function be negatively sloped appears much more critical. In fact, if the average cost function is increasing, then the negativity of the demand function is necessary in the following sense: Suppose to the contrary that there is some $b < p < d$ such that the demand function is positively sloped in a neighborhood of p. It is easy to show that such a p cannot belong to

[20] Suppose there is some $p < \tilde{d}$ such that $p = AC(f(p))$, and the demand and the average cost functions are tangent to each other. Then, for any $\lambda f(p)$, λ not equal to 1, but close to it, we can find a neighborhood of p such that the functions are not tangent to each other at any price in the neighborhood.

S.[21] If, however, the average cost is U-shaped, then Theorem 1 in NRC may go through even if the demand function is not negatively sloped.[22]

4 Free Entry

Let $M(r, n)$ denote a market with the demand function $rf(p)$ and n firms.

In this section we examine equilibria when market size is large, and there is free entry of firms, where free entry is formalized as there being inactive firms in equilibrium. Hence we examine n-firm single price equilibria where m ($< n$) firms are active (i.e. set the lowest price), and $n-m$ (> 0) firms are inactive (i.e. charge higher prices and have no demand). We then study the limiting equilibria as the market demand goes to infinity. We characterize the set of all prices p such that if r is sufficiently large, then, for $M(r, n)$, there is some free entry equilibrium where the market price is arbitrarily close to p.

Definition: $T = \{p$: there is a sequence $p(r)$ that converges to p such that for each sufficiently large r, there is an integer n and an equilibrium for the market $M(r, n)$ in which all active firms charge the lowest price $p(r)$, but not all firms set the price $p(r)\}$.

We focus on average cost functions that are *ultimately increasing*.

Assumption 7. There exists some smallest $\tilde{q}$ such that $AC(q)$ is strictly

[21] Suppose there is an equilibrium where all firms charge $p(n)$ which is sufficiently close to p. Then, for n large enough, the profit level of all such firms is close to zero. Since $f(p(n))$ is positively sloped at $p(n)$, and $p(n) > b$, one of the firms can deviate to $p(n) + \epsilon$, and, for ϵ small enough, obtain a positive profit that is independent of n. Thus for n large enough, deviating to $p(n) + \epsilon$ is profitable.

[22] Consider an U-shaped average cost function, and let the demand function be positively sloped over the interval $[p', p'']$, and negatively sloped otherwise. In order to rule out trivial cases, we assume that $c^* < p'$. Let $AC(q) > \overline{p}$, $\forall q \leq f(p'') - f(p')$. In that case $F = [0, \infty)$, and Theorem 1 in NRC goes through.

increasing for all $q > \tilde{q}$.

Note that Assumption 7 is satisfied by both the increasing, as well as the U-shaped average cost function. If $AC(q)$ is increasing, then $\tilde{q} = 0$, whereas if $AC(q)$ is U-shaped, then $\tilde{q} = q^*$.

Next let

$$d(r) = \text{argmax}_{\,p \text{ such that } AC(rf(p))=p} \;\; f(p).$$

Further, let $d^* = \lim_{r \to \infty} d(r)$.

Given Assumptions 1, 2, 3 and 7, we have the following lemma.

Lemma 1. *For $c^* < \hat{p}$, d^* is well defined. Moreover, if d^* is well defined, then $\hat{p} \geq d^* > c^*$.*

Proof. If $c^* < \hat{p}$, then, for r sufficiently large, $d(r)$ is well defined. Further, for r large enough, $rf(d(r)) > \tilde{q}$, so that $AC(q)$ is strictly increasing in q. Hence $d(r)$ is increasing in r. Further, $d(r) \leq \hat{p}$. Thus d^* exists, and, moreover, $d^* \leq \hat{p}$. Finally, since $d(r) \geq c^*$, and $d(r)$ is strictly increasing for r large, $c^* < d^*$. ∎

We need some further notations.

Lat $q(p)$ be the minimum q' such that, $AC(q') = p$ and $AC(q)$ is strictly negatively sloped at q'.

$V = \{p : \; q(p) \text{ is well defined.}\}$

$\tilde{F} = \{p : \; \exists$ some sequence $< p(r) >$ converging to p such that $p(r) \in F(r), \forall r.\}$[23]

We are finally in a position to characterize the set T.

[23] Clearly, $\tilde{F}$ is closed. Consider some sequence $< p_n >$ converging to p, where $p_n \in \tilde{p} \; \forall n$. Using the triangle inequality it is straightforward to show that $\exists$ some sequence $< p(r) >$ converging to p such that $p(r) \in F(r)$, $\forall r$.

Proposition 4. *Assume that* $c^* < \hat{p}$ *and Assumptions 1, 2, 3 and 7 hold. Then* $T = [c^*, d^*] \cap \overline{V} \cap \tilde{F}$.

Proof. Note that since $c^* < \hat{p}$, d^* is well defined from Lemma 1. Further, given Assumption 7, the interval $[c^*, d^*]$ is non-empty. Next suppose to the contrary that $p(r)$ converges to some p outside $[c^*, d^*] \cap \overline{V} \cap \tilde{F}$ as r increases, and for each sufficiently large r, there is an n and an equilibrium for $M(r, n)$ in which $p(r)$ is the lowest price, but not all firms charge $p(r)$.

Clearly, $p(r) \geq c^*$, $\forall r$, so $p \geq c^*$. We then argue that any equilibrium price $p(r) < d^*$. There are two cases to consider.

Case 1. $d^* = \hat{p}$. Define $p'(r)$ as the maximum p satisfying $rf(p) = q^*$. Clearly, $p'(r)$ is defined for r large enough. Further, $p'(r) < \hat{p}$.[24] Moreover, since $rf(p)$ is negatively sloped at $p'(r)$, $p'(r)$ is increasing in r. Hence $\lim_{r\to\infty} p'(r)$ is well defined. Also note that $\lim_{r\to\infty} p'(r) = \hat{p}$.[25] Since $\hat{p} > c^*$, for r sufficiently large, $\hat{p} > p'(r) > c^*$. Now suppose to the contrary that $p(r) \geq d^* = \hat{p}$. Then $p(r) \geq \hat{p} > p'(r) > c^*$. But then an inactive firm could undercut by charging $p'(r)$, and make a strictly positive profit.

Case 2. $d^* < \hat{p}$. Since $d^* < \hat{p}$, from Assumption 1(b), $f(d^*) > 0$. Thus for r sufficiently large, $rf(d^*) > \tilde{q}$. Moreover, since $AC(q)$ is strictly increasing for $q > \tilde{q}$, it follows that $AC(rf(d^*)) < d^*$. From continuity, for $\epsilon > 0$ small enough, $AC(rf(d^* - \epsilon)) < d^* - \epsilon$. Now suppose to the contrary that $p(r) \geq d^*$. Then $p(r) \geq d^* > AC(rf(d^*))$. But then, for $\epsilon > 0$ but sufficiently small, an inactive firm could deviate to price $d^* - \epsilon$, sell $rf(d^* - \epsilon)$ and earn a strictly positive profit.

Next consider some $p \notin \overline{V}$. Thus p is such that $\forall q'(p)$ satisfying $AC(q) = p$, $AC(q)$ is strictly positively sloped at $q'(p)$. In case such a $q'(p)$ does not exist, then $p > AC(q)$ for all q, and an inactive firm can match p and make

[24] Suppose not. Then, from Assumption 1(b), $q^* = rf(p'(r)) = 0$. Since $q^* > 0$, this is a contradiction.

[25] Suppose not. Let $\lim_{r\to\infty} p'(r) = \tilde{p} < \hat{p}$. Then $\lim_{r\to\infty} f(p'(r)) = f(\tilde{p}) > 0$. Hence, $q^* = \lim_{r\to\infty} rf(p'(r)) \to \infty$, which is a contradiction.

a gain. So suppose such a $q'(p)$ exists. For $p \in T$ there must be some market $M(r, n)$ and some free entry equilibrium such that all active firms charge p', where p' is arbitrarily close to p, and there are some inactive firms. Since $\overline{V}$ is closed, $p' \notin \overline{V}$ and $AC(q)$ is positively sloped at $q'(p')$. Since the active firms make non-negative profits, the output level of any active firm is less than equal to $q'(p')$. Then an inactive firm can match p' and earn a strictly positive profit.

We then consider some $p \notin \tilde{F}$. Suppose to the contrary that $p \in T$. Then we can find a sequence $< p(r) >$ converging to p such that $p(r)$ constitutes a free entry equilibrium for $rf(p)$. Then $p(r) \in F(r)$, $\forall r$, otherwise some of the inactive firms can deviate to some appropriate $p' > p(r)$, and make a positive profit. However, this implies that $p \in \tilde{F}$.

Consider any p such that $c^* < p < d^*$ and $p \in V \cap \tilde{F}$. We argue that any such p must be in the limit set. Since $p \in \tilde{F}$, there exists a sequence $< p(r_n) >$ converging to p such that $p(r_n) \in F(r_n)$, $\forall r_n$. Without loss of generality let $< p(r_n) > \subseteq V$. Consider $q(p(r_n))$ (since $p(r_n) \in V$, $AC(q)$ is negatively sloped at $q(p(r_n))$). Let $N(r_n)$ be the largest integer such that $N(r_n) < \frac{r_n f(p(r_n))}{q(p(r_n))}$. For r_n sufficiently large, $AC(\frac{r_n f(p(r_n))}{N(r_n)}) < p \leq AC(\frac{r_n f(p(r_n))}{N(r_n)+1})$. Let $N(r_n)$ firms each set the price $p(r_n)$ in the r_n-market and share demand equally, and let one firm set a higher price. Then the active firms all earn a positive profit. If one of them, or the inactive firms undercuts the price, then that firm must produce to meet a demand that exceeds $r_n \underline{f}$, where $\underline{f} = \min_{c^* \leq p' \leq p} f(p') > 0$. But as r_n gets large, $AC(r_n \underline{f})$ either approaches or exceeds $d^* > p(r_n)$. Also, if an inactive firm matches the lowest price, by the properties of $N(r_n)$ the firm at best has a profit of zero. Since $p(r_n) \in F(r_n)$, none of the firms can charge a higher price and gain. Thus for each sufficiently large r_n, $p(r_n)$ is an equilibrium price for r, and thus is in the limit set.

Finally, all p in $[c^*, d^*] \cap \overline{V} \cap \tilde{F} - (c^*, d^*) \cap V \cap \tilde{F}$ can be obtained as limits of appropriate sequences of $p(r)$. ∎

We then argue that if the average cost function is *ultimately decreasing*, then T is empty. Let Assumptions 1, 2, 3 and 6 hold. Suppose that there exists some minimum $\overrightarrow{q}$ such that $AC(q)$ is negatively sloped for all $q > \overrightarrow{q}$. Further, $\forall q$, let $AC(q) > \lim_{q' \to \infty} AC(q')$, . For r sufficiently large, $d(r)$ exists, $rf(d(r)) > \overrightarrow{q}$, and $d(r)$ satisfies $AC(rf(p)) = p$. Now, for any $n > 1$, any price above $d(r)$ can be undercut. Whereas if there is a single firm charging $d(r)$, then it can increase its price slightly and make a positive profit.

4.1 The Folk Theorem

We then use Proposition 4 to examine if the folk theorem holds in this framework or not, i.e. if $T = \{c^*\}$.

Proposition 5. *Suppose that $c^* < \hat{p}$ and Assumptions 1, 2, 3 and 5 hold.*

(i) If Assumption 7 holds, then the folk theorem fails to holds.

(ii) If $c^* = b$, then the folk theorem holds if and only if $AC(q)$ is constant and $\overline{p}$ is not well defined.

Proof. We consider two cases.

Case 1. Suppose $c^* < b$. Then $[c^*, b] \subseteq \overline{V}$. There are two sub-cases to consider:

1(a). First, consider the case where $\overline{p}$ is not well defined. We can then mimic the proof of Case 2(a) of Proposition 2 to argue that $\exists \epsilon(c^*) > 0$ such that $\exists p \in (c^*, c^* + \epsilon(c^*)]$, so that $f(p'(p)) \leq f(p)$, $\forall p'(p) > p$. W.l.o.g. let $\epsilon(c^*) < b$. But then $p \in F(r)$, $\forall F(r)$, and hence $p \in \tilde{F}$. Thus $T \neq \{c^*\}$.

1(b). Next suppose $\overline{p}$ is well defined. Then $f(\overline{p}) > f(c^*)$. Thus, $\forall r$, we can find some $p'(c^*, r) > c^*$ such that $f(p'(c^*, r)) > f(c^*)$, and $rf(p'(c^*, r)) - rf(c^*)$ is small enough such that $p'(c^*, r) > AC(rf(p'(c^*, r)) - rf(c^*))$. Thus $\exists \epsilon' > 0$, such that $\forall p \in [c^*, c^* + \epsilon']$, $\exists p'(p, r) > p$ such that $f(p'(p, r)) > f(p)$

and $[p'(p,r) - AC(rf(p'(p,r)) - rf(p))] > 0$. Thus such a p cannot be sustained as a free entry equilibrium for any r, however large, as one of inactive firms can deviate to $p'(p,r)$ and make a gain. Thus c^* cannot be sustained as a limit of free entry equilibrium price sequence $< p(r) >$.

Case 2. Suppose $c^* = b$. Then, from Assumption 5(ii), $AC(q)$ is either increasing, or constant. First consider the case where $AC(q)$ is increasing. Then no $p > b$ can be sustained as a free entry equilibrium for any r. Suppose not. Then, for any such p, one of the inactive firms can match this price and make a positive profit.

We next consider the case where $AC(q)$ is constant. Note that any single price equilibrium must involve all the firms charging c^*. Suppose $\overline{p}$ is not well defined. Then, $\forall p > c^*$, $f(c^*) > f(p)$. Thus, $\forall r$, there is a free entry equilibrium where all the active firms charge c^*. Further, there is no other free entry equilibrium. Next suppose $\overline{p}$ is well defined, so that $f(\overline{p}) > f(c^*)$. But then, given Assumption 4(ii), the active firms will have an incentive to deviate to $\overline{p}$. ■

Thus, for the class of average cost functions that are ultimately increasing, the folk theorem for the free entry case holds if and only if $AC(q)$ is constant and $\overline{p}$ is not defined. Next recall that if $AC(q)$ is ultimately decreasing (and $\forall q$, $AC(q) > \lim_{q' \to \infty} AC(q')$,) then no price can be sustained as a free entry equilibrium, so that the folk theorem fails. These results are in sharp contrast to that for the Cournot case when Novshek (1980) shows that the folk theorem goes through for U-shaped average cost functions.

4.2 Relating Proposition 4 to NRC

We then relate Proposition 4 to Theorem 2 in NRC. Recall that for negatively sloped demand functions, $F(r) = [0, \infty)$, $\forall r$, so that $\tilde{F} = [0, \infty)$. Further, for U-shaped average cost functions, $V = \{p : c^* < p < b\}$. Thus, if the demand function is negatively sloped and the average cost function is

U-shaped, $T = [c^*,\ \min\{b, d^*\}]$, so that the two characterizations coincide. Thus Proposition 4 extends Theorem 2 in NRC to demand functions that are not necessarily negatively sloped, and to average cost functions that are ultimately increasing.

We then use Proposition 4 to show, that for negatively sloped demand functions, the characterization in NRC hold for a larger class of average cost functions than the U-shaped one.

Define $b' = \max\{p : p = AC(q')$ where q' is a local maximizer of $AC(q)\}$.

Proposition 6. *Assume that $c^* < \hat{p}$ (so that d^* is well defined) and assumptions 1, 2 and 5 hold. If the demand function is negatively sloped and* $\max\{b', d^*\} \leq b$*, then* $T = [c^*,\ \min\{b, d^*\}]$.

Proof. There are two cases to consider.

Case 1. Let $b' \leq b$. Given Proposition 4, it is sufficient to observe that $V = \{p : c^* < p < b\}$.

Case 2. Let $d^* \leq b$. Given Proposition 4, it is sufficient to observe that $\{p : c^* < p < b\} \subseteq V$. ∎

Thus, the characterization of the limit equilibrium set T in NRC can be generalized to allow for a class of average cost functions that are ultimately increasing, and $\max\{b', d^*\} \leq b$.

5 Conclusion

In this paper we examine if the folk theorem of perfect competition goes through under Bertrand competition (where firms supply all demand). We allow for a large class of demand and cost functions where we essentially only assume that the demand function is continuous and intersects both the axes, and that the cost function is continuous (except possibly at the origin). We find that the folk theorem fails to hold for a large class of demand and cost functions. In fact, for the folk theorem to hold, it is in some sense necessary that the cost function be linear.

Inter alia, we also characterize the limit equilibrium sets and relate the characterizations obtained in this paper to those in NRC.

6 Appendix: Multiple Price Equilibria

In this appendix we allow for multiple price equilibria and examine if analogues of our earlier results go through.

6.1 Exogenous Entry

For the market $M(n)$, let $P(n) = (p_1(n), \cdots, p_n(n))$ denote a multiple price Bertrand equilibrium (MPE) with exogenous entry (so that all firms are active). For this case, the limit equilibrium set S' is defined as follows:

Definition: $S' = \{p$: there is a sequence $< P(n) >$ such that for $\forall \epsilon > 0$, $\exists n(\epsilon)$, so that for each $n > n(\epsilon)$, all elements of $P(n)$ belongs to an ϵ-neighborhood of $p\}$.

Proposition 7 below provides a partial characterization of S'.

Proposition 7. *Let Assumptions 1, 2, 3 and 4 hold.*

(i) If $b < \tilde{d}$, then $S' \supseteq \overline{(b, \tilde{d}) \cap F}$. Further, no $p < b$, and no $p > \tilde{d}$ belongs to S'.

(ii) If $b > \tilde{d}$, then S is empty.

(iii) Suppose $b = \tilde{d}$.

(a) Let $AC(q)$ be constant. If $\overline{p}$ is not well defined, then $S' = \{b\}$, whereas if $\overline{p}$ is well defined, then S' is empty.

(b) Suppose $AC(q)$ is not constant. If there exists some sequence $< p(n) >$ in F such that, $\forall n$, $p(n) < b$, and $< p(n) >$ converges to b, then $b \in S'$.

Proof: To begin with we argue that no price less than b, or greater than $\tilde{d}$ can belong in S'. Suppose not.

First let $p < \tilde{d}$. Consider some MPE, $P(n)$, of $M(n)$. Note that there is some active firm which has an output of at most $\frac{f(\tilde{p})}{n}$, which converges to zero as n goes to infinity. Thus if $p < b$, then for all sufficiently large n, there is some active firm charging $p(n)$ where $p(n) < AC(\frac{f(p(n))}{n})$, so that $p(n)$ cannot be an equilibrium price.

Next let $p > \tilde{d}$. Consider some MPE, $P(n)$, of $M(n)$. Note that there is some active firm whose profit, $[p(n) - AC(\frac{f(p(n))}{n})]\frac{f(p(n))}{n}$, is less than $(\hat{p} - c^*)\frac{f(\tilde{p})}{n}$, which, for n large, converges to zero. Moreover, from the definition of $\tilde{d}$, there exists p' such that $\tilde{d} < p' < p(n)$ $\forall p(n) \in \{P(n)\}$ and $p' > AC(f(p'))$. Undercutting to such a price p' yields a strictly positive profit that depends on p', but not on n. Thus, for n large, undercutting is strictly profitable for some active firm making a profit less than $(\hat{p} - c^*)\frac{f(\tilde{p})}{n}$.

We can then mimic the argument in Proposition 1 to argue that every price in the interval $(b, \tilde{d}) \cap F$ is in the limit set. We can similarly argue that, for $b < \tilde{d}$, all $p \in \overline{(b, \tilde{d}) \cap F} - (b, \tilde{d}) \cap F$, belongs to S'.

Next, let $b = \tilde{d} < \hat{p}$. Suppose average cost is constant. We first note that in any MPE the least price charged by the firms, say p, must be c^*. (Suppose not, i.e. $p > c^*$. Suppose that there are more than one firm charging p, then this price will be undercut by the firms charging p. Next suppose that there is exactly one firm charging p. Then, for n large, some of the other firms will have a profit close to zero, and will undercut p.) If $\overline{p}$ is not well defined then, $\forall p > c^*$, $f(c^*) > f(p)$. Thus, $\forall n$, there is an equilibrium where all the firms charge c^*. Further, there cannot be any other equilibrium. Next suppose $\overline{p}$ is well defined, so that $f(\overline{p}) > f(c^*)$. But then, the firms charging c^* will have an incentive to deviate to $\overline{p}$ when they earn a strictly positive profit.

Finally, for the case where $b = \tilde{d}$, but $AC(q)$ is not constant, we can mimic the proof in Proposition 1. ∎

Proposition 7 above is an analogue of Proposition 1 earlier. Note, however, that we only achieve a partial characterization of S'. The characterization would be complete if one can show that if $p \notin F$, then $p \notin S'$. Whether this is true is an open question.

We then argue that an analogue of Proposition 2 goes through.

Proposition 8. *Suppose Assumptions 1, 2, 3, 4 and 5 hold. Then $S' = \{c^*\}$, if and only if $AC(q)$ is constant and $\overline{p}$ is not well defined.*

Proof. **Case 1.** $c^* < b$. Consider some MPE for $M(n)$, that is arbitrarily close to c^*. Then there will be some firm that will be supplying at most $\frac{f(\tilde{p})}{n}$. Now suppose n is taken to infinity, and w.l.o.g. assume that the identity of the firm supplying at most $\frac{f(\tilde{p})}{n}$ remains the same. Then, for n sufficiently large, this firm makes a loss (since the average cost of this firm will be close to $b > c^*$). Thus $c^* \notin S'$.

Case 2. $c^* = b$. From Assumption 5(ii), $AC(q)$ is either constant, or increasing. To begin with we consider the case where $AC(q)$ is increasing, so that $b = c^* < \tilde{d}$. From Assumption 5(i), $\exists \tilde{d} > \epsilon(c^*) > 0$, such that $f(p)$ is monotonic over $[c^*, c^* + \epsilon(c^*)]$. We consider two sub-cases.

Case (2a). Suppose $\overline{p}$ is not well defined. Then we can mimic the argument in Proposition 2 to show that there exists $\tilde{d} > p > c^*$ such that $p \in F$, so that $p \in S'$.

Case (2b). We next consider the case where $\overline{p}$ is well defined. We can mimic the argument in Proposition 2 to claim that there exists $0 < \epsilon' \leq \epsilon(c^*)$ such that $\forall p \in [c^*, c^* + \epsilon']$, $\exists p'(p) > p$, such that $f(p'(p)) > f(p)$ and $[p'(p) - AC(f(p'(p)) - f(p))][f(p'(p)) - f(p)] > \delta > 0$. Consider some MPE for $M(n)$ where n is sufficiently large so that the maximum price charged is $c^* < \overline{p}(n) < c^* + \epsilon'$. The profit of all active firms are bounded above by $(\overline{p}(n) - c^*)f(\tilde{p})$. Thus for $\overline{p}(n)$ close to c^*, the profit of all active firms is less than δ, and have an incentive to deviate to $p'(\overline{p}(n))$.

Finally, if $AC(q)$ is constant, we can mimic the argument in Proposition 7. ∎

6.2 Free Entry

For the market $M(r,n)$, let $P(r) = (p_1(r), \cdots, p_n(r))$ denote a multiple price free entry Bertrand equilibrium. For this case, the limit equilibrium set T' is defined as follows:

Definition: $T' = \{p :$ there is a sequence $< P'(r) >$ such that $\forall \epsilon > 0$, $\exists r(\epsilon)$ so that $\forall r > r(\epsilon)$, there is an integer n and a free entry equilibrium for the market $M(r,n)$ in which all active firms charge prices that are within an ϵ-neighborhood of $p\}$.

We then provide a partial characterization of T'.

Proposition 9. *Assume that $c^* < \hat{p}$ and Assumptions 1, 2, 3 and 7 hold. Then $T' \supseteq [c^*, d^*] \cap \overline{V} \cap \tilde{F}$. Further, no $p < c^*$, or $p > d^*$, or $p \notin \overline{V}$ can be in T'.*

Proof. Clearly, in any free entry equilibrium, the price charged by any active firm must be at least c^*, so, for any $p \in T'$, $p \geq c^*$. We then argue that for any $p \in T'$, $p \leq d^*$. There are two cases to consider.

Case 1. $d^* = \hat{p}$. As in the proof of Proposition 4, for r sufficiently large $\exists p'(r)$ which is the maximum p satisfying $rf(p) = q^*$. Further, for r sufficiently large, $\hat{p} > p'(r) > c^*$. Now suppose to the contrary $p > d^*$. Then, for r large enough, the least price charged by the active firms, $p(r) \geq \hat{p} > p'(r) > c^*$. But then an inactive firm could undercut by charging $p'(r)$, and make a strictly positive profit.

Case 2. $d^* < \hat{p}$. As in the proof of Proposition 4, for r sufficiently large, $\exists \epsilon > 0$ small enough such that $AC(rf(d^* - \epsilon)) < d^* - \epsilon$. Now suppose to the contrary that $p > d^*$. Then for any free entry equilibrium where r is sufficiently large, the least price charged by the active firms is at least d^*. But then, for $\epsilon > 0$ but sufficiently small, an inactive firm could deviate to price $d^* - \epsilon$, sell $rf(d^* - \epsilon)$ and earn a strictly positive profit.

Next consider some $p \notin \overline{V}$. Then $\forall q'(p)$ satisfying $AC(q) = p$, $AC(q)$ is strictly positively sloped at $q'(p)$. First suppose such a $q'(p)$ exists. If $p \in T$, then there is some market $M(r, n)$ and some free entry equilibrium such that the least price charged by the active firms is p', where p' is arbitrarily close to p. Since $\overline{V}$ is closed, $AC(q)$ is positively sloped at $q'(p')$. Since the active firms make non-negative profits, the output level of any active firm is less than equal to $q'(p')$. Then an inactive firm can match p' and earn a strictly positive profit. Next suppose such a $q'(p)$ does not exist. Then $p > d^*$, and we can mimic the argument in case 2 above.

Finally, we can mimic the proof of Proposition 4 to claim that all p in $[c^*, d^*] \cap \overline{V} \cap \tilde{F}$ belongs to T'. ■

Note that Proposition 9 achieves a partial characterization of T'. For a full characterization one needs to show that if $p \notin \tilde{F}$, then $p \notin T'$. Whether this is true is an open question.

We finally write down an analogue of Proposition 5.

Proposition 10. *Suppose that $c^* < \hat{p}$ and Assumptions 1, 2, 3 and 5 hold.*

(i) If Assumption 7 holds (so that $c^* < b$), then the folk theorem fails to holds.

(ii) If $c^* = b$, then the folk theorem holds if and only if $AC(q)$ is constant and $\overline{p}$ is well defined.

Proof. **Case 1.** Suppose $c^* < b$. Then $[c^*, b] \subseteq \overline{V}$. There are two sub-cases to consider:

1(a). First, consider the case where $\overline{p}$ is not well defined. We can then mimic the proof of Proposition 5 to argue that $\exists b > p > c^*$, such that $p \in \tilde{F}$. But then $p \in T'$.

1(b). Next suppose $\overline{p}$ is well defined. Then $f(\overline{p}) > f(c^*)$. We can mimic the proof of Proposition 5 to argue that $\exists \epsilon' > 0$, such that $\forall p \in [c^*, c^* + \epsilon']$,

$\exists p'(p,r) > p$ such that $f(p'(p,r)) > f(p)$ and $[p'(p,r) - AC(rf(p'(p,r)) - rf(p))] > 0$. Consider some free entry equilibrium where the highest price charged by some active firm is $p < c^* + \epsilon'$. Thus such a p cannot be sustained as a free entry equilibrium for any r, however large, as one of inactive firms can deviate to $p'(p,r)$ and make a gain.

Case 2. Suppose $c^* = b$ so that, from Assumption 5(ii), $AC(q)$ is either increasing, or constant. First consider the case where $AC(q)$ is increasing. Suppose that there is some free entry equilibrium where the lowest price charged by some active firm is p, where $p > b$. Then one of the inactive firms can match this price and make a positive profit.

We next consider the case where $AC(q)$ is constant. Note that in any free entry equilibrium, the lowest price charged by the active firms must be c^*. Suppose $\overline{p}$ is not well defined, so that $\forall p > c^*$, $f(c^*) > f(p)$. Thus, $\forall r$, there is a unique free entry equilibrium where all the active firms charge c^*. Next suppose $\overline{p}$ is well defined, so that $f(\overline{p}) > f(c^*)$. But then, the firms charging c^* will have an incentive to deviate to $\overline{p}$ when they obtain a positive profit. ■

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