ALTERNATIVE FORMULATIONS FOR THE INCOME COMPONENT OF HDI

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In this note, we follow the notation of Anand and Sen (1996), "The Income Component of the Human Development Index", Section 3.2.

1. Constant Elasticity Formulations

Let the variable $A(y)$ be a proxy for achievements other than longevity and education. We posit that achievement $A(y)$ is a monotonic increasing and concave function of income y. Thus,

$$A'(y) > 0$$

$$A''(y) < 0 \quad \text{for all } y \geq 0.$$

As noted in Anand and Sen (1996), Section 2.3, the "discounted income" function $W(y)$ of HDRs 1991 - 1997 does not satisfy the second inequality for all $y \geq 0$ and is thus not concave in income. Moreover, the elasticity $\eta(y)$ of the marginal valuation function $W'(y)$ is neither constant nor increasing from 0 to 1 with income y. Indeed the elasticity $\eta(y)$ varies from ∞ to k between the endpoints of the income interval corresponding to the kth multiple of poverty line income y^*, for each integer k. Hence, the elasticity is neither monotonic increasing nor monotonic decreasing for $y \geq 0$ -- rather it jumps up to ∞ at the start of each multiple k of poverty line income y^*, after decreasing from ∞ to the finite number $(k - 1)$ in the $(k - 1)$th income interval.

We would like to propose formulations for the $A(y)$ function which require it to be concave throughout the income range, i.e. for which $A''(y) < 0$ for $y \geq 0$. We begin with the class of constant elasticity marginal valuation (CEMV) functions. By definition, the elasticity $\varepsilon(y)$ of the marginal achievement function $A'(y)$ is defined as

$$\varepsilon(y) = -\frac{d\log A'(y)}{d\log y} = -y\frac{A''(y)}{A'(y)}.$$

If we require $\varepsilon(y)$ to be constant for all y, then

$$-\frac{d\log A'(y)}{d\log y} = \varepsilon(y) \equiv \varepsilon, \text{ say.}$$

Integrating with respect to log y, we have

$$\log A'(y) = -\varepsilon \log y + c, \quad \text{where } c \text{ is a constant}$$

or $$A'(y) = \beta y^{-\varepsilon}, \quad \text{where } \beta \text{ is a positive constant } (\log\beta = c).$$

Now integrating the last equation with respect to y, we get

$$A(y) = \begin{cases} \beta \dfrac{y^{1-\varepsilon}}{1-\varepsilon} + \alpha, \ \mathit{for}\ \varepsilon \neq 1 \\ \\ \beta \log y + \alpha, \ \mathit{for}\ \varepsilon = 1 \end{cases}$$

This is the class of constant elasticity marginal achievement (CEMA) functions *A(y)*. Note that *A(y)* will be strictly concave if and only if $\varepsilon > 0$.

We consider three values of $\varepsilon > 0$ in turn. First, take the case of $\varepsilon = 1$, i.e. the case where the *A(y)* function is logarithmic up to a positive affine transformation. The constants α and β will be determined by specifying the value of the function *A(y)* for two different values of y. In the construction of HDI, the value $A(y_{\min}) = 0$ is assigned to the smallest income level $y_{\min}$, and the value $A(y_{\max}) = 1$ is assigned to the highest income level $y_{\max}$. These boundary conditions jointly determine α and β. Thus we have the two equations:

$$\beta \log y_{\min} + \alpha = 0$$

$$\beta \log y_{\max} + \alpha = 1.$$

Subtracting the first equation from the second gives:

$$\beta(\log y_{\max} - \log y_{\min}) = 1$$

or $$\beta = \frac{1}{(\log y_{\max} - \log y_{\min})}.$$

Substituting for β into the first equation gives:

$$\alpha = \frac{-\log y_{\min}}{(\log y_{\max} - \log y_{\min})}.$$

Hence, we obtain

$$A(y) = \frac{\log y - \log y_{\min}}{\log y_{\max} - \log y_{\min}}$$

as the complete functional form for the third component of HDI for the case of $\varepsilon = 1$.

Next, we consider the case of $\varepsilon = 2$, the value of ε used to calculate the gender-related development index (GDI) in the <u>HDRs</u>. In this case,

$$A(y) = \alpha - \beta y^{-1}.$$

Using the same boundary conditions of $A(y_{\min}) = 0$ and $A(y_{\max}) = 1$, we have two equations for the two unknowns α and β:

$$\alpha - \beta y_{\min}^{-1} = 0$$

$$\alpha - \beta y_{\max}^{-1} = 1.$$

Subtracting the first equation from the second gives:

$$-\beta(y_{\max}^{-1} - y_{\min}^{-1}) = 1$$

or $$\beta = \frac{1}{(y_{\min}^{-1} - y_{\max}^{-1})}.$$

Substituting for β into the first equation gives:

$$\alpha = \frac{y_{\min}^{-1}}{(y_{\min}^{-1} - y_{\max}^{-1})}.$$

Hence, we obtain

$$A(y) = \frac{y_{\min}^{-1} - y^{-1}}{y_{\min}^{-1} - y_{\max}^{-1}}$$

as the complete functional form for the third component of HDI for the case of $\varepsilon = 2$. ✓

Finally, we consider an intermediate value of ε, viz. $\varepsilon = 1.5$. Using the same boundary conditions as before, in a similar manner to the above we obtain

$$A(y) = \frac{y_{\min}^{-0.5} - y^{-0.5}}{y_{\min}^{-0.5} - y_{\max}^{-0.5}}$$

as the complete functional form for the third component of HDI for the case of $\varepsilon = 1.5$.

In practice, the value chosen for $y_{\max}$ in <u>HDR</u> is very large (PPP$40,000) in relation to that for $y_{\min}$ (PPP$100). Hence, $y_{\max}^{-1}$ and $y_{\max}^{-0.5}$ will be much smaller numbers than $y_{\min}^{-1}$ and $y_{\min}^{-0.5}$, respectively. For the case of $\varepsilon = 2$, for example, we can thus approximate *A(y)* as:

$$A(y) = \frac{y_{\min}^{-1} - y^{-1}}{y_{\min}^{-1}}$$

$$= 1 - \frac{y_{\min}}{y}.$$

✓ for y_{max} large.

The equivalent approximation for *A(y)* in the case of $\varepsilon = 1.5$ is:

$$A(y) = 1 - \frac{y_{\min}^{0.5}}{y^{0.5}}.$$

An alternative set of boundary conditions that will generate these forms exactly is:

$$A(y_{\min}) = 0$$

and $A(y) \rightarrow 1$ as $y \rightarrow \infty$.

In this case, *A(y)* becomes arbitrarily close to 1 as *y* gets indefinitely large -- but *A(y)* does not actually equal to 1 for any finite *y*.

Assuming that the first set of boundary conditions $A(y_{\min}) = 0$ and $A(y_{\max}) = 1$ hold, for the cases of $\varepsilon = 1$, $\varepsilon = 2$, and $\varepsilon = 1.5$ we have plotted the $A(y)$ functional form in Figure 1, Figure 2, and Figure 1.5, respectively. These figures also show the achievement levels $A(y)$ for a selection of different (PPP$) y values.

The alternative upper boundary condition that $A(y) \to 1$ as $y \to \infty$ is also used for the cases of $\varepsilon = 2$ and $\varepsilon = 1.5$. (This boundary condition is, of course, not possible to invoke for the case of $\varepsilon = 1$ because $\log y$ is not bounded above as $y \to \infty$.) Figure 2A and Figure 1.5A show the graphs of the $A(y)$ functional form when this alternative upper boundary condition is specified.

Figure 1

$$A(y) = (\log y - \log y_{min}) \;/\; (\log y_{max} - \log y_{min})$$

y_{min} = \$100

y_{max} = \$40,000

y (PPP$)	A(y)
100	0.0000
200	0.1157
400	0.2314
600	0.2991
800	0.3471
1,000	0.3843
2,000	0.5000
4,000	0.6157
6,000	0.6834
8,000	0.7314
10,000	0.7686
20,000	0.8843
40,000	1.0000

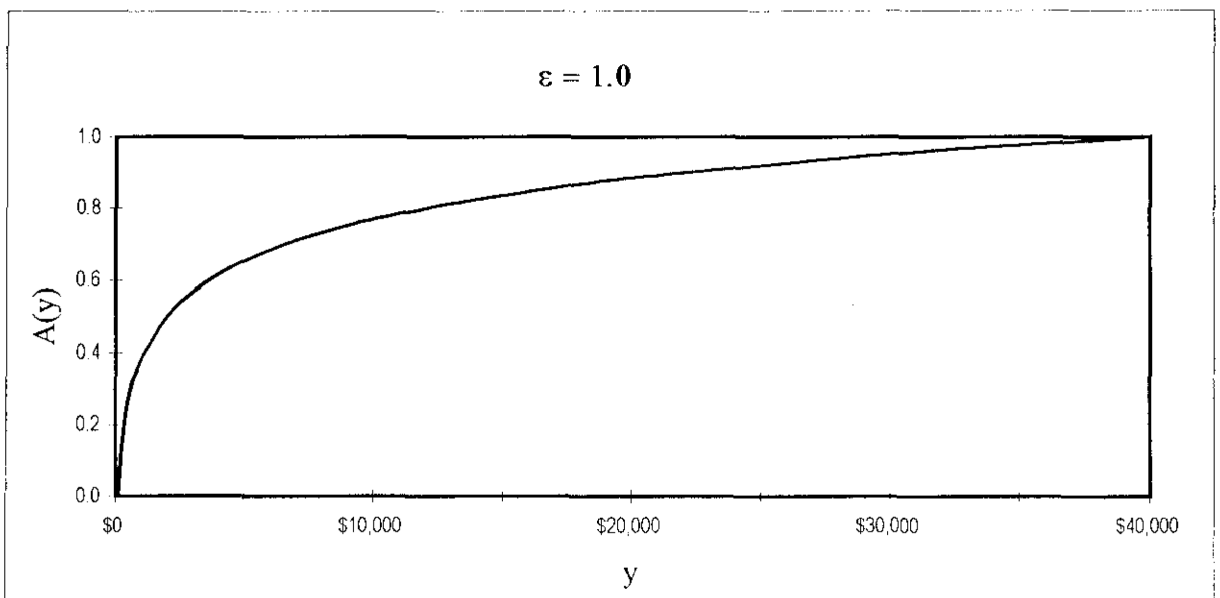


Figure 2

$$A(y) = (y_{min}^{-1} - y^{-1}) / (y_{min}^{-1} - y_{max}^{-1})$$

y_{min} = \$100

y_{max} = \$40,000

y (PPP$)	A(y)
100	0.0000
200	0.5013
400	0.7519
600	0.8354
800	0.8772
1,000	0.9023
2,000	0.9524
4,000	0.9774
6,000	0.9858
8,000	0.9900
10,000	0.9925
20,000	0.9975
40,000	1.0000

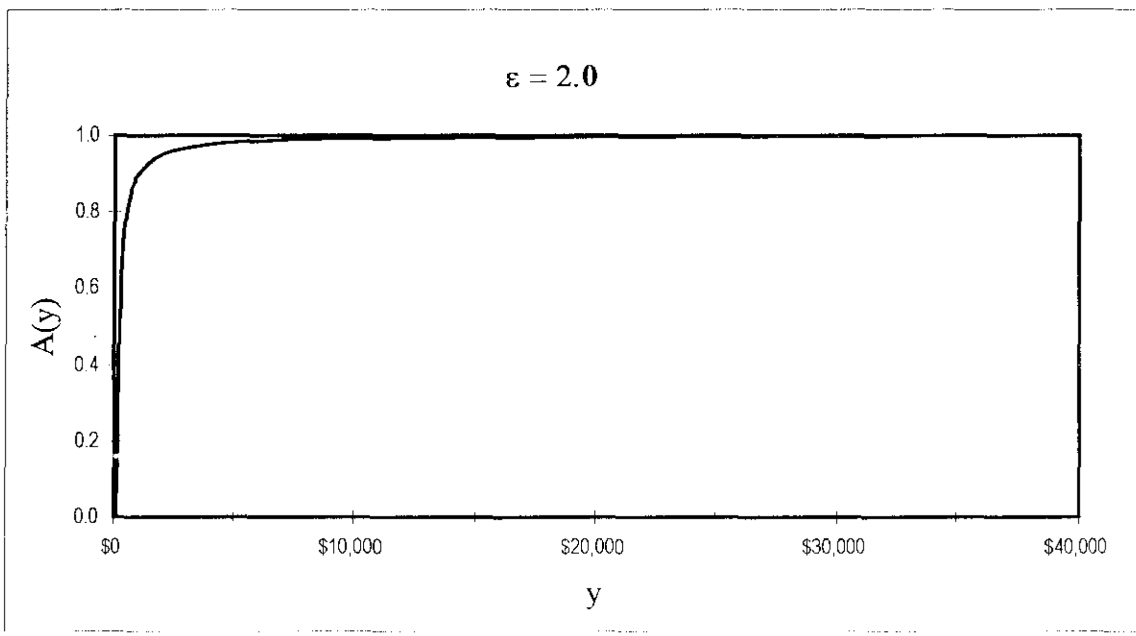


Figure 1.5

$$A(y) = (y_{min}^{-0.5} - y^{-0.5}) / (y_{min}^{-0.5} - y_{max}^{-0.5})$$

y_{min} = \$100

y_{max} = \$40,000

y (PPP$)	A(y)
100	0.0000
200	0.3083
400	0.5263
600	0.6229
800	0.6805
1,000	0.7198
2,000	0.8173
4,000	0.8862
6,000	0.9167
8,000	0.9349
10,000	0.9474
20,000	0.9782
40,000	1.0000

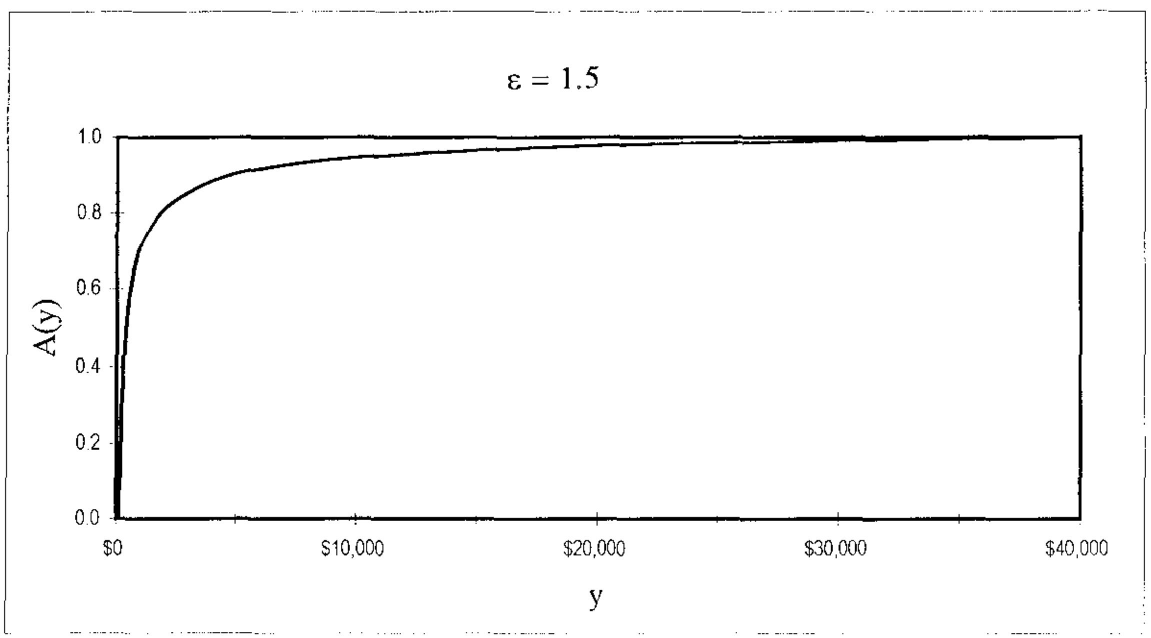


Figure 2A

$$A(y) = 1 - (y_{min} / y)$$

y_{min} = \$100

y_{max} = \$40,000

y (PPP$)	A(y)
100	0.0000
200	0.5000
400	0.7500
600	0.8333
800	0.8750
1,000	0.9000
2,000	0.9500
4,000	0.9750
6,000	0.9833
8,000	0.9875
10,000	0.9900
20,000	0.9950
40,000	0.9975

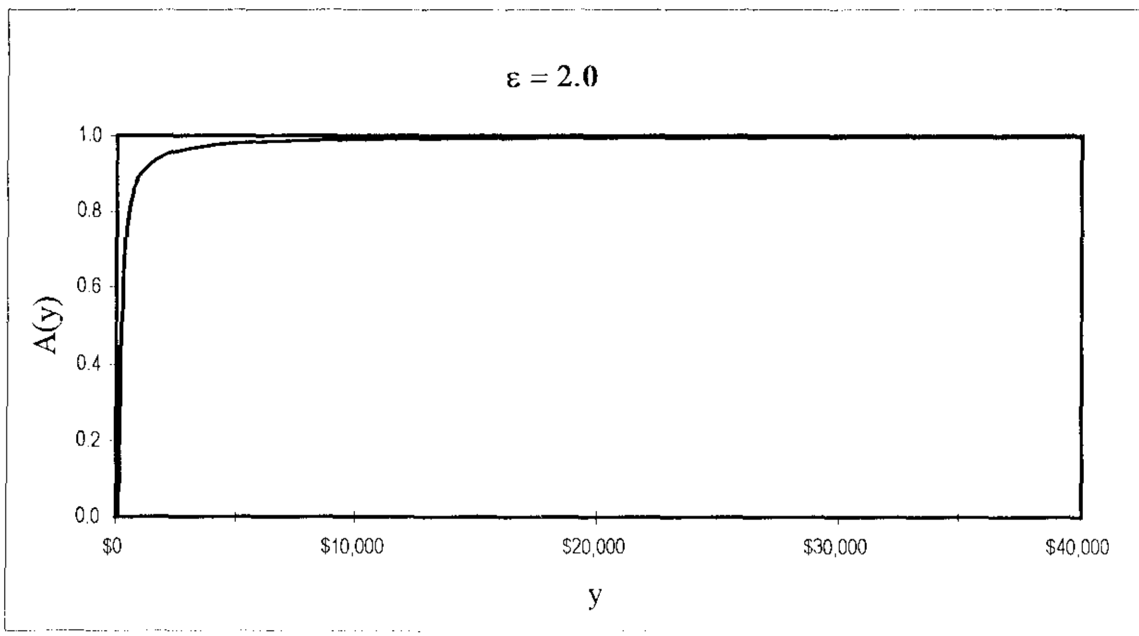


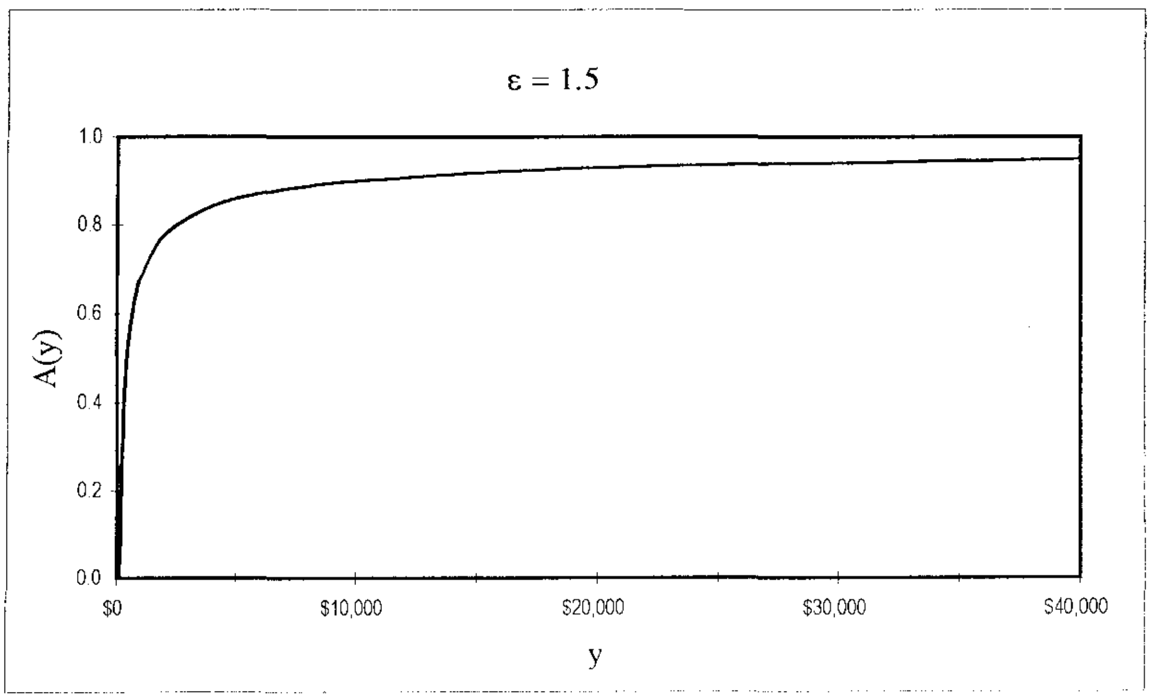
Figure 1.5A

$$A(y) = 1 - (y_{min} / y)^{0.5}$$

y_{min} = \$100

y_{max} = \$40,000

y (PPP$)	A(y)
100	0.0000
200	0.2929
400	0.5000
600	0.5918
800	0.6464
1,000	0.6838
2,000	0.7764
4,000	0.8419
6,000	0.8709
8,000	0.8882
10,000	0.9000
20,000	0.9293
40,000	0.9500



2. A Variable Elasticity Formulation for the Income Component of HDI

In Anand and Sen (1995), "Gender Inequality in Human Development: Theories and Measurement", we consider the class of constant absolute inequality aversion functions. For $A(y)$ to be a member of this class, it must satisfy the following equation for all y:

$$-\frac{A''(y)}{A'(y)} = \gamma\,, \quad \text{where } \gamma \text{ is a constant.}$$

Hence, the elasticity $\varepsilon(y)$ of the marginal achievement function $A'(y)$ is given by:

$$\begin{aligned} \varepsilon(y) &= -\frac{d\log A'(y)}{d\log y} \\ &= -y\frac{A''(y)}{A'(y)} \\ &= \gamma y. \end{aligned}$$

For positive γ, the elasticity $\varepsilon(y)$ thus increases linearly with y, from 0 to ∞.

The form of this variable elasticity function is obtained by integrating twice with respect to y the defining equation

$$\frac{A''(y)}{A'(y)} = -\gamma .$$

Integrating once, we get:

$$\log A'(y) = -\gamma y + c, \quad \text{where } c \text{ is a constant.}$$

Therefore,

$$A'(y) = e^{-\gamma y} \cdot e^{c} .$$

Integrating again with respect to y, we obtain:

$$A(y) = -\frac{1}{\gamma} e^{-\gamma y} \cdot e^{c} + \alpha , \quad \text{where } \alpha \text{ is a constant.}$$

Hence,

$$A(y) = \alpha - \beta e^{-\gamma y} , \quad \text{where } \beta = \frac{e^{c}}{\gamma} \text{ is a constant.}$$

For $A(y)$ to be a strictly concave function, i.e. for $A''(y) < 0$, we require that $\gamma > 0$.

As before we can specify boundary conditions to determine the values of α and β.

Let

$$A(y_{\min}) = 0$$

$$A(y_{\max}) = 1.$$

[An alternative specification of the second boundary condition, as before, would be that $A(y) \to 1$ as $y \to \infty$, or that $A(\infty) = 1$.] Thus we have

$$\alpha - \beta e^{-\gamma y_{\min}} = 0$$
$$\alpha - \beta e^{-\gamma y_{\max}} = 1.$$

These equations imply that

$$\beta = \frac{1}{(e^{-\gamma y_{\min}} - e^{-\gamma y_{\max}})}$$

and $$\alpha = \frac{e^{-\gamma y_{\min}}}{e^{-\gamma y_{\min}} - e^{-\gamma y_{\max}}}.$$

Hence,

$$A(y) = \frac{e^{-\gamma y_{\min}} - e^{-\gamma y}}{e^{-\gamma y_{\min}} - e^{-\gamma y_{\max}}}.$$

If the alternative specification of the second boundary condition, viz. $A(\infty) = 1$, is adopted, then $\alpha = 1$ and $\beta = e^{\gamma y_{\min}}$. In this case, the third component of HDI will be

$$A(y) = 1 - e^{-\gamma (y - y_{\min})}.$$

We now compute the values of $A(y)$ for alternative values of real GDP per capita y. Two separate exercises are reported below for two values of the parameter γ, viz. $\gamma = 0.00025$ and $\gamma = 0.0005$. The $A(y)$ values for selected y values, and plots of the $A(y)$ functional form, are shown in Figure VE1 and Figure VE2.

Figure VE1

$$A(y) = [\exp(-\gamma y_{min}) - \exp(-\gamma y)] / [\exp(-\gamma y_{min}) - \exp(-\gamma y_{max})]$$

y_{min} = \$100

y_{max} = \$40,000

y (PPP\$)	A(y)	
	$\gamma = 0.00025$	$\gamma = 0.0005$
100	0.0000	0.0000
200	0.0247	0.0488
400	0.0723	0.1393
600	0.1175	0.2212
800	0.1606	0.2953
1,000	0.2015	0.3624
2,000	0.3781	0.6133
4,000	0.6228	0.8577
6,000	0.7713	0.9477
8,000	0.8613	0.9807
10,000	0.9159	0.9929
20,000	0.9931	1.0000
40,000	1.0000	1.0000

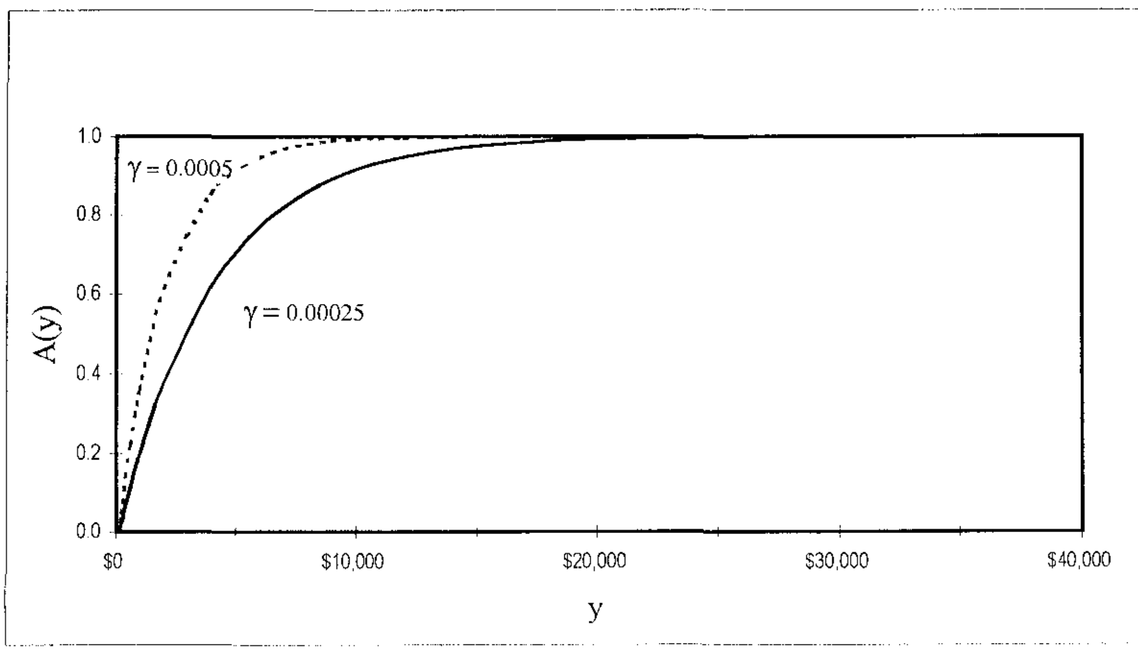


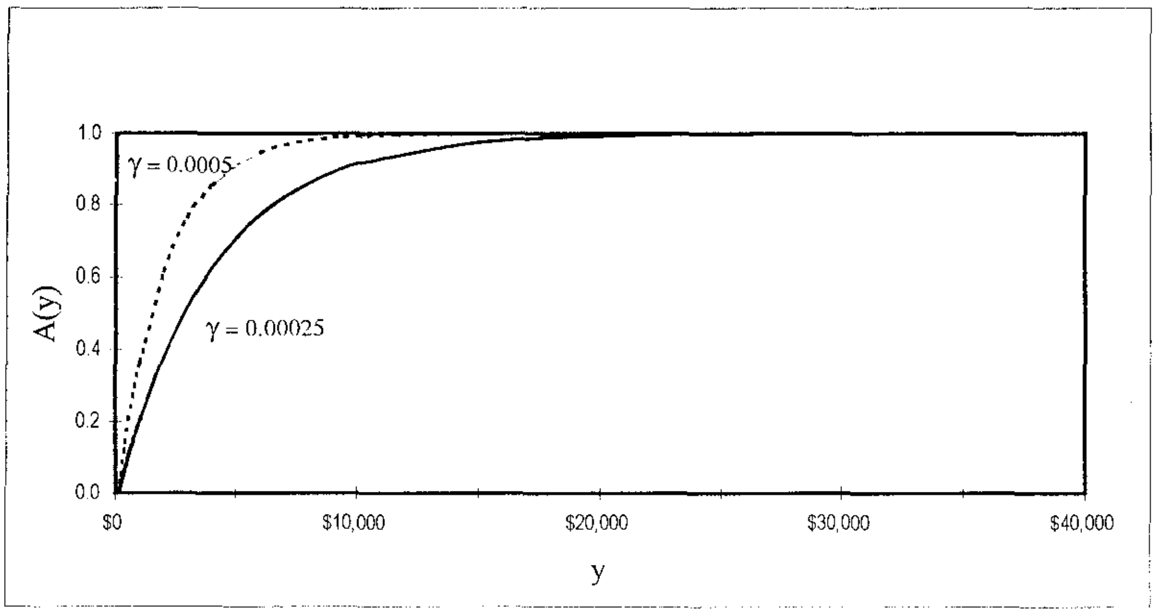
Figure VE2

$$A(y) = 1 - \exp[-\gamma(y - y_{min})]$$

y_{min} = $100

y_{max} = $40,000

y (PPP$)	A(y)	
	γ = 0.00025	γ = 0.0005
100	0.0000	0.0000
200	0.0247	0.0488
400	0.0723	0.1393
600	0.1175	0.2212
800	0.1605	0.2953
1,000	0.2015	0.3624
2,000	0.3781	0.6133
4,000	0.6228	0.8577
6,000	0.7712	0.9477
8,000	0.8612	0.9807
10,000	0.9158	0.9929
20,000	0.9931	1.0000
40,000	1.0000	1.0000



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Dear Sakiko,

I enclose the first of our four notes dealing with the most urgent - but not necessarily the most important - agenda item, to wit, what to do with the mess of the income component of HDR. The note "Alternative Formulations for the Income Component of HDI" deals both with "constant elasticity formulations" (pages 1-12) and "variable elasticity formulations" (pages 13-17). I had half promised to Selim to send this by the end of the week that has just ended, and while I have missed that deadline, I shall send it on Monday by Fed Ex so that you and Selim should get it by Tuesday. (Meanwhile I send this letter by fax to keep you and Selim informed of what's happening.)

Sudhir is currently away (mostly in China), and I shall go to Berlin this Wednesday. I shall be back by Sunday the 12th night and you can catch me the following week. I shall be in touch with Sudhir by phone; he will return here on 26 October. We shall be working together in England during the week of October 20 when I have to go to a meeting there (just after our New York meeting).

Regarding the different formulations, I should be inclined to go for the simplicity of a "constant elasticity" formulation. The case of "variable elasticity" (pages 13-17) is of great theoretical interest, particularly for keeping the "absolute inequality aversion" constant, but it is perhaps a little harder to grasp and may be somewhat more difficult to explain to the lay readers of the HDR (to which we shall have to give attention as the report nears the presentational phase).

Regarding value of ϵ, we could (Sudhir and I think) sensibly use $\epsilon = 2$, which we have already used (and defended) in the context of the GDI. The formulae are spelt out on pages 4-5. However, we should get the numerical tables for other values as well. For the "runs" I would suggest that we do the values for $\epsilon = 1$ (see pages 3-4 for the formulae) and for $\epsilon = 1.5$ (see pages 5-6 for the formulae), in addition to $\epsilon = 2$.

We can't go below $\epsilon = 0$ for reasons of strict concavity, and $\epsilon = 1$ is a good famous case (that of the log of income). Going above $\epsilon = 2$ may be useful to further flatten out the rewards for high income (without making the additional reward to be nil - which was Robert Summers' sensible objection to the earlier "cut off" approach). But I don't think it will be needed to go that far, given the flattening that $\epsilon = 2$ already produces (see Figure 2 on page 9 vis-a-vis Figure 1 on page 8).

Also, in general what a difference there would be with the use of a simple and uniform concave function, rather than the ad hoc step-wise approach with great complexity as well as violation of concavity at crucial points. As I see the treatment of HDR from the second report to the last (1991 to 1997), I cannot but cringe somewhat in this respect.

Please remind Selim that he had promised to send me some notes on "consumption" - any thing you can send will be of great use for us also. Phase 2 (consumption) seems like the most challenging part of our four-some task.

Regards,

Yours,

Amartya Sen