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Two-Sided Competition and Differentiation (with an Application to Media)

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Two-sided competition and differentiation (with an application to media)

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Abstract

We model a duopoly in which two-sided platforms compete on both sides of a two-sided market. Platforms (or intermediaries) select the quality they offer consumers, and the prices they charge to consumers and firms. In this model, non-trivial competition on *both* sides induces non-quasiconcave payoffs in one subgame. All equilibria are characterized. Under well-defined conditions, the unique equilibrium in pure strategies can be computed. Prices entail a discount on one side, a premium on the other one and the quality offered to consumers is distorted downward. When the pure-strategy equilibrium fails to exist, a mixed-strategy equilibrium is shown to always exist and the distributions are characterized. In this case, the market may be preempted *ex post*. The model may find applications in the media, internet trading platforms, the software industry or even the health care industry (HMO/PPO).

Keywords: two-sided market, vertical differentiation, industrial organization, platform competition. JEL Classification: C72, D43, D62, L13, L15.

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1 Introduction

"The only thing advertisers care about is circulation, circulation, circulation."

Edward J. Atorino, analyst
Fulcrum Global Partners, New York
June 17, 2004 (The Boston Globe).

In many markets, the standard *modus operandi* requires firms to satisfy two constituencies: for example, consumers on one side and advertisers on the other in the case of media, or policyholders and service providers for HMOs and PPOs. Their behaviour is typically analysed as that of platforms competing in a two-sided market. But unlike e-Bay, say, whose only purpose is to facilitate transactions between buyers and sellers, a medium provides an information (or entertainment) good to attract consumers. Likewise, HMOs and PPOs typically differ in the characteristics of the service they offer to their policyholders.[1] We develop a model of platform competition in which *a*) the quality of the consumer good is endogenous; and *b*) competition cannot be reduced to the sole problem of attracting consumers. That is, players compete on *both* sides of the market. This latter characteristic generates the main contributions of this paper. When competition is not trivial on *both* sides, the equilibrium differs markedly from the results typically found in the industrial organization literature.

The game has three stages: quality setting (inducing vertical differentiation), price setting to consumers and price setting to advertisers. All the equilibria of the game are characterized. A unique pure-strategy equilibrium exists only when advertising is not too lucrative (in a sense made precise). This requires showing existence of, and computing, the unique Nash equilibrium of a pricing game with non-quasiconcave payoffs. This problem falls outside the premises of standard existence theorems ([7], [4], [18]). Only [5] presents a general existence theorem, but is silent as to uniqueness and characterization. Beyond a well-defined threshold, the quality-adjusted price of the high-quality firm is so low that it preempts this side of the market, and consequently the other one as well. This fails to be an equilibrium as the excluded firm possesses a non-local deviation and

[1] For example, PPOs are known to offer access to a larger diversity of physicians, while HMOs put more emphasis on cost containment. The rest of the paper will make use of the media vernacular for concreteness so we will speak of consumers and advertisers, but the reader should bear in mind other applications such as the health care industry or software development.

monopolize the market. Then platforms must play in mixed strategies. Existence of an equilibrium is verified in spite of discontinuous payoff functions. Equilibrium strategies are fully characterized and entail a mass point.

Beyond the characterization result we show that, when a pure-strategy equilibrium exists, the optimal quality level of the top firm is lower than in a well-established benchmark (Shaked and Sutton,[23]). Quality and advertising become *substitutes* for the platforms. In the Shaked and Sutton problem, a high quality is a means of extracting consumer surplus at the cost of giving away market share to the competition. Here every consumer becomes more valuable because the platform can extract surplus from advertisers as well, therefore advertisers cross-subsidize consumers. Given lower prices, the quality level required to induce the marginal consumer to purchase from the high-quality platform decreases, hence the substitution effect. When playing in mixed strategies, the market may be preempted *ex post*, which is a distinct feature of two-sided markets in practice. In this model it owes not to a contraction of the consumer market but rather to an expansion of the other side, which induces more competition for consumers.

This article departs from two often-cited applied papers by Gabszewicz, Laussel and Sonnac ([10], hereafter GLS) and Dukes and Gal-Or ([8], now DGO) in the following manner. In these constructs, platforms act as bottlenecks between advertisers and consumers. They become monopolists in the advertising market, by each offering sole access to their respective set of consumers.[2] In GLS this is an immediate consequence of the specific form of 'multi-homing' assumption: an advertiser may place at most one (of two available) advert on each platform, which prevents placing both ads on the same medium. It is a *substantive* assumption with the consequence that price competition on the advertising side vanishes. It results in a neat computation of the unique and symmetric equilibrium. These outcomes cannot be replicated when competition is preserved on the advertising market.[3] In DGO the bottleneck effect owes to the additive (hence, separable) nature of the objective function. In the present paper, competition is re-introduced in the form of a 'single-homing' assumption: advertisers may place at most one ad. From [6] and [13] we know that multi-homing typically tames competition directly on the side that multi-homes, and there-

[2]In GLS, the revenue function at the advertising pricing stage of the game is *independent* of the competitor's price. There is no proper subgame at that stage of the game. In the same spirit, in DGO the bargaining stage is independent across media.

[3]For example, if advertisers were simply allowed to place two ads on the same medium if they so wished.

fore indirectly on the other side. In contrast, single-homing (for advertisers) implies competition on the advertising side, and therefore more intense competition for consumers. One could argue that whether agents multi-home or single-home should be determined as an equilibrium outcome, not debated as an assumption. In the Appendix we show this is a moot point. What is really important for the characteristics of an equilibrium is whether there exists competition on *both* sides. A general formulation should allow for *any* quantity to be purchased from any provider. Some specifications of the returns on advertising (e.g. Cobb-Douglas) lead to payoff functions for the platforms that are akin to the multi-homing assumption; that is, they shutdown competition in advertising. Other specifications do preserve competition, as does the simpler single-homing assumption. Thus relying on single-homing yields qualitatively identical results: what matters is that platforms compete for the *marginal* unit – in the case of single-homing, the only unit. Besides technical implications, single homing finds some empirical support in [15] in the context of German magazines.

In the next section we set the paper in the context of the relevant literature, then we introduce the model. Section 4 covers the characterization and Section 5 speaks to the role of externalities. All proofs are sent to the Appendix, as well as some additional technical material.

2 Literature

Rochet and Tirole ([19],[20]), Armstrong ([2]) and Caillaud and Jullien ([6]) are the seminal references when it comes to studying two-sided markets. The works closest to this paper are [10] and [8], which this paper complements. When a pure-strategy equilibrium exist, our results resemble DGO's equilibrium, which they call 'minimal differentiation'. However here it owes not to any nuisance cost of advertising, but to the increased value of each consumer. The quality distortion that obtains then is intermediate to the GLS equilibria of maximal or no differentiation, and varies smoothly with the size of the advertising market. In both these papers the only mixed-strategy equilibria that exist are trivial ones, and cannot result in *ex post* pre-emption. Armstrong and Weeds (2005) study public versus commercial TV broadcasting and also allow for a quality variable. In their model however, quality is not a strategic instrument since it enters the demand function as an (already) price-adjusted variable. In contrast to GLS, [9] take the locations as fixed. Gabszewicz and Wauthy ([12]) do consider endogenous costless quality, however with the option of multi-homing. [1] conduct a welfare analysis of the broadcasting market; advertising may be underprovided, depending on its nuisance cost and its expected benefit to advertisers. There is no direct competition between broadcasters for the advertisers business. In [17] it is shown that when advertising volumes feed back into the consumers' utility, it is *as if* platforms were competing in advertising ("pecuniary externality") however indirectly through the consumer demand for media (see also [22] for a more general formulation). In the context of healthcare, Bardey and Rochet ([3]) allow for competition for policy holders (consumers) through quality and prices, but there is no direct competition for service providers. [13] studies commitment problems in setting prices on one side of the platform to attract participants on the other one.

This work is also related to an older strand of the industrial organisation literature. Building on [11], [23] shows that when firms compete in a vertical differentiation model, their profits, prices and market shares are ranked according to their quality choices.

3 Model

There are two platforms, identified with the subscripts 1 and 2, and a continuum of consumers of mass 1 with private valuation b for their good. The benefit b is distributed on an interval $\left[\underline{\beta}, \overline{\beta}\right]$ following a continuous, uniform distribution. All consumers value quality in the sense of vertical differentiation – there is no ambiguity for consumers as to what quality is. Let $\theta \in \Theta = \left[\underline{\theta}, \overline{\theta}\right]$ denote the quality parameter of each good.

Consumers' net utility function is expressed as $u(b, \theta_i, p_i^R) = \theta_i b - p_i^R; \quad i = 1, 2$ when facing a price p_i^R, where the superscript R stands for 'reader'. Let $\mathbf{p}^R = \left(p_1^R, p_2^R\right), \theta = (\theta_1, \theta_2)$. Consumers buy at most one medium. When $\theta_1 > \theta_2$, define the measure

$$D_1^R\left(\mathbf{p}^R, \theta\right) \equiv Pr\left(\theta_1 \beta - p_1^R \geq \max\left\{0, \theta_2 \beta - p_2^R\right\}\right)$$

Hence consumers will purchase from provider 1 over provider 2 as long as $\beta \geq \max\left\{\hat{\beta} \equiv \frac{p_1^R - p_2^R}{\theta_1 - \theta_2}, \tilde{\beta} \equiv \frac{p_1^R}{\theta_1}\right\}$. Our first assumption is standard and rules out the trivial case in which the low-quality platform necessarily faces zero demand in the price game.

Assumption 1 $\overline{\beta} - 2\underline{\beta} > 0$

Advertisers have a profit function $\mathcal{A}(\mathbf{y}, x)$ separable in $x, \mathbf{y}$; $x \in \{0, 1\}$ denotes advertising consumption and $\mathbf{y}$ is a vector of variables orthogonal to x. These include any other action a platform may undertake. Let D_i denote the quality of platform i as perceived by the advertisers: the more consumers any advertiser can reach, the more they value an ad, and e be a scaling parameter.[4] For any $\hat{\mathbf{y}}$, advertisers may choose to purchase at most one unit of space at price p_i^A if $eD_i [A(\hat{\mathbf{y}}, 1) - A(\hat{\mathbf{y}}, 0)] - p_i^A = eD_i a - p_i^A \geq 0; \quad i = 1, 2$. That is, they derive an increase in (expected) profit a. The one unit limit is a convenient way of ensuring competition on this side of the market and can be interpreted as a tight liquidity constraint. Advertisers may value the benefit from advertising differently according to the parameter a, which is also uniformly distributed on $[\underline{\alpha}, \overline{\alpha}]$ with mass 1. They act as price takers and there is no strategic interaction between them. We assume neither constraint on advertising space, nor that advertising affects readership (see Remark 4).[5] The cost of running adverts is set at zero. Quality however is costly to provide and is modeled as an investment with cost $k\theta_i^2$, where we impose

Assumption 2 $k > \frac{(2\overline{\beta} - \underline{\beta})^2}{18\overline{\theta}}$

for an interior solution in the benchmark case.[6] Taken together, Assumptions 1 and 2 guarantee that the consumer market is covered in equilibrium, which greatly simplifies the analysis (see Section 7.2). We also rule out exogenous preemption on the advertising side, i.e.

Assumption 3 $\overline{\alpha} - 2\underline{\alpha} > 0$

Externality: The ranking of the platforms' market shares on the consumer side defines their relative quality on the other side. Given prices $\mathbf{p}^A = \left(p_1^A, p_2^A\right)$ and coverage $\mathbf{D}^R = (D_1^R, D_2^R)$, a producer purchases from 1 over 2, only if $eD_1 a - p_1^A \geq \max\left\{0, eD_2 a - p_2^A\right\}$. This decision rule generates the measure $Pr\left(eD_1 a - p_1^A \geq \max\left\{0, eD_2 a - p_2^A\right\}\right) \equiv q_1^A\left(\mathbf{p}^A, \mathbf{D}^R\right)$.

[4] The advertisers payoffs are independent of the consumers' 'identity' (preferences for media). This spares us a signaling game between consumers and advertisers: media consumption would then provide information about preferences for commoditities.

[5] A capacity constraint is either trivially exogenous, or endogenous as in [16], which may induce a quantity-setting game instead of the price game.

[6] In the absence of a sufficiently large parameter k the Shaked and Sutton boundary result prevails.

Game: Platforms first choose a quality level simultaneously. Given observed qualities, they each set prices to consumers, who make purchasing decisions, and in a third stage, to advertisers. Upon observing these prices, advertisers choose whether to purchase. This sequence captures the facts that *a*) consumer prices (cover prices or subscription rates) are more difficult to change than advertising rates, and *b*) readership is often reported to advertisers (*ex ante* and *ex post*, with potential rebates). It also affords us the use of the subgame-perfect equilibrium concept, while pricing to consumers and advertiser simultaneously would require to use a rational expectation framework. The three-stage game is denoted Γ. A platform collects revenues from both sides, with monies from either perfectly substitutable. For any medium $i = 1, 2$, the objective function takes the form

$$\Pi_i = D_i^R\left(\mathbf{p}^R, \theta\right) p_i^R - k\theta_i^2 + q_i^A\left(\mathbf{p}^A, \mathbf{D}^R\right) p_i^A = R_i(\mathbf{p}^A, \mathbf{p}^R, \theta) - k\theta_i^2 \tag{3.1}$$

4 Equilibrium characterization

We proceed in two steps, starting with the advertising market where the firms' behavior is not directly affected by quality choices.

4.1 Advertising market subgame

The last subgame replicates the result of the classical analysis of vertical differentiation. Let $e\Delta D^R = e(D_1^R - D_2^R)$ denote the difference in the platforms' quality. Then equilibrium payoffs take a simple form in the last stage, for which the proof is standard and therefore omitted.

Lemma 1 *Suppose $D_1^R \geq D_2^R$ w.l.o.g. There may be three pure strategy equilibria in the advertising market. When $D_1^R > D_2^R > 0$, the profit functions write $\overline{\Pi}_1^A = e\Delta D^R \left(\frac{2\overline{\alpha}-\underline{\alpha}}{3}\right)^2$; $\underline{\Pi}_2^A = e\Delta D^R \left(\frac{\overline{\alpha}-2\underline{\alpha}}{3}\right)^2$. When $D_1^R > D_2^R = 0$, platform 1 is a monopolist and its profits are $\Pi_1^{AM} = eD_1^R \left(\frac{\overline{\alpha}}{2}\right)^2$. For $D_1^R = D_2^R$, the Bertrand outcome prevails and platforms have zero profits.*

Following Lemma 1 the profit function (3.1) rewrites

$$\Pi_i = p_i^R D_i^R(\mathbf{p}^R, \theta) - k(\theta_i) + \Pi_i^A\left(e\Delta D^R(\mathbf{p}^R, \theta)\right) \tag{4.1}$$

on the equilibrium path, where consumer demand for the commodity takes the form $D_i^R = \overline{\beta} - \frac{p_i^R - p_j^R}{\Delta\theta}$, $D_j^R = \frac{p_i^R - p_j^R}{\Delta\theta} - \underline{\beta}$ for $\theta_i > \theta_j$ thanks to Assumptions 1 and 2. As usual, $\Delta\theta = \theta_i - \theta_j$ and for convenience $\overline{A} = \left(\frac{2\overline{\alpha}-\underline{\alpha}}{3}\right)^2$ and $\underline{A} = \left(\frac{\overline{\alpha}-2\underline{\alpha}}{3}\right)^2$.

Remark 1 *Consider now the Hotelling setup applied to the present model. Using GLS notation, at this stage of the game, profits would write*

$$\begin{aligned} \Pi_i &= (p_i - c)n_i + e\overline{A}(n_i - n_j) \\ \Pi_j &= (p_j - c)n_j + e\underline{A}(n_i - n_j) \end{aligned}$$

whenever $n_i > n_j$, where $n_i \equiv n_i(p_i, p_j)$ is i's consumer demand and p_i the consumer price. In GLS an equilibrium is always symmetric: $n_i = n_j$, in which case the second term is always naught. But this cannot be an equilibrium in our construct: any firm could alter its location marginally, lower its consumer price marginally at receive a first-order gain $eA(n_i - n_j)$, $A = \overline{A}, \underline{A}$. The reason is that profits in the present game are a function of the consumer demands of both *platforms, not just of one player.*

4.2 Consumer price subgame

From Lemma 1 three distinct configurations may arise on the equilibrium path. In the first case platform 1 dominates the consumer market, in the second one both share the consumer market equally and in the last one it is dominated by firm 2. Hence the profit function (4.1) of each firm $i = 1, 2$ rewrites

$$\Pi_i = p_i^R D_i^R(\mathbf{p}^R, \theta) - k\theta_i^2 + \begin{cases} \overline{\Pi}_i^A, & \text{if } D_i^R > D_j^R; \\ 0, & \text{if } D_i^R = D_j^R; \\ \underline{\Pi}_i^A, & \text{if } D_i^R < D_j^R. \end{cases} \tag{4.2}$$

This function is continuous with a kink at the profile of consumer prices $\tilde{\mathbf{p}}^R$ such that $D_1^R = D_2^R$. More importantly it is not quasi-concave, which follows from the externality generated by advertising revenue and induces discontinuous best responses. Thus the conditions of Theorem 2 of [7] are not met, and neither are those of [18]. The sufficient conditions (Proposition 1) of [4] also fail here as the sum of payoffs does not satisfy diagonal quasi-concavity, so their existence result cannot be readily applied.[7] Proceeding by construction it is nonetheless possible to show that a

[7]A recent contribution (subsequent to the writing of this paper) by Philippe Bich establishes existence by introducing a measure of the lack ofquasi-concavity that resembles ironing [5]. Our construction does remain essential in that we face a potential multiplicity of equilibria and seek a characterization.

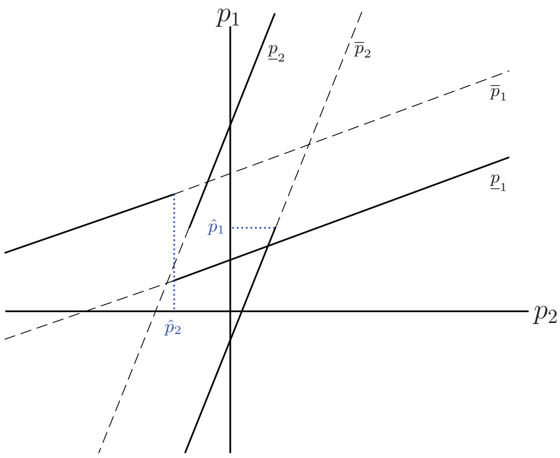

Figure 1: Best replies and unique equilibrium

unique equilibrium in pure strategies always exists. This demonstration is left to the Appendix, Section 7.3; here we discuss it briefly and focus on its outcome, which takes a simple form. At face value an equilibrium has a flavour of rational expectations, in that platforms must select actions that are 'consistent' with each other (for example, *both* must play as if $D_1^R > D_2^R$ or the converse), as well as compatible with an equilibrium. Such a rationality requirement is not necessary. First, from (4.2), it is immediate that any price profile $\mathbf{p}^R$ such that $D_1^R = D_2^R$ is dominated. Next we can define 'quasi best responses' corresponding to platforms playing as if either $D_1^R > D_2^R$ or $D_1^R < D_2^R$ (for example, $\overline{p}_2, \underline{p}_2$ in Figure 1), from which we can construct the true best replies – discontinuous at the points $\hat{p}_1, \hat{p}_2$. In a penultimate step, we derive a necessary and sufficient condition for existence, that is, for these best responses to intersect. Last, this condition is verified by construction for one of two candidate equilibria. To paraphrase Dasgupta and Maskin [7], the discontinuity of the best replies *is* essential: the discontinuity set is not trivial and even mixed strategies cannot restore this second candidate equilibrium. This is depicted in Figure 1. Thus we need not call on the rational expectation framework. Elimination of weakly dominated strategies is sufficient to rule it out and play a less strenuous Nash equilibrium. For example, an outcome such that $\theta_1 > \theta_2$ and $D_1 < D_2$ entails playing a weakly dominated strategy for player 2: if she finds it attractive to reduce her price that much, so must player 1. But the intuitive reasoning whereby the low-quality firm may find it profitable to behave very aggressively in order to access large advertising revenue does not hold true (see Lemma 4 in the Appendix).[8] Jumping to the result, we can state

Proposition 1 ***Consumer prices.*** *Let $\theta_1 > \theta_2$ w.l.o.g. There may be two possible configurations arising in the consumer price subgame. For each, there exists a unique Nash equilibrium in pure strategies characterised as*

- *For* $\Delta\theta > \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}$

$$p_1^{R*} = \frac{1}{3}\left[\Delta\theta\left(2\overline{\beta}-\underline{\beta}\right)+2e\left(\underline{A}-2\overline{A}\right)\right]$$
$$p_2^{R*} = \frac{1}{3}\left[\Delta\theta\left(\overline{\beta}-2\underline{\beta}\right)+2e\left(2\underline{A}-\overline{A}\right)\right]$$

- *If* $\Delta\theta \leq \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}$

$$p_1^{R*} = \frac{\Delta\theta\overline{\beta}}{2} - 2e\overline{A}$$
$$p_2^{R*} = 0$$

Consumer prices include a discount as platforms engage in cross-subsidisation. The lure of advertising revenue intensifies the competition for consumers because they become more valuable than just for their willingness to pay for the information good. Unlike in [23], $\overline{\beta}-2\underline{\beta} > 0$ is not sufficient to afford the low-quality firm some positive demand: $\Delta\theta$, defined in the first stage, may be too narrow to sustain two firms. That is, the high-quality platform may choose to act so as to exclude firm 2 endogenously.

4.3 First-stage actions

In the first stage, platforms face the profit function (4.2), which they each maximise by choice of their quality variable θ_i. That is, each of them solves

Problem 1

$$\max_{\theta_i\in\left[\underline{\theta},\overline{\theta}\right]} p_i^{R*}D_i^R\left(\theta_i,\theta_j^*,\mathbf{p}^{R*}\right)+\Pi_i^A\left(e,\Delta D^R(\mathbf{p}^{R*},\theta_i,\theta_j^*)\right)-k\theta_i^2$$

subject to

$$\hat{\beta} = \frac{p_i^R - p_j^R}{\theta_i-\theta_j} \in \left[\underline{\beta},\overline{\beta}\right] \tag{4.3}$$

[8] That is, playing $\theta_i < \theta_2$ but offering a very low price p_i^R so that $D_i^R > D_j^R$.

The constraint does not limit quality choices *per se* but is a natural restriction guaranteeing that the endogenous threshold $\hat{\beta}$ remain within the exogenous bounds $[\underline{\beta}, \overline{\beta}]$.[9] On the equilibrium path, Constraint (4.3) can be rearranged as a pair of inequalities: $\Delta\theta\left(2\overline{\beta} - \underline{\beta}\right) + 2e(\overline{A} + \underline{A}) \geq 0$ and $\Delta\theta\left(\overline{\beta} - 2\underline{\beta}\right) - 2e(\overline{A} + \underline{A}) \geq 0$. Only the second one is constraining. Let $C \equiv \left[2e\left(\overline{A} + \underline{A}\right)\right]^2$; then the objective function of firm 1 reads

$$\Pi_1 = \begin{cases} \frac{1}{9}\left(\Delta\theta(2\overline{\beta} - \underline{\beta})^2 + B_1 + \frac{C}{\Delta\theta}\right) - k\theta_1^2, & \text{if } \Delta\theta > \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}; \\ \frac{1}{9}\left(\Delta\theta(2\overline{\beta} - \underline{\beta})^2 + B_1 + \sqrt{C}(\overline{\beta} - 2\underline{\beta})\right) - k\theta_1^2, & \text{if } \Delta\theta \leq \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}} \end{cases} \tag{4.4}$$

where $B_1 = (2\overline{\beta} - \underline{\beta})2e\left(2\underline{A} - \overline{A}\right) + 3e\left(\overline{\beta} + \underline{\beta}\right)\overline{A}$ is a constant. The second line of the definition of Π_1 rules out the artificial case of firm 1 facing a demand larger than the whole market. It is derived by taking $\frac{C}{\Delta\theta}$ as fixed at its lowest value, that is, where $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}} = \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}$. For platform 2, profits are

$$\Pi_2 = \begin{cases} \frac{1}{9}\left(\Delta\theta(\overline{\beta} - 2\underline{\beta})^2 + B_2 + \frac{C}{\Delta\theta}\right) - k\theta_2^2, & \text{if } \Delta\theta(\overline{\beta} - 2\underline{\beta}) > 2e(\overline{A} + \underline{A}); \\ 0, & \Delta\theta(\overline{\beta} - 2\underline{\beta}) \leq 2e(\overline{A} + \underline{A}) \text{ and } \theta_2 = 0; \\ -k\theta_2^2, & \Delta\theta(\overline{\beta} - 2\underline{\beta}) \leq 2e(\overline{A} + \underline{A}) \text{ and } \theta_2 > 0; \end{cases} \tag{4.5}$$

with $B_2 = (\overline{\beta} - 2\underline{\beta})2e\left(\underline{A} - 2\overline{A}\right) + 3e\left(\overline{\beta} + \underline{\beta}\right)\underline{A}$. Some difficulty may arise in solving this problem as the profit functions are not necessarily well-behaved. Section 7.5 of the Appendix studies the profit function $\Pi_1(\theta_1, \theta_2)$ in the details necessary to support our results. In particular it identifies a threshold C^f such that the function admits a binding first-order condition if C does not exceed C^f. We first focus on this case. This is illustrated in Figure 2 (the higher curve corresponds to the complementary case of $C > C^f$). The solid lines represent the first part, and the dashed ones the second part, of (4.4).

4.3.1 Pure-strategy equilibrium

For $C < C^f$ the function $\Pi_1(.,.)$ remains increasing (and concave) on the portion *beyond* $\theta_1 = \tilde{\theta}(e) \equiv \underline{\theta} + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$, where it admits a maximiser. To ensure this is the case, Assumption 3 is strengthened and turned into

[9] $\theta_i \to \theta_j \Rightarrow \hat{\beta} \to \infty$.

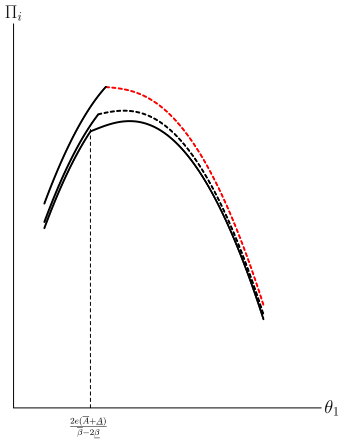

Figure 2: Profit functions for different values of advertising

Assumption 4 $e < \bar{e} \equiv \min \left\{ 1, \left(\frac{(2\overline{\beta}-\underline{\beta})^2}{27k} - \underline{\theta} \right) \frac{\overline{\beta}-2\underline{\beta}}{2(\overline{A}+\underline{A})} \right\}$. [10]

Assumption 4 guarantees that when $\hat{\theta}_1$ solves the first-order condition, we have $\Delta\theta \geq \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ so that both platforms operate (see Proposition 1). Collecting the results from Lemma 1 and Proposition 1, and letting platform 1 be the high-quality medium w.l.o.g., we can finally state

Proposition 2 ***Pure-strategy equilibrium.*** *Suppose Assumption 4 holds. The game* Γ *admits a unique equilibrium in pure strategies in which both platforms operate and choose different qualities. It is characterised by the triplet of profiles* $\left(\mathbf{p}^{*R}, \mathbf{p}^{*A}, \theta^{*},\right)$ *defined by Proposition 1, Lemma 1, and the optimal actions* $\theta_2^* = \underline{\theta}$ *and* $\theta_1^* = \hat{\theta}_1$*, where* $\hat{\theta}_1$ *uniquely solves*

$$(2\overline{\beta} - \underline{\beta})^2 = 18k\theta_1 + \frac{C}{(\Delta\theta)^2} \tag{4.6}$$

We label the term $\frac{C}{(\Delta\theta)^2}$ the 'cross-market effect': it acts as an incentive to reduce quality. In condition (4.6), firm 1 trades off the marginal benefit of quality (the left-hand side) not only with its marginal investment cost but also with the marginal advertising profit that it must forego because of higher consumer prices induced by higher quality (the RHS). Given that it markets a lesser good, platform 2 selects $\underline{\theta}$ to mitigate the price war. This is the Differentiation Principle at work, but here it is subsumed by the cross-market effect.

[10] This arises from the condition $(\hat{\theta}_1 - \underline{\theta})(\overline{\beta} - 2\underline{\beta}) > (\theta_1^f - \underline{\theta})(\overline{\beta} - 2\underline{\beta}) \geq \sqrt{C}$, where $\theta_1^f = \frac{(2\overline{\beta}-\underline{\beta})^2}{27k}$ is defined in Section 7.5.

Proof: The optimality of $\theta_2^* = \underline{\theta}$ and $\theta_1^* = \hat{\theta}_1$ is established by Lemma 5, in the Appendix (Section 7.6). The rest of the claim follows immediately under Assumption 4. ■

Proposition 2 can be appended with an immediate corollary, for which we omit the proof.

Corollary 1 *If a subgame perfect Nash equilibrium of* Γ *exists, platforms may also play in mixed strategies.*

This is a simple coordination game where each event $\theta_i^* = \underline{\theta}$ or $\theta_i = \hat{\theta}$ is a mass-point.

4.3.2 Mixed strategies

When Assumption 4 is not satisfied, the necessary first-order condition (4.6) fails to hold entirely. As can be seen on Figure 2, the high-quality medium would like to pick the point $\tilde{\theta}(e) \equiv \underline{\theta} + \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}$, where $\Pi_1(.,.)$ reaches is maximum. At that point its rival is excluded ($\Delta\theta$ is low enough), and firm 1 still extracts as much surplus from consumers as it can without losing its status as monopolist. But then firm 2 can 'leap' over it and become the monopolist at a negligible incremental cost $k\theta_1^2 - k(\theta_1 + \varepsilon)^2$. Intuitively, when advertising returns are large enough every consumer becomes very valuable to both platforms. It is not immediate that the game admits a mixed strategy equilibrium, for the payoff correspondences are not upper-hemicontinuous and their sum is not necessarily so either.[11] Nonetheless it is possible to show that a mixed-strategy equilibrium always exists, which we do in Section 7.7 of the Appendix. [24] provides an appealing approach to characterize mixed strategies in a problem of entry with sunk cost, but it does not quite apply here. Indeed there is no proper entry stage and the payoffs depend not just on the ranking of the firms' decisions (θ_1, θ_2), but on the difference $\theta_1 - \theta_2$. In particular, playing $\theta_i = \underline{\theta}$ cannot be interpreted as a decision to not enter the market because $\Pi_i(\underline{\theta}_i, \theta_j) > 0$ for θ_j such that $\Delta\theta > \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}$. Let $H_i(\theta_i)$ be the probability distribution over i's play and $h_i(.)$ the corresponding density, Θ_i^N the relevant support of H_i and θ_i^c the upper bound of the support. Let also $R_i(\theta_i, \theta_j)$ denote the revenue accruing to i. We claim

Proposition 3 *There exists a pair of distributions* H_i, $i = 1, 2$ *on* $\Theta_i^N = \{\underline{\theta}_i\} \cup \left[\tilde{\theta}_i(e), \theta_i^c\right]$, $i = 1, 2$

[11] Here we face a more standard problem of payoff discontinuity, as addressed by [7].

satisfying

$$H_i(\underline{\theta}_i)\int_{\Theta_j^N} R_i(\underline{\theta}_i,\theta_j)dH_j^*(\theta_j)+\int_{\theta_i'=\theta_j}^{\theta_i^c} R_i(\theta_i,\theta_j)d(H_i(\theta_i)\times H_j^*(\theta_j))=k\int_{\tilde{\theta}_i}^{\theta_i^c}\theta_i^2 d(H_i(\theta_i)\times H_j^*(\theta_j)) \tag{4.7}$$

with

$$H_i^*(s)\begin{cases}\in(0,1), & s=\underline{\theta}_i;\\ =1, & s=\theta_i^c.\end{cases}$$

and

$$h_i(s)=0,\quad s\in\left(\underline{\theta}_i,\tilde{\theta}_i(e)\right)$$

and θ_i^c *defined in Lemma 9.*

Noticeably $\Theta_i^N\subset[\underline{\theta},\theta^c]$ and platforms place some mass at the lower bound $\underline{\theta}$. Indeed it is obvious from the profit function (4.5) that playing any $\theta_i\in\left(\underline{\theta},\tilde{\theta}(e)\right)$ is strictly dominated by selecting the lower bound. This is because $\Pi_i(\underline{\theta}_i,\theta_j)>0$ for $\theta_j>\tilde{\theta}(e)$: if j plays anything in the support Θ_j^N but $\underline{\theta}_j$, i necessarily derives positive profits by playing $\underline{\theta}_i$. It follows that no mass is assigned on the dominated range $\left(\underline{\theta},\tilde{\theta}(e)\right)$. Expected profits are naught of course.

Remark 2 *The existence and characterisation of a mixed-strategy equilibrium is useful beyond completeness. Suppose that consumer prices are exogenously fixed at zero, as in the broadcasting world. Then a pure-strategy equilibrium can never exist. The formal statement and proof of this claim are left to the Appendix, Section 7.9.*

Next we want to understand the impact of the externality e on players' behaviour and on the breakdown of the equilibrium.

5 The role of externalities

The results of the preceding analysis are first contrasted with standard ones well established in the literature [23].

5.1 Quality distortion and advertising revenue

A goal of this paper is to understand the behavior of quality in the presence of cross-market externalities. To study this problem we take to be [23] (hereafter S&S) to be the benchmark. It is easy to adapt their model to allow for costly quality.

Proposition 4 ***Quality distortion.*** *In any pure-strategy equilibrium of the game* Γ*, quality is lower than it would be absent advertising.*

Differentiation is known to soften price competition, but advertising revenue puts emphasis back on market share. This leads to more intense price competition for consumers. Lower consumer prices uniformly relax the need to provide costly quality: at lower prices, the marginal consumer demands a lesser product to make a purchase. More precisely, given any quality, in the second stage firms *must* offer a discount to consumers. The extent of that discount, given fixed quality, is determined by profits to be collected on the advertising market: it increases in the advertising profits. In the quality-setting stage, the high-quality firm can *further* increase this discount by lowering quality: its consumer price is $p_1^* = \frac{1}{3}\left[\Delta\theta(2\overline{\beta} - \underline{\beta}) + 2e(\underline{A} - 2\overline{A})\right]$, while that of its rival is $p_2^* = \frac{1}{3}\left[\Delta\theta(\overline{\beta} - 2\underline{\beta}) + 2e(2\underline{A} - \overline{A})\right]$. This may go on until the quality spread is so narrow that firm 2 faces preemption ($\Delta\theta \leq 2e(\overline{A} + \underline{A})/(\overline{\beta} - 2\underline{\beta})$ – which is not an equilibrium). This phenomenon resembles that observed in industries such as software or game development: a widely used operating system need not provide the highest intrinsic quality because it supports so many applications.

5.2 Taxonomies: externalities and differentiation

Consistent with our claim to contrast our results against those of the current literature, we present a taxonomy of outcomes in two parts.

Proposition 5 ***Taxonomy I:*** $\mathbf{k} > \mathbf{0}$. *Let* $\theta_1^* > \theta_2^*$ *w.l.o.g.,*

For $e = 0$ *The equilibrium is that of Problem 4 (adapted from S&S) with both firms operating;*

For $\bar{e} > e > 0$ *The equilibrium is characterised by Proposition 2;*

For $e > \bar{e}$ *There is no pure-strategy equilibrium. Proposition 3 applies.*

and

Proposition 6 ***Taxonomy II:*** $\mathbf{k} = \mathbf{0}$. *Let* $\theta_1^* > \theta_2^*$. *There exists some* $\hat{e} > \bar{e} > 0$ *such that*

For $\hat{e} > e > 0$ *Maximum differentiation obtains with both firms having positive demand;*

For $e > \hat{e}$ *There is no pure-strategy equilibrium. Proposition 3 applies as well.*

Note that the case $\hat{e} > e > 0, k = 0$ yields maximal differentiation, as in S&S and is therefore equivalent to GLS (but of course the equilibrium is not symmetric). However this is only true as long as e is bounded below $\hat{e}$; that is, advertising is not too lucrative. Beyond that point, the incentives *resemble* those of GLS, that is, firms seek to increase market share on the consumer side, but competition is too stiff for a pure-strategy equilibrium to be sustained – as in Propositions 2 and 3. Some comparative statics are informative, for which the derivations are collated in the Appendix, Section 7.12. Let $\theta_1 > \theta_2$ w.l.o.g. At an equilibrium $\left(\theta^*, \mathbf{p}^{*R}, \mathbf{p}^{*A}\right)$, $\frac{d\theta_2}{de} = 0$, but $\frac{d\theta_1}{de} < 0$ and $\frac{d^2\theta_1}{de^2} < 0$ $\frac{dp_1^A}{de} > \frac{dp_2^A}{de} > 0$ and $\frac{dp_1^R}{de} < \frac{dp_2^R}{de} < 0$. The presence of a second source of revenue not only depresses the quality of the consumer good, it does increasingly so as the advertising market becomes more valuable. Price competition is correspondingly more intense in the consumer market, but less in the advertising market, where platforms become more differentiated.

Remark 3 *Either Maximal (or minimal) Differentiation can obtain when* $k = 0$ and GLS' multi-homing assumption is imposed. *That is, the equivalence between vertical and horizontal differentiation is preserved in the GLS construct (see [10]). It no longer is if competition prevails on both sides. As shown in Remark 1, no symmetric pure-strategy equilibrium can exist in a Hotelling model with single-homing. That is, Maximal Differentiation cannot hold – and neither does minimal differentiation.*

Remark 4 *The model ignores whatever disutility consumers may suffer from advertising. Introducing such disutility would extend the range of parameters on which the pure-strategy equilibrium can be sustained, as it reduces the value of advertising to the platform. It otherwise does not modify the results qualitatively. To see why, rewrite the consumers' utility function as* $u_i = \theta_i b - p_i^R - \delta q_i^A$. *Advertising demand is defined as before, but suppose* $\theta_1 > \theta_2$, *consumer demands are* $D_1^R = \overline{\beta} - \frac{\Delta p^R + \delta \Delta q^A}{\Delta\theta}$ *and* $D_2^R = \frac{\Delta p^R + \delta \Delta q^A}{\Delta\theta} - \underline{\beta}$. *The new term is* $\delta \Delta q^A$*: the utility impact of the difference in advertising levels. For firm 2 to operate,* $D_2^R > 0 \Leftrightarrow \Delta p^R > \Delta\theta\underline{\beta} - \delta q^A$, *as opposed to* $\Delta p^R > \Delta\theta\underline{\beta}$*: firm 1 can price closer to firm 2, which still has positive market share. In equilibrium, this implies that the spread* $\Delta\theta$ *(which governs equilibrium prices) can also be narrower. A disutility function reduces the value of the marginal advertiser from the perspective of the platform. In other words, it modifies the rate of substitution between surplus extraction from consumers and from advertisers.*

5.3 Properties of the mixed-strategy equilibrium

Although the distributions H_1, H_2 do not lend themselves to easy interpretation, more can be said about the nature of the equilibrium. Here we claim

Proposition 7 *Suppose $e > \bar{e}$. When no platform plays at the lower bound $\underline{\theta}$, the market is necessarily monopolised* ex post. *Otherwise both operate.*

The dominated firm loses its investment $k\theta_i^2$ because the length of the interval $[\tilde{\theta}(e), \theta^c]$ is not sufficient to accommodate two firms because $(\theta^c - \tilde{\theta}(e)) < 2e(\overline{A} + \underline{A})/(\overline{\beta} - 2\underline{\beta})$. So either monopolisation or the competitive situation may be an *ex post* outcome, which fits some industry patterns.[12] The results suggest an alternative rationale for an increased concentration in markets such as newsprint, radio broadcasting or internet trading. According to this model, some players may be driven out not because of a market contraction on the consumer side, but because of an *expansion* on the other one. In addition *ex post* profits in mixed-strategy case are not monotonically ranked: consider the play $\langle \underline{\theta}_1, \theta_2^c \rangle$, which implies $\Pi_1 > \Pi_2 = 0$ although $\theta_2 > \theta_1$. This result also compares favourably to some media idiosyncrasies, where the higher-quality shows do not necessarily yield higher profits.

6 Conclusion

This paper has developed an analysis of differentiation in a duopoly of two-sided platforms, where competition prevails on *both* sides of the market. In this case, market share on one side not only induces a ranking, but also a premium to being the better platform, on the other side. This exacerbates the competition for consumers.

Restoring competition on both sides of the two-sided market yields markedly different outcomes, as compared to those typically found in the applied literature. When a pure-strategy equilibrium exists, maximum differentiation is hampered because too costly in terms of market share, but minimum differentiation cannot be an equilibrium either. Qualities can come so close to each other that the low-quality platform becomes strictly dominated, at which point the equilibrium breaks down. Then platforms play in mixed strategies. In addition, equivalence of horizontal and vertical differentiation breaks down here.

[12] Only New York City has more than one significant newspaper, for example. Or there is one largely dominant online trading website.

Our ability to compute an equilibrium rests on the simple structure chosen, and in particular on the assumption of independence between goods' consumption decisions and media consumption decisions. This implies that advertisers only take into account the average consumer and care only about coverage. Media strive to segment consumer markets (using real or perceived correlation between media and commodity consumption) to better serve their advertisers; that is, media consumption may be used as a *signal* of other consumptions choices. They also operate in conglomerates. These important characteristics are so far left out and for future research.

7 Appendix

The Appendix contains some additional material as well the proofs of the propositions developed in the main text. We begin by arguing in favor of the single-homing assumption this paper rests on.

7.1 Justifying single-homing

Consider an advertiser (for concreteness) contemplating purchasing quantities q_1, q_2 from two intermediaries (platforms), given some prices p_1, p_2. Let the payoff function be $av(\mathbf{y}; D_1q_1, D_2q_2),\ i = 1, 2$, where $a \in [\underline{\alpha}, \overline{\alpha}] \subset \mathbb{R}$ is the advertisers' type and D_i the consumer coverage of platform i. We suppose that $v(.)$ is increasing and concave in each of its argument, that its third derivatives are positive (complete monotonicity) and that $v_{q_1q_2} \geq 0$. Suppressing the dependence on the vector $\mathbf{y}$, given D_1, D_2, an advertiser's problem is

Problem 2

$$\max_{q_1,q_2} av(D_1q_1, D_2q_2) - \sum_i p_iq_i \quad s.t. \quad \sum_i p_iq_i \leq M$$

where M is an exogenous "resource" constraint. This may be thought as a liquidity constraint or a manager's advertising budget. The first-order conditions yield $v_{q_1}(.,.) = \frac{p_1}{p_2}\frac{D_2}{D_1}v_{q_2}(.,.)$, which we can invert to obtain $q_1 \equiv h\left(\frac{p_1}{p_2}\frac{D_2}{D_1}v_{q_2}\right)$. The function $h(.) \equiv (v_{q_1})^{-1}$ is the inverse of the marginal payoff, so it is decreasing and convex. From here we recover a demand function $q_1(q_2, p_1, p_2, D_1, D_2)$ for advertiser a. Similarly for commodity 2. We can also verify that

$$\frac{dq_1}{dp_1} = \frac{\frac{v_{q_2}}{p_2}\frac{D_2}{D_1}}{v_{q_1q_1}D_1 - \frac{p_1}{p_2}D_2v_{q_2q_1}} < 0 \quad and \quad \frac{dq_1}{dp_2} = \frac{-\frac{D_2}{D_1}\frac{p_1}{p_2^2}v_{q_2}}{v_{q_1q_1}D_1 - v_{q_2q_1}\frac{p_1}{p_2}D_2} > 0$$

by simple differentiation of the first-order conditions. So the demand (for each advertiser a) behaves in standard fashion. Integrating q_i over the set of participating advertisers yields advertising demands $Q_i, \; i = 1, 2$

$$Q_i \equiv \int_{\alpha}^{\overline{\alpha}} q_i dG(z).$$

If we presume of the Inada condition at $q_i = 0$ we can guarantee ourselves full coverage: $Q_i \equiv \int_{\underline{\alpha}}^{\overline{\alpha}} q_i dG(z)$. Now the platform's problem, given D_1, D_2, is

Problem 3

$$\max_{p_i} \quad p_i \int_{\underline{\alpha}}^{\overline{\alpha}} q_i dG(z), \; \forall \; i$$

with (pointwise) first-order conditions

$$h(.) + p_i h'(.) \left[\frac{v_{q_j}}{p_j} \frac{D_j}{D_i} + \frac{p_i}{p_j} D_j v_{q_j q_i} \frac{dq_i}{dp_i} \right] = 0 \qquad (7.1)$$

for each type α. We know that $h'(.) < 0$ and differentiating the first-order condition with respect to p_i is enough to show that $\frac{v_{q_j}}{p_j} \frac{D_j}{D_i} + \frac{p_i}{p_j} D_j v_{q_j q_i} \frac{dq_i}{dp_i} > 0$. More importantly, equation (7.1) directly implies that $p_i \equiv p_i(p_j)$. To be obvious, rewrite (7.1)

$$h(.) + \frac{p_i}{p_j} D_j h'(.) \left[\frac{v_{q_j}}{D_i} + p_i v_{q_j q_i} \frac{dq_i}{dp_i} \right] = 0.$$

That is, (7.1) characterizes a *reaction function*, unlike in the GLS case where p_i is *independent* of p_j. So, there is competition (in prices) at the advertising stage, and each firm's profits depend on *both* D_i and D_j. In special cases of the form $v(q_1, q_2) = D_1 q_1^{\gamma} D_2 q_2^{1-\gamma}$, for example, the reader can verify that we revert to the GLS construct: price competition on the advertising side (in the last stage of the game) is circumvented. Similarly if the advertiser's payoffs are additively separable (as in DGO). Thus what matters for competition to be preserved on the advertising side is *not* the single-homing assumption, but the specification of payoff functions and the resource constraint.

7.2 The sufficiency of Assumptions 1 and 2 for market coverage

Suppose there is no externality and denote the equilibrium quality levels are given by $\theta_1^0 = \frac{1}{2k} \left(\frac{2\overline{\beta} - \underline{\beta}}{3} \right)^2 > \theta_2^0 = \underline{\theta}$.[13] The condition for a covered market is $\frac{\overline{\beta} - 2\underline{\beta}}{3} (\theta_1^0 - \theta_2^0) \leq \underline{\beta} \theta_2^0$ (see for

[13] Refer section 5 for details of this equilibrium.

example, Tirole, (1988)). Substituting for the values of θ_1^0, θ_2^0 and re-arranging, the market is covered for $k \geq \frac{1}{2\underline{\theta}}\left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2\left(\frac{\overline{\beta}-2\underline{\beta}}{\overline{\beta}-\underline{\beta}}\right)$, which is necessarily satisfied by Assumption 2. It follows that both firms operate and the relevant demand functions in the consumer market are the competitive ones. It will be obvious that it is satisfied in an equilibrium of this game.

7.3 Existence and characterization of a unique (price) equilibrium in pure strategies

Denote $\Delta\theta = \theta_1 - \theta_2$, $\overline{A} = \left(\frac{2\overline{\alpha}-\underline{\alpha}}{3}\right)^2$ and $\underline{A} = \left(\frac{\overline{\alpha}-2\underline{\alpha}}{3}\right)^2$. The space $P_i^R \times P_{-i}^R$ of action profiles (prices) can be divided into three regions: region I such that $D_i^R > D_{-i}^R$, region II such that $D_i^R < D_{-i}^R$ and region III such that $D_i^R = D_i^R$. We begin with

Definition 1 *For $i = 1, 2$, the platforms' 'quasi-best responses' are defined as the solution to the problem* $\max_{p_i^R} \Pi_i\left(p_i^R, D_i^R(\mathbf{p}^R, \theta); \Pi_i^A(D_i^R, D_j^R)\right)$*, where the profit function is defined by (4.2). Therefore, letting $\theta_1 > \theta_2$ w.l.o.g,*

$$p_1^R\left(p_2^R\right) = \begin{cases} \underline{p}_1^R\left(p_2^R\right) = \frac{1}{2}\left(p_2^R + \Delta\theta\overline{\beta} - 2e\overline{A}\right), & \text{if } D_1^R > D_2^R; \\ \frac{1}{2}\left(p_2^R + \Delta\theta\overline{\beta}\right), & \text{if } D_1^R = D_2^R; \\ \overline{p}_1^R\left(p_2^R\right) = \frac{1}{2}\left(p_2^R + \Delta\theta\overline{\beta} + 2e\underline{A}\right), & \text{if } D_1^R < D_2^R; \end{cases}$$

and

$$p_2^R\left(p_1^R\right) = \begin{cases} \underline{p}_2^R\left(p_1^R\right) = \frac{1}{2}\left(p_2^R - \Delta\theta\underline{\beta} - 2e\overline{A}\right), & \text{if } D_1^R < D_2^R; \\ \frac{1}{2}\left(p_1^R - \Delta\theta\underline{\beta}\right), & \text{if } D_1^R = D_2^R; \\ \overline{p}_2^R\left(p_1^R\right) = \frac{1}{2}\left(p_2^R - \Delta\theta\underline{\beta} + 2e\underline{A}\right), & \text{if } D_1^R > D_2^R; \end{cases}$$

While it is always possible to find some point where 'quasi-best responses' intersect (e.g. such that both play as if $D_1^R < D_2^R$), it by no means defines an equilibrium. Doing so assumes that in some sense platforms coordinate on a particular market configuration – for example, such that $D_1^R < D_2^R$, which may not been immune from unilateral deviation. To find the equilibrium, if it exists, we first need to pin down the firms' true best replies.

Lemma 2 *Let $\theta_1 > \theta_2$ w.l.o.g. There exists a pair of actions $(\hat{p}_1, \hat{p}_2)$ such that the best response correspondences are defined as*

$$p_1^R\left(p_2^R\right) = \begin{cases} \underline{p}_1^R\left(p_2^R\right), & \text{for } p_2 \geq \hat{p}_2; \\ \overline{p}_1^R\left(p_2^R\right), & \text{for } p_2 < \hat{p}_2; \end{cases} \tag{7.2}$$

and

$$p_2^R\left(p_1^R\right) = \begin{cases} \overline{p}_2^R\left(p_1^R\right), & \textit{for } p_1 < \hat{p}_1; \\ \underline{p}_2^R\left(p_1^R\right), & \textit{for } p_1 \geq \hat{p}_1; \end{cases} \tag{7.3}$$

Lemma 2 thus defines the 'true' best-response of each player. It says that platform 1, for example, prefers responding with $\underline{p}_1^R\left(p_2^R\right)$ for any prices $p_2 \geq \hat{p}_2$ and switches to $\overline{p}_1^R\left(p_2^R\right)$ otherwise. The best reply correspondence is discontinuous at that point where platforms are indifferent between being the dominant platform and not, that is, between the combination of prices $\left(\underline{p}_i^R(p_j^R), p_i^A(\underline{p}_i^R)\right)$ and $\left(\overline{p}_i^R(p_j^R), p_i^A(\overline{p}_i^R)\right)$.

Proof: First notice from (4.2) that playing a profile $\tilde{\mathbf{p}}^R$ such that $D_1^R = D_2^R$ can never be a best reply. When $D_1^R = D_2^R$ advertising profits Π_i^A are nil for both platforms. So both players have a deviation strategy $p_i^R + \varepsilon$ in either direction since $\overline{\Pi}_i^A > \underline{\Pi}_i^A > 0,\ i = 1, 2$ as soon as $D_i^R \neq D_{-i}^R$. Maximising the profit function (4.2) taking p_{-i} as fixed leaves us with two 'quasi-reaction correspondences', for each competitor, depending on whether $D_1^R > D_2^R$ or the converse. Player i's profit function can be rewritten $\Pi_i\left(p_1^R(p_2^R), p_2^R; \Pi_i^A\right)$. Depending on firm 2's decision, platform 1's profit is either

$$\Pi_1 = \begin{cases} \Pi_1\left(\underline{p}_1^R(p_2^R), p_2^R; \Pi_i^A\right) = \Pi_1\left(\frac{1}{2}\left(p_2^R + \Delta\theta\overline{\beta} - 2e\overline{A}\right), p_2^R; \Pi_i^A\right), & \text{or;} \\ \Pi_1\left(\overline{p}_1^R(p_2^R), p_2^R; \Pi_i^A\right) = \Pi_1\left(\frac{1}{2}\left(p_2^R + \Delta\theta\overline{\beta} + 2e\underline{A}\right), p_2^R; \Pi_i^A\right). \end{cases}$$

Define $g_1(p_2^R) \equiv \Pi_1\left(\overline{p}_1^R(p_2^R), p_2^R; \Pi_i^A\right) - \Pi_1\left(\underline{p}_1^R(p_2^R), p_2^R; \Pi_i^A\right)$. This quantity is the difference in profits generated by firm 1 when it chooses one 'quasi-best response' over the other, as a function of the consumer price set by firm 2. For p_2^R sufficiently low, $g_1(.) > 0$. This function is continuous and a.e differentiable, for it is the sum of two continuous, differentiable functions. Using the definitions of equilibrium advertising profits (in Lemma 1), it is immediate to compute $\frac{dg_1}{dp_2^R} = \frac{d\Pi_1^A(\overline{p}_1^R, p_2^R)}{dp_2^R} - \frac{d\Pi_1^A(\underline{p}_1^R, p_2^R)}{dp_2^R} < 0$, and $\frac{d^2 g_1}{d(p_2^R)^2} = 0$, whence there exists a point $\hat{p}_2^R$ such that $g_1(\hat{p}_2^R) = 0$. At $\hat{p}_2^R$, $\Pi_i\left(\underline{p}_1^R(\hat{p}_2^R), \hat{p}_2^R\right) = \Pi_i\left(\overline{p}_1^R(\hat{p}_2^R), \hat{p}_2^R\right)$ and platform 1 is indifferent between these two profit functions, that is between either best response $\underline{p}_1^R(\hat{p}_2^R)$ or $\overline{p}_1^R(\hat{p}_2^R)$. The same follows for platform 2, which defines $\hat{p}_1^R$. Computing the profit functions, it is immediate that

$$\Pi_1\left(\underline{p}_1^R(p_2^R), p_2^R; \Pi_i^A\right) \geq \Pi_1\left(\overline{p}_1^R(p_2^R), p_2^R; \Pi_i^A\right) \Leftrightarrow p_2^R \geq \hat{p}_2^R \equiv -\left(\Delta\theta\underline{\beta} + e(\overline{A} - \underline{A})\right)$$

and

$$\Pi_2\left(p_1^R, \underline{p}_2^R(p_1^R); \Pi_i^A\right) \geq \Pi_2\left(p_1^R, \overline{p}_2^R(p_1^R); \Pi_i^A\right) \Leftrightarrow p_1^R \geq \hat{p}_1^R \equiv \Delta\theta\overline{\beta} - e(\overline{A} - \underline{A})$$

■

For each firm, its action must be an element of the best reply correspondence and these correspondences must intersect. We define a condition that captures both these features, and will show next that it is both necessary and sufficient for an equilibrium to exist. From the 'quasi-best responses', an equilibrium candidate is a pair of prices such that

$$\left(p_1^{*R}, p_2^{*R}\right) = \begin{cases} \underline{p}_1^R\left(p_2^R\right) \cap \overline{p}_2^R\left(p_1^R\right), & \text{if } D_1^R > D_2^R \text{ or;} \\ \overline{p}_1^R\left(p_2^R\right) \cap \underline{p}_2^R\left(p_1^R\right), & \text{if } D_1^R < D_2^R; \end{cases}$$

An equilibrium exists only if these intersections are non-empty. Together, the definitions of a best-response profile (relations (7.2) and (7.3)) and of an equilibrium candidate sum to

Condition 1 *Either*

$$\hat{p}_1^R \geq p_1^{*R} \text{ and } \hat{p}_2^R \leq p_2^{*R}$$

or

$$\hat{p}_1^R \leq p_1^{*R} \text{ and } \hat{p}_2^R \geq p_2^{*R}$$

or both.

Consider an action profile $\mathbf{p}^{*R}$ satisfying this condition; from Lemma 2 each p_i^{*R} is an element of i's best response. Now, for it to be an equilibrium, players must choose reaction functions that intersect. This is exactly what Condition 1 requires. For example, the first pair of inequalities tells us that player 1's optimal action has to be low enough and simultaneously that of 2 must be high enough. When they hold, player 2's reaction correspondence is necessarily continuous until 1 reaches the maximiser p_1^{*R}, and similarly for 1's best reply. Then

Lemma 3 *Condition 1 is necessary and sufficient for at least one equilibrium* $\mathbf{p}^{*R} = \left(p_1^{*R}, p_2^{*R}\right)$ *to exist. When both inequalities are satisfied, the game admits two equilibria.*

When Condition 1 holds, the Nash correspondence $p_1^R(p_2^R) \times p_2^R(p_1^R)$ has a closed graph and standard theorems apply. The potential multiplicity of equilibria owes to the discontinuity of the best-reply correspondences.

Proof: Since player i's action set is $P_i^R \subseteq \mathbb{R}$, it is compact and convex. For each platform this can be partitioned into two subsets $\underline{P}_i^R = \left[p_i^{R,min}, \hat{p}_i^R\right]$ and $\overline{P}_i^R = \left[\hat{p}_i^R, p_i^{R,max}\right]$, on which the best-response correspondences defined by (7.2) and (7.3) are continuous for each platform i. It can be verified that the profit function (4.2) is concave in its own argument in each of these domains, but it is not immune from non-local deviation (that is, not quasi-concave over the whole set P_i^R. Consider any equilibrium candidate $\left(p_1^{*R}, p_2^{*R}\right)$. By construction it is defined as the intersection of the 'quasi-best responses', which is not necessarily an equilibrium. But when Condition 1 holds, following the definitions given by equations (7.2) and (7.3), either $p_1^{*R} \in \underline{p}_1^R(p_2^R)$ and $p_2^{*R} \in \overline{p}_2^R(p_1^R)$, or $p_1^{*R} \in \overline{p}_1^R(p_2^R)$ and $p_2^{*R} \in \underline{p}_2^R(p_1^R)$ (or both, if two equilibria exist). Thus at the point $\left(p_1^{*R}, p_2^{*R}\right)$ the reaction correspondences necessarily intersect at least once, whence the Nash correspondence has a closed graph and the Kakutani fixed-point theorem applies. To show necessity, suppose a pair $\left(p_1^{*R}, p_2^{*R}\right)$ is a Nash equilibrium. By definition, $p_2^R\left(p_1^R\right) \cap p_1^R\left(p_2^R\right) \neq \varnothing$, and by Lemma 2, either $\left(p_1^{*R}, p_2^{*R}\right) = \underline{p}_1^R\left(p_2^R\right) \cap \overline{p}_2^R\left(p_1^R\right)$ or $\left(p_1^{*R}, p_2^{*R}\right) = \overline{p}_1^R\left(p_2^R\right) \cap \underline{p}_2^R\left(p_1^R\right)$, or both if two equilibria exist. For the first equality to hold, the first line of Condition 1 must hold, and for the second one, the second line of Condition 1 must be satisfied. ■

Let $C = \left[2e\left(\overline{A} + \underline{A}\right)\right]^2 = \left[2e\left(\left(\frac{2\overline{\alpha}-\underline{\alpha}}{3}\right)^2 + \left(\frac{\overline{\alpha}-2\underline{\alpha}}{3}\right)^2\right)\right]^2$. Condition 1 provides us with a pair of easy-to-verify conditions in terms of prices. Thus we can establish

Lemma 4 ***Existence***. *An equilibrium in pure strategies of the consumer price subgame always exists. It is unique and located in region I.*

Proof: First construct a candidate equilibrium as follows. Suppose that platforms maximise $\Pi_1^H = p_1^R D_1^R(\mathbf{p^R}, \theta) - k\theta_1^2 + \overline{\Pi}_1^A$ and $\Pi_2^H = p_2^R D_2^R(\mathbf{p^R}, \theta) - k\theta_2^2 + \underline{\Pi}_2^A$, respectively. Solving for the first-order conditions laid out in Definition 1 yields

$$p_1^{*R} = \tfrac{1}{3}\left[\Delta\theta\left(2\overline{\beta} - \underline{\beta}\right) + 2e\left(\underline{A} - 2\overline{A}\right)\right]$$
$$p_2^{*R} = \tfrac{1}{3}\left[\Delta\theta\left(\overline{\beta} - 2\underline{\beta}\right) + 2e\left(2\underline{A} - \overline{A}\right)\right]$$

From equilibrium prices it is straightforward to compute consumer demand: $D_1^R = \frac{1}{3\Delta\theta}\left[\Delta\theta(2\overline{\beta} - \underline{\beta}) + \sqrt{C}\right]$ and $D_2^R = \frac{1}{3\Delta\theta}\left[\Delta\theta(\overline{\beta} - 2\underline{\beta}) - \sqrt{C}\right]$, hence the restriction $D_2^R > 0$ provided $\Delta\theta > \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ and

$$p_1^{*R} = \tfrac{\Delta\theta\overline{\beta}}{2} - e2\overline{A}$$
$$p_2^{*R} = 0$$

otherwise. When $D_i^R = 0$ $i = 1, 2$, p_j^R is determined by platform j's reaction correspondence only. Thus it easy to verify that the first line of Condition 1 is satisfied and that $\left(p_1^{*R}, p_2^{*R}\right)$ indeed constitutes an equilibrium by Lemma 3. This equilibrium *always* exists because $\hat{p}_1^R \geq p_1^{*R}$ and $\hat{p}_2^R \leq p_2^{*R}$ are always satisfied. Indeed, either both hold when both platforms are active, for $\Delta\theta\left(\overline{\beta}+\underline{\beta}\right)+e\left(\overline{A}+\underline{A}\right)\geq 0$ is always true, or $p_2^{*R}=0>\hat{p}_2^R$ and $\hat{p}_1^R>p_1^{*R}$ can be immediately verified when only firm 1 is active. Another candidate equilibrium $\left(p_1^{**R},p_2^{**R}\right)$ can be constructed by letting platform 1 play as if $\Pi_1^L=p_1^R D_1^R(\mathbf{p^R},\theta)-k\theta_1^2+\underline{\Pi}_1^A$ and platform 2 as if $\Pi_2^L=p_2^R D_2^R(\mathbf{p^R},\theta)-k\theta_2^2+\overline{\Pi}_2^A$, whence

$$p_1^{**R}=\tfrac{1}{3}\left[\Delta\theta\left(2\overline{\beta}-\underline{\beta}\right)+2e\left(2\underline{A}-\overline{A}\right)\right]$$
$$p_2^{**R}=\tfrac{1}{3}\left[\Delta\theta\left(\overline{\beta}-2\underline{\beta}\right)+2e\left(\underline{A}-2\overline{A}\right)\right]$$

with $D_1^R=\frac{1}{3}\left[(2\overline{\beta}-\underline{\beta})^2-\sqrt{C}\right]$ and $D_2^R=\frac{1}{3}\left[(\overline{\beta}-2\underline{\beta})^2+\sqrt{C}\right]$, therefore $D_1^R>0$ if $\Delta\theta>\frac{\sqrt{C}}{2\overline{\beta}-\underline{\beta}}$. Notice that an equilibrium such that

$$p_1^{*R}=0$$
$$p_2^{*R}=-\tfrac{\Delta\theta\underline{\beta}}{2}-e2\overline{A}$$

cannot exist, for these prices are not best response to each other. At the price-setting stage, the cost of quality is sunk, so for $\theta_1>\theta_2$ there always exists some price $p_1^R\geq p_2^R$ such that consumers prefer purchasing from platform 1. Then when both firms are active Condition 1 holds as long as $\Delta\theta\left(\overline{\beta}+\underline{\beta}\right)-e\left(\overline{A}+\underline{A}\right)\leq 0$. Given that $\Delta\theta\geq\frac{\sqrt{C}}{2\overline{\beta}-\underline{\beta}}$, take the lower bound and substitute into the second line of Condition 1. Recalling $\sqrt{C}=2e(\overline{A}+\underline{A})$,

$$e(\overline{A}+\underline{A})\left(\frac{2(\overline{\beta}+\underline{\beta})}{\overline{\beta}-2\underline{\beta}}-1\right)>0,\ \forall\underline{\beta}\geq 0$$

which violates the second pair of inequalities of the necessary Condition 1. So the second candidate can never be an equilibrium. For completeness, Condition 1 is also sufficient to rule out deviations from the pairs $\left(p_1^{*R},p_2^{*R}\right)$ and $\left(p_1^{**R},p_2^{**R}\right)$. The SOC of the profit function (4.2) is satisfied at prices p_i^{*R} and p_i^{**R} $\forall i,\forall p_{-i}^R$, there cannot be any local deviation. Consider now deviations involving inconsistent actions, that is, such that both platforms maximise either $p_i^R D_i^R(p^R,\theta)-k\theta_i^2+\overline{\Pi}_i^A$ or $p_i^R D_i^R(p^R,\theta)-k\theta_i^2+\underline{\Pi}_i^A$. Since $\left(p_1^{*R},p_2^{*R}\right)$ always exists, the first line of Condition 1 always holds. It immediately follows from (7.2) and (7.3) that $\overline{p}_1^R\left(p_2^R\right)\cap\overline{p}_2^R\left(p_1^R\right)=\varnothing$ and $\underline{p}_1^R\left(p_2^R\right)\cap\underline{p}_2^R\left(p_1^R\right)=\varnothing$ as well. ■

7.4 Proof of Proposition 1

Directly from Lemma 4, which establishes existence and uniqueness of this equilibrium. In particular no such alternative equilibrium can exist when $\Delta\theta < \frac{\sqrt{C}}{2\overline{\beta}-\underline{\beta}}$. Consider such a situation, then the prices

$$p_1^R = \frac{\Delta\theta\overline{\beta}}{2} - 2e\overline{A}$$
$$p_2^R = 0$$

do form an equilibrium for they satisfies Condition 1. But the pair

$$p_1^R = 0$$
$$p_2^R = -\frac{\Delta\theta\underline{\beta}}{2} - 2e\overline{A}$$

cannot be best responses to each other. At the price-setting stage, the cost of quality is sunk. So with $\theta_1 > \theta_2$, there always exists some price $p_1^R \geq p_2^R$ such that consumers prefer purchasing from platform 1.

7.5 Analysis of the high-quality firm's profit function

In the sequel $\theta_1 > \theta_2$ without loss of generality. The profit function $\Pi_1(.,.)$ is obviously continuous for $\theta_1 < \underline{\theta} + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ or the converse. Furthermore, assume $e < \infty$, then

Claim 1 *The function Π_1 is continuous for $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$*

Proof: For ease of notation, let $\Pi_1 = \Pi_1^L$ for all $\Delta\theta \geq \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ and $\Pi_1 = \Pi_1^R$ otherwise. These are the definitions of $\Pi_1(\theta_1, \underline{\theta})$ to the left and the right of the point such that $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ for any pair (θ_1, θ_2). To the left platform 1 is a monopolist whose profits Π_1^L are necessarily bounded. The function is defined as $\Pi_1^L : \Theta_1 \times \Theta_2 \subseteq \mathbb{R}^2 \mapsto \mathbb{R}$, therefore Theorem 4.5 in Haaser and Sullivan ([14], page 66) applies: a mapping from a metric space into another metric space is continuous if and only if the domain is closed when the range is closed. So $\Pi_1^L(\theta_1, \theta_2)$ is continuous at $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$, and is necessary the left-hand limit of the same function Π_1^L. Now consider a sequence θ_1^n such that $\Delta\theta > \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ converging to $\frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ from above for some fixed θ_2 . This sequence exists and always converges for $\Theta_1 \subseteq \mathbb{R}$ is complete. As $e < \infty$ and $\overline{A}$ and $\underline{A}$ are necessarily bounded, C is finite so there is some n and some arbitrarily small δ such that $\Pi_1^R(\theta_1^n, \theta_2) - \Pi_1^L(\theta_2 + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}, \theta_2) < \delta$. That is, $\lim_{\theta_1^n \to \theta_2 + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}} \Pi_1^R(\theta_1^n) = \Pi_1^L(\theta_2 + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}, \theta_2)$. Hence Π_1 is continuous for $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$. ∎

$\Pi_1(.,.)$ being the difference of two convex functions, its exact shape is affected by that of these two primitives. Indeed, when C becomes large enough, it is no longer well behaved.

Claim 2 *There exists some* $C^f \equiv \left[\frac{(2\overline{\beta}-\underline{\beta})^2}{27k} - \underline{\theta}\right]^2 \left(\frac{(2\overline{\beta}-\underline{\beta})^2}{3}\right)$ *such that* $\Pi_1(.,.)$ *admits a binding first-order condition for* $C \leq C^f$ *only. When* $C > C^f$*, its maximum is reached at the kink:* $\theta_1 = \underline{\theta} + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$.

Proof: Seeking first-order conditions of $\Pi_1(.,.)$ with respect to θ_1 yields

$$\frac{\partial \Pi_1}{\partial \theta_1} = \begin{cases} \left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - 2k\theta_1 = 0, & \text{for } \Delta\theta \leq \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}; \\ \left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - \frac{C}{(3\Delta\theta)^2} - 2k\theta_1 = 0, & \text{for } \Delta\theta > \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}} \text{ and } C \leq C^f; \\ \left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - \frac{C}{(3\Delta\theta)^2} - 2k\theta_1 < 0, & \text{for } \Delta\theta > \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}} \text{ and } C > C^f; \end{cases} \tag{7.4}$$

When binding, the second line of system (7.4) can be rearranged as $\left(2\overline{\beta}-\underline{\beta}\right)^2 = \phi(\theta_1)$, with slope $\phi'(\theta_1) = 18k - \frac{2C}{(\Delta\theta)^3}$. Since $\Delta\theta > 0$, this FOC has at most two solutions: one where $\phi'(\theta_1) < 0$ and the other with $\phi'(\theta_1) > 0$. The SOC requires $\phi'(\theta_1) \geq 0$ for the FOC to identify a maximiser, so there exists a unique local maximiser of Π_1, denoted $\hat{\theta}_1$. Let θ_1^0 be the (unique) maximiser of the first line of system (7.4). It is immediate that $\hat{\theta}_1 < \theta_1^0$ and consequently $\theta_1^0 - \theta_2 \leq \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$, $\theta_1 \in BR_1(\theta_2)$ can never be true. That is, the two statements of the first line of (7.4) cannot be simultaneously satisfied: firm 1 would not play the first line of (4.4), but the second one. We rewrite:

$$\frac{\partial \Pi_1}{\partial \theta_1} = \left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - 2k\theta_1 > 0; \quad for\ \Delta\theta \leq \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$$

Recall that the profit function is continuous, so it does not jump anywhere. Because Π_1 is monotonically increasing below $\hat{\theta}_1$ and the SOC is monotonic beyond $\hat{\theta}_1$, it is concave for $C \leq C^f$ and $\hat{\theta}_1$ is a global maximiser. The binding first-order condition defines a function $C(\theta_1, \theta_2) \equiv (\Delta\theta)^2 \left[(2\overline{\beta}-\underline{\beta})^2 - 18k\theta_1\right]$, whence $\frac{dC(.)}{d\theta_1} = 0 \Leftrightarrow \theta_1^f = \frac{(2\overline{\beta}-\underline{\beta})^2}{27k}$. Substituting back into $C(\theta_1, \theta_2)$ gives the cut-off value $C^f \equiv \left[\frac{(2\overline{\beta}-\underline{\beta})^2}{27k} - \theta_2\right]^2 \left(\frac{(2\overline{\beta}-\underline{\beta})^2}{3}\right)$. When $C > C^f$, the first-order condition (7.4) is everywhere negative, hence

$$\frac{d\Pi_1}{d\theta_1}\Big|_{\theta_1 < \underline{\theta} + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}} > 0$$

$$\frac{d\Pi_1}{d\theta_1}\Big|_{\theta_1 > \underline{\theta} + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}} < 0$$

While this profit function is not differentiable for $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$, it has been established that it is nonetheless continuous for any such pair (θ_1, θ_2). It is monotonic on either side of $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$, so that $\hat{\theta}_1$ such that $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$, is the unique maximum of $\Pi_1(\theta_1, \theta_2)$ given some fixed θ_2. ■

Last in this section we examine the behavior of the quality variable θ_1 when the first-order condition (7.4) does bind.

Claim 3 *Let $\hat{\theta}_1$ solve $\left(2\overline{\beta} - \underline{\beta}\right)^2 - \frac{C}{(\Delta\theta)^2} - 18k\theta_1 = 0$, then $\frac{d\hat{\theta}_1}{de} < 0$ and $\frac{d\hat{\theta}_1}{dk} < 0$.*

Proof: Differentiate the first-order condition (7.4); after some manipulations we can write

$$\frac{d\theta_1^*}{de} = \frac{8\Delta\theta e(\overline{A} + \underline{A})}{2\left(2e(\overline{A} + \underline{A})\right)^2 - 18k(\Delta\theta)^3}$$

$\frac{d\theta_1^*}{de} \geq (\leq) 0 \Leftrightarrow 2C - 18k(\Delta\theta)^3 = -(\Delta\theta)^3\phi'(\theta_1)|_{\theta_1=\theta_1^*} \geq (\leq) 0$ so that $\frac{d\theta_1^*}{de} < 0$ (assuming the SOC holding strictly at θ_1^*, otherwise $\frac{d\theta_1^*}{de}$ is not defined and we need to consider the left derivative). The second statement is similar: differentiate the first-order condition of (4.4) to find $2C(\Delta\theta)^{-3}\frac{d\theta_1}{dk} - 18\theta_1 - 18k\frac{d\theta_1}{dk} = 0$, which is rearranged as $\frac{d\theta_1}{dk} = \frac{18\theta_1(\Delta\theta)^3}{2C-18k(\Delta\theta)^3}$. The denominator is exactly the SOC of (4.4), which we know to hold, multiplied by $(\Delta\theta)^3$. ■

7.6 Proof of Proposition 2

We begin by characterising the first-stage actions

Lemma 5 *Let $\theta_1 > \theta_2$ w.l.o.g. and Assumption 4 hold. Optimal actions consist of $\theta_2^* = \underline{\theta}$ and $\theta_1^* = \hat{\theta}_1$, where $\hat{\theta}_1$ uniquely solves*

$$(2\overline{\beta} - \underline{\beta})^2 = 18k\theta_1 + \frac{C}{(\Delta\theta)^2} \tag{7.5}$$

Both platforms operate.

Proof: First off the following simplifies the analysis and lets us focus on platform 1's problem.

Claim 4 *In any pure-strategy Nash equilibrium (θ_1^*, θ_2^*) such that $\theta_1^* > \theta_2^*$, $\theta_2^* = \underline{\theta}$ necessarily.*

Proof: Assume the FOC (7.4) binds so that $\theta_1^* = \hat{\theta}_1$. Computing the slope of the profit function Π_2 yields

$$\frac{d\Pi_2}{d\theta_2} = \begin{cases} -(\overline{\beta} - 2\underline{\beta})^2 + \frac{C}{(\Delta\theta)^2} - 2k\theta_2 < -2k\theta_2, & \text{if } \Delta\theta(\overline{\beta} - 2\underline{\beta}) > \sqrt{C}; \\ -2k\theta_2, & \text{if } \Delta\theta(\overline{\beta} - 2\underline{\beta}) \leq \sqrt{C}. \end{cases}$$

whence it is immediate that $\frac{d\Pi_2}{d\theta_2}|_{\theta_2>\underline{\theta}} < \frac{d\Pi_2}{d\theta_2}|_{\underline{\theta}} < 0$. ■

Next delineate an impossibility. When C is said to be 'large' the profit function $\Pi_1(.,.)$ is no longer well behaved, as shown in Section 7.5. This leads to

Lemma 6 *Let* $\theta_1 > \theta_2$ *w.l.o.g. and* $C \geq C^f \equiv \left[\frac{(2\overline{\beta}-\underline{\beta})^2}{27k} - \theta_2\right]^2 \left(\frac{(2\overline{\beta}-\underline{\beta})^2}{3}\right)$*, a Nash equilibrium in pure strategies cannot exist.*

Proof: Follows directly from Claims 4 and 2 in Section 7.5. Any pair $\left(\theta_2 + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}, \theta_2\right)$ cannot be an equilibrium because firm 2 can 'jump' and assume the monopolist's role at incremental cost $k\varepsilon^2$. ■

In line with the previous section of the Appendix, firm 1's first-order condition reads $\left(2\overline{\beta} - \underline{\beta}\right)^2 - \frac{C}{(\Delta\theta)^2} - 18k\theta_1 = 0$ and admits a unique maximiser $\hat{\theta}_1$. This analysis does not yet identify an equilibrium of this game but only platform 1's behaviour, taking that of firm 2 fixed. Suppose firm 1 plays $\hat{\theta}_1$; by Claim 4, platform 2 cannot increase its quality to any $\theta_2 \in \left(\underline{\theta}, \hat{\theta}_1\right)$. So the pair $\left(\hat{\theta}_1, \underline{\theta}\right)$ is an equilibrium as long as firm 2 cannot 'jump' over firm 1 and become the high-quality firm. It will necessarily do so if platform 1 turns out to be a monopolist. To guarantee firm 2 operates we need $(\hat{\theta}_1 - \underline{\theta})(\overline{\beta} - 2\underline{\beta}) > \sqrt{C}$ – Assumption 4 must holds. When firm 2 does operate, the smallest 'leap' it can undertake is such that $\tilde{\theta}_2 \geq \hat{\theta}_1 + \varepsilon$. Hence the no-deviation condition is $\Pi_2\left(\hat{\theta}_1, \underline{\theta}\right) \geq \Pi_2\left(\hat{\theta}_1, \hat{\theta}_1 + \varepsilon\right)$, or

$$\begin{aligned}
(\hat{\theta}_1 - \underline{\theta})(\overline{\beta} - 2\underline{\beta})^2 + B_2 + \tfrac{C}{(\hat{\theta}_1-\underline{\theta})} &\geq B_1 + \sqrt{C}(\overline{\beta} - 2\underline{\beta}) - 9k(\hat{\theta}_1 + \varepsilon)^2 \\
(\hat{\theta}_1 - \underline{\theta})\left[(\overline{\beta} - 2\underline{\beta})^2 + (2\overline{\beta} - \underline{\beta})^2\right] - 18k\hat{\theta}_1^2 + B_2 &\geq B_1 + \sqrt{C}(\overline{\beta} - 2\underline{\beta}) - 9k(\hat{\theta}_1 + \varepsilon)^2 \\
(\hat{\theta}_1 - \underline{\theta})\left[(\overline{\beta} - 2\underline{\beta})^2 + (2\overline{\beta} - \underline{\beta})^2\right] - 9k\hat{\theta}_1^2 + B_2 &\geq B_1 + \sqrt{C}(\overline{\beta} - 2\underline{\beta})
\end{aligned}$$

using the FOC $(2\overline{\beta} - \underline{\beta})^2 - 18k\hat{\theta}_1 - \frac{C}{(\hat{\theta}_1-\underline{\theta})^2} = 0$ and the fact that $k\hat{\theta}_1\underline{\theta} = k\underline{\theta}^2 = 0$ (by assumption). Noting $\hat{\theta}_1 - \underline{\theta} > \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$, this condition is generically satisfied. ■

7.7 Existence of a mixed-strategy equilibrium

Proposition 8 *A mixed-strategy equilibrium of the game* Γ *always exists.*

This assertion holds trivially by Corollary 1 when Assumption 4 holds. The balance focuses on the case where it fails. The conditions of the Proposition guarantee that the market is covered – this is Assumption 3. We need some preliminaries to establish the Proposition.

Denote $\tilde{\theta} = \underline{\theta} + \frac{\sqrt{C}}{\overline{\beta} - 2\underline{\beta}}$ from now on. It is not immediate that the game Γ admits a mixed-strategy equilibrium, for the payoffs are not everywhere continuous. First define by θ_1^c the threshold such that $\Pi_1(\theta_1^c, \underline{\theta}) = 0$ when $\theta_1 > \theta_2$. This point exists and exceeds $\tilde{\theta}_1$ because $\frac{d\Pi_1}{d\theta_1}|_{\theta_1 > \tilde{\theta}_1} < 0$ and the cost function is convex. Neither platform will want to exceed that threshold, so we restrict the set of pure actions over which firms randomise to be $[\underline{\theta}, \theta_i^c] \subseteq \Theta_i,\ i = 1, 2$. Next, any distribution over this set must assign zero mass to any $\theta_i \in (\underline{\theta}, \tilde{\theta})$ by Claim 4: any action in this interval is dominated by either $\underline{\theta}$ or $\tilde{\theta}$. For $[\tilde{\theta}, \theta_i^c]$ large enough (and $\theta_2 \geq \tilde{\theta}$) there may be outcomes such that $\Delta\theta > \frac{\sqrt{C}}{\overline{\beta} - 2\underline{\beta}}$, in which case both platforms are active, or $\Delta\theta \leq \frac{\sqrt{C}}{\overline{\beta} - 2\underline{\beta}}$, in which case only the high-quality firm operates. Take $\theta_1 > \theta_2 > \underline{\theta}$ and suppose $\Delta\theta > \frac{\sqrt{C}}{\overline{\beta} - 2\underline{\beta}}$ and $\Pi_1 > \Pi_2 > 0$. Let θ_2 increase, both Π_1 and Π_2 vary smoothly. But while $\lim_{\theta_2^n \uparrow \theta_1} \Pi_1 = \Pi_1 > 0$, $\lim_{\theta_2^n \downarrow \theta_1} \Pi_1 = -k\theta_1^2$, and similarly for firm 2. Both payoff functions are discontinuous at the point $\theta_1 = \theta_2$. In this case neither the payoffs nor their sum are even upper-hemicontinous. Following Dasgupta and Maskin's (1986) Theorem 5, it is first necessary to characterise the discontinuity set. If it has Lebesgue measure zero, a mixed-strategy equilibrium does exist. Consider the case where $\theta_1 \geq \theta_2$ w.l.o.g. and define $\Upsilon_0 = \left\{(\theta_1, \theta_2) | \theta_1 = \theta_2, \theta_i \in [\tilde{\theta}_i, \theta_i^c]\ \forall i\right\}$, the set on which the payoffs are discontinuous. Further define the probability measure $\mu(\theta_1, \theta_2)$ over the set $\Theta^N = \{\underline{\theta}_1\} \cup [\tilde{\theta}_1, \theta_1^c] \times \{\underline{\theta}_2\} \cup [\tilde{\theta}_2, \theta_2^c]$. It is immediate that Υ_0 has Lebesgue measure zero, so that $\Pr\left((\theta_1, \theta_2) \in \Upsilon_0\right) = 0$. Next we claim

Lemma 7 *Suppose $\theta_1 = \theta_2 = \underline{\theta}$, an equilibrium in mixed strategies exists in the consumer price subgame.*

As each platform's payoffs are bounded below at zero and only one of them can operate (except at $p_1^R = p_2^R$), their sum is almost everywhere continuous, except for the set of pairs $(p_1^R = p_2^R)$, which has measure zero.

Proof: Let $\theta_1 = \theta_2 = \underline{\theta}$. The sum of profits $\Pi = \Pi_1 + \Pi_2$ is almost everywhere continuous. Either $\Pi = \Pi_1 > 0\ \forall p_1^R < p_2^R$, or $\Pi = \Pi_2 > 0\ \forall p_1^R > p_2^R$, both of which are continuous except at $p_1^R = p_2^R$, where $\Pi = \Pi_1 + \Pi_2 = 0$. But the set $\Psi = \left\{(p_1^R, p_2^R) | p_1^R = p_2^R, (p_1^R, p_2^R) \in \mathbb{R}^2\right\}$ has Lebesgue measure zero. Theorem 5 of Dasgupta and Maskin (1986) directly applies and guarantees existence of an equilibrium in mixed strategies. ■

Therefore the pair $\theta_1 = \theta_2 = \underline{\theta}$ may be part of an equilibrium of the overall game. Then Proposition 8 asserts that a mixed-strategy equilibrium of the game Γ exists, which can now be easily proven.

Proof: We only need showing that the payoff functions Π_i $i = 1, 2$ are lower-hemicontinuous in their own argument θ_i. Without loss of generality, fix $\theta_1 > \theta_2$. We know that Π_1 is continuous for any $\theta_1 > \theta_2$ (refer Section 7.5). From Claim 4 it is immediate that Π_2 is continuous for $\theta_1 > \theta_2$. Last, for $i = 1, 2$

$$\Pi_i = \begin{cases} 0, & \text{if } \theta_1 = \theta_2 = \underline{\theta}; \\ -k\theta_i^2, & \text{if } \theta_1 = \theta_2 > \underline{\theta}. \end{cases}$$

that is, $\Pi_i,\ i = 1, 2$ is l.h.c. Since $(\theta_2, \theta_1)\, s.t\ \theta_2 = \theta_1 \in \Upsilon_0$, Theorem 5 in Dasgupta and Maskin (1986) can be applied, whence an equilibrium in mixed strategies must exist. ■

7.8 Proof of Proposition 3

Let θ_i^c denote the upper bound of the support of the distribution of the pure action space, a precise definition of which will soon be provided. Let $H_i(\theta_i)$ be the distribution over i's pure actions $\theta_i \in \{\underline{\theta}\} \cup \left[\tilde{\theta}, \theta^c\right]$. For any equilibrium mixing probability $H_2^*(\theta_2)$, write the expected profit of firm 1 as

$$\begin{aligned} \mathbb{E}_{\theta_2}[\Pi_1] &= \int \Pi_1(\underline{\theta}_1, \theta_2) d(H_1 \times H_2^*) + \int_{\tilde{\theta}_1}^{\theta_1' = \theta_2} \Pi_1(\theta_1, \theta_2) d(H_1 \times H_2^*) + \int_{\theta_1' = \theta_2}^{\theta_1^c} \Pi_1(\theta_1, \theta_2) d(H_1 \times H_2^*) \\ &= H_1(\underline{\theta}_1) \int \Pi_1(\underline{\theta}_1, \theta_2) d(H_2^*) + \int_{\tilde{\theta}_1}^{\theta_1' = \theta_2} \Pi_1(\theta_1, \theta_2) d(H_1 \times H_2^*) + \int_{\theta_1' = \theta_2}^{\theta_1^c} \Pi_1(\theta_1, \theta_2) d(H_1 \times H_2^*) \end{aligned}$$

with possibly an atom at $\underline{\theta}_1$. With probability $\int_{\tilde{\theta}_1}^{\theta_1' = \theta_2} d(H_1 \times H_2^*)$ it plays $\theta_1 > \underline{\theta}$ such that medium 2 is the dominant firm ($\theta_2 \geq \theta_1$); in this case, $\Pi_1(\theta_1, \theta_2) = -k\theta_1^2 < 0$. With probability $\int_{\theta_1' = \theta_2}^{\theta_1^c} d(H_1 \times H_2^*)$ it is the dominant firm (the second integral). We first claim

Lemma 8 *There is a mass point at $\underline{\theta}_i$. More precisely, $\forall\ i,\ H_i(\underline{\theta}_i) \in (0, 1)$.*

Proof: Suppose $H_1(\underline{\theta}_1) = 1$, then $\arg\max \mathbb{E}_{\theta_1}[\Pi_2(\underline{\theta}_1, \theta_2)] = \tilde{\theta}_2$, so $H_2(\underline{\theta}_2) = 0$ and $H_2(\theta_2)$ assigns full mass at $\tilde{\theta}_2 : h_2(\tilde{\theta}_2) = 1$. But then firm 1 should play some $\theta_1 > \tilde{\theta}_2$ and become the monopolist for sure. If $H_1(\underline{\theta}_1) = 0$, then 1 necessarily plays on $\left[\tilde{\theta}, \theta^c\right]$ and playing $\tilde{\theta}_2$ is a dominated strategy for firm 2. It therefore assigns no mass at this point. But then $\forall\ \theta_2 \in \left(\tilde{\theta}_2, \theta_2^c\right]$, $\Pi_1(\underline{\theta}_1, \theta_2) > 0$ and platform 1 should shift some mass to $\underline{\theta}_1$. ■

The equilibrium conditions write $\forall \theta_i \in \Theta_i^N$,

$$\begin{aligned} &\mathbb{E}_{\theta_j}[\Pi_i(\theta_i, \theta_j)] = \Pi_i(\underline{\theta}_i, \tilde{\theta}_j) \\ &\Pi_i(\underline{\theta}_i, \tilde{\theta}_j) = 0 \end{aligned} \tag{7.6}$$

The first line asserts that i's expected payoff cannot be worse than if not investing for sure, in which case j's best response is $\tilde{\theta}_j$. The second one sates that if not investing for sure, a platform can only expect zero profits. Thus expected profits in the mixed-strategy equilibrium must be zero. We next need to determine the upper bound θ_i^c of the support of $H_i(\theta_i)$ for each platform $i = 1, 2$. As a consequence of Lemma 8 it solves either

$$\Pi_i(\underline{\theta}_j, \theta_i^c) = 0$$

or

$$\Pi_i(\tilde{\theta}_j, \theta_i^c) = 0$$

hence

Lemma 9 $\theta_i^c = \max\left\{\theta_i' | \Pi_i(\underline{\theta}_j, \theta_i') = 0, \Pi_i(\tilde{\theta}_j, \theta_i') = 0\right\}$

Proof: Let θ_i' solve $\Pi_i(\underline{\theta}_j, \theta_i') = 0$ and θ_i'' solve $\Pi_i(\tilde{\theta}_j, \theta_i'') = 0$. Suppose $\theta_i' < \theta_i''$ and $\theta_i^c = \theta_i'$: there is a measure $\theta_i'' - \theta_i'$ on which i places zero weight. Then j should shift at least some weight to $\theta_i' + \epsilon$, $\epsilon > 0$ and small, to obtain $\mathbb{E}_{\hat{H}(\theta_i)}[\Pi_j] > 0 = \mathbb{E}_{\theta_i}[\Pi_j(\theta_i, \theta_j)]$ (where $\hat{H}(.)$ is an alternative distribution). Clearly this extends to any $\theta_i \in [\theta_i', \theta_i'')$. ■

Rewriting the equilibrium condition (7.6), $\forall\ \theta_i \in \Theta_i^N$,

$$H_i(\underline{\theta}_i) \int_{\Theta_j^N} R_i(\underline{\theta}_i, \theta_j) dH_j^*(\theta_j) + \int_{\theta_i'=\theta_j}^{\theta_i^c} R_i(\theta_i, \theta_j) d(H_i(\theta_i) \times H_j^*(\theta_j)) = k \int_{\tilde{\theta}_i}^{\theta_i^c} \theta_i^2 d(H_i(\theta_i) \times H_j^*(\theta_j))$$

where $R_i(\theta_i, \theta_j)$ stands for platform i's revenue (gross of costs). Hence Proposition 3, the proof of which we complete below.

Proof: Existence is established by Proposition 8. For any play θ_j, total revenue $R_i(\theta_i, \theta_j)$ is decreasing in $\theta_i \in \Theta_i^N \setminus \underline{\theta}_i$ – refer Conditions (4.4) and (4.5). Thus for any distribution $H_i(\theta_i) \times H_j^*(\theta_j)$ the LHS is bounded as well, and decreasing in θ_i. ■

7.9 No charge to consumers (Remark 2)

In this section we study a constrained version of the problem, namely, when the consumer price is exogenously fixed at zero. First we show that

Lemma 10 *When consumer prices are identical a pure strategy equilibrium cannot exist.*

Proof: Given $p_1^R = p_2^R$, consumer demand is given by

$$D_i^R = \begin{cases} 1, & \text{if } \theta_i > \theta_j; \\ \frac{1}{2}, & \text{if } \theta_i = \theta_j; \text{ and} \\ 0, & \text{if } \theta_i < \theta_j. \end{cases}$$

for $i \neq j,\ i = 1, 2$, whence platform i faces payoffs

$$\Pi_i = \begin{cases} eD_i^R \left(\frac{\alpha}{2}\right)^2 - k\theta_i^2 \geq 0, & \text{if } \theta_i > \theta_j \geq \underline{\theta}; \\ -k\theta_i^2 \leq 0, & \text{if } \underline{\theta} \leq \theta_i \leq \theta_j; \end{cases}$$

Any profile $\theta_1 = \theta_2$ can never be an equilibrium. Suppose so, then $D_1 = D_2$ and platforms are Bertrand competitors in the advertising market, realising $-k\theta_i^2 \leq 0$ each. When $-k\theta_i^2 < 0$, firm i possesses a unilateral deviation: set $\theta_i = \underline{\theta}$. When $-k\theta_i^2 = 0$, it also possesses a unilateral deviation: set $\theta_i > \underline{\theta}$. ■

Therefore we have

Proposition 9 *Fix* $p_1^R = p_2^R = 0$. *A pure-strategy equilibrium does not exist. A mixed-strategy equilibrium exists and is characterised as in Proposition 3.*

Proof: Directly from Lemma 10. Existence is established in Proposition 8. ■

Proposition 10 admits a straightforward corollary.

Corollary 2 *Fix* $p_1^R = p_2^R = 0$. *Platforms are monopolists on the advertising side except at* $\theta_1 = \theta_2 = \underline{\theta}$, *where* $p_1^A = p_2^A = 0$.

We refer to this as an *irrelevance* result: the other player's action can be disregarded in the advertising market. This result differs from GLS however in that only one firm operates.

Proof: The proof follows directly from the fact that firms necessarily play a mixed strategy equilibrium. Since $\theta_i \in \Theta_i^N\ \forall i$, each event but $\theta_1 = \theta_2 = \underline{\theta}$ has probability zero – see the atom from Proposition 3. Therefore, except at $\underline{\theta}$, platform will be a monopolist in the advertising market with certainty and the cost $k\theta_i^2$ is sunk. Else they are Bertrand competitors. ■

7.10 Elements of Proof of Proposition 4 – unique subgame perfect equilibrium of the Shaked and Sutton model

In the Shaked and Sutton (1982) model there exists a unique equilibrium in the price subgame. In the first stage of the game, firms solve

Problem 4

$$\max_{\theta_i \in \Theta_i} \; p_i^* D_i(\mathbf{p}^*, \theta_i, \theta_j^*) - k\theta_i^2$$

for $i = 1, 2$ and with demand $D_1 = \frac{1}{3}\left(2\overline{\beta} - \underline{\beta}\right)$, $D_2 = \frac{1}{3}\left(\overline{\beta} - 2\underline{\beta}\right)$ and prices $p_1 = \frac{\Delta\theta}{3}\left(2\overline{\beta} - \underline{\beta}\right)$, $p_2 = \frac{\Delta\theta}{3}\left(\overline{\beta} - 2\underline{\beta}\right)$, respectively. This problem is concave $\forall i$, and, given equilibrium prices p_i^* $\forall i$, has obvious maximisers $\theta_2^0 = \underline{\theta}$ and $\theta_1^0 = \frac{1}{2k}\left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2$ with $\theta_1^0 < \overline{\theta}$ thanks to $k > \frac{(2\overline{\beta}-\underline{\beta})^2}{18\overline{\theta}}$. These individually optimal maximisers also form a Nash equilibrium, for although $\Pi_1\left(\theta_1^0, \theta_2^0\right) > \Pi_2\left(\theta_1^0, \theta_2^0\right)$ $\forall k > 0$ [14], it is also true that

Claim 5 *$\nexists\ \tilde{\theta}_2 > \theta_1^0$ such that $\Pi_2\left(\theta_1^0, \tilde{\theta}_2\right) \geq \Pi_2\left(\theta_1^0, \theta_2^0\right)$.*

Proof: Consider a deviation $\tilde{\theta}_2 = \theta_1^0 + \epsilon$, ϵ arbitrarily small. We can compute firm 2 profit from this deviation as $\Pi_2\left(\theta_1^0, \tilde{\theta}_2\right) = \epsilon\left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - k\tilde{\theta}_2^2 < 0$ and the marginal profit $\left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - 2k(\theta_1^0 + \epsilon) < 0$. ■

This is the equilibrium characterisation of the benchmark model. To complete the proof of Proposition 4, observe that firm 1's first-order condition in the benchmark problem reads $\left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - 2k\theta_1^0 = 0$ while that of Problem 1 is $\left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - 2k\hat{\theta}_1 = \frac{C}{(3\Delta\theta)^2} > 0$. Therefore $\hat{\theta}_1 < \theta_1^0$. ■

7.11 Proof of Propositions 5 and 6

For lines 1 and 3 the proof follows directly from Propositions 2 and 4, as well as the analysis of $\Pi_1(.,.)$ in Section 7.5. When $k = 0$, because quality is a sunk cost in the original model, nothing is altered until platforms' have to choose their quality variable. That is, the analysis of the third and second stages remains valid. In the first stage, they now face profit functions

$$\begin{aligned} \Pi_1 &= \frac{1}{9}\left[\Delta\theta(2\overline{\beta} - \underline{\beta})^2 + B_1 + \frac{C}{\Delta\theta}\right] \\ \Pi_2 &= \frac{1}{9}\left[\Delta\theta(\overline{\beta} - 2\underline{\beta})^2 + B_2 + \frac{C}{\Delta\theta}\right] \end{aligned}$$

subject to constraint (4.3) as well. As in the proof of Lemma 5, it is useful to rewrite them as

$$\Pi_1 = \begin{cases} \frac{1}{9}\left[\Delta\theta(2\overline{\beta} - \underline{\beta})^2 + B_1 + \frac{C}{\Delta\theta}\right], & \text{if } \Delta\theta > \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}} \text{ and ;} \\ \frac{1}{9}\left[\Delta\theta(2\overline{\beta} - \underline{\beta})^2 + B_1 + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}\right], & \text{if } \Delta\theta \leq \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}. \end{cases}$$

[14] We can readily compute these profits with closed form solutions: $\Pi_1 > \Pi_2 \Leftrightarrow \left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2\left[\frac{1}{2k}\left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - \underline{\theta}\right] > \left(\frac{\overline{\beta}-2\underline{\beta}}{3}\right)^2\left[\frac{1}{2k}\left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - \underline{\theta}\right]$, which can be re-arranged as $\theta_1^0\left[\frac{1}{2} + \frac{2}{3}\frac{\overline{\beta}\underline{\beta}}{\overline{\beta}^2 - \underline{\beta}^2}\right] > \underline{\theta}$, and always holds

and

$$\Pi_2 = \begin{cases} \frac{1}{9}\left[\Delta\theta(\overline{\beta}-2\underline{\beta})^2 + B_2 + \frac{C}{\Delta\theta}\right], & \text{if } \Delta\theta > \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}} \text{ and ;} \\ \frac{1}{9}\left[\Delta\theta(\overline{\beta}-2\underline{\beta})^2 + B_2 + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}\right], & \text{if } \Delta\theta \le \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}. \end{cases}$$

The necessary FOC of the first line of Π_1 identifies a *minimiser* for firm 1 for any $\theta_1 < \theta_2$. Therefore when a FOC binds, firm 1 has a strict incentive to jump to the boundary $\overline{\theta}$. Meanwhile as before, $\frac{d\Pi_2}{d\theta_2} < 0$ as in the proof of Claim 4. Hence maximal differentiation obtains when the FOC binds, that is, when C is not too large. Let $\hat{e}$ denote the corresponding threshold on e; it is immediate that $\hat{e} > \overline{e}$ since $C(e)$ is increasing in e and k is naught here. When $e > \hat{e}$, a FOC fails to bind entirely for firm 1 – it is clearly negative. Hence firm 1 would like to set $\theta_1 = \theta_2 + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ and firm 2, $\theta_2 = \underline{\theta}$. But of course this fails to be an equilibrium as firm 2 is excluded. The analysis of Propositions 8 and 3 carries through. ■

7.12 Comparative statics

Uniqueness of the subgame-perfect equilibrium renders the comparative statics exercise valid. For the first line, recall that $\theta_2^* = \underline{\theta}$ is a strictly dominant strategy when an equilibrium exists, whence θ_2^* is independent of e. The second series of statements is stated and proven in Section 7.5. To show concavity, differentiate $\frac{d\theta_1^*}{de}$ once more and rearrange to find

$$\frac{d^2\theta_1^*}{de^2} = \frac{8\left(\overline{A}+\underline{A}\right)}{[-(\Delta\theta)^3\phi']^2}\left[\frac{d\theta_1^*}{de}e\left((\Delta\theta)^3 18k + 2C\right) - \Delta\theta\left(\phi' + 4C\right)\right]$$

Since $\phi' \ge 0$ it is immediate that $\frac{d^2\theta_1^*}{de^2} < 0$. The behavior of advertising prices obtains from their equilibrium definition:

$$\frac{dp_1^A}{de} = \left((\overline{\beta}+\underline{\beta}) + 4e\frac{\Delta\theta - e\frac{d\theta_1^*}{de}}{(\Delta\theta)^2}\right)\sqrt{\overline{A}} > \left((\overline{\beta}+\underline{\beta}) + 4e\frac{\Delta\theta - e\frac{d\theta_1^*}{de}}{(\Delta\theta)^2}\right)\sqrt{\underline{A}} = \frac{dp_2^A}{de} > 0.$$

Ascertaining the behaviour of consumer prices is equally simple:

$$\frac{dp_1^R}{de} = \frac{1}{3}\left[\frac{d\theta_1^*}{de}(2\overline{\beta}-\underline{\beta}) + 2(\underline{A}-2\overline{A})\right] < \frac{1}{3}\left[\frac{d\theta_1^*}{de}(\overline{\beta}-2\underline{\beta}) + 2(2\underline{A}-\overline{A})\right] = \frac{dp_2^R}{de} < 0.$$

7.13 Proof of Proposition 7

When e is large enough platform 1 (the high-quality firm) prefers playing such that $\Delta\theta = \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}} \equiv z(e)$ for any θ_2 (and θ_1 not so large as to induce negative profits). Its payoffs when $\Delta\theta \le z(e)$ are given by the second line of (4.4), where $B_1(e) = 2e(2\overline{\beta} - \underline{\beta})(2\underline{A} - \overline{A})$. This can be re-arranged as

$$\pi_1(e,\theta) = \frac{1}{9}\left[\Delta\theta\left(2\overline{\beta} - \underline{\beta}\right)^2 + 2e[\underline{A}(5\overline{\beta} - 4\underline{\beta}) - \overline{A}(\overline{\beta} + \underline{\beta})]\right] - k\theta_1^2$$

for $\Delta\theta \leq z(e)$ and

$$\pi_1(e,\theta) = \frac{1}{9}\left[\Delta\theta\left(2\overline{\beta} - \underline{\beta}\right)^2 + B_1(e) + \frac{[2e(\overline{A} + \underline{A})]^2}{\Delta\theta}\right] - k\theta_1^2$$

if $\Delta\theta > z(e)$. Let $\overline{\pi}_1(e,\theta) = \max \pi_1(e,\theta)$ for any pair $\theta_1 > \theta_2$ such that $\Delta\theta = z(e)$. This is an upper bound on firm 1's profits for any play by firm 2. Clearly $\overline{\pi}_1(e,\theta)$ is maximised for $\theta_2 = \underline{\theta}$. Recall that we denote the corresponding value of θ_1 by $\tilde{\theta}_1$. For any e and θ_2, $\frac{\partial\pi_1(e,\theta)}{\partial\theta_1} > 0$ when $\Delta\theta < z(e)$ and $\frac{\partial\overline{\pi}_1(e,\theta)}{\partial\theta_1} < 0$ when $\Delta\theta = z(e)$ and $\theta_2 > \underline{\theta}$. Therefore $\overline{\pi}_1(e,\theta)$ reaches zero for some value $\theta_1' \leq \theta_1^c$. Thus no firm will play out of these bounds. More precisely,

$$\begin{aligned} \tfrac{\partial\pi_1(e,\theta)}{\partial\theta_1} &= \tfrac{2\overline{\beta}-\underline{\beta}}{9} - 2k\theta_1 > 0, \quad \text{when } \Delta\theta < z(e) \text{ and} \\ \tfrac{\partial\overline{\pi}_1(e,\theta)}{\partial\theta_1} &= \tfrac{2\overline{\beta}-\underline{\beta}}{9} - 2k\theta_1 < 0, \quad \text{for } \Delta\theta = z(e),\ \theta_2 > \underline{\theta}. \end{aligned}$$

with $\max \frac{\partial\pi_1(e,\theta)}{\partial\theta_1}$ reached for $\theta_2 = \underline{\theta}$. Since $\arg\max\ \Pi_1(\theta_1,\theta_2) > \tilde{\theta}_1$ when $\theta_2 > \underline{\theta}$, it follows that

$$\frac{\partial\pi_1(e,\theta)}{\partial\theta_1} < \left|\frac{\partial\overline{\pi}_1(e,\theta)}{\partial\theta_1}\right|$$

and therefore $\mid \tilde{\theta}_1 - \theta_1^c \mid < z(e)$. ■

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