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Overdemand and Underdemand in Economies with Indivisible Goods and Unit Demand *

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Abstract

We study an economy where a collection of indivisible goods are sold to a set of buyers who want to buy at most one good. We characterize the set of Walrasian equilibrium price vectors in such an economy using sets of overdemanded and underdemanded goods. Further, we give characterizations for the minimum and the maximum Walrasian equilibrium price vectors of this economy. Using our characterizations, we give a sufficient set of rules that generates a broad class of ascending and descending auctions in which truthful bidding is an ex post Nash equilibrium.

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1 Introduction

The classical Arrow-Debreu model (Arrow and Debreu, 1954) for studying competitive equilibrium assumes goods to be divisible (commodities). But economies with indivisible goods are common in many types of markets such as housing markets, job markets, and auctions with goods like spectrum licenses. This paper investigates economies with indivisible goods under the assumption that buyers have unit demand, i.e., every buyer can buy at most one good. The unit demand assumption is common, for example, in settings of housing and job markets.

In case of quasi-linear utilities of buyers, the existence of a Walrasian equilibrium is guaranteed, and the set of Walrasian equilibrium price vectors form a complete lattice (Shapley and Shubik, 1972). In this work, we characterize the Walrasian equilibrium price vectors using the notions of *overdemanded* and *underdemanded* sets of goods. A set of goods is (weakly) overdemanded at a price vector if the number of buyers who demand goods only from that set is greater than (or equal to) the number of goods in that set. A set of goods is (weakly) underdemanded at a price vector if their prices are positive and the number of buyers who demand goods from that set is less than (or equal to) the number of goods in that set. Our first result is that a price vector is a Walrasian equilibrium price vector if and only if there is no set of overdemanded and underdemanded goods at that price vector. Since overdemanded goods indicate the presence of excess demand and underdemand goods indicate the presence of excess supply, our characterization reflects the intuition that a price vector is a Walrasian equilibrium price vector if and only if there is no excess supply and no excess demand.

The notion of overdemanded and underdemanded sets of goods has been studied in the context of designing iterative auctions, where prices either increase monotonically (ascending auctions) or decrease monotonically (descending auctions). Demange et al. (1986) use the notion of overdemanded goods to design an ascending auction that terminates at the minimum Walrasian equilibrium price vector. Analogously, Sotomayor (2002) uses the notion of underdemanded goods, albeit a slightly different notion from ours, to design a descending auction that terminates at the maximum Walrasian equilibrium price vector. Both papers, however, do not make any connection between these notions.

Gul and Stacchetti (2000) consider a model where they allow a buyer to buy more than one good and having *gross substitutes* valuations. In such a model, a Walrasian equilibrium price vector is guaranteed to exist, and the set of Walrasian equilibrium price vectors form a complete lattice (Gul and Stacchetti, 1999). For such a model, they provide a generalization of Hall's theorem (Hall, 1935), which results in a necessary condition for a Walrasian equilibrium (see Theorem 3 in Gul and Stacchetti (2000)). Therefore, they do not characterize the set of Walrasian equilibrium price vectors, which is what we will do for our model. Also, Hall's theorem is not enough to characterize the Walrasian equilibrium price vectors in our model. We show that we can use Hall's theorem to guarantee only one of the two conditions required for a Walrasian equilibrium.

We will also characterize the minimum and the maximum Walrasian equilibrium price vectors. A price vector is the minimum Walrasian equilibrium price vector if and only if no set of goods is overdemanded and no set of goods is weakly underdemanded at this price vector. Similarly, absence of sets of weakly overdemanded goods and underdemanded goods completely characterizes the maximum Walrasian equilibrium price vector.

A motivation for this work is the growing literature on *ascending combinatorial auctions* (Cramton et al., 2006). Ascending auctions are preferred over their sealed-bid counterparts for a variety of reasons. For the model considered in this work, an ascending auction will typically maintain a price for every good, and monotonically increase them based on the *bids* of buyers. The terminating condition in such ascending auctions is a Walrasian equilibrium. Moreover, terminating at the minimum Walrasian equilibrium price vector ensures that *truthful bidding* is an ex post Nash equilibrium for the bidders - this is a standard result in this literature (see for example Bikhchandani et al. (2002)), and can be understood from the fact that the minimum Walrasian equilibrium price vector corresponds to the payments of the strategy-proof Vickrey-Clarke-Groves mechanism in our model (Leonard, 1983). However, there are several *algorithms*, both Demange et al. (1986) and Sankaran (1994) describe such methods, that can be interpreted as an ascending auction that terminates at the minimum Walrasian equilibrium. An objective of this work is to give a broad class of ascending and also descending auctions that terminate at the minimum Walrasian equilibrium price vector.

Using our main characterization results, we give a sufficient set of *rules* that generates a broad class of ascending and descending auctions. Our class of auctions include the ascending auctions in Demange et al. (1986) and Sankaran (1994). Thus, our results unify the existing ascending auctions in the literature. Further, we hope that our broad class of ascending and descending auctions will be useful in identifying specific auctions in this class that are easier to implement in practice than the existing auctions.

The rest of the paper is organized as follows. In Section 2, we formally describe the model. Section 3 describes the concepts of overdemanded and underdemanded goods. Section 4 gives the characterizations of the different regions of price vectors. In Section 5, we discuss some implications of our characterization results, and we conclude in Section 6.

2 The Model

There is a set of indivisible goods $N = \{0, 1, \ldots, n\}$ for sale to a set of buyers $M = \{1, \ldots, m\}$. Each buyer can be assigned to at most one good. The good 0 is a dummy good which can be assigned to more than one buyer. The value of buyer $i \in M$ on good $j \in N$ is v_{ij}, assumed to be a non-negative real number. Every buyer has zero value on the dummy good. A **feasible allocation** μ assigns every buyer $i \in M$ a good $\mu_i \in N$ such that no good in $N \setminus \{0\}$ is assigned to more than one buyer. Note that a feasible allocation assigns every buyer a good (may be the dummy good), but some goods may not be assigned to any buyer. We say good $j \in N$ is **unassigned** in μ if there exists no buyer $i \in M$ with $\mu_i = j$. Let Γ be the set of all feasible allocations. An **efficient allocation** is a feasible allocation $\mu^* \in \Gamma$ satisfying $\sum_{i \in M} v_{i\mu_i^*} \geq \sum_{i \in M} v_{i\mu_i}$ for all $\mu \in \Gamma$.

A price vector $\mathbf{p} \in \mathbb{R}_+^{n+1}$ assigns every good $j \in N$ a nonnegative price p_j with $p_0 = 0$. We assume quasi-linear utilities. Given a price vector $\mathbf{p}$, the payoff of buyer $i \in M$ on good $j \in N$ at price vector $\mathbf{p}$ is $v_{ij} - p_j$. The **demand set** of buyer i at price vector $\mathbf{p}$ is $D_i(\mathbf{p}) = \{j \in N : v_{ij} - p_j \geq v_{ik} - p_k \ \forall \ k \in N\}$.

Definition 1 *A* **Walrasian equilibrium** *(WE) is a price vector* $\boldsymbol{p}$ *and a feasible allocation* μ *such that*

$$\mu_i \in D_i(\boldsymbol{p}) \qquad \text{for all } i \in M \tag{\textbf{WE-1}}$$

and

$$p_j = 0 \qquad \text{for all } j \in N \text{ that are unassigned in } \mu. \tag{\textbf{WE-2}}$$

If $(\boldsymbol{p}, \mu)$ *is a WE, then* $\boldsymbol{p}$ *is a* **Walrasian equilibrium price vector** *and* μ *is a* **Walrasian equilibrium allocation***.*

It is well known that every Walrasian equilibrium allocation is efficient (Shapley and Shubik, 1972). Moreover, the set of WE price vectors is non-empty [1], and forms a complete lattice (Shapley and Shubik, 1972) [2]. This implies the existence of a unique minimum WE price vector ($\mathbf{p}^{min}$) and a unique maximum WE price vector ($\mathbf{p}^{max}$). Of all the WE price vectors, $\mathbf{p}^{min}$ stands out since it corresponds to the Vickrey-Clarke-Groves (VCG) payments of buyers assigned to them in a WE (Leonard, 1983). The VCG payment is defined as the *externality* of a buyer on the remaining buyers (Vickrey, 1961; Clarke, 1971; Groves, 1973), and the VCG mechanism is an efficient and *strategy-proof* mechanism. In this work, we aim to characterize all WE price vectors, in particular $\mathbf{p}^{min}$ and $\mathbf{p}^{max}$, and discuss its implications.

For any two price vectors $\mathbf{p}$ and $\mathbf{p}'$ of equal dimension, we write $\mathbf{p} = \mathbf{p}'$ if $p_j = p'_j$ for all j. If there exists a j at which $p_j \neq p'_j$, then we write $\mathbf{p} \neq \mathbf{p}'$. If $p_j \geq p'_j$ for all j then we write $\mathbf{p} \geq \mathbf{p}'$ or $\mathbf{p}' \leq \mathbf{p}$. If $\mathbf{p} \geq \mathbf{p}'$ but $\mathbf{p} \neq \mathbf{p}$, then we write $\mathbf{p} \gneq \mathbf{p}'$ or $\mathbf{p}' \lneq \mathbf{p}$. If there exists some j for which $p_j < p'_j$, then we write $\mathbf{p} \ngeq \mathbf{p}'$ or $\mathbf{p}' \nleq \mathbf{p}$.

[1] The existence of a WE price vector follows from a standard linear programming duality argument and is proved in Shapley and Shubik (1972). The efficient allocation problem can be written as a standard one-to-one assignment problem, which always has an optimal solution (Leonard, 1983; Bikhchandani and Ostroy, 2002). So, the dual of such a linear program will also always have an optimal solution. Such optimal dual solutions correspond to the WE price vectors (Bikhchandani and Ostroy, 2002). Moreover, conditions **WE-1** and **WE-2** are complementary slackness conditions corresponding to these primal and dual linear programs.

[2] We can see that the set of WE price vectors is a special type of lattice. Given a WE $(\mathbf{p}, \mu)$, the inequalities that define this price vector are $p_j - p_{\mu_i} \geq [v_{ij} - v_{i\mu_i}]$ for all $i \in M$ and for all $j \in N$. These are lines in two dimensions, and they are either parallel to one of the axes (since $p_0 = 0$ for all price vectors) or at 45 degrees to both the axes in that dimension.

3 Overdemand and Underdemand

We define **demanders** of a set of goods $S \subseteq (N \setminus \{0\})$ at price vector $\mathbf{p}$ as $U(S, \mathbf{p}) = \{i \in M : D_i(\mathbf{p}) \cap S \neq \emptyset\}$. We define the **exclusive demanders** of a set of goods $S \subseteq (N \setminus \{0\})$ at price vector $\mathbf{p}$ as $O(S, \mathbf{p}) = \{i \in M : D_i(\mathbf{p}) \subseteq S\}$. Clearly, for every $\mathbf{p}$ and every $S \subseteq (N \setminus \{0\})$, we have $O(S, \mathbf{p}) \subseteq U(S, \mathbf{p})$. We denote the cardinality of a finite set S as $\#S$. Given a price vector $\mathbf{p}$, define $N^+(\mathbf{p}) = \{j \in N : p_j > 0\}$. By definition $0 \notin N^+(\mathbf{p})$ for any $\mathbf{p}$.

Definition 2 *A set of goods S is* **overdemanded** *at price vector $\boldsymbol{p}$ if $S \subseteq (N \setminus \{0\})$ and $\#O(S, \boldsymbol{p}) > \#S$. A set of goods S is* **weakly overdemanded** *at price vector $\boldsymbol{p}$ if $S \subseteq (N \setminus \{0\})$ and $\#O(S, \boldsymbol{p}) \geq \#S$.*

The notion of overdemanded sets of goods can be found in Demange et al. (1986) and Sankaran (1994), who use it as a basis for the design of ascending auctions for the model of our paper. For settings where a buyer can buy more than one good, the notion of overdemanded goods has been generalized in Gul and Stacchetti (2000), de Vries et al. (2007), and Mishra and Parkes (2007), who also use it as a basis for the design of ascending auctions for general models.

Definition 3 *A set of goods S is* **underdemanded** *at price vector $\boldsymbol{p}$ if $S \subseteq N^+(\boldsymbol{p})$ and $\#U(S, \boldsymbol{p}) < \#S$. A set of goods S is* **weakly underdemanded** *at price vector $\boldsymbol{p}$ if $S \subseteq N^+(\boldsymbol{p})$ and $\#U(S, \boldsymbol{p}) \leq \#S$.*

The notion of underdemanded sets of goods can be found in Sotomayor (2002) [3], who uses it to design descending auctions for our model.

Both concepts give us an idea about the imbalance of supply and demand in the economy, albeit differently. A measure of total demand on a set of goods is obtained by counting the number of exclusive demanders of these goods in the notion of sets of overdemanded goods and by counting the number of demanders of these goods in the notion of sets of underdemanded goods. However, the dummy good is never part of a set of overdemanded goods and zero priced goods, which always includes the dummy good, are never part of sets of underdemanded goods. In some sense, the existence of sets of overdemanded (underdemanded) goods at a price vector indicates that there is excess demand (supply) in the economy. Since both overdemanded and underdemanded sets of goods may exist at a given price vector, excess demand and excess supply can exist simultaneously in the economy.

[3] There is a small difference between our definition of underdemanded goods and the definition in Sotomayor (2002). Sotomayor (2002) assumes the existence of a dummy buyer who demands every good with zero price and who can be allocated more than one good. Then, a set of goods S is underdemanded in Sotomayor (2002) at a price vector $\mathbf{p}$ if *every* good in N is demanded by a buyer (may be the dummy buyer), $S \subseteq N^+(\mathbf{p})$, and $\#U(S, \mathbf{p}) < \#S$.

	Goods		
Buyers	0	1	2
1	0	8	4
2	0	6	3
3	0	1	1

Table 1: An Illustrating Example

$\mathbf{p}$	$D_1(\mathbf{p})$	$D_2(\mathbf{p})$	$D_3(\mathbf{p})$	OD	UD	Weakly OD	Weakly UD
$(0,0,0)$	$\{1\}$	$\{1\}$	$\{1,2\}$	$\{1\},\{1,2\}$	$\emptyset$	$\{1\},\{1,2\}$	$\emptyset$
$(0,\frac{\epsilon}{2},\epsilon)$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\},\{1,2\}$	$\{2\}$	$\{1\},\{1,2\}$	$\{2\}$
$(0,\epsilon,3+\epsilon)$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\},\{1,2\}$	$\{2\}$	$\{1\},\{1,2\}$	$\{2\}$
$(0,4,1)$	$\{1\}$	$\{1,2\}$	$\{0,2\}$	$\emptyset$	$\emptyset$	$\{1\},\{1,2\}$	$\emptyset$
$(0,4-\epsilon,1+\epsilon)$	$\{1\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{1\},\{1,2\}$	$\{2\},\{1,2\}$
$(0,5,\epsilon)$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\},\{1,2\}$	$\{1\}$	$\{2\},\{1,2\}$	$\{1\}$
$(0,5,2)$	$\{1\}$	$\{1,2\}$	$\{0\}$	$\emptyset$	$\emptyset$	$\{1\},\{1,2\}$	$\{2\},\{1,2\}$
$(0,5,2+\epsilon)$	$\{1\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{1\},\{1,2\}$	$\{2\},\{1,2\}$
$(0,6,1+\epsilon)$	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\{1\}$	$\{2\},\{1,2\}$	$\{1\},\{1,2\}$
$(0,6,4)$	$\{1\}$	$\{0,1\}$	$\{0\}$	$\emptyset$	$\{2\}$	$\{1\}$	$\{2\},\{1,2\}$
$(0,7,3)$	$\{1,2\}$	$\{0,2\}$	$\{0\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{1\},\{1,2\}$
$(0,7+\epsilon,\epsilon)$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\},\{1,2\}$	$\{1\}$	$\{2\},\{1,2\}$	$\{1\}$
$(0,8,2)$	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\{1\}$	$\{2\},\{1,2\}$	$\{1\},\{1,2\}$
$(0,7+\epsilon,3+\epsilon)$	$\{1,2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{1,2\}$	$\emptyset$	$\{1\},\{2\},\{1,2\}$

Table 2: Illustration of overdemand (OD) and underdemand (UD) for the example in Table 1

We give an example to illustrate these notions. Suppose there are three buyers and two goods. Valuations of buyers are given in Table 1. The minimum WE price vector for the example in Table 1 is $\mathbf{p}^{min} = (0,4,1)$. The maximum WE price vector for the example in Table 1 is $\mathbf{p}^{max} = (0,7,3)$. In Table 2, we give various price vectors and describe our notions at those price vectors. In the table it holds that $0 < \epsilon < 1$.

The following insights from Table 2 are worth noting.

- At price vector $(0,0,0)$ no set of goods is underdemanded since $N^+((0,0,0)) = \emptyset$. But sets of underdemanded goods may exist at low price vectors (for example good 2 is underdemanded at price vector $(0,\frac{\epsilon}{2},\epsilon)$). In general, the existence of sets of underdemanded or overdemanded goods depends on the *relative* prices of goods, in addition to the entire price vector.

- Price vectors $(0,4,1)$, $(0,5,2)$ and $(0,7,3)$ are WE price vectors. No set of goods is overdemanded and no set of goods is underdemanded at these price vectors. Moreover, at the minimum WE price vector $(0,4,1)$ no set of goods is weakly underdemanded. Similarly, at the maximum WE price vector $(0,7,3)$ no set of goods is weakly overdemanded. These are no coincidences. In Theorems 1, 2, and 3 we formalize these relations of sets of overdemanded and underdemanded goods with WE price vectors.

- At low enough price vectors, we expect sets of overdemanded goods to exist. In Table 2, we see that at price vectors below the minimum WE price vector $(4,1)$, there is always an overdemanded set of goods. Similarly, at price vectors above the maximum WE price vector $(7,3)$ there is always an underdemanded set of goods. We formalize these results in Corollaries 1 and 2.

4 Characterization Results

In this section, we give a characterization of the price vector space. Our characterization of the WE price vectors is based on the notions of sets of overdemanded and underdemanded goods. Further, together with the notions of sets of weakly overdemanded and weakly underdemanded goods, we characterize the minimum and maximum WE price vectors and the price vectors that are not WE price vectors.

Define $M^+(\mathbf{p}) = \{i \in M : 0 \notin D_i(\mathbf{p})\}$ for any price vector $\mathbf{p}$. Notice that $M^+(\mathbf{p}) = O(N \setminus \{0\}, \mathbf{p})$. Now, consider the following lemmas.

Lemma 1 *Suppose no set of goods is overdemanded. Then there exists a feasible allocation in which every buyer is assigned a good from his demand set.*

Proof: Since $N \setminus \{0\}$ is not overdemanded, $\#(N \setminus \{0\}) \geq \#O(N \setminus \{0\}, \mathbf{p}) = \#M^+(\mathbf{p})$. Consider $S \subseteq M^+(\mathbf{p})$. Let $T = \cup_{i \in S} D_i(\mathbf{p})$. Since $0 \notin T$ and T is not overdemanded, we get $\#T \geq \#O(T, \mathbf{p}) \geq \#S$. Using Hall's theorem (Hall, 1935), there is a feasible allocation in which every buyer i in $M^+(\mathbf{p})$ can be assigned a good in $D_i(\mathbf{p})$, and every buyer in $M \setminus M^+(\mathbf{p})$ can be assigned the dummy good 0, which is in his demand set. ■

Lemma 2 *Suppose no set of goods is underdemanded. Then there exists a feasible allocation in which every good in $N^+(\boldsymbol{p})$ is assigned to a buyer who is a demander of that good.*

Proof: Since $N^+(\mathbf{p})$ is not underdemanded, $\#N^+(\mathbf{p}) \leq \#U(N^+(\mathbf{p}), \mathbf{p}) \leq \#M$. Consider $T \subseteq N^+(\mathbf{p})$. Let $S = U(T, \mathbf{p})$. Since T is not underdemanded, $\#T \leq \#U(T, \mathbf{p}) = \#S$. Using Hall's theorem (Hall, 1935), there is a feasible allocation in which every good in $N^+(\mathbf{p})$ can be assigned to a buyer who is a demander of that good, and the remaining buyers can be assigned the dummy good. ■

The absence of only overdemanded or only underdemanded sets of goods cannot guarantee a WE price vector. For instance, consider an example with a single good and three buyers with values 10, 6, and 3. A WE price is any price between 6 and 10. At any price higher than 10, the good is not overdemanded but it is not a WE price. Similarly, at any price between 3 and 6, the good is not underdemanded but it is not a WE price. In some sense, Lemma 1 says that condition (**WE-1**) in Definition 1 is satisfied in the absence of overdemanded goods, but condition (**WE-2**) may be violated. Similarly, Lemma 2 says that condition (**WE-2**) in Definition 1 is satisfied in the absence of underdemanded goods, but condition (**WE-1**) may be violated. However, the WE prices can be precisely characterized by the absence of both overdemanded and underdemanded sets of goods.

Theorem 1 *A price vector* $\boldsymbol{p}$ *is a WE price vector if and only if no set of goods is overdemanded and no set of goods is underdemanded at* $\boldsymbol{p}$*.*

Proof: Suppose $\mathbf{p}$ is a WE price vector. By condition (**WE-2**), there exists a feasible allocation in which every good in $N^+(\mathbf{p})$ can be assigned to a unique demander of that good. Hence no set of goods is underdemanded. If some set of goods, say, $S \subseteq N \setminus \{0\}$, is overdemanded, then condition (**WE-1**) will fail for some buyer in $O(S, \mathbf{p})$ in every feasible allocation, which is not true since $\mathbf{p}$ is a WE price vector. Hence no set of goods can be overdemanded.

Suppose now that no set of goods is overdemanded and no set of goods is underdemanded at price vector $\mathbf{p}$. By Lemma 1 there is a non-empty set of feasible allocations Γ^* that allocates every buyer a good from his demand set. Choose an allocation $\mu \in \Gamma^*$ for which the number of goods from $N^+(\mathbf{p})$ that is allocated in μ is maximal over all the allocations in Γ^*. Let us call such an allocation a maximal allocation in Γ^*. Let $T^0 = \{j \in N^+(\mathbf{p}) : \mu_i \neq j \ \forall \ i \in M\}$. If $T^0 = \emptyset$, then by definition $(\mathbf{p}, \mu)$ is a WE. We will show that T^0 is empty. Assume for contradiction that T^0 is not empty.

We first show that for every buyer $i \in M$, if $\mu_i \notin N^+(\mathbf{p})$ then $T^0 \cap D_i(\mathbf{p}) = \emptyset$. Assume for contradiction for some $i \in M$ with $\mu_i \notin N^+(\mathbf{p})$ there exists $j \in T^0 \cap D_i(\mathbf{p})$. In that case, we can construct a new allocation μ' in which $\mu'_i = j$ and $\mu'_k = \mu_k$ for all $k \neq i$. Allocation μ' is in Γ^* and assigns one good more from $N^+(\mathbf{p})$ than μ does. This is a contradiction since μ is a maximal allocation in Γ^*. As a result of this, the demanders of T^0 are assigned to goods in $N^+(\mathbf{p}) \setminus T^0$. Let $X^0 = U(T^0, \mathbf{p})$. So, $X^0 \subseteq \{i \in M : \mu_i \in N^+(\mathbf{p}) \setminus T^0\}$. Now, for any $k \geq 0$, consider a sequence $(T^0, X^0, T^1, X^1, \ldots, T^k, X^k)$, where for every $1 \leq q \leq k$, T^q is the set of goods assigned to buyers in X^{q-1} in μ and $X^q = U(\cup_{r=0}^{q} T^r, \mathbf{p}) \setminus U(\cup_{r=0}^{q-1} T^r, \mathbf{p})$. Note that by definition $T^q \cap T^r = \emptyset$ for every $q \neq r$.

We show that if $T^q \neq \emptyset$ and $T^q \subseteq N^+(\mathbf{p})$ for all $0 \leq q \leq k$, then there exists $T^{k+1} \neq \emptyset$ such that $T^{k+1} \subseteq N^+(\mathbf{p})$ and $T^{k+1} \cap T^q = \emptyset$ for all $0 \leq q \leq k$. By definition of X^q, $0 \leq q \leq k$, and T^q, $1 \leq q \leq k$,

$$\begin{aligned}\#U(\cup_{q=0}^{k} T^q, \mathbf{p}) &= \#U(\cup_{q=0}^{k-1} T^q, \mathbf{p}) + \#X^k \\ &= \sum_{q=0}^{k} \#X^q \\ &= \sum_{q=1}^{k} \#T^q + \#X^k. \end{aligned} \tag{1}$$

Since $T^0, \dots, T^k$ are disjoint and $\cup_{q=0}^{k} T^q \subseteq N^+(\mathbf{p})$ is not underdemanded, we have

$$\#U(\cup_{q=0}^{k} T^q, \mathbf{p}) \geq \sum_{q=0}^{k} \#T^q. \tag{2}$$

Using equations (1) and (2), we get $\#X^k \geq \#T^0$. Since T^0 is non-empty, X^k is non-empty. Define T^{k+1} as the set of goods assigned to buyers in X^k in μ. Hence T^{k+1} is non-empty. By definition $T^{k+1} \cap T^q = \emptyset$ for every $0 \leq q \leq k$. To show that $T^{k+1} \subseteq N^+(\mathbf{p})$, assume for contradiction that there exists a buyer $i_k \in X^k$ such that $\mu_{i_k} \notin N^+(\mathbf{p})$. By definition of X^k, i_k should demand some good $j_k \in T^k$. Now consider the sequence $(i_k, j_k, i_{k-1}, j_{k-1}, \dots, i_0, j_0)$, where for every $0 \leq q \leq k-1$, i_{q-1} is the buyer assigned to good j_q in μ (note that $i_{q-1} \in X^{q-1}$ by definition) and j_{q-1} is a good demanded by i_{q-1} from T^{q-1} (such a good exists by the definition of X^{q-1} and T^{q-1}). Now, construct an allocation μ' with $\mu'_{i_q} = j_q$ for all $0 \leq q \leq k$ and $\mu'_i = \mu_i$ for any $i \notin \{i_0, \dots, i_k\}$. Clearly, $\mu' \in \Gamma^*$. By assigning i_k to j_k, μ' assigns one good more from $N^+(\mathbf{p})$ than μ does, contradicting the fact that μ is a maximal allocation in Γ^*. Hence $T^{k+1} \subseteq N^+(\mathbf{p})$. This process can be repeated infinitely many times starting from T^0. So $(T^0, T^1, \dots)$ is an infinite sequence such that $T^q \cap T^r = \emptyset$ for every $q \neq r$, $T^q \neq \emptyset$ for all q, and $T^q \subseteq N^+(\mathbf{p})$ for all q. This is a contradiction since $N^+(\mathbf{p})$ is finite. So, $T^0 = \emptyset$, and therefore $(\mathbf{p}, \mu)$ is a WE. ■

The characterization in Theorem 1 shows that given a price vector and the demand sets of buyers, it is possible to check if the given price vector is a WE price vector by checking for the existence of overdemanded and underdemanded sets of goods. In some sense this is a generalization of Hall's theorem (Hall, 1935) for our model.

In contrast to Definition 1, the characterization in Theorem 1 does not require to compute a feasible allocation to check if a price vector is a WE price vector. Theorem 1 uses only demand set information of buyers to characterize the WE price vectors. Further, it enables us to characterize the minimum and the maximum WE price vectors (Theorems 2 and 3).

Notice that absence of overdemanded goods requires that there is no excess demand in a weak sense, since we only count the exclusive demanders in checking for overdemanded goods. Similarly, absence of underdemanded goods requires that there is no excess supply in a weak sense, since zero priced goods are not counted while checking for underdemanded goods. Theorem 1 assures the existence of a Walrasian equilibrium at a price vector if there is neither excess demand nor excess supply. This provides a direct economic interpretation of our result.

Given the lattice structure of the set of WE price vectors, one is tempted to think that a precise characterization of the minimum and maximum WE price vectors is possible, where we relax the notions of excess demand and excess supply. We do this in the next two theorems. In some sense, these theorems provide further generalizations of Hall's theorem for our model.

Theorem 2 *A price vector $\mathbf{p}$ is equal to $\mathbf{p}^{min}$ if and only if there is no overdemanded set of goods and no weakly underdemanded set of goods at $\mathbf{p}$.*

Proof: Suppose $\mathbf{p} = \mathbf{p}^{min}$. By Theorem 1, no set of goods is overdemanded at $\mathbf{p}$. We need to show that no set of goods is weakly underdemanded. Assume for contradiction that a set of goods, say, $S \subseteq N$, is weakly underdemanded. By definition $S \subseteq N^+(\mathbf{p})$ and $\#U(S, \mathbf{p}) \leq \#S$. Since $\mathbf{p}$ is a WE price vector, every good in S is assigned to a buyer in his demand set at price vector $\mathbf{p}$. So, $\#U(S, \mathbf{p}) \geq \#S$. This implies that $\#U(S, \mathbf{p}) = \#S$. Since $S \subseteq N^+(\mathbf{p})$, we can decrease the price of goods in S by a sufficiently small amount so that no buyer in $M \setminus U(S, \mathbf{p})$ demands a good from S. Buyers in $U(S, \mathbf{p})$ will continue to demand goods from S after such a price decrease. Thus, the new price vector is a WE price vector, and is smaller than $\mathbf{p} = \mathbf{p}^{min}$. This is a contradiction since $\mathbf{p}^{min}$ is the unique minimum WE price vector.

Now, we assume that no set of goods is overdemanded and no set of goods is weakly underdemanded at a price vector $\mathbf{p}$. Applying Theorem 1, $\mathbf{p}$ is a WE price vector. Assume for contradiction that $\mathbf{p} \neq \mathbf{p}^{min}$. By definition of $\mathbf{p}^{min}$, $p_j \geq p_j^{min}$ for all $j \in N$ and there exists a set of goods $S = \{j \in N : p_j > p_j^{min}\}$. By our assumption $S \neq \emptyset$. For all $j \in S$, it holds that $p_j > p_j^{min} \geq 0$, implying $S \subseteq N^+(\mathbf{p})$. Because S is not weakly underdemanded,

$$\#U(S, \mathbf{p}) > \#S. \tag{3}$$

Since prices of goods in S strictly decrease from $\mathbf{p}$ to $\mathbf{p}^{min}$ but remain the same for goods in $N \setminus S$, buyers in $U(S, \mathbf{p})$ will only demand goods from S at price vector $\mathbf{p}^{min}$. Using equation (3), we can write $\#O(S, \mathbf{p}^{min}) \geq \#U(S, \mathbf{p}) > \#S$. This means S is overdemanded at price vector $\mathbf{p}^{min}$. This is a contradiction by Theorem 1. ■

At a WE price vector, every good with positive price is allocated to some demander of that good. Hence, the number of demanders of such a set of positive price goods is at least equal to the number of goods in that set. Absence of weakly underdemanded goods at a WE price vector implies that for a set of goods with positive price, there is some buyer not allocated to these goods who demands a good from that set. This provides an alternate interpretation of Theorem 2. Also, the characterization of the minimum WE price vector gives us an idea about the existence of overdemanded and weakly underdemanded sets of goods in other regions of price vector space.

Corollary 1 *If $\mathbf{p} \ngeq \mathbf{p}^{min}$, then there exists an overdemanded set of goods. Further, if $\mathbf{p} \nleq \mathbf{p}^{min}$, then there exists a weakly underdemanded set of goods.*

Proof: Suppose $\mathbf{p} \ngeq \mathbf{p}^{min}$. Let $S = \{j \in N : p_j < p_j^{min}\}$. Since $\mathbf{p} \ngeq \mathbf{p}^{min}$, $S \neq \emptyset$. Further, because $p_j^{min} > p_j \geq 0$ for all $j \in S$, $S \subseteq N^+(\mathbf{p}^{min})$. Since prices of goods in S decrease from $\mathbf{p}^{min}$ to $\mathbf{p}$ while prices of goods in $N \setminus S$ do not decrease, $U(S, \mathbf{p}^{min}) \subseteq O(S, \mathbf{p})$. So, $\#O(S, \mathbf{p}) \geq \#U(S, \mathbf{p}^{min}) > \#S$, where the last inequality follows from Theorem 2 (S is not weakly underdemanded at $\mathbf{p}^{min}$). Hence S is overdemanded at $\mathbf{p}$.

Now, suppose $\mathbf{p} \nleq \mathbf{p}^{min}$. Define $S' = \{j \in N : p_j > p_j^{min}\}$. Because $\mathbf{p} \nleq \mathbf{p}^{min}$, $S' \neq \emptyset$. Further, since $p_j > p_j^{min} \geq 0$ for all $j \in S'$, $S' \subseteq N^+(\mathbf{p})$. Since prices of goods in S' decrease from $\mathbf{p}$ to $\mathbf{p}^{min}$ while prices of goods in $N \setminus S'$ do not decrease, $U(S', \mathbf{p}) \subseteq O(S', \mathbf{p}^{min})$. So, $\#U(S', \mathbf{p}) \leq \#O(S', \mathbf{p}^{min}) \leq \#S'$, where the last inequality follows from Theorem 2 (S' is not overdemanded at $\mathbf{p}^{min}$). Hence S' is weakly underdemanded at $\mathbf{p}$. ■

In every region of the price vector space with respect to $\mathbf{p}^{min}$, Corollary 1 shows if an overdemanded set of goods or a weakly underdemanded set of goods always exists in that region.

To identify regions in the price vector space where underdemanded goods and weakly overdemanded goods can be guaranteed, we give a characterization of the maximum WE price vector.

Theorem 3 *A price vector $\mathbf{p}$ is equal to $\mathbf{p}^{max}$ if and only if there is no weakly overdemanded set of goods and no underdemanded set of goods at $\mathbf{p}$.*

Proof: Let $\mathbf{p} = \mathbf{p}^{max}$. By Theorem 1, no set of goods is underdemanded. We will show that no set of goods is weakly overdemanded. Assume for contradiction that for some $\emptyset \neq S \subseteq N \setminus \{0\}$, S is weakly overdemanded. So, $\#O(S, \mathbf{p}) \geq \#S$. Since $\mathbf{p}$ is a WE price vector, S cannot be overdemanded. Hence, $\#O(S, \mathbf{p}) = \#S$. By definition of WE, any WE allocation should assign buyers in $O(S, \mathbf{p})$ goods from S. Since buyers in $O(S, \mathbf{p})$ do not demand the dummy good, their payoff is positive. Hence, by increasing the price of goods in S by a sufficiently small amount, buyers in $O(S, \mathbf{p})$ will continue to demand the same goods in S at the higher price, and we will reach a higher WE price vector. This is a contradiction since $\mathbf{p} = \mathbf{p}^{max}$ is the unique maximum WE price vector.

Now, assume that no set of goods is weakly overdemanded and no set of goods is underdemanded at $\mathbf{p}$. Using Theorem 1, $\mathbf{p}$ is a WE price vector. Assume for contradiction $\mathbf{p} \neq \mathbf{p}^{max}$. By definition of $\mathbf{p}^{max}$, $p_j \leq p_j^{max}$ for all $j \in N$ and there exists a non-empty set of goods $S = \{j \in N : p_j < p_j^{max}\}$. Since S is not weakly overdemanded at $\mathbf{p}$, we can write

$$\#O(S, \mathbf{p}) < \#S. \tag{4}$$

By increasing prices from $\mathbf{p}$ to $\mathbf{p}^{max}$, prices of goods in $N \setminus S$ do not increase but prices of goods in S increase. This means buyers in $M \setminus O(S, \mathbf{p})$ will not have goods from S in their demand set at $\mathbf{p}^{max}$. Using equation (4) we can write $\#U(S, \mathbf{p}^{max}) \leq \#O(S, \mathbf{p}) < \#S$. Since prices of goods in S increase, $S \subseteq N^+(\mathbf{p}^{max})$. Hence, S is underdemanded at $\mathbf{p}^{max}$. This is a contradiction. ■

Consider a WE price vector and a set of goods that are allocated in that WE. If this set of goods is not weakly overdemanded, then some of the buyers allocated to these goods must demand a good not in this set of goods. This provides an alternate interpretation of Theorem 3. Analogous to Corollary 1, we have the following corollary.

Corollary 2 *If $\boldsymbol{p} \ngeq \boldsymbol{p}^{max}$, then there exists a weakly overdemanded set of goods. Further, if $\boldsymbol{p} \nleq \boldsymbol{p}^{max}$, then there exists an underdemanded set of goods.*

Proof: The proof is analogous to Corollary 1 except that we make use of Theorem 3 instead of Theorem 2. ■

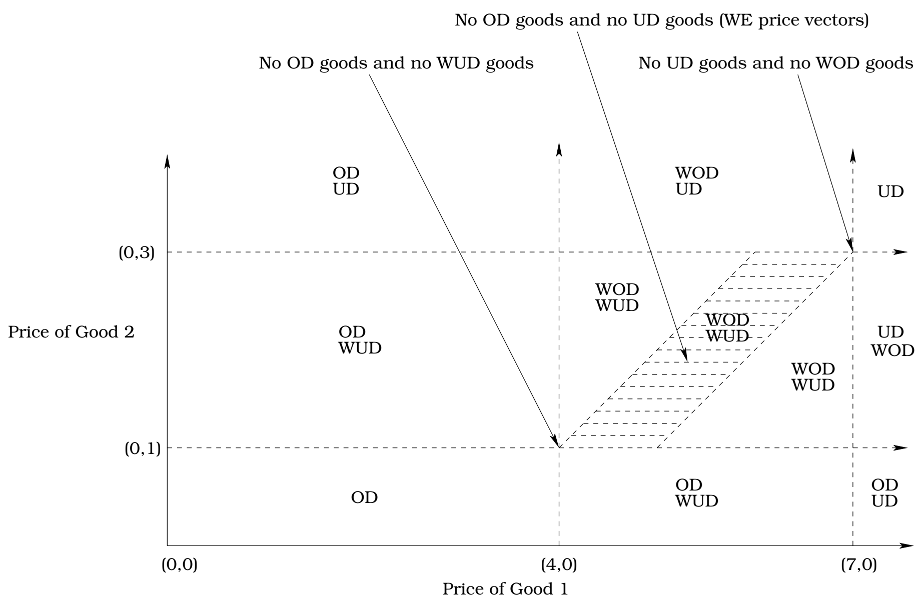

Figure 1: Various regions of the price vector space for the example in Table 1

The results in the paper so far are illustrated in Figure 1 for the example in Table 1. The labelling in various regions of the figure indicates whether (weakly) overdemanded sets of goods ((W)OD) and (weakly) underdemanded sets of goods ((W)UD) exist at all price vectors in these regions. By Theorem 1, there is no set of overdemanded and underdemanded goods in the lattice corresponding to the WE price vector region in Figure 1. The minimum and the maximum WE price vectors are characterized by Theorems 2 and 3, respectively. The other regions in Figure 1 are labelled using Corollaries 1 and 2. For example, for every price vector in the upper-right corner, an underdemanded set of goods exist, whereas for every price vector in the lower-left corner, an overdemanded set of goods exist. The reader can also see how different price vectors in Table 2 lie in various regions of Figure 1. Notice that once every set of goods is weakly underdemanded, then no set of goods can be overdemanded. This happens, for example when all prices are set equal or above the highest valuation of the goods. Also, there exist regions (upper-left and lower-right corners in Figure 1) where sets of underdemanded and overdemanded goods co-exist.

We can say something more about various price vectors than what the results in Corollaries 1 and 2 seem to indicate. If we decrease the prices of positive price goods at the minimum WE price vector by an equal amount such that no price goes below zero, then at the new price vector no weakly underdemanded goods exist. But, by Corollary 1, some set of goods is overdemanded. So, if $\mathbf{p}^{min} \neq \mathbf{0}$, then there is some non-zero price vector $\mathbf{p} \lneq \mathbf{p}^{min}$ where no set of goods is weakly underdemanded but some set of goods is overdemanded. This argument illustrates that we can draw a piecewise linear path from the minimum WE price vector to the zero price vector along which no set of goods is weakly underdemanded but some set of goods is overdemanded.

Similarly, if we increase the prices of positive price goods by an equal amount from the maximum WE price vector, no set of goods is weakly overdemanded at the new price vector, but some set of goods is underdemanded. So, the 45 degree straight line from the maximum WE price vector in the north-east direction is a set of (infinite) price vectors where no set of goods is weakly overdemanded but some set of goods is underdemanded.

Our earlier results do not say anything about the structure of the sets of overdemanded and underdemanded goods. In Table 2, we can see that a good can be both part of an overdemanded set of goods and an underdemanded set of goods at some price vector, e.g. at price vector $(0, \epsilon, 3 + \epsilon)$, good 2 is underdemanded, and is also in the overdemanded set $\{1, 2\}$. But this anomaly is absent if we consider minimal overdemanded and minimal underdemanded sets of goods. The following theorem reconciles these ideas.

Theorem 4 *If a good is part of a minimal overdemanded set of goods at a price vector, then it cannot be part of a minimal weakly underdemanded set of goods at that price vector. Similarly, if a good is part of a minimal weakly overdemanded set of goods at a price vector, then it cannot be part of a minimal underdemanded set of goods at that price vector.*

Proof: Consider any price vector $\mathbf{p}$. Let S^u be a minimal weakly underdemanded set of goods and let S^o be a minimal overdemanded set of goods at the price vector $\mathbf{p}$. We will show that $S^u \cap S^o = \emptyset$. Since S^u is weakly underdemanded at $\mathbf{p}$, $\#O(S^u, \mathbf{p}) \leq \#U(S^u, \mathbf{p}) \leq \#S^u$. This shows that S^u is not overdemanded at $\mathbf{p}$. So, $S^u \neq S^o$. Assume for contradiction $S^u \cap S^o \neq \emptyset$. There are three cases to consider.

Case 1: $S^o \subsetneq S^u$. Since S^u is minimal weakly underdemanded at $\mathbf{p}$ and $S^u \setminus S^o$ is non-empty, $S^u \setminus S^o$ is not weakly underdemanded. So, we can write

$$\#U(S^u \setminus S^o, \mathbf{p}) > \#(S^u \setminus S^o). \tag{5}$$

Since S^u is weakly underdemanded we get

$$\#U(S^u, \mathbf{p}) \leq \#S^u. \tag{6}$$

Since S^o is overdemanded we get

$$\#O(S^o, \mathbf{p}) > \#S^o. \tag{7}$$

Now, since $S^o \subsetneq S^u$ and using equations (6) and (7)

$$\begin{aligned}\#(S^u \setminus S^o) &= \#S^u - \#S^o \\ &> \#U(S^u, \mathbf{p}) - \#O(S^o, \mathbf{p}) \\ &\geq \#U(S^u \setminus S^o, \mathbf{p}).\end{aligned}$$

The last inequality comes from the fact that $O(S^o, \mathbf{p}) \cup U(S^u \setminus S^o, \mathbf{p}) \subseteq U(S^u, \mathbf{p})$. Using equation (5), we get a contradiction.

Case 2: $S^u \subsetneq S^o$. Since S^o is minimal overdemanded and $S^o \setminus S^u$ is not empty, $S^o \setminus S^u$ is not overdemanded. This gives us

$$\#O(S^o \setminus S^u, \mathbf{p}) \leq \#(S^o \setminus S^u). \tag{8}$$

Now, since $S^u \subsetneq S^o$ and using equations (6) and (7)

$$\begin{aligned}\#(S^o \setminus S^u) &= \#S^o - \#S^u \\ &< \#O(S^o, \mathbf{p}) - \#U(S^u, \mathbf{p}) \\ &\leq \#O(S^o \setminus S^u, \mathbf{p}).\end{aligned}$$

The last inequality comes from the fact that $O(S^o, \mathbf{p}) \subseteq O(S^o \setminus S^u, \mathbf{p}) \cup U(S^u, \mathbf{p})$. Using equation (8), we get a contradiction.

Case 3: $S^u \cap S^o = T$, $T \neq S^u$, $T \neq S^o$, and T is non-empty. Since S^u is minimal weakly underdemanded, $S^u \setminus T$ is not weakly underdemanded. This gives us

$$\#U(S^u \setminus T, \mathbf{p}) > \#(S^u \setminus T). \tag{9}$$

Similarly, $S^o \setminus T$ is not overdemanded, which gives us

$$\#O(S^o \setminus T, \mathbf{p}) \leq \#(S^o \setminus T). \tag{10}$$

Denote $Y = O(S^o, \mathbf{p}) \setminus O(S^o \setminus T, \mathbf{p})$. From the definition of Y, every buyer in Y demands goods from S^o only but at least some good from T. Hence, $Y \cap U(S^u \setminus T, \mathbf{p}) = \emptyset$. This results in the following set of inequalities by the definition of Y and using equations (7), (9) and (10)

$$
\begin{aligned}
\#U(S^u, \mathbf{p}) &\geq \#U(S^u \setminus T, \mathbf{p}) + \#Y \\
&> \#(S^u \setminus T) + \#O(S^o, \mathbf{p}) - \#O(S^o \setminus T, \mathbf{p}) \\
&> \#S^u - \#T + \#S^o - \#(S^o \setminus T) \\
&= \#S^u.
\end{aligned}
$$

The last inequality follow from the fact that $T \subsetneq S^o$ and $T \subsetneq S^u$. It implies that S^u is not weakly underdemanded. This is a contradiction.

Using an analogous proof, it can be shown that if a good is part of a minimal weakly overdemanded set of goods, then it cannot be part of a minimal underdemanded set of goods. ■

5 Implications of Characterization Results

Our characterizations, besides being of theoretical interest, has some implications in some practical applications. These applications mainly arise in contexts where the minimum or the maximum WE price vector is used to price the goods. We describe some of these applications below, and implications of our characterization result in these applications.

5.1 Marginal Economies

Marginal economies, in which either a single buyer or a single good is removed from the original economy, play a vital role in various game theoretic solutions. For example, the payment of a buyer in the VCG mechanism can be computed by analyzing the marginal economy corresponding to that buyer. Also, marginal payoff vectors are focal point of many cooperative game solutions (e.g., the Shapley value).

In general, we denote an economy with goods $A \subseteq N$ with $0 \in A$ and buyers $B \subseteq M$ as $E(A, B)$ (i.e., only goods in A and buyers in B are present). Denote as $\mathbf{p}^{min}(A, B)$ and $\mathbf{p}^{max}(A, B)$ the minimum and the maximum WE price vectors of economy $E(A, B)$, respectively. Also, for any price vector $\mathbf{p} \in \mathbb{R}_+^{|A|}$ the vector of components of $\mathbf{p}$ except the j^{th} component p_j is denoted as $\mathbf{p}_{-j}$. Using our earlier results we show next how the lattice of WE price vectors shifts in marginal economies.

Theorem 5 *For every $A \subseteq N$ with $0 \in A$ and $B \subseteq M$,*

(a) $\boldsymbol{p}^{min}(A, B \setminus \{i\}) \leq \boldsymbol{p}^{min}(A, B) \leq \boldsymbol{p}^{max}(A, B \setminus \{i\}) \leq \boldsymbol{p}^{max}(A, B)$ *for all $i \in B$,*

(b) $\boldsymbol{p}_{-j}^{min}(A, B) \leq \boldsymbol{p}^{min}(A \setminus \{j\}, B) \leq \boldsymbol{p}_{-j}^{max}(A, B) \leq \boldsymbol{p}^{max}(A \setminus \{j\}, B)$ *for all $j \in A$.*

Proof: **Proof of (a)**: For some $i \in B$, consider the marginal economy $E(A, B \setminus \{i\})$. By Theorem 2, no set of goods is overdemanded and no set of goods is weakly underdemanded at $\mathbf{p}^{min}(A, B)$ in economy $E(A, B)$. By removing buyer i, no set of goods is overdemanded at $\mathbf{p}^{min}(A, B)$ in economy $E(A, B \setminus \{i\})$. Now, consider a set of goods S which has positive prices in $\mathbf{p}^{min}(A, B)$. Since S is not weakly underdemanded, we can write $\#B(S, \mathbf{p}^{min}(A, B)) > \#S$, and so $\#B(S, \mathbf{p}^{min}(A, B)) \geq \#S + 1$. In economy $E(A, B \setminus \{i\})$ the demand of buyers in $B \setminus \{i\}$ do not change at $\mathbf{p}^{min}(A, B)$. Hence the number of demanders of S in economy $E(A, B \setminus \{i\})$ is equal to $\#B(S, \mathbf{p}^{min}(A, B)) - 1 \geq \#S$. Hence S is not underdemanded at $\mathbf{p}^{min}(A, B)$ in economy $E(A, B \setminus \{i\})$. Since no set of goods is overdemanded and no set of goods is underdemanded at $\mathbf{p}^{min}(A, B)$ in economy $E(A, B \setminus \{i\})$, $\mathbf{p}^{min}(A, B)$ is a WE price vector of economy $E(A, B \setminus \{i\})$ (due to Theorem 1). By the lattice structure of the WE price vector space, we get that $\mathbf{p}^{min}(A, B \setminus \{i\}) \leq \mathbf{p}^{min}(A, B) \leq \mathbf{p}^{max}(A, B \setminus \{i\})$.

By Theorem 3, no set of goods is weakly overdemanded and no set of of goods is underdemanded at $\mathbf{p}^{max}(A, B)$ in economy $E(A, B)$. By removing a buyer $i \in B$, no set of goods is weakly overdemanded at $\mathbf{p}^{max}(A, B)$ in economy $E(A, B \setminus \{i\})$. By Corollary 2, $\mathbf{p}^{max}(A, B \setminus \{i\}) \leq \mathbf{p}^{max}(A, B)$.

Proof of (b): For some $j \in A$, consider the marginal economy $E(A \setminus \{j\}, B)$. By Theorem 3, no set of goods is underdemanded and no set of goods is weakly overdemanded at $\mathbf{p}^{max}(A, B)$ in economy $E(A, B)$. By removing a good j no set of goods is underdemanded in economy $E(A \setminus \{j\}, B)$ at $\mathbf{p}^{max}_{-j}(A, B)$. Now consider $S \subseteq (A \setminus \{j, 0\})$. Let K be the exclusive demanders of S at $\mathbf{p}^{max}_{-j}(A, B)$ in economy $E(A \setminus \{j\}, B)$. Buyers who are the exclusive demanders of $S \cup \{j\}$ at $\mathbf{p}^{max}(A, B)$ in economy $E(A, B)$ are the buyers from K plus the exclusive demanders of good j. With respect to economy $E(A, B)$, we can write $\#O(S \cup \{j\}, \mathbf{p}^{max}(A, B)) = \#O(\{j\}, \mathbf{p}^{max}(A, B)) + \#K < \#S + 1$ (since $S \cup \{j\}$ is not weakly overdemanded at $\mathbf{p}^{max}(A, B)$). Since $\{j\}$ is not weakly overdemanded at $\mathbf{p}^{max}(A, B)$, we get $\#O(\{j\}, \mathbf{p}^{max}(A, B)) = 0$. Therefore, $\#K < \#S + 1$, and so $\#K \leq \#S$. Hence S is not overdemanded at $\mathbf{p}^{max}_{-j}(A, B)$ in economy $E(A \setminus \{j\}, B)$. By Theorem 1, $\mathbf{p}^{max}_{-j}(A, B)$ is a WE price vector of economy $E(A \setminus \{j\}, B)$. By the lattice structure of the set of WE price vectors, we get $\mathbf{p}^{min}(A \setminus \{j\}, B) \leq \mathbf{p}^{max}_{-j}(A, B) \leq \mathbf{p}^{max}(A \setminus \{j\}, B)$.

By Theorem 2, no set of goods is weakly underdemanded at $\mathbf{p}^{min}(A, B)$ in economy $E(A, B)$. By removing a good $j \in A$, no set of goods is weakly underdemanded at $\mathbf{p}^{min}_{-j}(A, B)$ in economy $E(A \setminus \{j\}, B)$. By Corollary 1, $\mathbf{p}^{min}_{-j}(A, B) \leq \mathbf{p}^{min}(A \setminus \{j\}, B)$. ∎

We remark that part of Theorem 5 is proved in Gul and Stacchetti (1999) (Theorem 7 in their paper). In a general model where buyers have *gross substitutes* valuation functions, which is satisfied in our model, Gul and Stacchetti (1999) show that for all $A \subseteq N$, and for all $B \subseteq M$: (a) $\mathbf{p}^{min}(A, B \setminus \{i\}) \leq \mathbf{p}^{min}(A, B)$ and $\mathbf{p}^{max}(A, B \setminus \{i\}) \leq \mathbf{p}^{max}(A, B)$ for all $i \in B$; (b) $\mathbf{p}^{min}_{-j}(A, B) \leq \mathbf{p}^{min}(A \setminus \{j\}, B)$ and $\mathbf{p}^{max}_{-j}(A, B) \leq \mathbf{p}^{max}(A \setminus \{j\}, B)$ for all $j \in A$. Our results in Theorem 5 are more general than this for the unit demand setting, in the sense that we also show that $\mathbf{p}^{min}(A, B)$ is a WE price vector of economy $E(A, B \setminus \{i\})$

for all $i \in B$ and $\mathbf{p}^{max}_{-j}(A,B)$ is a WE price vector of economy $E(A \setminus \{j\}, B)$ for all $j \in A$. This is not valid for the general model with gross substitutes valuations. Moreover, our proofs use the characterization results above and are very different from the proofs given in Gul and Stacchetti (1999).

As a corollary to Theorem 5, we have the following result (Corollary 3 is essentially the new contribution of Theorem 5 with respect to Gul and Stacchetti (1999)).

Corollary 3 *Consider any $A \subseteq N$ with $0 \in A$ and $B \subseteq M$. $\boldsymbol{p}^{min}(A,B)$ is a WE price vector of economy $E(A, B \setminus \{i\})$ for all $i \in B$ and $\boldsymbol{p}^{max}_{-j}(A,B)$ is a WE price vector of economy $E(A \setminus \{j\}, B)$ for all $j \in A$.*

To summarize Theorem 5, by removing a buyer from the economy (essentially reducing demand), the WE price vector lattice shifts *downwards*. Similarly, by removing a good from the economy (essentially reducing supply), the WE price vector lattice shifts *upwards* (in a dimension that is one less than the dimension of the original lattice). So, the standard intuitions of economics that prices decrease with lowering of demand and increase with lowering of supply continue to hold in our model.

Connections between WE price vectors and the VCG payments of buyers can be made using Corollary 3. To remind, the VCG mechanism chooses an efficient allocation and asks every buyer to pay his externality on other buyers. This allocation and payment scheme makes it a *strategy-proof* and efficient mechanism. It can be shown, using standard linear programming duality arguments, that if $\mathbf{p}^{min}(N,M)$ is a WE price vector of economy $E(N,M)$ and therefore by Corollary 3 also a WE price vector of the marginal economy $E(N, M \setminus \{i\})$ for every $i \in M$, then the VCG payment of every buyer $i \in M$ is $p^{min}_{\mu_i}(M,N)$, where μ is an efficient allocation of economy $E(M,N)$ (Leonard, 1983). This proves that payments in a Walrasian equilibrium corresponding to the minimum Walrasian price vector are precisely the VCG payments of buyers. But we can also relate the VCG payment of a buyer to the maximum WE price vector of a marginal economy corresponding to that buyer using Corollary 3.

Proposition 1 *For every buyer $i \in M$ it holds that his VCG payment is equal to $p^{max}_{\mu_i}(N, M \setminus \{i\})$, where μ is an efficient allocation chosen by the VCG mechanism.*

Proof: For any $A \subseteq N$ with $0 \in A$ and $B \subseteq M$, define $V(A,B)$ as the total value of the buyers in an efficient allocation of economy $E(A,B)$ and let $P(B,\mathbf{p})$ be the total payoff of the buyers in B at price vector $\mathbf{p}$. If $\mathbf{p}$ is a WE price vector of economy $E(A,B)$, then $V(A,B) = P(B,\mathbf{p}) + \sum_{j \in A} p_j$ (this can be deduced from standard linear programming arguments, see for example Bikhchandani and Ostroy (2002)). Now, consider an efficient allocation μ of economy $E(N,M)$. The claim clearly holds for buyer $i \in M$ if $\mu_i = 0$. For

$\mu_i \neq 0$, the VCG payment of buyer $i \in M$ can be written as

$$\begin{aligned}
p_i^{VCG} &= V(N, M \setminus \{i\}) - V(N \setminus \{\mu_i\}, M \setminus \{i\}) \\
&= P(M \setminus \{i\}, \mathbf{p}^{max}(N, M \setminus \{i\})) + \sum_{j \in N} p_j^{max}(N, M \setminus \{i\}) \\
&\quad - P(M \setminus \{i\}, \mathbf{p}^{max}(N, M \setminus \{i\})) - \sum_{j \in N \setminus \{\mu_i\}} p_j^{max}(N, M \setminus \{i\}) \\
&= p_{\mu_i}^{max}(N, M \setminus \{i\}),
\end{aligned}$$

since by Corollary 3 it holds that $\mathbf{p}_{-\mu_i}^{max}(N, M \setminus \{i\})$ is a WE price vector of economy $E(N \setminus \{\mu_i\}, M \setminus \{i\})$. ■

Since the VCG payments correspond to $\mathbf{p}^{min}(N, M)$, we have the following corollary of Proposition 1.

Corollary 4 *Let μ be an efficient allocation of economy $E(N, M)$. Then $p_{\mu_i}^{min}(N, M) = p_{\mu_i}^{max}(N, M \setminus \{i\})$ for all $i \in M$, and $p_j^{min}(N, M) = 0$ for all $j \in N$ that is unassigned in μ.*

Proposition 1 gives an alternative interpretation of the VCG payment of a buyer. The VCG payment of a buyer is the maximum payment that can be received in a WE in the marginal economy without him for the good assigned to him in the VCG mechanism. Corollary 4 relates the minimum WE price vector of an economy to the maximum WE price vector of its marginal economies corresponding to buyers. Such a relationship between the maximum WE price vector and the minimum WE price vector of marginal economies corresponding to goods does not hold. This can be verified from the example in Table 1.

5.2 Existing Iterative Auctions

Iterative auctions, where prices monotonically increase (ascending auctions) or decrease (descending auctions) are practical and transparent methods to sell goods. The design of iterative auctions for our model has been studied earlier - ascending auctions can be found in Demange et al. (1986) and Sankaran (1994), whereas descending auctions can be found in Sotomayor (2002) and Mishra and Veeramani (2006) [4]. These auctions terminate at a WE price vector - the auctions in Demange et al. (1986), Sankaran (1994), and Mishra and Veeramani (2006) terminate at the minimum WE price vector, while the auction in Sotomayor (2002) terminates at the maximum WE price vector [5]. Moreover, the underlying price adjustment in these auctions is based on the ideas of overdemanded and

[4]The auction in Mishra and Veeramani (2006) is an ascending auction for a procurement (production) economy. An ascending auction in a procurement economy translates to a descending auction in our model.

[5]Since minimum WE price vector corresponds to the VCG payments, the auctions in Demange et al. (1986), Sankaran (1994), and Mishra and Veeramani (2006) have *truthful bidding* in an equilibrium, whereas buyers can *manipulate* the auction in Sotomayor (2002).

underdemanded sets of goods. Interestingly, the papers on ascending auctions do not talk about underdemanded sets of goods and use the notion overdemanded sets of goods only. Similarly, the papers on descending auctions do not talk about overdemanded sets of goods and use the notion of (weakly) underdemanded sets of goods only. The terminating conditions in these auctions are absence of overdemanded sets of goods for ascending auctions and absence of underdemanded sets of goods for descending auctions. Still, these auctions terminate at an extreme WE price vector. Our results can be used to explain why this is possible. In the rest of this section, we assume valuations of buyers and prices to be integers.

Consider the following class of ascending auctions:

- S0 Start the auction at a price vector $\mathbf{p}$ where no set of goods is weakly underdemanded (by Corollary 1, $\mathbf{p} \leq \mathbf{p}^{min}$);
- S1 Collect demand sets of buyers and check if an overdemanded set of goods exist;
- S2 If no overdemanded set of goods exist, then stop (by Theorem 2, this is the minimum WE price vector);
- S3 Else increase prices of goods such that no set of goods is weakly underdemanded at the new price vector, and repeat from Step (S1).

The auctions in Demange et al. (1986) and Sankaran (1994) are such auctions, though they do not mention this explicitly. Both these auctions start from the zero price vector [6]. At the zero price vector, no set of goods is weakly underdemanded. In Step (S3), Demange et al. (1986) increase prices by unity for goods in a minimal overdemanded set, whereas Sankaran (1994) increases prices by unity for goods in an overdemanded set, which he finds using a labeling algorithm of graph theory. Both the price adjustments ensure that no set of goods is weakly underdemanded after the price increase (i.e., satisfy Step (S3)), and we stay below the minimum WE price vector (by Corollary 1).

The descending auctions share an analogous feature. Consider the following class of descending auctions:

- S0 Start the auction at a price vector $\mathbf{p}$ where no set of goods is weakly overdemanded (by Corollary 2, $\mathbf{p} \geq \mathbf{p}^{max}$);
- S1 Collect demand sets of buyers and check if an underdemanded set of goods exist;
- S2 If no underdemanded set of goods exist, then stop (by Theorem 3, this is the maximum WE price vector);
- S3 Else decrease prices of goods such that no set of goods is weakly overdemanded at the new price vector, and repeat from Step (S1).

[6] To be precise, they use the reserve price of every good as the starting price, which is assumed to be zero in our model.

The auction in Sotomayor (2002) starts from a very high price vector where every buyer demands only the dummy good. Hence no set of goods is weakly overdemanded. By decreasing prices by unity for goods in a minimal underdemanded set, no set of goods is weakly overdemanded after the price decrease, and the price in the auction stays above the maximum WE price vector.

This class of descending auctions can be modified to terminate at the minimum WE price vector. Such auctions have to start from a price vector where no set of goods is overdemanded (by Corollary 2 such a price vector is above the minimum WE price vector). These auctions should stop if no set of goods is weakly underdemanded, and price decrease should be such that no set of goods is overdemanded at the new price vector.

Thus, our characterization results unify the existing iterative auctions by bringing them under a broad class of auctions. We hope that this will be useful in identifying more iterative auctions from this class which are easier to implement in practice than the auctions known in the literature.

Finaly, a note on the incentive properties of these auctions. It is well known that submitting true demand sets in each iteration of ascending and descending auctions that terminate at the minimum WE price vector is an ex post Nash equilibrium (Bikhchandani et al., 2002). This can be reconciled from the fact that the minimum WE price vector corresponds to the VCG payments of buyers in our setting (Leonard, 1983). Hence, all auctions discussed in this section that terminate at the minimum WE price vector share this incentive property.

6 Conclusions

We characterize the Walrasian equilibrium price vectors for economies with indivisible goods and unit demand. Our characterizations are based on the notions of overdemanded sets of goods and underdemanded sets of goods. These notions also lead to characterizations of extreme points of the Walrasian equilibrium price vector space. As a consequence of these characterizations, we are able to classify the space of price vectors into regions where (weakly) overdemanded and (weakly) underdemanded goods are guaranteed to exist. We discuss some implications of such a classification, including how the space of Walrasian equilibrium price vectors looks in marginal economies and how it forms the underlying basis of iterative auction design.

A generalization of our characterizations, for settings where buyers can be assigned more than one good, is a useful direction of future research. However, Walrasian equilibrium may fail to exist in such general settings, except under specific types of valuations called *gross substitutes* valuations (Gul and Stacchetti, 1999). Gross substitutes valuations not only ensure existence of Walrasian equilibrium, but also ensure that the space of Walrasian equilibrium price vectors form a lattice (Gul and Stacchetti, 1999). Under gross substitutes valuations, the concept of overdemanded goods has been generalized in Gul and Stacchetti (2000), where they design ascending auctions using this concept. It remains to be seen whether our characterizations can be extended to gross substitutes valuations.

Another line of future research is to identify specific auctions from the broad class of auctions described in Section 5.2, and compare them (say in terms of computation and communication overhead or some parameter that is relevant in practice) with the existing auctions in the literature.

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Overdemand and Underdemand in Economies with Indivisible Goods and Unit Demand *

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Abstract

We study an economy where a collection of indivisible goods are sold to a set of buyers who want to buy at most one good. We characterize the set of Walrasian equilibrium price vectors in such an economy using sets of overdemanded and underdemanded goods. Further, we give characterizations for the minimum and the maximum Walrasian equilibrium price vectors of this economy. Using our characterizations, we give a sufficient set of rules that generates a broad class of ascending and descending auctions in which truthful bidding is an ex post Nash equilibrium.

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1 Introduction

The classical Arrow-Debreu model (Arrow and Debreu, 1954) for studying competitive equilibrium assumes goods to be divisible (commodities). But economies with indivisible goods are common in many types of markets such as housing markets, job markets, and auctions with goods like spectrum licenses. This paper investigates economies with indivisible goods under the assumption that buyers have unit demand, i.e., every buyer can buy at most one good. The unit demand assumption is common, for example, in settings of housing and job markets.

In case of quasi-linear utilities of buyers, the existence of a Walrasian equilibrium is guaranteed, and the set of Walrasian equilibrium price vectors form a complete lattice (Shapley and Shubik, 1972). In this work, we characterize the Walrasian equilibrium price vectors using the notions of *overdemanded* and *underdemanded* sets of goods. A set of goods is (weakly) overdemanded at a price vector if the number of buyers who demand goods only from that set is greater than (or equal to) the number of goods in that set. A set of goods is (weakly) underdemanded at a price vector if their prices are positive and the number of buyers who demand goods from that set is less than (or equal to) the number of goods in that set. Our first result is that a price vector is a Walrasian equilibrium price vector if and only if there is no set of overdemanded and underdemanded goods at that price vector. Since overdemanded goods indicate the presence of excess demand and underdemand goods indicate the presence of excess supply, our characterization reflects the intuition that a price vector is a Walrasian equilibrium price vector if and only if there is no excess supply and no excess demand.

The notion of overdemanded and underdemanded sets of goods has been studied in the context of designing iterative auctions, where prices either increase monotonically (ascending auctions) or decrease monotonically (descending auctions). Demange et al. (1986) use the notion of overdemanded goods to design an ascending auction that terminates at the minimum Walrasian equilibrium price vector. Analogously, Sotomayor (2002) uses the notion of underdemanded goods, albeit a slightly different notion from ours, to design a descending auction that terminates at the maximum Walrasian equilibrium price vector. Both papers, however, do not make any connection between these notions.

Gul and Stacchetti (2000) consider a model where they allow a buyer to buy more than one good and having *gross substitutes* valuations. In such a model, a Walrasian equilibrium price vector is guaranteed to exist, and the set of Walrasian equilibrium price vectors form a complete lattice (Gul and Stacchetti, 1999). For such a model, they provide a generalization of Hall's theorem (Hall, 1935), which results in a necessary condition for a Walrasian equilibrium (see Theorem 3 in Gul and Stacchetti (2000)). Therefore, they do not characterize the set of Walrasian equilibrium price vectors, which is what we will do for our model. Also, Hall's theorem is not enough to characterize the Walrasian equilibrium price vectors in our model. We show that we can use Hall's theorem to guarantee only one of the two conditions required for a Walrasian equilibrium.

We will also characterize the minimum and the maximum Walrasian equilibrium price vectors. A price vector is the minimum Walrasian equilibrium price vector if and only if no set of goods is overdemanded and no set of goods is weakly underdemanded at this price vector. Similarly, absence of sets of weakly overdemanded goods and underdemanded goods completely characterizes the maximum Walrasian equilibrium price vector.

A motivation for this work is the growing literature on *ascending combinatorial auctions* (Cramton et al., 2006). Ascending auctions are preferred over their sealed-bid counterparts for a variety of reasons. For the model considered in this work, an ascending auction will typically maintain a price for every good, and monotonically increase them based on the *bids* of buyers. The terminating condition in such ascending auctions is a Walrasian equilibrium. Moreover, terminating at the minimum Walrasian equilibrium price vector ensures that *truthful bidding* is an ex post Nash equilibrium for the bidders - this is a standard result in this literature (see for example Bikhchandani et al. (2002)), and can be understood from the fact that the minimum Walrasian equilibrium price vector corresponds to the payments of the strategy-proof Vickrey-Clarke-Groves mechanism in our model (Leonard, 1983). However, there are several *algorithms*, both Demange et al. (1986) and Sankaran (1994) describe such methods, that can be interpreted as an ascending auction that terminates at the minimum Walrasian equilibrium. An objective of this work is to give a broad class of ascending and also descending auctions that terminate at the minimum Walrasian equilibrium price vector.

Using our main characterization results, we give a sufficient set of *rules* that generates a broad class of ascending and descending auctions. Our class of auctions include the ascending auctions in Demange et al. (1986) and Sankaran (1994). Thus, our results unify the existing ascending auctions in the literature. Further, we hope that our broad class of ascending and descending auctions will be useful in identifying specific auctions in this class that are easier to implement in practice than the existing auctions.

The rest of the paper is organized as follows. In Section 2, we formally describe the model. Section 3 describes the concepts of overdemanded and underdemanded goods. Section 4 gives the characterizations of the different regions of price vectors. In Section 5, we discuss some implications of our characterization results, and we conclude in Section 6.

2 The Model

There is a set of indivisible goods $N = \{0, 1, \ldots, n\}$ for sale to a set of buyers $M = \{1, \ldots, m\}$. Each buyer can be assigned to at most one good. The good 0 is a dummy good which can be assigned to more than one buyer. The value of buyer $i \in M$ on good $j \in N$ is v_{ij}, assumed to be a non-negative real number. Every buyer has zero value on the dummy good. A **feasible allocation** μ assigns every buyer $i \in M$ a good $\mu_i \in N$ such that no good in $N \setminus \{0\}$ is assigned to more than one buyer. Note that a feasible allocation assigns every buyer a good (may be the dummy good), but some goods may not be assigned to any buyer. We say good $j \in N$ is **unassigned** in μ if there exists no buyer $i \in M$ with $\mu_i = j$. Let Γ be the set of all feasible allocations. An **efficient allocation** is a feasible allocation $\mu^* \in \Gamma$ satisfying $\sum_{i \in M} v_{i\mu_i^*} \geq \sum_{i \in M} v_{i\mu_i}$ for all $\mu \in \Gamma$.

A price vector $\mathbf{p} \in \mathbb{R}_+^{n+1}$ assigns every good $j \in N$ a nonnegative price p_j with $p_0 = 0$. We assume quasi-linear utilities. Given a price vector $\mathbf{p}$, the payoff of buyer $i \in M$ on good $j \in N$ at price vector $\mathbf{p}$ is $v_{ij} - p_j$. The **demand set** of buyer i at price vector $\mathbf{p}$ is $D_i(\mathbf{p}) = \{j \in N : v_{ij} - p_j \geq v_{ik} - p_k \ \forall \ k \in N\}$.

Definition 1 *A* **Walrasian equilibrium** *(WE) is a price vector* $\boldsymbol{p}$ *and a feasible allocation* μ *such that*

$$\mu_i \in D_i(\boldsymbol{p}) \qquad \text{for all } i \in M \tag{\textbf{WE-1}}$$

and

$$p_j = 0 \qquad \text{for all } j \in N \text{ that are unassigned in } \mu. \tag{\textbf{WE-2}}$$

If $(\boldsymbol{p}, \mu)$ *is a WE, then* $\boldsymbol{p}$ *is a* **Walrasian equilibrium price vector** *and* μ *is a* **Walrasian equilibrium allocation**.

It is well known that every Walrasian equilibrium allocation is efficient (Shapley and Shubik, 1972). Moreover, the set of WE price vectors is non-empty [1], and forms a complete lattice (Shapley and Shubik, 1972) [2]. This implies the existence of a unique minimum WE price vector ($\mathbf{p}^{min}$) and a unique maximum WE price vector ($\mathbf{p}^{max}$). Of all the WE price vectors, $\mathbf{p}^{min}$ stands out since it corresponds to the Vickrey-Clarke-Groves (VCG) payments of buyers assigned to them in a WE (Leonard, 1983). The VCG payment is defined as the *externality* of a buyer on the remaining buyers (Vickrey, 1961; Clarke, 1971; Groves, 1973), and the VCG mechanism is an efficient and *strategy-proof* mechanism. In this work, we aim to characterize all WE price vectors, in particular $\mathbf{p}^{min}$ and $\mathbf{p}^{max}$, and discuss its implications.

For any two price vectors $\mathbf{p}$ and $\mathbf{p}'$ of equal dimension, we write $\mathbf{p} = \mathbf{p}'$ if $p_j = p'_j$ for all j. If there exists a j at which $p_j \neq p'_j$, then we write $\mathbf{p} \neq \mathbf{p}'$. If $p_j \geq p'_j$ for all j then we write $\mathbf{p} \geq \mathbf{p}'$ or $\mathbf{p}' \leq \mathbf{p}$. If $\mathbf{p} \geq \mathbf{p}'$ but $\mathbf{p} \neq \mathbf{p}$, then we write $\mathbf{p} \gneq \mathbf{p}'$ or $\mathbf{p}' \lneq \mathbf{p}$. If there exists some j for which $p_j < p'_j$, then we write $\mathbf{p} \ngeq \mathbf{p}'$ or $\mathbf{p}' \nleq \mathbf{p}$.

[1] The existence of a WE price vector follows from a standard linear programming duality argument and is proved in Shapley and Shubik (1972). The efficient allocation problem can be written as a standard one-to-one assignment problem, which always has an optimal solution (Leonard, 1983; Bikhchandani and Ostroy, 2002). So, the dual of such a linear program will also always have an optimal solution. Such optimal dual solutions correspond to the WE price vectors (Bikhchandani and Ostroy, 2002). Moreover, conditions **WE-1** and **WE-2** are complementary slackness conditions corresponding to these primal and dual linear programs.

[2] We can see that the set of WE price vectors is a special type of lattice. Given a WE $(\mathbf{p}, \mu)$, the inequalities that define this price vector are $p_j - p_{\mu_i} \geq [v_{ij} - v_{i\mu_i}]$ for all $i \in M$ and for all $j \in N$. These are lines in two dimensions, and they are either parallel to one of the axes (since $p_0 = 0$ for all price vectors) or at 45 degrees to both the axes in that dimension.

3 Overdemand and Underdemand

We define **demanders** of a set of goods $S \subseteq (N \setminus \{0\})$ at price vector $\mathbf{p}$ as $U(S, \mathbf{p}) = \{i \in M : D_i(\mathbf{p}) \cap S \neq \emptyset\}$. We define the **exclusive demanders** of a set of goods $S \subseteq (N \setminus \{0\})$ at price vector $\mathbf{p}$ as $O(S, \mathbf{p}) = \{i \in M : D_i(\mathbf{p}) \subseteq S\}$. Clearly, for every $\mathbf{p}$ and every $S \subseteq (N \setminus \{0\})$, we have $O(S, \mathbf{p}) \subseteq U(S, \mathbf{p})$. We denote the cardinality of a finite set S as $\#S$. Given a price vector $\mathbf{p}$, define $N^+(\mathbf{p}) = \{j \in N : p_j > 0\}$. By definition $0 \notin N^+(\mathbf{p})$ for any $\mathbf{p}$.

Definition 2 *A set of goods S is* **overdemanded** *at price vector $\boldsymbol{p}$ if $S \subseteq (N \setminus \{0\})$ and $\#O(S, \boldsymbol{p}) > \#S$. A set of goods S is* **weakly overdemanded** *at price vector $\boldsymbol{p}$ if $S \subseteq (N \setminus \{0\})$ and $\#O(S, \boldsymbol{p}) \geq \#S$.*

The notion of overdemanded sets of goods can be found in Demange et al. (1986) and Sankaran (1994), who use it as a basis for the design of ascending auctions for the model of our paper. For settings where a buyer can buy more than one good, the notion of overdemanded goods has been generalized in Gul and Stacchetti (2000), de Vries et al. (2007), and Mishra and Parkes (2007), who also use it as a basis for the design of ascending auctions for general models.

Definition 3 *A set of goods S is* **underdemanded** *at price vector $\boldsymbol{p}$ if $S \subseteq N^+(\boldsymbol{p})$ and $\#U(S, \boldsymbol{p}) < \#S$. A set of goods S is* **weakly underdemanded** *at price vector $\boldsymbol{p}$ if $S \subseteq N^+(\boldsymbol{p})$ and $\#U(S, \boldsymbol{p}) \leq \#S$.*

The notion of underdemanded sets of goods can be found in Sotomayor (2002) [3], who uses it to design descending auctions for our model.

Both concepts give us an idea about the imbalance of supply and demand in the economy, albeit differently. A measure of total demand on a set of goods is obtained by counting the number of exclusive demanders of these goods in the notion of sets of overdemanded goods and by counting the number of demanders of these goods in the notion of sets of underdemanded goods. However, the dummy good is never part of a set of overdemanded goods and zero priced goods, which always includes the dummy good, are never part of sets of underdemanded goods. In some sense, the existence of sets of overdemanded (underdemanded) goods at a price vector indicates that there is excess demand (supply) in the economy. Since both overdemanded and underdemanded sets of goods may exist at a given price vector, excess demand and excess supply can exist simultaneously in the economy.

[3] There is a small difference between our definition of underdemanded goods and the definition in Sotomayor (2002). Sotomayor (2002) assumes the existence of a dummy buyer who demands every good with zero price and who can be allocated more than one good. Then, a set of goods S is underdemanded in Sotomayor (2002) at a price vector $\mathbf{p}$ if *every* good in N is demanded by a buyer (may be the dummy buyer), $S \subseteq N^+(\mathbf{p})$, and $\#U(S, \mathbf{p}) < \#S$.

	Goods		
Buyers	0	1	2
1	0	8	4
2	0	6	3
3	0	1	1

Table 1: An Illustrating Example

$\mathbf{p}$	$D_1(\mathbf{p})$	$D_2(\mathbf{p})$	$D_3(\mathbf{p})$	OD	UD	Weakly OD	Weakly UD
$(0,0,0)$	$\{1\}$	$\{1\}$	$\{1,2\}$	$\{1\},\{1,2\}$	$\emptyset$	$\{1\},\{1,2\}$	$\emptyset$
$(0,\frac{\epsilon}{2},\epsilon)$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\},\{1,2\}$	$\{2\}$	$\{1\},\{1,2\}$	$\{2\}$
$(0,\epsilon,3+\epsilon)$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\},\{1,2\}$	$\{2\}$	$\{1\},\{1,2\}$	$\{2\}$
$(0,4,1)$	$\{1\}$	$\{1,2\}$	$\{0,2\}$	$\emptyset$	$\emptyset$	$\{1\},\{1,2\}$	$\emptyset$
$(0,4-\epsilon,1+\epsilon)$	$\{1\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{1\},\{1,2\}$	$\{2\},\{1,2\}$
$(0,5,\epsilon)$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\},\{1,2\}$	$\{1\}$	$\{2\},\{1,2\}$	$\{1\}$
$(0,5,2)$	$\{1\}$	$\{1,2\}$	$\{0\}$	$\emptyset$	$\emptyset$	$\{1\},\{1,2\}$	$\{2\},\{1,2\}$
$(0,5,2+\epsilon)$	$\{1\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{1\},\{1,2\}$	$\{2\},\{1,2\}$
$(0,6,1+\epsilon)$	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\{1\}$	$\{2\},\{1,2\}$	$\{1\},\{1,2\}$
$(0,6,4)$	$\{1\}$	$\{0,1\}$	$\{0\}$	$\emptyset$	$\{2\}$	$\{1\}$	$\{2\},\{1,2\}$
$(0,7,3)$	$\{1,2\}$	$\{0,2\}$	$\{0\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{1\},\{1,2\}$
$(0,7+\epsilon,\epsilon)$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\},\{1,2\}$	$\{1\}$	$\{2\},\{1,2\}$	$\{1\}$
$(0,8,2)$	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\{1\}$	$\{2\},\{1,2\}$	$\{1\},\{1,2\}$
$(0,7+\epsilon,3+\epsilon)$	$\{1,2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{1,2\}$	$\emptyset$	$\{1\},\{2\},\{1,2\}$

Table 2: Illustration of overdemand (OD) and underdemand (UD) for the example in Table 1

We give an example to illustrate these notions. Suppose there are three buyers and two goods. Valuations of buyers are given in Table 1. The minimum WE price vector for the example in Table 1 is $\mathbf{p}^{min} = (0, 4, 1)$. The maximum WE price vector for the example in Table 1 is $\mathbf{p}^{max} = (0, 7, 3)$. In Table 2, we give various price vectors and describe our notions at those price vectors. In the table it holds that $0 < \epsilon < 1$.

The following insights from Table 2 are worth noting.

- At price vector $(0, 0, 0)$ no set of goods is underdemanded since $N^+((0, 0, 0)) = \emptyset$. But sets of underdemanded goods may exist at low price vectors (for example good 2 is underdemanded at price vector $(0, \frac{\epsilon}{2}, \epsilon)$). In general, the existence of sets of underdemanded or overdemanded goods depends on the *relative* prices of goods, in addition to the entire price vector.

- Price vectors $(0, 4, 1)$, $(0, 5, 2)$ and $(0, 7, 3)$ are WE price vectors. No set of goods is overdemanded and no set of goods is underdemanded at these price vectors. Moreover, at the minimum WE price vector $(0, 4, 1)$ no set of goods is weakly underdemanded. Similarly, at the maximum WE price vector $(0, 7, 3)$ no set of goods is weakly overdemanded. These are no coincidences. In Theorems 1, 2, and 3 we formalize these relations of sets of overdemanded and underdemanded goods with WE price vectors.

- At low enough price vectors, we expect sets of overdemanded goods to exist. In Table 2, we see that at price vectors below the minimum WE price vector $(4, 1)$, there is always an overdemanded set of goods. Similarly, at price vectors above the maximum WE price vector $(7, 3)$ there is always an underdemanded set of goods. We formalize these results in Corollaries 1 and 2.

4 Characterization Results

In this section, we give a characterization of the price vector space. Our characterization of the WE price vectors is based on the notions of sets of overdemanded and underdemanded goods. Further, together with the notions of sets of weakly overdemanded and weakly underdemanded goods, we characterize the minimum and maximum WE price vectors and the price vectors that are not WE price vectors.

Define $M^+(\mathbf{p}) = \{i \in M : 0 \notin D_i(\mathbf{p})\}$ for any price vector $\mathbf{p}$. Notice that $M^+(\mathbf{p}) = O(N \setminus \{0\}, \mathbf{p})$. Now, consider the following lemmas.

Lemma 1 *Suppose no set of goods is overdemanded. Then there exists a feasible allocation in which every buyer is assigned a good from his demand set.*

Proof: Since $N \setminus \{0\}$ is not overdemanded, $\#(N \setminus \{0\}) \geq \#O(N \setminus \{0\}, \mathbf{p}) = \#M^+(\mathbf{p})$. Consider $S \subseteq M^+(\mathbf{p})$. Let $T = \cup_{i \in S} D_i(\mathbf{p})$. Since $0 \notin T$ and T is not overdemanded, we get $\#T \geq \#O(T, \mathbf{p}) \geq \#S$. Using Hall's theorem (Hall, 1935), there is a feasible allocation in which every buyer i in $M^+(\mathbf{p})$ can be assigned a good in $D_i(\mathbf{p})$, and every buyer in $M \setminus M^+(\mathbf{p})$ can be assigned the dummy good 0, which is in his demand set. ■

Lemma 2 *Suppose no set of goods is underdemanded. Then there exists a feasible allocation in which every good in $N^+(\mathbf{p})$ is assigned to a buyer who is a demander of that good.*

Proof: Since $N^+(\mathbf{p})$ is not underdemanded, $\#N^+(\mathbf{p}) \leq \#U(N^+(\mathbf{p}), \mathbf{p}) \leq \#M$. Consider $T \subseteq N^+(\mathbf{p})$. Let $S = U(T, \mathbf{p})$. Since T is not underdemanded, $\#T \leq \#U(T, \mathbf{p}) = \#S$. Using Hall's theorem (Hall, 1935), there is a feasible allocation in which every good in $N^+(\mathbf{p})$ can be assigned to a buyer who is a demander of that good, and the remaining buyers can be assigned the dummy good. ■

The absence of only overdemanded or only underdemanded sets of goods cannot guarantee a WE price vector. For instance, consider an example with a single good and three buyers with values 10, 6, and 3. A WE price is any price between 6 and 10. At any price higher than 10, the good is not overdemanded but it is not a WE price. Similarly, at any price between 3 and 6, the good is not underdemanded but it is not a WE price. In some sense, Lemma 1 says that condition (**WE-1**) in Definition 1 is satisfied in the absence of overdemanded goods, but condition (**WE-2**) may be violated. Similarly, Lemma 2 says that condition (**WE-2**) in Definition 1 is satisfied in the absence of underdemanded goods, but condition (**WE-1**) may be violated. However, the WE prices can be precisely characterized by the absence of both overdemanded and underdemanded sets of goods.

Theorem 1 *A price vector $\mathbf{p}$ is a WE price vector if and only if no set of goods is overdemanded and no set of goods is underdemanded at $\mathbf{p}$.*

Proof: Suppose $\mathbf{p}$ is a WE price vector. By condition (**WE-2**), there exists a feasible allocation in which every good in $N^+(\mathbf{p})$ can be assigned to a unique demander of that good. Hence no set of goods is underdemanded. If some set of goods, say, $S \subseteq N \setminus \{0\}$, is overdemanded, then condition (**WE-1**) will fail for some buyer in $O(S, \mathbf{p})$ in every feasible allocation, which is not true since $\mathbf{p}$ is a WE price vector. Hence no set of goods can be overdemanded.

Suppose now that no set of goods is overdemanded and no set of goods is underdemanded at price vector $\mathbf{p}$. By Lemma 1 there is a non-empty set of feasible allocations Γ^* that allocates every buyer a good from his demand set. Choose an allocation $\mu \in \Gamma^*$ for which the number of goods from $N^+(\mathbf{p})$ that is allocated in μ is maximal over all the allocations in Γ^*. Let us call such an allocation a maximal allocation in Γ^*. Let $T^0 = \{j \in N^+(\mathbf{p}) : \mu_i \neq j \ \forall\ i \in M\}$. If $T^0 = \emptyset$, then by definition $(\mathbf{p}, \mu)$ is a WE. We will show that T^0 is empty. Assume for contradiction that T^0 is not empty.

We first show that for every buyer $i \in M$, if $\mu_i \notin N^+(\mathbf{p})$ then $T^0 \cap D_i(\mathbf{p}) = \emptyset$. Assume for contradiction for some $i \in M$ with $\mu_i \notin N^+(\mathbf{p})$ there exists $j \in T^0 \cap D_i(\mathbf{p})$. In that case, we can construct a new allocation μ' in which $\mu'_i = j$ and $\mu'_k = \mu_k$ for all $k \neq i$. Allocation μ' is in Γ^* and assigns one good more from $N^+(\mathbf{p})$ than μ does. This is a contradiction since μ is a maximal allocation in Γ^*. As a result of this, the demanders of T^0 are assigned to goods in $N^+(\mathbf{p}) \setminus T^0$. Let $X^0 = U(T^0, \mathbf{p})$. So, $X^0 \subseteq \{i \in M : \mu_i \in N^+(\mathbf{p}) \setminus T^0\}$. Now, for any $k \geq 0$, consider a sequence $(T^0, X^0, T^1, X^1, \ldots, T^k, X^k)$, where for every $1 \leq q \leq k$, T^q is the set of goods assigned to buyers in X^{q-1} in μ and $X^q = U(\cup_{r=0}^{q} T^r, \mathbf{p}) \setminus U(\cup_{r=0}^{q-1} T^r, \mathbf{p})$. Note that by definition $T^q \cap T^r = \emptyset$ for every $q \neq r$.

We show that if $T^q \neq \emptyset$ and $T^q \subseteq N^+(\mathbf{p})$ for all $0 \leq q \leq k$, then there exists $T^{k+1} \neq \emptyset$ such that $T^{k+1} \subseteq N^+(\mathbf{p})$ and $T^{k+1} \cap T^q = \emptyset$ for all $0 \leq q \leq k$. By definition of X^q, $0 \leq q \leq k$, and T^q, $1 \leq q \leq k$,

$$\begin{aligned}
\#U(\cup_{q=0}^{k} T^q, \mathbf{p}) &= \#U(\cup_{q=0}^{k-1} T^q, \mathbf{p}) + \#X^k \\
&= \sum_{q=0}^{k} \#X^q \\
&= \sum_{q=1}^{k} \#T^q + \#X^k.
\end{aligned} \tag{1}$$

Since $T^0, \ldots, T^k$ are disjoint and $\cup_{q=0}^{k} T^q \subseteq N^+(\mathbf{p})$ is not underdemanded, we have

$$\#U(\cup_{q=0}^{k} T^q, \mathbf{p}) \geq \sum_{q=0}^{k} \#T^q. \tag{2}$$

Using equations (1) and (2), we get $\#X^k \geq \#T^0$. Since T^0 is non-empty, X^k is non-empty. Define T^{k+1} as the set of goods assigned to buyers in X^k in μ. Hence T^{k+1} is non-empty. By definition $T^{k+1} \cap T^q = \emptyset$ for every $0 \leq q \leq k$. To show that $T^{k+1} \subseteq N^+(\mathbf{p})$, assume for contradiction that there exists a buyer $i_k \in X^k$ such that $\mu_{i_k} \notin N^+(\mathbf{p})$. By definition of X^k, i_k should demand some good $j_k \in T^k$. Now consider the sequence $(i_k, j_k, i_{k-1}, j_{k-1}, \ldots, i_0, j_0)$, where for every $0 \leq q \leq k-1$, i_{q-1} is the buyer assigned to good j_q in μ (note that $i_{q-1} \in X^{q-1}$ by definition) and j_{q-1} is a good demanded by i_{q-1} from T^{q-1} (such a good exists by the definition of X^{q-1} and T^{q-1}). Now, construct an allocation μ' with $\mu'_{i_q} = j_q$ for all $0 \leq q \leq k$ and $\mu'_i = \mu_i$ for any $i \notin \{i_0, \ldots, i_k\}$. Clearly, $\mu' \in \Gamma^*$. By assigning i_k to j_k, μ' assigns one good more from $N^+(\mathbf{p})$ than μ does, contradicting the fact that μ is a maximal allocation in Γ^*. Hence $T^{k+1} \subseteq N^+(\mathbf{p})$. This process can be repeated infinitely many times starting from T^0. So $(T^0, T^1, \ldots)$ is an infinite sequence such that $T^q \cap T^r = \emptyset$ for every $q \neq r$, $T^q \neq \emptyset$ for all q, and $T^q \subseteq N^+(\mathbf{p})$ for all q. This is a contradiction since $N^+(\mathbf{p})$ is finite. So, $T^0 = \emptyset$, and therefore $(\mathbf{p}, \mu)$ is a WE. ■

The characterization in Theorem 1 shows that given a price vector and the demand sets of buyers, it is possible to check if the given price vector is a WE price vector by checking for the existence of overdemanded and underdemanded sets of goods. In some sense this is a generalization of Hall's theorem (Hall, 1935) for our model.

In contrast to Definition 1, the characterization in Theorem 1 does not require to compute a feasible allocation to check if a price vector is a WE price vector. Theorem 1 uses only demand set information of buyers to characterize the WE price vectors. Further, it enables us to characterize the minimum and the maximum WE price vectors (Theorems 2 and 3).

Notice that absence of overdemanded goods requires that there is no excess demand in a weak sense, since we only count the exclusive demanders in checking for overdemanded goods. Similarly, absence of underdemanded goods requires that there is no excess supply in a weak sense, since zero priced goods are not counted while checking for underdemanded goods. Theorem 1 assures the existence of a Walrasian equilibrium at a price vector if there is neither excess demand nor excess supply. This provides a direct economic interpretation of our result.

Given the lattice structure of the set of WE price vectors, one is tempted to think that a precise characterization of the minimum and maximum WE price vectors is possible, where we relax the notions of excess demand and excess supply. We do this in the next two theorems. In some sense, these theorems provide further generalizations of Hall's theorem for our model.

Theorem 2 *A price vector* $\mathbf{p}$ *is equal to* $\mathbf{p}^{min}$ *if and only if there is no overdemanded set of goods and no weakly underdemanded set of goods at* $\mathbf{p}$*.*

Proof: Suppose $\mathbf{p} = \mathbf{p}^{min}$. By Theorem 1, no set of goods is overdemanded at $\mathbf{p}$. We need to show that no set of goods is weakly underdemanded. Assume for contradiction that a set of goods, say, $S \subseteq N$, is weakly underdemanded. By definition $S \subseteq N^+(\mathbf{p})$ and $\#U(S, \mathbf{p}) \leq \#S$. Since $\mathbf{p}$ is a WE price vector, every good in S is assigned to a buyer in his demand set at price vector $\mathbf{p}$. So, $\#U(S, \mathbf{p}) \geq \#S$. This implies that $\#U(S, \mathbf{p}) = \#S$. Since $S \subseteq N^+(\mathbf{p})$, we can decrease the price of goods in S by a sufficiently small amount so that no buyer in $M \setminus U(S, \mathbf{p})$ demands a good from S. Buyers in $U(S, \mathbf{p})$ will continue to demand goods from S after such a price decrease. Thus, the new price vector is a WE price vector, and is smaller than $\mathbf{p} = \mathbf{p}^{min}$. This is a contradiction since $\mathbf{p}^{min}$ is the unique minimum WE price vector.

Now, we assume that no set of goods is overdemanded and no set of goods is weakly underdemanded at a price vector $\mathbf{p}$. Applying Theorem 1, $\mathbf{p}$ is a WE price vector. Assume for contradiction that $\mathbf{p} \neq \mathbf{p}^{min}$. By definition of $\mathbf{p}^{min}$, $p_j \geq p_j^{min}$ for all $j \in N$ and there exists a set of goods $S = \{j \in N : p_j > p_j^{min}\}$. By our assumption $S \neq \emptyset$. For all $j \in S$, it holds that $p_j > p_j^{min} \geq 0$, implying $S \subseteq N^+(\mathbf{p})$. Because S is not weakly underdemanded,

$$\#U(S, \mathbf{p}) > \#S. \tag{3}$$

Since prices of goods in S strictly decrease from $\mathbf{p}$ to $\mathbf{p}^{min}$ but remain the same for goods in $N \setminus S$, buyers in $U(S, \mathbf{p})$ will only demand goods from S at price vector $\mathbf{p}^{min}$. Using equation (3), we can write $\#O(S, \mathbf{p}^{min}) \geq \#U(S, \mathbf{p}) > \#S$. This means S is overdemanded at price vector $\mathbf{p}^{min}$. This is a contradiction by Theorem 1. ■

At a WE price vector, every good with positive price is allocated to some demander of that good. Hence, the number of demanders of such a set of positive price goods is at least equal to the number of goods in that set. Absence of weakly underdemanded goods at a WE price vector implies that for a set of goods with positive price, there is some buyer not allocated to these goods who demands a good from that set. This provides an alternate interpretation of Theorem 2. Also, the characterization of the minimum WE price vector gives us an idea about the existence of overdemanded and weakly underdemanded sets of goods in other regions of price vector space.

Corollary 1 *If* $\mathbf{p} \ngeq \mathbf{p}^{min}$*, then there exists an overdemanded set of goods. Further, if* $\mathbf{p} \nleq \mathbf{p}^{min}$*, then there exists a weakly underdemanded set of goods.*

Proof: Suppose $\mathbf{p} \ngeq \mathbf{p}^{min}$. Let $S = \{j \in N : p_j < p_j^{min}\}$. Since $\mathbf{p} \ngeq \mathbf{p}^{min}$, $S \neq \emptyset$. Further, because $p_j^{min} > p_j \geq 0$ for all $j \in S$, $S \subseteq N^+(\mathbf{p}^{min})$. Since prices of goods in S decrease from $\mathbf{p}^{min}$ to $\mathbf{p}$ while prices of goods in $N \setminus S$ do not decrease, $U(S, \mathbf{p}^{min}) \subseteq O(S, \mathbf{p})$. So, $\#O(S, \mathbf{p}) \geq \#U(S, \mathbf{p}^{min}) > \#S$, where the last inequality follows from Theorem 2 (S is not weakly underdemanded at $\mathbf{p}^{min}$). Hence S is overdemanded at $\mathbf{p}$.

Now, suppose $\mathbf{p} \nleq \mathbf{p}^{min}$. Define $S' = \{j \in N : p_j > p_j^{min}\}$. Because $\mathbf{p} \nleq \mathbf{p}^{min}$, $S' \neq \emptyset$. Further, since $p_j > p_j^{min} \geq 0$ for all $j \in S'$, $S' \subseteq N^+(\mathbf{p})$. Since prices of goods in S' decrease from $\mathbf{p}$ to $\mathbf{p}^{min}$ while prices of goods in $N \setminus S'$ do not decrease, $U(S', \mathbf{p}) \subseteq O(S', \mathbf{p}^{min})$. So, $\#U(S', \mathbf{p}) \leq \#O(S', \mathbf{p}^{min}) \leq \#S'$, where the last inequality follows from Theorem 2 (S' is not overdemanded at $\mathbf{p}^{min}$). Hence S' is weakly underdemanded at $\mathbf{p}$. ■

In every region of the price vector space with respect to $\mathbf{p}^{min}$, Corollary 1 shows if an overdemanded set of goods or a weakly underdemanded set of goods always exists in that region.

To identify regions in the price vector space where underdemanded goods and weakly overdemanded goods can be guaranteed, we give a characterization of the maximum WE price vector.

Theorem 3 *A price vector* $\mathbf{p}$ *is equal to* $\mathbf{p}^{max}$ *if and only if there is no weakly overdemanded set of goods and no underdemanded set of goods at* $\mathbf{p}$*.*

Proof: Let $\mathbf{p} = \mathbf{p}^{max}$. By Theorem 1, no set of goods is underdemanded. We will show that no set of goods is weakly overdemanded. Assume for contradiction that for some $\emptyset \neq S \subseteq N \setminus \{0\}$, S is weakly overdemanded. So, $\#O(S, \mathbf{p}) \geq \#S$. Since $\mathbf{p}$ is a WE price vector, S cannot be overdemanded. Hence, $\#O(S, \mathbf{p}) = \#S$. By definition of WE, any WE allocation should assign buyers in $O(S, \mathbf{p})$ goods from S. Since buyers in $O(S, \mathbf{p})$ do not demand the dummy good, their payoff is positive. Hence, by increasing the price of goods in S by a sufficiently small amount, buyers in $O(S, \mathbf{p})$ will continue to demand the same goods in S at the higher price, and we will reach a higher WE price vector. This is a contradiction since $\mathbf{p} = \mathbf{p}^{max}$ is the unique maximum WE price vector.

Now, assume that no set of goods is weakly overdemanded and no set of goods is underdemanded at $\mathbf{p}$. Using Theorem 1, $\mathbf{p}$ is a WE price vector. Assume for contradiction $\mathbf{p} \neq \mathbf{p}^{max}$. By definition of $\mathbf{p}^{max}$, $p_j \leq p_j^{max}$ for all $j \in N$ and there exists a non-empty set of goods $S = \{j \in N : p_j < p_j^{max}\}$. Since S is not weakly overdemanded at $\mathbf{p}$, we can write

$$\#O(S, \mathbf{p}) < \#S. \tag{4}$$

By increasing prices from $\mathbf{p}$ to $\mathbf{p}^{max}$, prices of goods in $N \setminus S$ do not increase but prices of goods in S increase. This means buyers in $M \setminus O(S, \mathbf{p})$ will not have goods from S in their demand set at $\mathbf{p}^{max}$. Using equation (4) we can write $\#U(S, \mathbf{p}^{max}) \leq \#O(S, \mathbf{p}) < \#S$. Since prices of goods in S increase, $S \subseteq N^+(\mathbf{p}^{max})$. Hence, S is underdemanded at $\mathbf{p}^{max}$. This is a contradiction. ■

Consider a WE price vector and a set of goods that are allocated in that WE. If this set of goods is not weakly overdemanded, then some of the buyers allocated to these goods must demand a good not in this set of goods. This provides an alternate interpretation of Theorem 3. Analogous to Corollary 1, we have the following corollary.

Corollary 2 *If $\boldsymbol{p} \ngeq \boldsymbol{p}^{max}$, then there exists a weakly overdemanded set of goods. Further, if $\boldsymbol{p} \nleq \boldsymbol{p}^{max}$, then there exists an underdemanded set of goods.*

Proof: The proof is analogous to Corollary 1 except that we make use of Theorem 3 instead of Theorem 2. ■

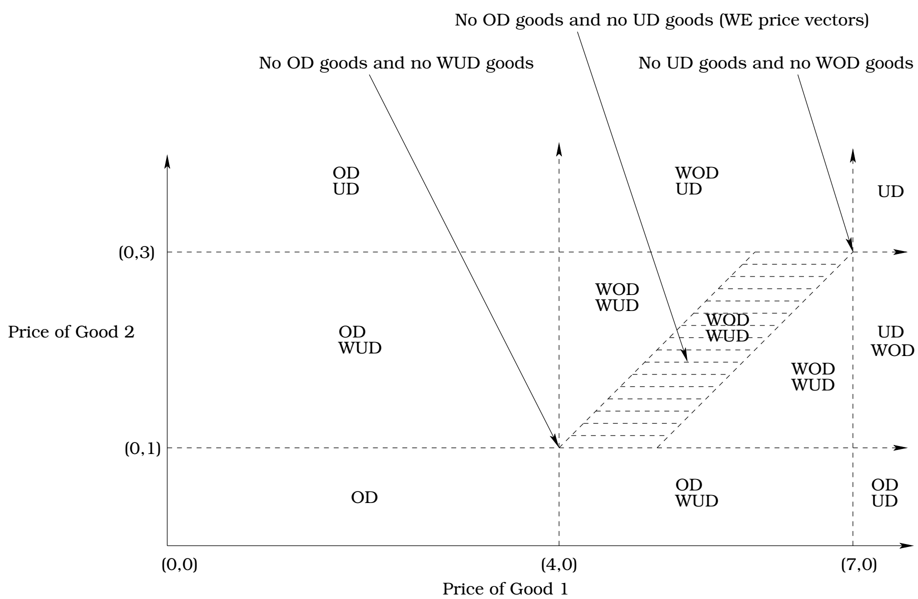

Figure 1: Various regions of the price vector space for the example in Table 1

The results in the paper so far are illustrated in Figure 1 for the example in Table 1. The labelling in various regions of the figure indicates whether (weakly) overdemanded sets of goods ((W)OD) and (weakly) underdemanded sets of goods ((W)UD) exist at all price vectors in these regions. By Theorem 1, there is no set of overdemanded and underdemanded goods in the lattice corresponding to the WE price vector region in Figure 1. The minimum and the maximum WE price vectors are characterized by Theorems 2 and 3, respectively. The other regions in Figure 1 are labelled using Corollaries 1 and 2. For example, for every price vector in the upper-right corner, an underdemanded set of goods exist, whereas for every price vector in the lower-left corner, an overdemanded set of goods exist. The reader can also see how different price vectors in Table 2 lie in various regions of Figure 1. Notice that once every set of goods is weakly underdemanded, then no set of goods can be overdemanded. This happens, for example when all prices are set equal or above the highest valuation of the goods. Also, there exist regions (upper-left and lower-right corners in Figure 1) where sets of underdemanded and overdemanded goods co-exist.

We can say something more about various price vectors than what the results in Corollaries 1 and 2 seem to indicate. If we decrease the prices of positive price goods at the minimum WE price vector by an equal amount such that no price goes below zero, then at the new price vector no weakly underdemanded goods exist. But, by Corollary 1, some set of goods is overdemanded. So, if $\mathbf{p}^{min} \neq \mathbf{0}$, then there is some non-zero price vector $\mathbf{p} \lneq \mathbf{p}^{min}$ where no set of goods is weakly underdemanded but some set of goods is overdemanded. This argument illustrates that we can draw a piecewise linear path from the minimum WE price vector to the zero price vector along which no set of goods is weakly underdemanded but some set of goods is overdemanded.

Similarly, if we increase the prices of positive price goods by an equal amount from the maximum WE price vector, no set of goods is weakly overdemanded at the new price vector, but some set of goods is underdemanded. So, the 45 degree straight line from the maximum WE price vector in the north-east direction is a set of (infinite) price vectors where no set of goods is weakly overdemanded but some set of goods is underdemanded.

Our earlier results do not say anything about the structure of the sets of overdemanded and underdemanded goods. In Table 2, we can see that a good can be both part of an overdemanded set of goods and an underdemanded set of goods at some price vector, e.g. at price vector $(0, \epsilon, 3 + \epsilon)$, good 2 is underdemanded, and is also in the overdemanded set $\{1, 2\}$. But this anomaly is absent if we consider minimal overdemanded and minimal underdemanded sets of goods. The following theorem reconciles these ideas.

Theorem 4 *If a good is part of a minimal overdemanded set of goods at a price vector, then it cannot be part of a minimal weakly underdemanded set of goods at that price vector. Similarly, if a good is part of a minimal weakly overdemanded set of goods at a price vector, then it cannot be part of a minimal underdemanded set of goods at that price vector.*

Proof: Consider any price vector $\mathbf{p}$. Let S^u be a minimal weakly underdemanded set of goods and let S^o be a minimal overdemanded set of goods at the price vector $\mathbf{p}$. We will show that $S^u \cap S^o = \emptyset$. Since S^u is weakly underdemanded at $\mathbf{p}$, $\#O(S^u, \mathbf{p}) \leq \#U(S^u, \mathbf{p}) \leq \#S^u$. This shows that S^u is not overdemanded at $\mathbf{p}$. So, $S^u \neq S^o$. Assume for contradiction $S^u \cap S^o \neq \emptyset$. There are three cases to consider.

Case 1: $S^o \subsetneq S^u$. Since S^u is minimal weakly underdemanded at $\mathbf{p}$ and $S^u \setminus S^o$ is non-empty, $S^u \setminus S^o$ is not weakly underdemanded. So, we can write

$$\#U(S^u \setminus S^o, \mathbf{p}) > \#(S^u \setminus S^o). \tag{5}$$

Since S^u is weakly underdemanded we get

$$\#U(S^u, \mathbf{p}) \leq \#S^u. \tag{6}$$

Since S^o is overdemanded we get

$$\#O(S^o, \mathbf{p}) > \#S^o. \tag{7}$$

Now, since $S^o \subsetneq S^u$ and using equations (6) and (7)

$$\begin{aligned}
\#(S^u \setminus S^o) &= \#S^u - \#S^o \\
&> \#U(S^u, \mathbf{p}) - \#O(S^o, \mathbf{p}) \\
&\geq \#U(S^u \setminus S^o, \mathbf{p}).
\end{aligned}$$

The last inequality comes from the fact that $O(S^o, \mathbf{p}) \cup U(S^u \setminus S^o, \mathbf{p}) \subseteq U(S^u, \mathbf{p})$. Using equation (5), we get a contradiction.

Case 2: $S^u \subsetneq S^o$. Since S^o is minimal overdemanded and $S^o \setminus S^u$ is not empty, $S^o \setminus S^u$ is not overdemanded. This gives us

$$\#O(S^o \setminus S^u, \mathbf{p}) \leq \#(S^o \setminus S^u). \tag{8}$$

Now, since $S^u \subsetneq S^o$ and using equations (6) and (7)

$$\begin{aligned}
\#(S^o \setminus S^u) &= \#S^o - \#S^u \\
&< \#O(S^o, \mathbf{p}) - \#U(S^u, \mathbf{p}) \\
&\leq \#O(S^o \setminus S^u, \mathbf{p}).
\end{aligned}$$

The last inequality comes from the fact that $O(S^o, \mathbf{p}) \subseteq O(S^o \setminus S^u, \mathbf{p}) \cup U(S^u, \mathbf{p})$. Using equation (8), we get a contradiction.

Case 3: $S^u \cap S^o = T$, $T \neq S^u$, $T \neq S^o$, and T is non-empty. Since S^u is minimal weakly underdemanded, $S^u \setminus T$ is not weakly underdemanded. This gives us

$$\#U(S^u \setminus T, \mathbf{p}) > \#(S^u \setminus T). \tag{9}$$

Similarly, $S^o \setminus T$ is not overdemanded, which gives us

$$\#O(S^o \setminus T, \mathbf{p}) \leq \#(S^o \setminus T). \tag{10}$$

Denote $Y = O(S^o, \mathbf{p}) \setminus O(S^o \setminus T, \mathbf{p})$. From the definition of Y, every buyer in Y demands goods from S^o only but at least some good from T. Hence, $Y \cap U(S^u \setminus T, \mathbf{p}) = \emptyset$. This results in the following set of inequalities by the definition of Y and using equations (7), (9) and (10)

$$
\begin{aligned}
\#U(S^u, \mathbf{p}) &\geq \#U(S^u \setminus T, \mathbf{p}) + \#Y \\
&> \#(S^u \setminus T) + \#O(S^o, \mathbf{p}) - \#O(S^o \setminus T, \mathbf{p}) \\
&> \#S^u - \#T + \#S^o - \#(S^o \setminus T) \\
&= \#S^u.
\end{aligned}
$$

The last inequality follow from the fact that $T \subsetneq S^o$ and $T \subsetneq S^u$. It implies that S^u is not weakly underdemanded. This is a contradiction.

Using an analogous proof, it can be shown that if a good is part of a minimal weakly overdemanded set of goods, then it cannot be part of a minimal underdemanded set of goods. ■

5 Implications of Characterization Results

Our characterizations, besides being of theoretical interest, has some implications in some practical applications. These applications mainly arise in contexts where the minimum or the maximum WE price vector is used to price the goods. We describe some of these applications below, and implications of our characterization result in these applications.

5.1 Marginal Economies

Marginal economies, in which either a single buyer or a single good is removed from the original economy, play a vital role in various game theoretic solutions. For example, the payment of a buyer in the VCG mechanism can be computed by analyzing the marginal economy corresponding to that buyer. Also, marginal payoff vectors are focal point of many cooperative game solutions (e.g., the Shapley value).

In general, we denote an economy with goods $A \subseteq N$ with $0 \in A$ and buyers $B \subseteq M$ as $E(A, B)$ (i.e., only goods in A and buyers in B are present). Denote as $\mathbf{p}^{min}(A, B)$ and $\mathbf{p}^{max}(A, B)$ the minimum and the maximum WE price vectors of economy $E(A, B)$, respectively. Also, for any price vector $\mathbf{p} \in \mathbb{R}_+^{|A|}$ the vector of components of $\mathbf{p}$ except the j^{th} component p_j is denoted as $\mathbf{p}_{-j}$. Using our earlier results we show next how the lattice of WE price vectors shifts in marginal economies.

Theorem 5 *For every $A \subseteq N$ with $0 \in A$ and $B \subseteq M$,*

(a) $\boldsymbol{p}^{min}(A, B \setminus \{i\}) \leq \boldsymbol{p}^{min}(A, B) \leq \boldsymbol{p}^{max}(A, B \setminus \{i\}) \leq \boldsymbol{p}^{max}(A, B)$ *for all* $i \in B$,

(b) $\boldsymbol{p}_{-j}^{min}(A, B) \leq \boldsymbol{p}^{min}(A \setminus \{j\}, B) \leq \boldsymbol{p}_{-j}^{max}(A, B) \leq \boldsymbol{p}^{max}(A \setminus \{j\}, B)$ *for all* $j \in A$.

Proof: **Proof of (a)**: For some $i \in B$, consider the marginal economy $E(A, B \setminus \{i\})$. By Theorem 2, no set of goods is overdemanded and no set of goods is weakly underdemanded at $\mathbf{p}^{min}(A, B)$ in economy $E(A, B)$. By removing buyer i, no set of goods is overdemanded at $\mathbf{p}^{min}(A, B)$ in economy $E(A, B \setminus \{i\})$. Now, consider a set of goods S which has positive prices in $\mathbf{p}^{min}(A, B)$. Since S is not weakly underdemanded, we can write $\#B(S, \mathbf{p}^{min}(A, B)) > \#S$, and so $\#B(S, \mathbf{p}^{min}(A, B)) \geq \#S + 1$. In economy $E(A, B \setminus \{i\})$ the demand of buyers in $B \setminus \{i\}$ do not change at $\mathbf{p}^{min}(A, B)$. Hence the number of demanders of S in economy $E(A, B \setminus \{i\})$ is equal to $\#B(S, \mathbf{p}^{min}(A, B)) - 1 \geq \#S$. Hence S is not underdemanded at $\mathbf{p}^{min}(A, B)$ in economy $E(A, B \setminus \{i\})$. Since no set of goods is overdemanded and no set of goods is underdemanded at $\mathbf{p}^{min}(A, B)$ in economy $E(A, B \setminus \{i\})$, $\mathbf{p}^{min}(A, B)$ is a WE price vector of economy $E(A, B \setminus \{i\})$ (due to Theorem 1). By the lattice structure of the WE price vector space, we get that $\mathbf{p}^{min}(A, B \setminus \{i\}) \leq \mathbf{p}^{min}(A, B) \leq \mathbf{p}^{max}(A, B \setminus \{i\})$.

By Theorem 3, no set of goods is weakly overdemanded and no set of of goods is underdemanded at $\mathbf{p}^{max}(A, B)$ in economy $E(A, B)$. By removing a buyer $i \in B$, no set of goods is weakly overdemanded at $\mathbf{p}^{max}(A, B)$ in economy $E(A, B \setminus \{i\})$. By Corollary 2, $\mathbf{p}^{max}(A, B \setminus \{i\}) \leq \mathbf{p}^{max}(A, B)$.

Proof of (b): For some $j \in A$, consider the marginal economy $E(A \setminus \{j\}, B)$. By Theorem 3, no set of goods is underdemanded and no set of goods is weakly overdemanded at $\mathbf{p}^{max}(A, B)$ in economy $E(A, B)$. By removing a good j no set of goods is underdemanded in economy $E(A \setminus \{j\}, B)$ at $\mathbf{p}^{max}_{-j}(A, B)$. Now consider $S \subseteq (A \setminus \{j, 0\})$. Let K be the exclusive demanders of S at $\mathbf{p}^{max}_{-j}(A, B)$ in economy $E(A \setminus \{j\}, B)$. Buyers who are the exclusive demanders of $S \cup \{j\}$ at $\mathbf{p}^{max}(A, B)$ in economy $E(A, B)$ are the buyers from K plus the exclusive demanders of good j. With respect to economy $E(A, B)$, we can write $\#O(S \cup \{j\}, \mathbf{p}^{max}(A, B)) = \#O(\{j\}, \mathbf{p}^{max}(A, B)) + \#K < \#S + 1$ (since $S \cup \{j\}$ is not weakly overdemanded at $\mathbf{p}^{max}(A, B)$). Since $\{j\}$ is not weakly overdemanded at $\mathbf{p}^{max}(A, B)$, we get $\#O(\{j\}, \mathbf{p}^{max}(A, B)) = 0$. Therefore, $\#K < \#S + 1$, and so $\#K \leq \#S$. Hence S is not overdemanded at $\mathbf{p}^{max}_{-j}(A, B)$ in economy $E(A \setminus \{j\}, B)$. By Theorem 1, $\mathbf{p}^{max}_{-j}(A, B)$ is a WE price vector of economy $E(A \setminus \{j\}, B)$. By the lattice structure of the set of WE price vectors, we get $\mathbf{p}^{min}(A \setminus \{j\}, B) \leq \mathbf{p}^{max}_{-j}(A, B) \leq \mathbf{p}^{max}(A \setminus \{j\}, B)$.

By Theorem 2, no set of goods is weakly underdemanded at $\mathbf{p}^{min}(A, B)$ in economy $E(A, B)$. By removing a good $j \in A$, no set of goods is weakly underdemanded at $\mathbf{p}^{min}_{-j}(A, B)$ in economy $E(A \setminus \{j\}, B)$. By Corollary 1, $\mathbf{p}^{min}_{-j}(A, B) \leq \mathbf{p}^{min}(A \setminus \{j\}, B)$. ■

We remark that part of Theorem 5 is proved in Gul and Stacchetti (1999) (Theorem 7 in their paper). In a general model where buyers have *gross substitutes* valuation functions, which is satisfied in our model, Gul and Stacchetti (1999) show that for all $A \subseteq N$, and for all $B \subseteq M$: (a) $\mathbf{p}^{min}(A, B \setminus \{i\}) \leq \mathbf{p}^{min}(A, B)$ and $\mathbf{p}^{max}(A, B \setminus \{i\}) \leq \mathbf{p}^{max}(A, B)$ for all $i \in B$; (b) $\mathbf{p}^{min}_{-j}(A, B) \leq \mathbf{p}^{min}(A \setminus \{j\}, B)$ and $\mathbf{p}^{max}_{-j}(A, B) \leq \mathbf{p}^{max}(A \setminus \{j\}, B)$ for all $j \in A$. Our results in Theorem 5 are more general than this for the unit demand setting, in the sense that we also show that $\mathbf{p}^{min}(A, B)$ is a WE price vector of economy $E(A, B \setminus \{i\})$

for all $i \in B$ and $\mathbf{p}^{max}_{-j}(A, B)$ is a WE price vector of economy $E(A \setminus \{j\}, B)$ for all $j \in A$. This is not valid for the general model with gross substitutes valuations. Moreover, our proofs use the characterization results above and are very different from the proofs given in Gul and Stacchetti (1999).

As a corollary to Theorem 5, we have the following result (Corollary 3 is essentially the new contribution of Theorem 5 with respect to Gul and Stacchetti (1999)).

Corollary 3 *Consider any $A \subseteq N$ with $0 \in A$ and $B \subseteq M$. $\boldsymbol{p}^{min}(A, B)$ is a WE price vector of economy $E(A, B \setminus \{i\})$ for all $i \in B$ and $\boldsymbol{p}^{max}_{-j}(A, B)$ is a WE price vector of economy $E(A \setminus \{j\}, B)$ for all $j \in A$.*

To summarize Theorem 5, by removing a buyer from the economy (essentially reducing demand), the WE price vector lattice shifts *downwards*. Similarly, by removing a good from the economy (essentially reducing supply), the WE price vector lattice shifts *upwards* (in a dimension that is one less than the dimension of the original lattice). So, the standard intuitions of economics that prices decrease with lowering of demand and increase with lowering of supply continue to hold in our model.

Connections between WE price vectors and the VCG payments of buyers can be made using Corollary 3. To remind, the VCG mechanism chooses an efficient allocation and asks every buyer to pay his externality on other buyers. This allocation and payment scheme makes it a *strategy-proof* and efficient mechanism. It can be shown, using standard linear programming duality arguments, that if $\mathbf{p}^{min}(N, M)$ is a WE price vector of economy $E(N, M)$ and therefore by Corollary 3 also a WE price vector of the marginal economy $E(N, M \setminus \{i\})$ for every $i \in M$, then the VCG payment of every buyer $i \in M$ is $p^{min}_{\mu_i}(M, N)$, where μ is an efficient allocation of economy $E(M, N)$ (Leonard, 1983). This proves that payments in a Walrasian equilibrium corresponding to the minimum Walrasian price vector are precisely the VCG payments of buyers. But we can also relate the VCG payment of a buyer to the maximum WE price vector of a marginal economy corresponding to that buyer using Corollary 3.

Proposition 1 *For every buyer $i \in M$ it holds that his VCG payment is equal to $p^{max}_{\mu_i}(N, M \setminus \{i\})$, where μ is an efficient allocation chosen by the VCG mechanism.*

Proof: For any $A \subseteq N$ with $0 \in A$ and $B \subseteq M$, define $V(A, B)$ as the total value of the buyers in an efficient allocation of economy $E(A, B)$ and let $P(B, \mathbf{p})$ be the total payoff of the buyers in B at price vector $\mathbf{p}$. If $\mathbf{p}$ is a WE price vector of economy $E(A, B)$, then $V(A, B) = P(B, \mathbf{p}) + \sum_{j \in A} p_j$ (this can be deduced from standard linear programming arguments, see for example Bikhchandani and Ostroy (2002)). Now, consider an efficient allocation μ of economy $E(N, M)$. The claim clearly holds for buyer $i \in M$ if $\mu_i = 0$. For

$\mu_i \neq 0$, the VCG payment of buyer $i \in M$ can be written as

$$\begin{aligned} p_i^{VCG} &= V(N, M \setminus \{i\}) - V(N \setminus \{\mu_i\}, M \setminus \{i\}) \\ &= P(M \setminus \{i\}, \mathbf{p}^{max}(N, M \setminus \{i\})) + \sum_{j \in N} p_j^{max}(N, M \setminus \{i\}) \\ &\quad - P(M \setminus \{i\}, \mathbf{p}^{max}(N, M \setminus \{i\})) - \sum_{j \in N \setminus \{\mu_i\}} p_j^{max}(N, M \setminus \{i\}) \\ &= p_{\mu_i}^{max}(N, M \setminus \{i\}), \end{aligned}$$

since by Corollary 3 it holds that $\mathbf{p}_{-\mu_i}^{max}(N, M \setminus \{i\})$ is a WE price vector of economy $E(N \setminus \{\mu_i\}, M \setminus \{i\})$. ■

Since the VCG payments correspond to $\mathbf{p}^{min}(N, M)$, we have the following corollary of Proposition 1.

Corollary 4 *Let μ be an efficient allocation of economy $E(N, M)$. Then $p_{\mu_i}^{min}(N, M) = p_{\mu_i}^{max}(N, M \setminus \{i\})$ for all $i \in M$, and $p_j^{min}(N, M) = 0$ for all $j \in N$ that is unassigned in μ.*

Proposition 1 gives an alternative interpretation of the VCG payment of a buyer. The VCG payment of a buyer is the maximum payment that can be received in a WE in the marginal economy without him for the good assigned to him in the VCG mechanism. Corollary 4 relates the minimum WE price vector of an economy to the maximum WE price vector of its marginal economies corresponding to buyers. Such a relationship between the maximum WE price vector and the minimum WE price vector of marginal economies corresponding to goods does not hold. This can be verified from the example in Table 1.

5.2 Existing Iterative Auctions

Iterative auctions, where prices monotonically increase (ascending auctions) or decrease (descending auctions) are practical and transparent methods to sell goods. The design of iterative auctions for our model has been studied earlier - ascending auctions can be found in Demange et al. (1986) and Sankaran (1994), whereas descending auctions can be found in Sotomayor (2002) and Mishra and Veeramani (2006) [4]. These auctions terminate at a WE price vector - the auctions in Demange et al. (1986), Sankaran (1994), and Mishra and Veeramani (2006) terminate at the minimum WE price vector, while the auction in Sotomayor (2002) terminates at the maximum WE price vector [5]. Moreover, the underlying price adjustment in these auctions is based on the ideas of overdemanded and

[4]The auction in Mishra and Veeramani (2006) is an ascending auction for a procurement (production) economy. An ascending auction in a procurement economy translates to a descending auction in our model.

[5]Since minimum WE price vector corresponds to the VCG payments, the auctions in Demange et al. (1986), Sankaran (1994), and Mishra and Veeramani (2006) have *truthful bidding* in an equilibrium, whereas buyers can *manipulate* the auction in Sotomayor (2002).

underdemanded sets of goods. Interestingly, the papers on ascending auctions do not talk about underdemanded sets of goods and use the notion overdemanded sets of goods only. Similarly, the papers on descending auctions do not talk about overdemanded sets of goods and use the notion of (weakly) underdemanded sets of goods only. The terminating conditions in these auctions are absence of overdemanded sets of goods for ascending auctions and absence of underdemanded sets of goods for descending auctions. Still, these auctions terminate at an extreme WE price vector. Our results can be used to explain why this is possible. In the rest of this section, we assume valuations of buyers and prices to be integers.

Consider the following class of ascending auctions:

- S0 Start the auction at a price vector $\mathbf{p}$ where no set of goods is weakly underdemanded (by Corollary 1, $\mathbf{p} \leq \mathbf{p}^{min}$);
- S1 Collect demand sets of buyers and check if an overdemanded set of goods exist;
- S2 If no overdemanded set of goods exist, then stop (by Theorem 2, this is the minimum WE price vector);
- S3 Else increase prices of goods such that no set of goods is weakly underdemanded at the new price vector, and repeat from Step (S1).

The auctions in Demange et al. (1986) and Sankaran (1994) are such auctions, though they do not mention this explicitly. Both these auctions start from the zero price vector [6]. At the zero price vector, no set of goods is weakly underdemanded. In Step (S3), Demange et al. (1986) increase prices by unity for goods in a minimal overdemanded set, whereas Sankaran (1994) increases prices by unity for goods in an overdemanded set, which he finds using a labeling algorithm of graph theory. Both the price adjustments ensure that no set of goods is weakly underdemanded after the price increase (i.e., satisfy Step (S3)), and we stay below the minimum WE price vector (by Corollary 1).

The descending auctions share an analogous feature. Consider the following class of descending auctions:

- S0 Start the auction at a price vector $\mathbf{p}$ where no set of goods is weakly overdemanded (by Corollary 2, $\mathbf{p} \geq \mathbf{p}^{max}$);
- S1 Collect demand sets of buyers and check if an underdemanded set of goods exist;
- S2 If no underdemanded set of goods exist, then stop (by Theorem 3, this is the maximum WE price vector);
- S3 Else decrease prices of goods such that no set of goods is weakly overdemanded at the new price vector, and repeat from Step (S1).

[6] To be precise, they use the reserve price of every good as the starting price, which is assumed to be zero in our model.

The auction in Sotomayor (2002) starts from a very high price vector where every buyer demands only the dummy good. Hence no set of goods is weakly overdemanded. By decreasing prices by unity for goods in a minimal underdemanded set, no set of goods is weakly overdemanded after the price decrease, and the price in the auction stays above the maximum WE price vector.

This class of descending auctions can be modified to terminate at the minimum WE price vector. Such auctions have to start from a price vector where no set of goods is overdemanded (by Corollary 2 such a price vector is above the minimum WE price vector). These auctions should stop if no set of goods is weakly underdemanded, and price decrease should be such that no set of goods is overdemanded at the new price vector.

Thus, our characterization results unify the existing iterative auctions by bringing them under a broad class of auctions. We hope that this will be useful in identifying more iterative auctions from this class which are easier to implement in practice than the auctions known in the literature.

Finaly, a note on the incentive properties of these auctions. It is well known that submitting true demand sets in each iteration of ascending and descending auctions that terminate at the minimum WE price vector is an ex post Nash equilibrium (Bikhchandani et al., 2002). This can be reconciled from the fact that the minimum WE price vector corresponds to the VCG payments of buyers in our setting (Leonard, 1983). Hence, all auctions discussed in this section that terminate at the minimum WE price vector share this incentive property.

6 Conclusions

We characterize the Walrasian equilibrium price vectors for economies with indivisible goods and unit demand. Our characterizations are based on the notions of overdemanded sets of goods and underdemanded sets of goods. These notions also lead to characterizations of extreme points of the Walrasian equilibrium price vector space. As a consequence of these characterizations, we are able to classify the space of price vectors into regions where (weakly) overdemanded and (weakly) underdemanded goods are guaranteed to exist. We discuss some implications of such a classification, including how the space of Walrasian equilibrium price vectors looks in marginal economies and how it forms the underlying basis of iterative auction design.

A generalization of our characterizations, for settings where buyers can be assigned more than one good, is a useful direction of future research. However, Walrasian equilibrium may fail to exist in such general settings, except under specific types of valuations called *gross substitutes* valuations (Gul and Stacchetti, 1999). Gross substitutes valuations not only ensure existence of Walrasian equilibrium, but also ensure that the space of Walrasian equilibrium price vectors form a lattice (Gul and Stacchetti, 1999). Under gross substitutes valuations, the concept of overdemanded goods has been generalized in Gul and Stacchetti (2000), where they design ascending auctions using this concept. It remains to be seen whether our characterizations can be extended to gross substitutes valuations.

Another line of future research is to identify specific auctions from the broad class of auctions described in Section 5.2, and compare them (say in terms of computation and communication overhead or some parameter that is relevant in practice) with the existing auctions in the literature.

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