What Model for Entry in First-Price Auctions? A Nonparametric Approach *

Vadim Marmer
University of British Columbia

Artyom Shneyerov
Concordia University, CIREQ, and CIRANO

Pai Xu
University of Hong Kong

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Abstract

We develop a nonparametric approach that allows for discrimination among alternative models of entry in first-price auctions. Three models of entry are considered: those of Levin and Smith (1994), Samuelson (1985), and a new model in which the information received at the entry stage is imperfectly correlated with bidder valuations. We derive testable restrictions of these models based on how the pro-competitive *selection effect* shifts bidder valuation quantiles in response to an increase in the number of potential bidders.

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1 Introduction

Entry in auctions is an important topic. It is clear that equilibrium allocations and efficiency are affected by who enters and why. At present, however, there is no consensus on what model of entry best describes what occurs in most markets. Recent empirical papers,

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recognizing that it is necessary to endogenize entry in modeling auctions, have offered structural models. Since one attractive feature of the auction literature is that nonparametric identification can be achieved under certain assumptions, it would be ideal if models of entry could similarly be distinguished nonparametrically. The main contribution of this paper is to offer a more general model of entry that nests those proposed in earlier theoretical work; and then to offer nonparametric tests of these models. To the best of our knowledge, this is the first paper to adopt a nonparametric approach first developed in Laffont and Vuong (1996) and Guerre, Perrigne, and Vuong (2000, GPV hereafter) to endogenous entry in first-price auctions.

We consider two models of entry proposed in the theoretical literature, Levin and Smith (1994, LS hereafter) and Samuelson (1985, S hereafter), as well as a new model, the *affiliated model of entry* (AME).Most of the empirical auction literature is based on the LS model. In that model, potential bidders are initially uninformed about of their valuations of the good, but may become informed and then submit a bid at a cost. In equilibrium, potential entrants randomize their entry decisions and earn zero expected profit.

In the S model, bidders make their entry decisions *after* they have learned their valuations. The entry cost is interpreted solely as the cost of preparing a bid, and bidders choose to enter if their valuations exceed a certain cutoff. The set of entrants is therefore a select sample biased towards bidders with higher valuations.

Both the LS and S models are stylized to capture the amount of information available to bidders at the entry stage: no information is available in LS, whereas the information is perfect in S. However, the S model also has what we call the *selection effect*, which has not been studied previously. We show that as the number of potential bidders increases, those who enter, tend to have larger valuations.

Our new AME model also allows for selective entry, but dispenses with the stark assumption that potential bidders know their valuations *perfectly* at the entry stage, as in S, thus sharing with the LS model a costly valuation discovery stage. This model formally nests the LS model. At the entry stage, the potential bidders each observe a private signal correlated with their as yet unknown valuation of the good. Based on this private signal, a bidder may learn the valuation upon incurring entry cost k. A bidder who has entered will bid only if the valuation exceeds the reserve price. Although the signals may be informative about the valuations, unlike in the S model, they are not perfectly informative. Both the LS and S models can be viewed as limit cases of our model: the LS model corresponds to uninformative signals, whereas the S model corresponds to perfectly informative signals.

How can one test for the presence of the selection effect in first-price auctions with endogenous entry? We build on the insight in Haile, Hong, and Shum (2003) and propose the use of exogenous variation in the number of potential bidders, N, as the basis for such a test.[1] Let $Q^*(\tau|N)$ be the τ-th quantile of *active* bidders' valuations conditional on N.

[1] Haile, Hong, and Shum (2003) consider a different model in which bidders' valuations may have a common component.

We show that, in the AME model, the selection effect manifests itself as the effect of N on $Q^*(\tau|N)$: in the face of greater potential competition, some potential entrants who may be less efficient in the auction will choose not to enter, and accordingly, the quantiles of those who do enter increase: $Q^*(\tau|N) \leq Q^*(\tau|N')$ for $N' > N$. We show that the inequality is strict in the S model, while $Q^*(\tau|N)$ does not depend on N in the LS model, so that there is *no* selection effect. Following the approach of GPV, we show that the quantiles, $Q^*(\tau|N)$, can be nonparametrically identified in any AME model if the number of potential bidders and all bids in each auction are observed. Our tests are based on an asymptotically normal estimator of $Q^*(\tau|N)$ that we develop by following an approach similar to Marmer and Shneyerov (forthcoming).

Models that are in some respects similar to our AME have been considered in the literature. For example, Hendricks, Pinkse, and Porter (2003) estimate a bidding model for off-shore oil. They sketch a model of entry that is in some respects similar to ours, however, with a common-value component. The focus of their paper, however, is not on entry but rather on testing an equilibrium model of bidding. This model is also outlined in the concluding section of Ye (2007).

A closely related paper is Li and Zheng (2009), who develop Bayesian semiparametric approach for the LS and S models.[2] They focus on estimation, and address neither non-parametric identification nor nonparametric testing of the entry models. In this regard, our results are complementary.

An important contribution of Li and Zheng (2009) is that they identify the *entry effect*. This effect exists in addition to the usual competition effect: as the number of potential bidders increases, the probability of entry falls, which may lead to a reduction in the auction price. Using data on lawn mowing contracts, they estimate a variant of the LS model, and find that, under certain conditions, it may be optimal to restrict the competition. In fact, their counterfactual experiment shows that, for a typical auction, it is not optimal to allow more than three bidders to participate.

The entry effect is counter-competitive and therefore operates in the direction opposite to the pro-competitive selection effect. Which of the two is dominant depends on the application. We illustrate our approach using the same data as in Li and Zheng (2009). For a certain subset of auctions, our nonparametric test favors the new AME we propose. We then find that, in this application, the selection effect can overturn the entry effect.

The selection effect identified in this paper has been subsequently studied in Einav and Esponda (2008), Roberts and Sweeting (2010b,a), and Coviello and Mariniello (2010). Einav and Esponda (2008) and Roberts and Sweeting (2010a) investigate the effects of bidder asymmetries; the latter paper considers second-price and open outcry auctions. Roberts and Sweeting (2010b) study sequential sales mechanisms in an environment with selective entry. Coviello and Mariniello (2010) explore how information revelation interacts with selective entry in Italian public procurements.

[2] Li (2005) develops a general parametric approach for auctions with entry.

Endogenous entry in auctions has been a subject of a number of recent papers. Bajari and Hortacsu (2003), for example, have investigated entry and bidding in eBay auctions using a common value framework. They implemented a Bayesian estimation method employing a dataset of mint and proof sets of US coins. The magnitude of the entry cost is estimated, and expected seller revenues are simulated under different reserve prices. Athey, Levin, and Seira (2011) estimate a model of timber auctions with costly entry. The entry cost is assumed to be the private information of potential bidders, who are selected into an entrant pool based on their entry cost draws. Krasnokutskaya and Seim (forthcoming) explore bid preference programs and bidder participation using California data. The latter paper also adopts the LS model, but in an asymmetric context in which small firms are treated preferentially. Bajari, Hong, and Ryan (2010) propose a parametric likelihood-based estimation strategy in the presence of multiple equilibria, and apply it to highway procurement auctions, using the LS model.

The paper proceeds as follows. Section 2 presents the AME and discusses the selection effect. Section 3 contains results on nonparametric identification. Our testing framework is presented in Section 4. Section 5 contains an empirical application, and Section 6 concludes.

2 Affiliated model of entry

In our model, N risk-neutral potential bidders in a first-price auction observe (at no cost) a signal that is correlated with the true value of the object, and make costly entry decisions based on this signals. The model nests as special cases the LS model and the S model, as noted in the introduction. The LS model arises when the signals and valuations are statistically independent. The S model arises when the signals are perfectly informative about valuations.

The affiliated model is formally described as follows. The game begins with the entry stage at which nature draws signal and valuation tuples (V_i, S_i) independently from the same distribution with joint cumulative distribution function (CDF) $F(v, s)$ and probability density function (PDF) $f(v, s)$ with support $[\underline{v}, \bar{v}]^2$. We assume that the marginal distributions of V_i and S_i are the same, and the marginal CDF is denoted as $F(\cdot)$.[3] At the entry stage, potential bidders privately observe their S_i's at no cost, but they do not observe V_i. The latter become observable to those potential bidders who have paid an entry cost $k \geq 0$. The bidders who have paid this cost are called *entrants*, and only they are eligible to bid. Moreover, only those with valuations at or above the reserve price r actually submit a bid. These bidders are called *active*. The reserve price is assumed to be binding, $r \in (\underline{v}, \bar{v})$.[4]

[3] This is a normalization and is without loss of generality. Any strictly increasing transformation of the signals is observationally equivalent. We adopt this normalization because, in the S model, the signals are perfectly informative.

[4] This case is relevant in many applications. In practice, sellers frequently use reserve prices.

The entry stage is followed by the bidding stage. Active bidders simultaneously and independently submit sealed bids. They do not know the number of active bidders, only the number of potential bidders, N. The good is awarded to the highest bidder. We assume that the signals are informative and that higher signals are "good news". Formally, we assume affiliation.[5]

Assumption 1 (Affiliation)**.** *For each bidder i, the variables (V_i, S_i) are affiliated: for any $z = (v, s)$ and $z' = (v', s')$,*

$$f\left(\max\left\{z, z'\right\}\right) f\left(\min\left\{z, z'\right\}\right) \geq f\left(z\right) f\left(z'\right). \tag{1}$$

Both the LS and S models can be viewed as limit cases of the AME. The former is formally nested since it corresponds to signals being independent of the valuations; this would effectively purify the mixed-strategy equilibrium. The latter corresponds to the other extreme, namely, perfectly informative signals.

We assume that

$$N \in \mathcal{N} \equiv \left\{\underline{N}, \underline{N}+1, ..., \bar{N}\right\},$$

where $2 \leq \underline{N} < \bar{N}$. With these preliminaries, a symmetric (perfect Bayesian) AME equilibrium is characterized in the following proposition.

Proposition 1. *A unique symmetric AME equilibrium is characterized by a signal cutoff $\bar{s}\left(N\right)$ such that only bidders with $S_i \geq \bar{s}\left(N\right)$ enter the auction, and a strictly increasing, continuously differentiable bidding strategy $B\left(\cdot|N\right) : [r, \bar{v}] \to \left[r, \bar{b}\left(N\right)\right]$ given by the solution of the differential equation*

$$B'\left(v|N\right) = \left(v - B\left(v|N\right)\right) \frac{\left(N-1\right)\left(1 - F\left(\bar{s}\left(N\right)\right)\right) f\left(v|S_i \geq \bar{s}\left(N\right)\right)}{F\left(\bar{s}\left(N\right)\right) + \left(1 - F\left(\bar{s}\left(N\right)\right)\right) F\left(v|S_i \geq \bar{s}\left(N\right)\right)}, \tag{2}$$

subject to the boundary condition $B\left(r|N\right) = r$. The expected profit from bidding $\Pi\left(s, \bar{s}, N\right)$ as a function of the observed signal $S_i = s$ (gross of the entry cost) is given by

$$\Pi\left(s, \bar{s}, N\right) = \int_r^{\bar{v}} \left(1 - F\left(v|s\right)\right)\left[F\left(\bar{s}\right) + \left(1 - F\left(\bar{s}\right)\right) F\left(v|S_i \geq \bar{s}\right)\right]^{N-1} dv. \tag{3}$$

The function $\Pi\left(s, \bar{s}, N\right)$ is non-decreasing in s and increasing in $\bar{s}$. Also, $\Pi\left(s, s, N\right)$ is continuous and strictly increasing in s, and we further have the following:

(a) *If $\Pi\left(\bar{v}, \bar{v}, N\right) = \int_r^{\bar{v}} \left(1 - F\left(v|s\right)\right) dv < k$, then there is no entry: $\bar{s}\left(N\right) = \bar{v}$.*

(b) *If $k \leq \Pi\left(\underline{v}, \underline{v}, N\right)$, then there is entry with probability 1, or "full" entry: $\bar{s}\left(N\right) = \underline{v}$.*

[5] This definition follows Milgrom and Weber (1982).

(c) *If* $\Pi(\underline{v}, \underline{v}, N) < k < \Pi(\bar{v}, \bar{v}, N)$*, then there is "partial" entry:* $\bar{s}(N) \in (\underline{v}, \bar{v})$ *is determined as the unique* $\bar{s}$ *that solves*

$$k = \Pi(\bar{s}, \bar{s}, N). \tag{4}$$

(d) *The cutoff* $\bar{s}(N)$ *is constant on* $\{N \in \mathcal{N} : N \leq N_*\}$*, and is strictly increasing on* $\{N \in \mathcal{N} : N \geq N_*\}$*, where*

$$N_* = \max\{N \in \mathcal{N} : \Pi(\underline{v}, \underline{v}, N) \geq k\} \tag{5}$$

and $N_* = \underline{N}$ *if* $\Pi(\underline{v}, \underline{v}, \underline{N}) \leq k$*.*

Because the tuples (V_i, S_i) are independently drawn across bidders, the signals are irrelevant at the bidding stage and they do not appear as arguments in the bidding strategy. They are only relevant for the entry decisions: the equilibrium probability of entry is equal to $1 - F(\bar{s}(N))$, and the distribution of the number of active bidders is binomial, with parameters N and $1 - F(\bar{s}(N))$. The differential equation (2) follows from the equilibrium characterization of an auction with a random number of bidders in Harstad, Kagel, and Levin (1990), whereas that of the equilibrium cutoff $\bar{s}$ follows arguments similar to those in LS.

In the S model, the valuations and signals are perfectly correlated, $S_i = V_i$, and thus a joint density $f(v, s)$ does not exist. Nevertheless, this model can be thought of as a limit case of the affiliated model. In particular,

$$F(v|s) = \begin{cases} 1 & , \quad v \geq s \\ 0 & , \quad v < s \end{cases} , \quad F(v|S_i \geq s) = \begin{cases} \frac{F(v) - F(s)}{1 - F(s)} & , \quad v \geq s \\ 0 & , \quad v < s \end{cases} . \tag{6}$$

Substituting (6) into (3) yields

$$\Pi(s, \bar{s}, N) = (\bar{s} - r) F(\bar{s})^{N-1} + \int_{\bar{s}}^{s} F(v)^{N-1} dv \quad \text{for } s \geq \bar{s}. \tag{7}$$

In an equilibrium with entry, $\bar{s}(N)$ is determined as the unique solution of the equation

$$(\bar{s} - r) F(\bar{s})^{N-1} = k. \tag{8}$$

Its equilibrium is formally characterized in the following proposition, which in many respects parallels Proposition 1.

Proposition 2 (Samuelson (1985))**.** *A unique symmetric equilibrium of the S model is characterized by a signal cutoff* $\bar{s}(N)$*, such that only bidders with* $S_i \geq \bar{s}(N)$ *(or* $V_i \geq \bar{s}(N)$*) enter the auction, and a strictly increasing, continuously differentiable bidding strategy* $B(\cdot|N) : [\bar{s}(N), \bar{v}] \to [r, \bar{b}(N)]$ *given by the solution of the differential equation (2) subject to the boundary condition* $B[\bar{s}(N)|N] = r$*. The expected profit from bidding* $\Pi(s, \bar{s}, N)$ *as a function of the observed signal* $S_i = s$ *(gross of the entry cost) is given by (7). The function* $\Pi(s, \bar{s}, N)$ *is increasing in both* s *and* $\bar{s}$*. We further have:*

(a) *If* $\Pi(\bar{v}, \bar{v}, N) < k$*, then there is no entry:* $\bar{s}(N) = \bar{v}$*;*

(b) *otherwise, there is "partial" entry:* $\bar{s}(N) \in (\underline{v}, \bar{v})$ *is determined as the unique solution of equation (8).*

(c) *The cutoff* $\bar{s}(N)$ *is strictly increasing in* N*.*[6]

The monotonicity property of $\bar{s}(N)$ in Propositions 1 and 2 is the key property for our nonparametric testing approach. To gain intuition, note that as N increases, it becomes more difficult to win the auction; thus, for a given realization of the signal s, the expected equilibrium profit will decrease. Since the expected profit $\Pi(s, \bar{s})$ is increasing in s, in equilibrium bidders will increase their cutoffs to ensure that they cover their entry costs.

2.1 Selection effect

Because the signal cutoffs $\bar{s}(N)$ are increasing in N for $N \geq N_*$, the affiliation property implies that the distribution of valuations conditional on bidding,

$$F^*(v|N) = F(v|S_i \geq \bar{s}(N), V_i \geq r), \tag{9}$$

is non-increasing in N. The active bidders' valuation quantiles,

$$Q^*(\tau|N) = F^{*-1}(\tau|N) \equiv \inf\{v : F^*(v|N) \geq \tau\} \qquad \tau \in (0,1),$$

are therefore nondecreasing in N:

$$Q^*(\tau|\underline{N}) \leq \ldots \leq Q^*\left(\tau|\bar{N}\right). \tag{10}$$

Moreover, the *strict affiliation* of values and signals implies that, unless there is full entry for all $N \in \mathcal{N}$, the quantiles $Q^*(\tau|N)$ are *strictly* increasing in N for $N \geq N_*$.[7] In other words, as N increases, the active bidders' valuation quantiles also increase.[8] This is the essence of the pro-competitive *selection effect* $\Delta(\tau, N, N')$ in the AME under strict affiliation:

$$\Delta\left(\tau, N, N'\right) \equiv Q\left(\tau|N'\right) - Q(\tau|N) > 0 \qquad \text{for } N' > N. \tag{11}$$

In the LS model, the population of entrants has the same distribution of valuations as does the population of potential bidders, i.e.,

$$F^*(v|N) = \frac{F(v) - F(r)}{1 - F(r)} \qquad \text{for all } N \in \mathcal{N},$$

[6]Equation (8) implies that $\bar{s} \in (r + k, \bar{v})$. The bounds are sharp, as can be seen by the following example. Let $F(s) = [(s - \underline{v}) / (\bar{v} - \underline{v})]^\alpha$ with $\alpha > 0$. It can be seen that as $\alpha \to 0$, $\bar{s} \to r + k$, and as $\alpha \to \infty$, $\bar{s} \to \bar{v}$.

[7]Strict affiliation requires (1) to hold as a strict inequality, provided $v > v'$ and $s > s'$. See e.g. Avery (1998) for this definition of strict affiliation.

[8]As the S model never has full entry, the selection effect is always present in that model.

and there is *no* selection effect:

$$Q^*(\tau|\underline{N}) = ... = Q^*\left(\tau|\bar{N}\right). \tag{12}$$

The selection effect forms the basis of the nonparametric test that we develop in Section 4.2, where we also derive a testable restriction of the S model.

3 Nonparametric identification

We now show that the valuation quantiles $Q^*(\tau|N)$, which are necessary for the identification of the selection effect, are nonparametrically identifiable if the econometrician can observe the reserve price r and all bids in each auction.[9] The number of potential bidders N is also assumed to be observable. Both the probability of submitting a bid,

$$p(N) = P(S_i \geq \bar{s}(N), V_i \geq r), \tag{13}$$

and the distribution of the active bidders' bids,

$$\begin{aligned} G^*(b|N) &= F(\xi(b|N)|S_i \geq \bar{s}(N), V_i \geq r) \\ &= F^*(\xi(b|N)|N), \end{aligned} \tag{14}$$

are identifiable, where $\xi(\cdot|N) : \left[r, \bar{b}(N)\right] \to [v^*(N), \bar{v}]$ is the inverse bidding strategy, which is defined as the unique solution to the equation $B(\xi|N) = b$. To unify the notation for the AME and S models, let

$$v^*(N) \equiv \begin{cases} r, & \text{in AME model} \\ \bar{s}(N), & \text{in S model} \end{cases}$$

denote the lowest valuation of those bidders who enter. Thus, the support of $F^*(v|N)$ is $[v^*(N), \bar{v}]$. The probability density of $G^*(\cdot|N)$ is denoted as $g^*(\cdot|N)$. Our identification strategy follows GPV.

Proposition 3. *In both models, the inverse bidding strategy is identifiable for all $b \in \left[r, \bar{b}(N)\right]$ as*

$$\xi(b|N) = b + \frac{1}{N-1}\left(\frac{G^*(b|N)}{g^*(b|N)} + \frac{1-p(N)}{p(N)}\frac{1}{g^*(b|N)}\right), \tag{15}$$

and $B(v|N)$ is identifiable on $[v^*(N), \bar{v}]$. The quantiles of active bidders' valuations (and therefore the selection effect $\Delta(\tau, N', N)$) are also identifiable according to

$$Q^*(\tau|N) = \xi(q(\tau|N)|N),$$

where $q(\tau|N) \equiv G^{*-1}(b|N)$ are the quantiles of bids.

[9] See, however, Remark 2 below.

Remark 1. Proposition 3 implies that the distribution of active bidders' valuations is identifiable in both models as $F^* (v|N) = G^* \left(\xi^{-1} (v|N) |N\right)$, with support

$$\left[\xi (r|N), \xi \left(\bar{b} (N) |N\right)\right]. \tag{16}$$

The intuition for this result is as follows. Consider the first-order equilibrium conditions of the bidding game. A bidder with value v who submits a bid b has a probability of winning over a given rival equal to $1 - p (N) + p (N) G^* (b|N)$. As there are $N - 1$ identical rivals, it follows by independence that the probability of winning is $(1 - p (N) + p (N) G^* (b|N))^{N-1}$, and the expected profit is

$$(v - b) (1 - p (N) + p (N) G^* (b|N))^{N-1}.$$

Writing out the first-order condition, i.e. taking the derivative of the expected profit with respect to b and setting it equal to 0, gives the inverse bidding strategy (15).

Remark 2 (Identification using only winning bids)**.** The assumption that all bids are observable is not necessary. If only the winning bid is observable, then the distribution $G^* (b|N)$ can be recovered from the distribution of the winning bid $G^*_Y (b|N)$ according to the formula

$$G^*_Y (b|N) = \frac{(1 - p (N) + p (N) G^* (b|N))^N - (1 - p (N))^N}{1 - (1 - p (N))^N}.$$

Therefore, the inverse bidding strategy $\xi(b|N)$ as well as the quantiles of active bidders valuations $Q^*(\tau|N)$ are identifiable as in Proposition 3.

Note that (15) is parallel to equation (25) in GPV, who consider the identification in a standard first-price auction model ($k = 0$) with a binding reserve price. In the standard model, a failure to bid is attributed solely to the truncating effect of the reserve price, and $p (N) = 1 - F (r)$. If $k > 0$, then there is an additional channel of costly entry, and $p (N)$ is given by (13). As $\bar{s} (N)$ is nondecreasing in N,

$$p (N) \leq p \left(N'\right), \quad \text{for } N > N'.$$

This is another restriction that is shared by all models in this paper and will be tested empirically.

3.1 Additional identification results

The focus of our paper is on the identification of the selection effect using variation in N. However, in view of the identification results in GPV for the standard model ($k = 0$) obtained for N *fixed*, it is also worth investigating the question what is identified under this assumption in models with endogenous entry. The following proposition provides an answer to this question.

Proposition 4. *Consider absolutely continuous distribution $G^*(\cdot|N)$ with support $\left[r, \bar{b}(N)\right]$, and let $\rho(\cdot|N)$ be a discrete distribution.*

(a) *There exists an AME or S model that rationalizes the data, i.e. such that the induced equilibrium distribution of bids is $G^*(\cdot|N)$ and the distribution of the number of active bidders n is $\rho(\cdot|N)$, if and only if: (i) $\rho(\cdot|N)$ is Binomial with parameters $(N, p(N))$, (ii) the observed bids are i.i.d. according to $G^*(\cdot|N)$, and (iii) the function $\xi(\cdot|N)$ defined in (15) is strictly increasing on $\left[r, \bar{b}(N)\right]$ and its inverse is differentiable on $\left[\xi(r|N), \xi\left(\bar{b}(N)|N\right)\right]$.*

Furthermore, when conditions (i)-(iii) in (a) hold,

(b) *An AME rationalizes the data if and only if $\xi(r|N) = r$, which in turn holds if and only if $\lim_{b \downarrow r} g^*(b|N) = +\infty$.*[10] *Any AME is observationally equivalent to the standard model ($k = 0$); therefore $F(v) = F^*(v|N)$ is identifiable for $v \geq r$, while the entry cost is not identifiable.*

(c) *An S model rationalizes the data if and only if $\xi(r|N) > r$. The distribution $F(v)$ is identifiable according to*

$$F(v) = p(N) F^*(v|N) + 1 - p(N)$$

on $\left[\xi(r|N), \xi\left(\bar{b}(N)|N\right)\right]$, and k is identifiable according to (8):

$$k = (\xi(r|N) - r)(1 - p(N))^{N-1}.$$

Remark 3 (Entry Cost Bias)**.** It is clear that model misspecification will lead to a biased estimate of the entry cost. The direction of this bias can also be determined. Suppose that k is identifiable and is estimated according to LS.[11] However, the data are generated according to the AME under strict affiliation. Then k is estimated using the wrong (LS) factor $1 - F^*(v)$ instead of the correct (AME) factor $1 - F(v|\bar{s}(N))$ in (3) and (4). Since strict affiliation implies $1 - F^*(v) > 1 - F(v|s)$, the true k is smaller than the estimated one. The intuition for the existence of this bias is straightforward. The entry cost in the LS model is equal to the equilibrium expected profit of the average potential bidder, while it is equal to the expected profit of the marginal potential bidder (with signal $\bar{s}(N)$) in the AME. Because the signals and valuations are strictly affiliated, the marginal bidder has a smaller expected profit than the average bidder, and this leads to over estimation of the entry cost.

The following proposition shows that the variation in N enhances the identification of the S and LS models. In the S model, the set of v for which $F(v)$ is identified is

[10] This mirrors the result in Theorem 4(C2) in GPV.

[11] Here, we abstract from the sampling issues.

$[\bar{s}(\underline{N}), \bar{v}] \supset [\bar{s}(N), \bar{v}]$ for all $N > \underline{N}$. In the LS model, $F(v)$ is still identified if and only if $v \in [r, \bar{v}]$. However, now the entry cost is identifiable if $\mathcal{N}$ has at least three elements and $N_* < \bar{N}$. This is because $N_* < \bar{N}$ and the cutoff monotonicity property in Proposition 1 implies

$$p(\underline{N}) = ... = p(N_*) > ... > p(\bar{N}) \tag{17}$$

When N belongs to the flat segment, $N \leq N_*$, we are certain that bidders enter with probability 1 and non-participation is due to the truncating effect of the reserve price only, and therefore are able to identify $F(r) = 1 - p(\underline{N})$. When N belongs to the decreasing segment, $N > N_*$, we are certain that bidders are indifferent between entering or not, and are able to identify the entry cost from the indifference condition given the knowledge of $F(r)$.[12]

Proposition 5. *In the S model,*

(a) $F(v)$ *is identifiable for* $v \geq \xi(r|\underline{N})$.

(b) k *is identifiable according to* (8), $k = (\xi(r|N) - r)(1 - p(N))^{N-1}$.

In the LS model,

(c) $F(v)$ *is identifiable for* $v \geq r$.

(d) N_* *is identifiable as*

$$N_* = \min\{N \in \mathcal{N} : p(N) > p(N+1)\}. \tag{18}$$

(e) *If* $\mathcal{N}$ *has at least three elements and* $N_* \geq \underline{N} + 1$, k *is identifiable according to*

$$k = (1 - F(r)) \int_r^{\bar{v}} (1 - F^*(v))(1 - p(N) + p(N) F^*(v))^{N-1} dv. \tag{19}$$

Remark 4. Even if $\mathcal{N}$ has at least three elements, the variation in N is insufficient for identification of the entry cost in the LS model when $N_* = \underline{N}$. In this case, $p(N) < p(N')$ for all $N, N' \in \mathcal{N}$, and (19) holds for all $N \in \mathcal{N}$. Then only the ratio $k/(1 - F(r))$ is identifiable according to (19), but k and $F(r)$ are not separately identifiable. Also, the result in (d) above applies to the AME.

[12] In a recent working paper, Xu (2009) develops a nonparametric estimator of the entry cost for the S model.

4 Econometric implementation

In what follows, we allow for observable auction heterogeneity by introducing a vector of auction-specific covariates $x \in \mathcal{X} \subset \mathbb{R}^d$. We now assume that the distribution of (v, s) can change from auction to auction depending on the covariate value x, and is denoted by $F(v, s|x)$. Other objects are also indexed by x in a similar notation. The AME differential equation (2) for the bidding strategy takes the following form.

$$B'(v|N,x) = (v - B(v|N,x)) \times \frac{(N-1)(1 - F(\bar{s}(N,x)|x)) f(v|S_i \geq \bar{s}(N,x), x)}{F(\bar{s}(N,x)|x) + (1 - F(\bar{s}(N,x))) F(v|S_i \geq \bar{s}(N,x), x)}, \quad (20)$$

with the boundary condition $B(r(x)|N,x) = r(x)$, whereas the counterpart of equation (4) for the equilibrium cutoff $\bar{s}(N,x)$ is

$$\int_{r(x)}^{\bar{v}(x)} (1 - F(v|\bar{s}, x)) (F(\bar{s}|x) + (1 - F(\bar{s}|x)) F(v|S_i \geq \bar{s}, x))^{N-1} dv = k(x), \quad (21)$$

where $k(x)$ denotes the entry cost function.

4.1 Data generating process (DGP)

We assume that a sample of L auctions is available, and index the auctions by $l = 1, .., L$. Each auction is characterized by the vector of covariates $x_l \in \mathcal{X}$. We assume that the covariates x_l are drawn independently for each auction from a distribution with density $\varphi(\cdot)$. Conditional on $x_l = x$, the number of potential bidders N_l is drawn independently from the distribution $\pi(N|x)$. It is assumed that $\pi(\cdot|x)$ has support $\mathcal{N} = \{\underline{N}, ..., \bar{N}\}$.[13] The entry cost $k(x)$ is assumed to be a deterministic function of x. There is a binding reserve price r_l, and it is observable.[14]

Conditional on $x_l = x$ and $N_l = N$, the valuations V_{il} of potential bidders $i = 1, ..., N_l$ are drawn independently from a distribution with density $f(\cdot|x)$ that does not depend on N. The support of V_{il} is $[\underline{v}(x), \bar{v}(x)]$, where $i = 1, \ldots, N_l$. These valuations are unobservable.

Central to our approach is the assumption that, conditional on x_l, the number of potential bidders N_l is *exogenous*. This assumption allows us to use the variation in the number of potential bidders for testing purposes. In Section 5, we explain why this assumption is plausible in the context of our empirical application.

Assumption 2. *Conditional on x_l, V_{il} and N_l are independent.*

[13] For simplicity, we assume that the support does not depend on x. The results continue to hold even without this assumption.

[14] Even when the binding reserve price is not observable, it is identifiable by the minimum bid.

The bid b_{il} corresponding to the valuation V_{il} is generated according to the bidding strategy

$$b_{il} = B\left(V_{il}|N_l, x_l\right),$$

where $B\left(\cdot|N, x\right)$ is a solution to (20). Decisions to submit a bid, $y_{il} \in \{0, 1\}$, are generated according to the cutoff strategy

$$y_{il} = 1 \text{ if } S_{il} \geq \bar{s}\left(N_l, x_l\right) \text{ and } V_{il} \geq r_l.$$

The bidding strategy B and the cutoff function $\bar{s}$ depend on the model's primitives f and k through the equilibrium conditions of each model; neither B nor $\bar{s}$ is available in closed form.

A full list of the technical econometric assumptions on the DGP required for our results is given in Assumption 3 of Appendix B.

4.2 Hypotheses

Our main hypotheses tests are based on the identification of the selection effect in the active bidders' valuation quantiles. Recall that, for $\tau \in (0, 1)$, $Q^*\left(\tau|N, x\right)$ denotes the conditional quantile function of the distribution of active bidders' valuations. Conditional on $x_l = x$, models' restrictions (10) and (12) now take the following form.

$$\begin{aligned} H_{AME} &: Q^*\left(\tau|\underline{N}, x\right) \leq ... \leq Q^*\left(\tau|\bar{N}, x\right). \qquad (22) \\ H_{LS} &: Q^*\left(\tau|\underline{N}, x\right) = ... = Q^*\left(\tau|\bar{N}, x\right). \end{aligned}$$

In the S model, $F^*\left(v|N, x\right)$ is derived by truncation from the common "parent" distribution $F\left(v|x\right)$:

$$\begin{aligned} F^*\left(v|N, x\right) &= \frac{F\left(v|x\right) - F\left(\bar{s}\left(N, x\right)|x\right)}{1 - F\left(\bar{s}\left(N, x\right)\right)} \\ &= \frac{F\left(v|x\right) - \left(1 - p\left(N, x\right)\right)}{p\left(N, x\right)}. \qquad (23) \end{aligned}$$

Because the distribution F does not depend on N, this leads to the restriction

$$p\left(N, x\right) F^*\left(v|N, x\right) + 1 - p\left(N, x\right) = p\left(N', x\right) F^*\left(v|N', x\right) + 1 - p\left(N', x\right). \qquad (24)$$

To express restriction (24) in terms of quantiles, we define

$$\beta\left(\tau, N, x\right) = 1 - \frac{p\left(\bar{N}, x\right)}{p\left(N, x\right)}\left(1 - \tau\right). \qquad (25)$$

Since $p\left(\bar{N}, x\right) \leq p\left(N, x\right)$, it follows that $0 \leq \beta\left(\tau, N, x\right) \leq 1$ for all $\tau \in [0, 1]$, and $\beta\left(\tau, N, x\right)$ can therefore be interpreted as a legitimate transformation of the quantile order τ.[15] One

[15] Although any other fixed value of N can be used in place of $\bar{N}$ in the definition of β, the choice $N = \bar{N}$ ensures that β takes on values in the zero-one interval.

can easily show that in the S model, condition (24) implies the following restriction in terms of the transformed quantiles.

$$H_S : Q^* \left(\beta \left(\tau, \underline{N}, x\right) | \underline{N}, x\right) = \ldots = Q^* \left(\beta \left(\tau, \bar{N}, x\right) | \bar{N}, x\right). \tag{26}$$

In this paper, we consider the independent testing of H_{LS}, H_{AME}, and H_S against their corresponding unrestricted alternatives.[16] In addition, we also test whether the entry probabilities $p(N, x)$ are non-increasing in N, as implied by all of the models. In this case, for a given value of x, the null hypothesis is

$$H_p : 1 > p\left(\underline{N}, x\right) \geq \ldots \geq p\left(\bar{N}, x\right) > 0. \tag{27}$$

Hypothesis H_p is also tested against its corresponding unrestricted alternative.

The fact that the equilibrium probabilities of submitting a bid decline with the number of potential bidders is probably a common feature of many other models of entry. Whenever a model with costly entry is used to explain why some potential bidders do not bid, an alternative explanation must be confronted. Following Paarsch (1997), even with no entry cost, non-participation can still be explained by the fact that some bidders draw valuations below the reserve price. In that case, however, the probability of bidding is equal to probability $P(V_i \leq r|x)$ and therefore does not depend on the number of potential bidders under Assumption 2. Thus, the null hypothesis of costless entry ($k = 0$) can be formulated as

$$H_{k=0} : p\left(\underline{N}, x\right) = \ldots = p\left(\bar{N}, x\right). \tag{28}$$

Note that because $k = 0$ is formally included as a special case of LS, H_{LS} is also a testable restriction of the standard model.[17]

We must stress that our testing approach relies on conditional independence of V_{il} and N_l (Assumption 2). In reality, this assumption may be violated. For example, unobserved heterogeneity or the presence of multiple equilibria (see below) may result in a more complicated DGP that violates Assumption 2.[18] It is therefore desirable to have a nonparametric test of this assumption, and we implement one such test. Observe that Assumption 2 implies that the distribution of the number of active bidders conditional on (N, x) is binomial:

$$\rho(n|N, x) = \binom{N}{n} p(N, x)^n (1 - p(N, x))^{N-n}. \tag{29}$$

Because the conditional probability $\rho(n|N, x)$ is directly identifiable, we can test the binomial assumption accordingly. In Section 4.4, we propose a test for this binomial restriction.

[16] Our testing framework is designed to capture the selection effect, and the resulting tests may not be consistent against all conceivable alternative models.

[17] This result also follows from equation (6) in Guerre, Perrigne, and Vuong (2009), who consider identification in a more general model with risk-averse bidders.

[18] Krasnokutskaya (forthcoming) explores identification and estimation in first-price auctions under unobserved heterogeneity. In a recent working paper, Roberts (2009) proposes an alternative approach. He shows that, to the extent that the reserve price is optimally chosen by the sellers, conditioning on the reserve controls for unobserved heterogeneity.

Remark 5 (Multiple Equilibria)**.** As can be seen from the proof of Proposition 1, in the AME, there are also asymmetric equilibria, where a subset of $\min\{N, N_*\}$ bidders enters with probability one.[19,20] These equilibria also induce a binomial distribution of the number of active bidders, conditional on (N, x): for $n = 0, 1, \ldots, \min\{N, N_*\}$,

$$\tilde{\rho}(n|N, x) = \binom{\min\{N, N_*\}}{n} (1 - F(r|x))^n F(r|x)^{\min\{N,N_*\}-n}.$$

However, this binomial distribution is different from that corresponding to the symmetric equilibrium in (29), whenever there is partial entry for some $N \in \mathcal{N}$. This is because partial entry means that $N_* < \bar{N}$, and therefore $\min\{N, N_*\} < N$ and $1 - F(r|x) > p(N, x)$, which implies $\tilde{\rho}(n|N, x) \neq \rho(n|N, x)$. If such asymmetric equilibria are present in the data with probability $\alpha(N, x) > 0$ for some $N > N_*$, then the conditional distribution of the number of active bidders is a *mixture* of two different binomial distributions $\rho(n|N, x)$ and $\tilde{\rho}(n|N, x)$:

$$\alpha(N, x)\,\tilde{\rho}(n|N, x) + [1 - \alpha(N, x)]\,\rho(n|N, x),$$

and therefore itself is *not* binomial. Our binomial test can thus be used to uncover the presence of such asymmetric equilibria in the data.

4.3 Nonparametric estimation of quantiles

In this section, we present our nonparametric estimation method for $Q^*(\tau|N, x)$. This method is based on the fact that, because the bidding strategies are increasing, the quantiles of valuations $Q^*(\tau|N, x)$ and bids $q^*(\tau|N, x)$, where

$$q^*(\tau|N, x) = G^{*-1}(\tau|N, x),$$

are linked through the (inverse) bidding strategy

$$Q^*(\tau|N, x) = \xi(q^*(\tau|N, x)\,|N, x).$$

Because both $\xi(b|N, x)$ and $q^*(\tau|N, x)$ can be estimated nonparametrically, we consider a natural plug-in estimator:

$$\hat{Q}^*(\tau|N, x) = \hat{\xi}(\hat{q}^*(\tau|N, x)\,|N, x), \tag{30}$$

where $\hat{\xi}(b|N, x)$ and $\hat{q}^*(\tau|N, x)$ denote the nonparametric estimators of $\xi(b|N, x)$ and $q^*(\tau|N, x)$, respectively, which we describe below.

[19] Since a subset of firms enters with probability 1, there is no selective entry in these equilibria.

[20] The equilibria of this form do not exist in the S model because the bidders with $v = \underline{v}$ make zero expected profit from bidding, and are thus unable to recover the entry cost. It is unknown under which conditions multiple equilibria exist in the S model (Tan and Yilankaya (2006) obtain some results for second-price auctions).

Recalling that the inverse bidding strategy $\xi(b|N,x)$ is given by

$$\xi(b|N,x) = b + \frac{1 - p(N,x) + p(N,x)\, G^*(b|N,x)}{(N-1)p(N,x)\, g^*(b|N,x)}$$

(see Proposition 3), our estimator $\hat{\xi}(b|N,x)$ is obtained by replacing $p(N,x)$, $G^*(b|N,x)$, and $g^*(b|N,x)$ with their corresponding nonparametric estimators, $\hat{p}(N,x)$, $\hat{G}^*(b|N,x)$, and $\hat{g}^*(b|N,x)$. The conditional quantile $q^*(\tau|N,x)$ is estimated by inverting the nonparametric estimator for the bids CDF estimator $\hat{G}^*(b|N,x)$,

$$\hat{q}^*(\tau|N,x) = \hat{G}^{*-1}(\tau|N,x) = \inf\left\{b : \hat{G}^*(b|N,x) \geq \tau\right\}.$$

Our nonparametric estimators for $g^*(b|N,x)$, $G^*(b|N,x)$, and $p(N,x)$ are based on the kernel method. More specifically, we employ the following estimators:

$$\begin{aligned}
\hat{\pi}(N|x) &= \frac{\sum_{l=1}^{L} 1\{N_l = N\} \prod_{k=1}^{d} K\left(\frac{x_{kl}-x_k}{h}\right)}{\sum_{l=1}^{L} \prod_{k=1}^{d} K\left(\frac{x_{kl}-x_k}{h}\right)}, \text{ and} \\
\hat{p}(N,x) &= \frac{\sum_{l=1}^{L} n_l 1\{N_l = N\} \prod_{k=1}^{d} K\left(\frac{x_{kl}-x_k}{h}\right)}{N\sum_{l=1}^{L} 1\{N_l = N\} \prod_{k=1}^{d} K\left(\frac{x_{kl}-x_k}{h}\right)},
\end{aligned}$$

where h is the bandwidth parameter, K is a kernel function satisfying Assumption 4 in Appendix C, and $n_l = \sum_{i=1}^{N_l} y_{il}$ is the number of active bidders in auction l. As the probability of observing N conditional on $x_l = x$ and the probability of submitting a bid conditional on $N_l = N$ and $x = x_l$ can be written as $\pi(N|x) = E[1\{N_l = N\}|x]$ and $p(N,x) = E[n|N,x]/N$ respectively, their estimators are standard nonparametric regression estimators.

In Proposition 6 in Appendix C we show that the estimator $\hat{p}(N,x)$ is asymptotically normal and derive its asymptotic variance

$$V_p(N,x) = \left(\int K(u)^2\, du\right)^d \frac{p(N,x)(1-p(N,x))}{N\pi(N|x)\varphi(x)}.$$

Moreover, the estimators $\hat{p}(N,x)$ are asymptotically independent for any distinct $N, N' \in \left\{\underline{N}, ...\bar{N}\right\}$ and x, x' in the interior of $\mathcal{X}$.

The proposed estimators of g^* and G^* are

$$\begin{aligned}
\hat{g}^*(b|N,x) &= \frac{\sum_{l=1}^{L} \sum_{i=1}^{N_l} y_{il} 1\{N_l = N\} K\left(\frac{b_{il}-b}{h}\right) \prod_{k=1}^{d} K\left(\frac{x_{kl}-x_k}{h}\right)}{NLh^{d+1}\hat{p}(N,x)\,\hat{\pi}(N|x)\,\hat{\varphi}(x)}, \text{ and} \\
\hat{G}^*(b|N,x) &= \frac{\sum_{l=1}^{L} \sum_{i=1}^{N_l} y_{il} 1\{N_l = N\} 1(b_{il} \leq b) \prod_{k=1}^{d} K\left(\frac{x_{kl}-x_k}{h}\right)}{NLh^{d}\hat{p}(N,x)\,\hat{\pi}(N|x)\,\hat{\varphi}(x)},
\end{aligned}$$

where $\hat{\varphi}(x)$ is the usual multivariate kernel density estimator of the PDF φ.[21] The transformed quantiles needed for tests of the S model, can be estimated by $\hat{Q}^*(\hat{\beta}(\tau, N, x)|N, x)$, where $\hat{\beta}(\tau, N, x)$ is a plug-in estimator of $\beta(\tau, N, x)$ in (25):

$$\hat{\beta}(\tau, N, x) = 1 - \frac{\hat{p}(\bar{N}, x)}{\hat{p}(N, x)}(1 - \tau).$$

In Appendix C, we prove that the estimators $\hat{Q}^*(\tau|N, x)$ and $\hat{Q}^*(\hat{\beta}(\tau, N, x)|N, x)$ are consistent and asymptotically normal. More specifically, we prove that, under certain technical but standard econometric assumptions, $\sqrt{Lh^{d+1}}(\hat{Q}^*(\tau|N, x) - Q^*(\tau|N, x))$ is asymptotically normal with mean zero and the variance given by

$$V_Q(N, \tau, x) = \left(\int K(u)^2 du\right)^{d+1} \frac{(1 - p(N, x)(1 - \tau))^2}{(N-1)^2 N p^3(N, x) g^{*3}(q^*(\tau|N, x)|N, x)\pi(N|x)\varphi(x)}.$$

Further, the estimation of $\beta(\tau, N, x)$ has no effect on the asymptotic distribution of the transformed quantiles estimator $\hat{Q}^*(\hat{\beta}(\tau, N, x)|N, x)$, since, as we also show in the Appendix, $\hat{\beta}(\tau, N, x)$ converges at a faster rate than $\hat{Q}^*(\cdot|N, x)$. A consistent estimator $\hat{V}_Q(N, \tau, x)$ can be obtained by replacing $p(N, x)$, $q^*(\tau|N, x)$, and the other unknown functions by their estimators. Moreover, for any distinct $N, N' \in \{\underline{N}, ...\bar{N}\}$, $\tau, \tau' \in \Upsilon$, and x, x' in the interior of $\mathcal{X}$, the estimators $\hat{Q}^*(\tau|N, x)$ are asymptotically independent, as are the estimators $\hat{Q}^*(\hat{\beta}(\tau, N, x)|N, x)$. These results are contained in Proposition 7 in Appendix C.

Remark 6 (Circumventing the curse of dimensionality)**.** Although nonparametric methods are attractive in principle, they may be unsuitable for applications with small samples and a high-dimensional set of covariates. Indeed, most auction papers having made nonparametric identification arguments, almost always follow the approach of Haile, Hong, and Shum (2003) and apply them to "homogenized" bids, thus finessing the covariate problem. Unfortunately, a standard trick of assuming linearity in the covariates, estimating a first-stage regression model to get rid of the covariates, and then using "homogenized" bids in the second stage does not work in our model with endogenous entry. However, following the suggestion in Paarsch and Hong (2006, Chapter 3.3), a variant of the quantile single index model can be applied to reduce the curse of dimensionality in our setting. See Appendix G.

4.4 Tests

In this section, we propose nonparametric tests based on the models' restrictions. (Refer to Table 1 for a summary of these tests.) Recall the definition of the selection effect (11)

[21]The estimators $\hat{g}^*$ and $\hat{G}^*$ are standard kernel estimators of the conditional density and conditional CDF, see Pagan and Ullah (1999), equation (2.127) on page 58.

Table 1: Test statistics

Hypothesis	$\hat{\Delta}$	Test statistic
AME	$\hat{Q}^*(\tau\|N,x) - \hat{Q}^*(\tau\|N',x)$	$\sqrt{Lh^{d+1}}\sum_{N=\underline{N}}^{\bar{N}}\sum_{N'=N}^{\bar{N}}\frac{\left[\hat{\Delta}(\tau,N,N',x)\right]_+}{\hat{\sigma}(\tau,N,N',x)}$
LS	$\hat{Q}^*(\tau\|N,x) - \hat{Q}^*(\tau\|N',x)$	$\sqrt{Lh^{d+1}}\sum_{N=\underline{N}}^{\bar{N}}\sum_{N'=N}^{\bar{N}}\frac{\left\|\hat{\Delta}(\tau,N,N',x)\right\|}{\hat{\sigma}(\tau,N,N',x)}$
S	$\hat{Q}^*\left(\hat{\beta}(\tau,N,x)\|N,x\right) - \hat{Q}^*\left(\hat{\beta}(\tau,N',x)\|N',x\right)$	$\sqrt{Lh^{d+1}}\sum_{N=\underline{N}}^{\bar{N}}\sum_{N'=N}^{\bar{N}}\frac{\left\|\hat{\Delta}(\tau,N,N',x)\right\|}{\hat{\sigma}(\tau,N,N',x)}$
Monotone Prob. of Entry	$\hat{p}(N',x) - \hat{p}(N,x)$	$\sqrt{Lh^{d}}\sum_{N=\underline{N}}^{\bar{N}}\sum_{N'=N}^{\bar{N}}\frac{\left[\hat{\Delta}_p(N,N',x)\right]_+}{\hat{\sigma}_p(N,N',x)}$
Costless Entry	$\hat{p}(N',x) - \hat{p}(N,x)$	$\sqrt{Lh^{d}}\sum_{N=\underline{N}}^{\bar{N}}\sum_{N'=N}^{\bar{N}}\frac{\left\|\hat{\Delta}_p(N,N',x)\right\|}{\hat{\sigma}_p(N,N',x)}$
Binomial	$\hat{\rho}(n\|N,x) - \left(\begin{array}{c} N \\ n \end{array}\right)\hat{p}(N,x)^n(1-\hat{p}(N,x))^{N-n}$	$\sqrt{Lh^{d}}\sum_{N=\underline{N}}^{\bar{N}}\sum_{n=0}^{N}\frac{\left\|\hat{\Delta}_{Binomial}(n,N,x)\right\|}{\hat{\sigma}_{Binomial}(n,N,x)}$

from Section 2.1. Conditional on $x_l = x$, the estimated selection effect is

$$\hat{\Delta}\left(\tau, N, N', x\right) = \hat{Q}^*\left(\tau|N, x\right) - \hat{Q}^*\left(\tau|N', x\right).$$

We propose to test the affiliated model using the following statistic

$$T^{AME}\left(x\right) = \sup_{\tau\in\Upsilon} T^{AME}\left(\tau, x\right),$$

where $\Upsilon = \{\tau_1, ..., \tau_k\} \subset [0, 1]$ is a finite set of quantile values, and $T^{AME}\left(\tau, x\right)$ is defined as

$$T^{AME}\left(\tau, x\right) = \sqrt{Lh^{d+1}} \sum_{N=\underline{N}}^{\bar{N}} \sum_{N'=N}^{\bar{N}} \frac{\left[\hat{\Delta}\left(\tau, N, N', x\right)\right]_+}{\hat{\sigma}\left(\tau, N, N', x\right)}.$$

In the above expression,

$$\hat{\sigma}^2\left(\tau, N, N', x\right) = \hat{V}_Q\left(b, N, x\right) + \hat{V}_Q\left(b, N', x\right) \tag{31}$$

is the estimator of the asymptotic variance of $\hat{\Delta}\left(\tau, N, N', x\right)$, and the function $[\cdot]_+$ is defined as

$$[u]_+ = \begin{cases} u, & u \geq 0, \\ 0, & u < 0. \end{cases}$$

Note that under the hypothesis H_{AME}, $Q^*\left(\tau|N, x\right) - Q^*\left(\tau|N', x\right) \leq 0$ for $N \leq N'$, and through the function $[\cdot]_+$, the statistic $T^{AME}\left(\tau, x\right)$ captures only the pairwise differences between the sample quantiles corresponding to different N's that violate the model restriction. We rescale each term by its standard error to give more weight to the difference in quantiles that are estimated more precisely.

The hypothesis H_{AME} is rejected for large values of $T^{AME}\left(x\right)$. Thus, the asymptotic size α test is

$$\text{Reject } H_{AME} \text{ when } T^{AME}\left(x\right) > c^{AME}_{L,1-\alpha}(x), \tag{32}$$

where $c^{AME}_{L,1-\alpha}(x)$ is an appropriately chosen critical value. We suggest computing the critical values $c^{AME}_{L,1-\alpha}(x)$ using the bootstrap procedure described below.

To generate bootstrap samples, we first draw randomly with replacement, L auctions from the original sample of auctions $\{(N_l, x_l) : l = 1, \ldots, L\}$. In the second step, we draw bootstrap bids randomly with replacement from the bid data corresponding to each selected auction. Thus, if auction $\bar{l}$ is selected in the first step, then, in the second step, we draw $N_{\bar{l}}$ values from the set $\{b_{i\bar{l}} : i = 1, \ldots, N_{\bar{l}}\}$. Here we redefine the b_{il}'s so that they take values in the augmented set $\{\mathbb{R}_{++}, \text{"no bid"}\}$. Let $\{(b^\dagger_{1l}, \ldots, b^\dagger_{N^\dagger_l l}, N^\dagger_l, x^\dagger_l) : l = 1, \ldots, L\}$ be a bootstrap sample, and M be the number of bootstrap samples. Then, in each bootstrap sample $m = 1, \ldots, M$, compute $T^{\dagger,AME}_m\left(x\right) = \sup_{\tau\in\Upsilon} T^{\dagger AME}_m\left(\tau, x\right)$, where

$$T^{\dagger,AME}_m\left(\tau, x\right) = \sqrt{Lh^{d+1}} \sum_{N=\underline{N}}^{\bar{N}} \sum_{N'=N}^{\bar{N}} \frac{\left[\hat{\Delta}^\dagger_m\left(\tau, N, N', x\right) - \hat{\Delta}\left(\tau, N, N', x\right)\right]_+}{\hat{\sigma}^\dagger_m\left(\tau, N, N', x\right)}.$$

Here, $\hat{\Delta}_m^{\dagger}(\tau, N, N', x)$ is the bootstrap analogue of the selection effect $\hat{\Delta}(\tau, N, N', x)$:

$$\hat{\Delta}_m^{\dagger}\left(\tau, N, N', x\right) = \hat{Q}_m^{*,\dagger}(\tau|N, x) - \hat{Q}_m^{*,\dagger}\left(\tau|N', x\right),$$

$\hat{Q}_m^{*,\dagger}(\tau|N, x)$ is the bootstrap analogue of estimator $\hat{Q}^*(\tau|N, x)$ for the bootstrap sample m, and $\hat{\sigma}_m^{\dagger}(\tau, N, N', x)$ is the bootstrap analogue of $\hat{\sigma}(\tau, N, N', x)$ computed using (31) in bootstrap sample m. To compute the required statistics, we also define the bootstrap variable $y_{il}^{\dagger}$, where $y_{il}^{\dagger} = 0$ if $b_{il}^{\dagger}$ takes the value "no bid", and $y_{il}^{\dagger} = 1$ otherwise. The critical value $c_{L,1-\alpha}^{AME}(x)$ is the $(1-\alpha)$-th sample quantile of $\{T_m^{\dagger,AME} : m = 1, \dots M\}$.

A similar approach is used to test the restrictions of the LS and S models and those of the standard model (no costly entry). In the case of the LS model, the statistic is given by $T^{LS}(x) = \sup_{\tau \in \Upsilon} T^{LS}(\tau, x)$, where $T^{LS}(\tau, x)$ is constructed similarly to $T^{AME}(\tau, x)$ but with the function $[\cdot]_+$ replaced by the absolute value $|\cdot|$. This is because the LS model's restrictions are equalities, and we need to capture the deviations from H_{LS} in either direction. For the S model, we define the quantile difference by using the transformed quantiles,

$$\hat{\Delta}\left(\tau, N, N', x\right) = \hat{Q}^*\left(\hat{\beta}(\tau, N, x)|N, x\right) - \hat{Q}^*\left(\hat{\beta}(\tau, N', x)|N', x\right),$$

and then proceed exactly as before.

The hypothesis H_p in (27) can be tested using the statistic

$$\begin{aligned} T_p(x) &= \sqrt{Lh^d} \sum_{N=\underline{N}}^{\bar{N}} \sum_{N'=N}^{\bar{N}} \frac{\left[\hat{\Delta}_p\left(N, N', x\right)\right]_+}{\hat{\sigma}_p\left(N, N', x\right)}, \text{ where} \\ \hat{\Delta}_p\left(N, N', x\right) &= \hat{p}\left(N', x\right) - \hat{p}(N, x). \end{aligned}$$

Here, $\hat{\sigma}_p^2$ is the estimator of the asymptotic variance of $\hat{\Delta}_p$. The null hypothesis of costless entry in (28) can be tested using a statistic constructed similarly to $T_p(x)$ but with the function $[\cdot]_+$ replaced by the absolute value $|\cdot|$.

To test the binomial restriction in (29), let

$$\hat{\Delta}_{Binomial}(n, N, x) = \hat{\rho}(n|N, x) - \begin{pmatrix} N \\ n \end{pmatrix} \hat{p}(N, x)^n (1 - \hat{p}(N, x))^{N-n}$$

be the deviation of the directly estimated probability of entry from the corresponding binomial probability, where

$$\hat{\rho}(n|N, x) = \frac{\sum_{l=1}^{L} 1\{n_l = n\} 1\{N_l = N\} \prod_{k=1}^{d} K\left(\frac{x_{kl} - x_k}{h}\right)}{\sum_{l=1}^{L} 1\{N_l = N\} \prod_{k=1}^{d} K\left(\frac{x_{kl} - x_k}{h}\right)} \tag{33}$$

is a consistent nonparametric estimator of $\hat{\rho}(n|N, x)$. To test the binomial restriction, we employ the following statistic:

$$T_{Binomial}(x) = \sqrt{Lh^d} \sum_{N=\underline{N}}^{\bar{N}} \sum_{n=0}^{N} \frac{\left|\hat{\Delta}_{Binomial}(n, N, x)\right|}{\hat{\sigma}_{Binomial}(n, N, x)},$$

where $\hat{\sigma}^2_{Binomial}(n, N, x)$ is the plug-in estimator of the asymptotic variance of the difference $\hat{\Delta}_{Binomial}(n, N, x)$ derived in Appendix C.

As described above, we test each hypothesis for a given value x of the covariates. The reason for this is that a null hypothesis can be true for one value of x and false for another. If, however, one wishes to test a hypothesis over a range of values of x, then our approach can be modified as follows. In the case of H_{AME}, for example, one can select a grid of values of x from the interval of interest and compute $T^{AME}(x)$ for each value in the grid. $T^{AME}(x)$ can then be averaged over the chosen grid of values. The critical values for the test can be simulated using the averaged over the grid bootstrap statistic $T_m^{\dagger,AME}(x)$. Alternatively, one can use the maximum of $T^{AME}(x)$ over the chosen grid for x. In this case, the critical values should be simulated using the maximum of $T_m^{\dagger,AME}(x)$ over the same grid.

We establish the asymptotic validity of the bootstrap tests in Appendix E. Note that to ensure a valid bootstrap approximation, the bootstrap statistics must use re-centered differences. Thus, the statistic $T^{AME}(\tau, x)$ is based on $\hat{\Delta}(\tau, N, N', x)$, and its bootstrap counterpart $T_m^{\dagger,AME}(\tau, x)$ is based on the re-centered differences $\hat{\Delta}_m^{\dagger}(\tau, N, N', x) - \hat{\Delta}(\tau, N, N', x)$. This is to ensure that the bootstrap critical values are simulated from the null distribution.

A Monte Carlo study of our test is contained in Appendix F. We find that our proposed tests are close in size to the nominal levels, and also have good power properties.

5 Empirical application

We use the same data as those in Li and Zheng (2009, LZ hereafter) in our empirical application. This dataset consists of 540 auctions for "mowing highway right-of-way" maintenance jobs held by the TDoT from January 2001 to December 2003. We observe all of the bids and the identities of the bidders. Moreover, the available data contain information on the project characteristics, include the engineer's estimates, the length of the contract (number of working days), the number of items in the proposal, the acreage of full-width mowing, the acreage of other mowing, whether it is a state job, and whether the job is on an interstate highway.[22] LZ provide further details of this dataset. We focus only on those aspects relevant for this study.

Importantly, we observe the list of eligible bidders (planholders) for each auction. In the nomenclature of this paper, these eligible bidders constitute the potential bidders. A firm becomes a planholder through the following process. All projects to be auctioned are advertised by the TDoT three to six weeks prior to the auction date. These advertisements include the engineer's estimate, a brief summary of the project, and the location and type of work involved. However, they lack detailed schedules of the work items that are revealed only in the construction plans. Interested firms then submit a request for plans and bidding proposals, and these documents contain the project specifics (such as the item schedule).

[22] Refer to Table 1 in LZ for the basic summary statistics of the data.

The list of planholders is made available to all bidders prior to the bid submission deadline. All bids must be submitted by the deadline in a sealed envelope. Submitted bids and active bidders' identities are released only after bid opening. In this application, we do not observe the reserve price.

Our testing framework shares a number of assumptions with that in LZ, the first of which is the exogeneity of the number of potential bidders. They note that "only (and usually all of) those contractors in the mowing sub-industry who are located in the same county as where the job is or nearby counties would request the official bidding proposals".[23] Thus, the exogeneity of N is supported by the local nature of participating firms and the variation in the number of local firms across counties.[24]

Next, we also assume that the firms are ex-ante symmetric. LZ argue that the local participating firms are likely to be small and not much differentiated from one another. However, dynamic considerations such as those in Jofre-Bonet and Pesendorfer (2003) may lead to asymmetries even within such firms. To address this issue, LZ construct the backlog variable as the amount of incomplete work, measured in dollars, that remains from the previously won projects. This backlog variable contains substantial variation, but has statistically insignificant effects on both the bidding and bid-submission decisions (see Tables 3 and 4 in LZ).

Before turning to our nonparametric tests, we first discuss the importance of the effects of various auction characteristics on bid levels and the decisions to submit a bid found in the random-effect and probit regressions in LZ.[25] In the former regression, the dependent variable is log(bid), where bid is the amount of the bid in dollars. The size of the project has a strongly positive effect on the bids, and clearly, is the most important variable. The impact of the other variables is much smaller. The effect of project size on the probability of submitting a bid is also positive, although not statistically significant. This positive effect may reflect the fact that bidders have some information when deciding whether to submit a bid, which further implies the possibility of selective entry.

The next potentially important variable is the number of items in the construction plan, which captures the complexity of the project. This variable is statistically significant in both regressions, although its degree of variation is very small. The standard deviation is 0.82, and the mean is 2.01. This is not surprising, as lawn mowing jobs are relatively simple. The estimates for "acreage of other mowing" are also statistically significant in both regressions. However, the effects are small: a 1% increase in the "acreage of other mowing" is associated with only a 0.0075% decrease in the bids.

In the implementation of our testing approach, we are confronted with the usual bias and variance trade-off of our nonparametric estimators. Including a greater number of variables reduces the bias, but at the same time increases the sample variability of the

[23]See the first paragraph on page 6 of LZ.

[24]The assumption of an exogenous number of potential bidders has also been employed in other empirical work on auctions. See, for example, Athey, Levin, and Seira (2011).

[25]Refer to Tables 3 and 4 in LZ for details.

Table 2: Test results: Hypotheses, test statistics, and critical values corresponding to asymptotic significance levels 0.10, 0.05, and 0.01

Hypothesis	Test statistic	Critical values		
		0.10	0.05	0.01
Costless Entry	14.50	13.56	14.92	17.05
Monotone Prob. of Entry	2.86	9.00	10.40	13.35
Binomial	327.27	391.53	470.76	759.94
AME model	8.40	37.09	41.45	49.99
LS model	64.87	51.72	55.09	63.01
S model	60.22	67.82	71.23	78.90

estimators. The reduced-form regressions in LZ indicate that, apart from project size, the auction characteristics are unlikely to matter much. In fact, their structural estimates suggest that the project size is the only significant factor.

We perform our tests conditional on the median project size, $x = \$138,000$.[26] We consider a grid Υ of 200 uniformly spaced quantile values with $\tau_{\min} = 0.05$ and $\tau_{\max} = 0.95$, and use the tri-weight kernels. The bandwidth chosen is $h = 1.978 \times 1.06 \times (\text{std.err.}) \times (\text{sample size})^{-(1/5)}$. A well-understood practical issue is that nonparametric estimators suffer from a substantial loss of precision when the sample size is very small. When we tried to include all of the auctions, however, the estimates of the quantiles $Q^* (\tau|N, x)$ were highly erratic. Because the data are sparse, for some N, the estimated probabilities $\hat{\pi} (N|x)$ are very close to zero.[27] To ensure the stability of our estimators, we decided to exclude those N where the number of observations with $X_i \in [x - h, x + h]$ is less than 100. The working sample ultimately consisted of auctions with 8-14 potential bidders, and all of the results discussed in the following were obtained using this smaller sample.

The results of our tests are reported in Table 2. We first test the standard model ($k = 0$), i.e. the equality of bid submission probabilities across N. The null hypothesis is rejected at the 10% significance level. We then test the prediction shared by all remaining models considered in this paper. As can be seen from Figure 1, the bid submission probabilities $p(N, x)$ are apparently non-increasing in N. The formal test of the monotonicity restriction

[26] We believe our choice of median project size is natural and reasonable. The models are not expected to vary with x in this application.

[27] The estimated probability of N, conditional on x, is presented in Figure 3. There are no auctions with 19 or 21 potential bidders conditional on $x = 138,000$. Note also that there remains considerable variation in N.

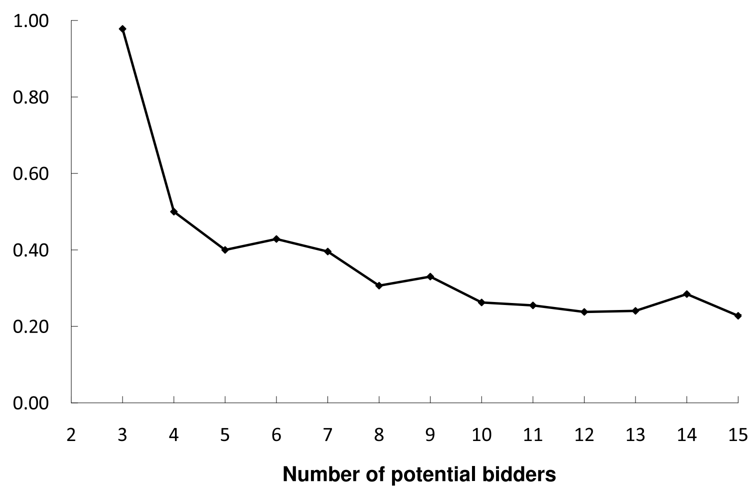

Figure 1: Estimated probabilities of bidding

on the bidding probabilities fails to reject the null at any reasonable significance level.[28] The DGP passes the binomial test, thereby offering no evidence of multiple equilibria or unobserved auction heterogeneity in the data.

The quantile test fails to reject the AME restriction $Q^*(\tau|N,x) \leq Q^*(\tau|N',x)$ for $N \geq N'$ uniformly over $\tau \in \Upsilon$.[29] We now turn to tests of the two extreme models: LS and S. We first note that the LS model is rejected even at a 1% significance level. We therefore reach a different conclusion from that in LZ, who find a better fit for the LS model. The S model passes all of our tests. However, based on the p-values of the S and AME tests, the AME has more empirical support in this application.

5.1 Should the TDoT restrict competition?

An interesting empirical exercise is to fit a parametric affiliated model to our data and then to compare its predictions to those in LZ, who investigate, in a counterfactual experiment, the effect of the number of potential bidders on the procurement price. In the LS model they estimated, the entry effect overwhelms the direct effect of potential competition. Indeed, for a typical project in the sample, they found it optimal to limit the competition to only

[28] It is worth noting that the effect of the number of potential bidders is statistically significant in the probit regression in LZ. Having more potential bidders reduces the bid submission rate. For example, increasing N by 1 reduces the odds of submitting a bid by about 4%.

[29] Note that here, the order of N and N' is reversed because, in our application, we are dealing with low-bid auctions.

$N = 3$ potential bidders. An interesting question is whether the selection effect is strong enough to overturn this conclusion.

To answer it, we estimate a simple parametric AME model and the resultant counterfactual effect of N on the auction price. We follow Bajari, Hong, and Ryan (2010) and normalize the costs and signals by the engineer's estimate. The joint distribution of costs and signals is specified to (truncated) normal in the spirit of Bajari (2001):

$$\frac{c_{il}}{x_l} = \mu_0 + \varepsilon_{il}, \qquad \frac{s_{il}}{x_l} = \mu_0 + \eta_{il},$$

where

$$(\varepsilon_{il}, \eta_{il}) \sim N\left(\mathbf{0}, \sigma_0^2 \begin{pmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{pmatrix}\right).$$

Since the reserve price r is unobservable, we treat it as an unknown parameter.[30] The vector of parameters to be estimated is thus given by

$$\theta_0 = \left(\mu_0, \sigma_0^2, \rho_0, k_0, r_0\right),$$

where k_0 and r_0 are the normalized entry cost and reserve price respectively.

To estimate θ_0, we adopt a variant of the computationally convenient approach of Hong and Shum (2002). Let $q^*(\tau|N, x; \theta_0)$ be the quantiles of bids predicted by the model. Since the bidding strategy $B(c|N, x; \theta_0)$ is monotone-increasing in c, the quantiles are easy to compute numerically using the relationship

$$q^*(\tau|N, x; \theta_0) = B\left[Q^*(\tau|N, c; \theta_0)\,|N, x; \theta_0\right].$$

Our estimation method is based on matching the observed and predicted quantiles of the bid distribution. Let $\tau_1, \ldots, \tau_M$ be M quantile values. For any $\theta \in \Theta$, define

$$m_l(\theta) = \frac{1}{N_l} \sum_{i=1}^{N_l} \begin{pmatrix} y_{il}\left(\tau_1 - 1\left(b_{il} \le q^*(\tau_1|N_l, x_l; \theta)\right)\right) \\ \vdots \\ y_{il}\left(\tau_M - 1\left(b_{il} \le q^*(\tau_M|N_l, x_l; \theta)\right)\right) \\ y_{il} - p(N_l, x_l; \theta) \end{pmatrix}$$

It is easy to verify that

$$E\left[y_{il}\left(\tau - 1\left\{b_{il} \le q^*(\tau|N_l, x_l; \theta_0)\right\}\right)|N_l, x_l\right] = 0,$$

such that the true value θ_0 satisfies the moment condition $Em(\theta_0) = 0$. Note that we have $M + 1$ moment conditions for the five-dimensional vector θ_0, so the model is overidentified if $M > 4$.

[30] Alternatively, this feature can be taken into account by considering a model with a secret reserve price as in Li and Perrigne (2003). However, this complicates significantly the derivation of the equilibrium bidding strategy.

Table 3: Estimates of AME structural parameters

Parameter	Estimate	Standard error
Mean of normalized cost, μ	0.970*	0.132
Standard deviation, σ	0.162*	0.055
Correlation, ρ	0.720*	0.181
Normalized reserve price, r	1.311*	0.073
Normalized entry cost, k	0.023*	0.005
GMM objective	21.10	

* Significant at 5%

Our generalized method of moments (GMM) estimator $\hat{\theta}$ is based on the moment condition $Em(\theta_0) = 0$ and is defined as

$$\begin{aligned} \hat{\theta} &= \arg\min_{\theta \in \Theta} J_L(\theta), \text{ where} \\ J_L(\theta) &= \left(\frac{1}{\sqrt{L}} \sum_{l=1}^{L} m_l(\theta) \right)' W_L \left(\frac{1}{\sqrt{L}} \sum_{l=1}^{L} m_l(\theta) \right), \end{aligned}$$

and W_L is a weighting matrix. By standard GMM results,

$$\sqrt{L} \left(\hat{\theta} - \theta_0 \right) \to_d N \left(0, \left(D_0' W D_0 \right)^{-1} D_0' W \Omega_0 W D_0 \left(D_0' W D_0 \right)^{-1} \right),$$

where

$$\Omega_0 = E \left[m_l(\theta_0) \, m_l(\theta_0)' \right], \qquad D_0 = \left(\frac{\partial}{\partial \theta} E \left[m_l(\theta) \right] \right)_{\theta = \theta_0},$$

and the optimal weight matrix should satisfy $W_L \to_p \Omega_0^{-1}$. Further details are provided in Appendix H. When the model is correctly specified, the GMM objective $J_L(\hat{\theta})$ computed using the optimal weight matrix is asymptotically χ^2-distributed with $M - 4$ degrees of freedom, and can be used to test the fit of the model.

The estimation results are given in Table 3. We match 19 equally spaced quantile values $\{0.05, 0.1, ..., 0.95\}$ for $N = 8, ..., 14$ (i.e. we chose the same values for N as those in the previous section). As the estimate $\hat{\rho} = 0.72$ indicates, the signals contain a substantial amount of information concerning the construction cost. This is realistic because, as we

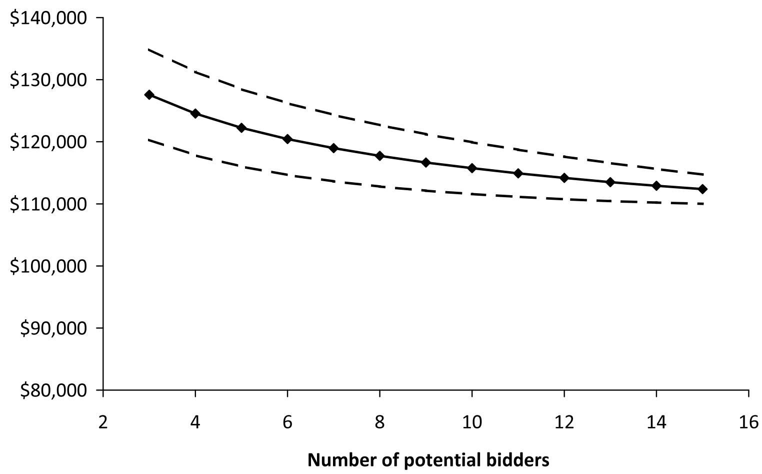

Figure 2: Estimated median cost conditional on entry (solid line) and its 95% confidence band (dashed lines)

have remarked previously, lawn-mowing jobs are relatively small and simple in nature, meaning that the plans are likely to be quite informative about the true project cost. The estimate of the entry cost $\hat{k} = 2.31\%$ is given as a percentage of the project size for a median-sized project, $x = \$138,000$.

The affiliated model is expected to yield a smaller estimate of the entry cost than the LS model. Indeed, the entry cost in the latter is equal to the equilibrium expected profit of the average potential bidder, whereas it is equal to the expected profit of the marginal potential bidder (with signal $\bar{s}(N)$) in the former. Because the signals in the AME are positively correlated with the costs, the marginal bidder has a smaller entry cost than the average bidder.[31]

Formally, the fit of the model is assessed by the J test. Since we have fitted 19 quantiles, the J statistic has a limiting χ^2 distribution with 14 degrees of freedom. The model fits reasonably well. The J test p-value is 0.17, and therefore our parametric model is not rejected.

In the counterfactual experiment, we find that the selection effect leads to a *reduction* in the procurement price. Figures 2 and 3 depict the counterfactual effect of the number of potential bidders on the median entrant's cost and price for a median-sized project.[32] The

[31] Also see Remark 3.

[32] LZ perform their counterfactual experiment for the 123rd auction in the sample, which has a project size close to the mean value of $165,349. As in our nonparametric test, we have decided here to use the median-sized project. Our results take the same form for other project sizes, including that considered by LZ.

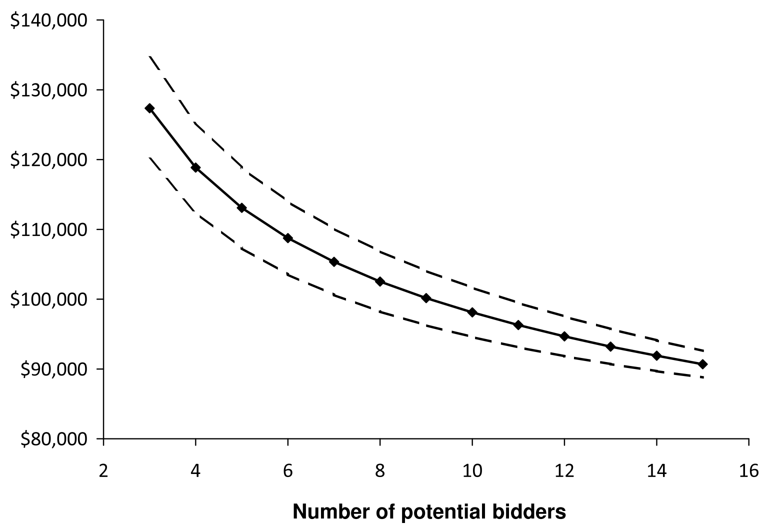

Figure 3: Counterfactual price (solid line) and its 95% confidence band (dashed lines)

graphs demonstrate the selection effect at work. Increasing the number of potential bidders by 1 results, on average, in a 1% reduction in the cost and 3% reduction in the price.[33] Therefore, our results suggest that, in order to reduce the cost of procurement, the TDoT should *not* limit the competition in lawn-mowing auctions.

6 Concluding remarks

In this paper, we propose nonparametric tests to discriminate among alternative models of entry in first-price auctions. Those considered are: (a) the Levin and Smith (1994) (LS) model with randomized entry strategies; (b) the Samuelson (1985) (S) model that assumes that bidders are perfectly informed about their valuations at the entry stage, and selected into the pool of entrants based on this information; and (c) a new model (the affiliated entry model, or AME), which allows for selective entry but in a less stark form than that in Samuelson (1985). More specifically, our model captures the selection effect by assuming that bidders receive signals that are informative about their valuations and make entry decisions based on these signals.

We estimate an AME and find a strong degree of correlation between costs and signals. We then repeat the counterfactual experiment of Li and Zheng (2009), and, using our new model, find that increasing the number of potential bidders would be desirable: on average, the price falls by 1% with each additional potential bidder in the auction.

[33] The confidence bands in the graphs are computed using the parametric bootstrap.

Our work can be extended in a number of directions. First, it would be interesting to extend our approach to other auction mechanisms and data structures as in Athey and Haile (2002).[34] Second, our models of entry do not fully endogenize the information acquisition process, the nature of which may have important consequences for auction design (see, e.g., Persico (2000) and Bergemann and Valimaki (2002)). Also, incorporating risk aversion, as Guerre, Perrigne, and Vuong (2009) and Campo, Guerre, Perrigne, and Vuong (forthcoming) do in the standard first-price auction model, would clearly be very interesting.

Another extension would be to allow for bidder asymmetries, as in a recent working paper by Krasnokutskaya and Seim (forthcoming). The obvious difficulty here would be the necessity to deal with multiple equilibria head-on. Bajari, Hong, and Ryan (2010) obtain a number of identification results in this direction, and estimate a parametric model with multiple equilibria for highway procurement auctions. Another extension would be to incorporate dynamic features, as in Jofre-Bonet and Pesendorfer (2003). These extensions are left for future research.

[34]See Athey and Haile (2007) for a survey of the literature.

Appendix

A Proofs of the results in Sections 2 and 3

Proof of Proposition 1. Observe that, if the bidders follow an entry strategy prescribed by the cutoff strategy $\bar{s}$, the auction in the second state of the game is an IPV auction with a random number of bidders. Specifically, the distribution of the number of bidders is binomial with parameters N and $1 - F(\bar{s}(N))$, and the distribution of entrants' valuations is $F(v|S_i \geq \bar{s}(N))$. Differential equation (2) follows directly from Theorem 1 in Harstad, Kagel, and Levin (1990). The expected profit (3) follows from the Envelope Theorem: by independence, the bidder with valuation v has the probability of winning the auction

$$(F(\bar{s}) + 1 - F(\bar{s}) F(v|S_i \geq \bar{s}))^{N-1} \equiv \lambda(v, \bar{s})^{N-1},$$

therefore his expected profit in the auction is $\int_r^v \lambda(x, \bar{s})^{N-1} dx$. At the entry stage, the expected profit is

$$\begin{aligned}
\Pi(s, \bar{s}, N) &= \int_r^{\bar{v}} \int_r^v \lambda(x, \bar{s})^{N-1} dx dF(v|s) \\
&= \int_r^{\bar{v}} (1 - F(v|s)) \lambda(v, \bar{s})^{N-1} dv,
\end{aligned}$$

where the last line follows from integration by parts. This proves (3).

Next, $\Pi(s, \bar{s}, N)$ is increasing in $\bar{s}$ because $\lambda(v, \cdot)$ is increasing: for all $v \in [\underline{v}, \bar{v})$,

$$\begin{aligned}
\frac{\partial \lambda(v, \bar{s})}{\partial \bar{s}} &= f(\bar{s}) + \frac{\partial}{\partial \bar{s}} P\{V_i \leq v, S_i \geq \bar{s}\} \\
&= f(\bar{s}) - f(\bar{s}) F(v|\bar{s}) > 0.
\end{aligned}$$

Also, $\Pi(s, \bar{s}, N)$ is non-decreasing in s because the affiliation property implies that the conditional CDF $F(v|S_i \geq s)$ is non-increasing in s. We see that $\Pi(s, s, N)$ is continuous increasing in s, and (a)-(c) follow straightforwardly in a symmetric equilibrium. The result in (a) follows since if $\Pi(\bar{v}, \bar{v}, N) \leq k$, then a bidder with $v = \bar{v}$, the highest valuation possible, is unable to recover the entry cost even when he is the single bidder in the auction. The result in (b) follows since if $\Pi(\underline{v}, \underline{v}, N) \geq k$, then a bidder with $v = \underline{v}$, the lowest valuation possible, is able to recover the entry cost even when all his rivals enter with probability 1. To show (c), note that if $\Pi(\underline{v}, \underline{v}, N) < k < \Pi(\bar{v}, \bar{v}, N)$, then strict monotonicity of $\Pi(s, s, N)$ in s implies the equation $\Pi(\bar{s}, \bar{s}, N) = k$ has a unique solution $\bar{s}(N) \in (\underline{v}, \bar{v})$. Then entering if $S_i > \bar{s}(N)$ and not entering if $S_i < \bar{s}(N)$ is clearly an equilibrium because $\Pi(S_i, \bar{s}, N) > k$ in the first case and $\Pi(S_i, \bar{s}, N) < k$ in the second case.

Lastly, observe that $\lambda(v, \bar{s}) \in (0, 1)$ in any equilibrium with with entry, which is our maintained assumption. Therefore $\Pi(s, s, N)$ is decreasing in N $(N \geq 2)$. Consequently, $\bar{s}(N)$ is constant on $\{N : N \leq N_*\}$ and is strictly increasing on $\{N : N \geq N_*\}$. ■

Proof of Proposition 3. For every $b \in \left[r, \bar{b}(N)\right]$, we have

$$\begin{aligned} G^*(b|N) &= F(\xi(b|N)|N, S_i \geq \bar{s}(N), V_i \geq r) \\ &= \frac{F(\xi(b|N)|S_i \geq \bar{s}(N)) - F(r|S_i \geq \bar{s}(N))}{1 - F(r|S_i \geq \bar{s}(N))}. \end{aligned} \tag{34}$$

where $\xi(\cdot|N) : \left[r, \bar{b}(N)\right] \to \mathbb{R}_+$ denotes the inverse bidding strategy, defined as the unique solution to the equation $B(\xi|N) = b$. Since the probability of submitting a bid is

$$\begin{aligned} p(N) &= P(S_i \geq \bar{s}(N), V_i \geq r) \\ &= P(S_i \geq \bar{s}(N)) P(V_i \geq r|S_i \geq \bar{s}(N)) \\ &= (1 - F(\bar{s}(N)))(1 - F(r|S_i \geq \bar{s}(N))), \end{aligned}$$

we see that

$$p(N) G^*(b|N) = (1 - F(\bar{s}(N)))[F(\xi(b|N)|S_i \geq \bar{s}(N)) - F(r|S_i \geq \bar{s}(N))]$$

After simplification,

$$1 - p(N) + p(N) G^*(b|N) = F(\bar{s}(N)) + (1 - F(\bar{s}(N))) F(\xi(b|N)|S_i \geq \bar{s}(N)).$$

Differentiating (34) with respect to b gives the density of bids

$$g^*(b|N) = \frac{f(\xi(b|N)|S_i \geq \bar{s}(N))}{1 - F(r|S_i \geq \bar{s}(N))} \xi'(b|N),$$

and we see that

$$p(N) g^*(b|N) = (1 - F(\bar{s}(N))) f(\xi(b|N)|S_i \geq \bar{s}(N)) \xi'(b|N).$$

Therefore,

$$\frac{1 - p(N) + p(N) G^*(b|N)}{p(N) g(b|N)} = \frac{F(\bar{s}(N)) + (1 - F(\bar{s}(N))) F(\xi(b|N)|S_i \geq \bar{s}(N))}{(1 - F(\bar{s}(N))) f(\xi(b|N)|S_i \geq \bar{s}(N))} \frac{1}{\xi'(b|N)}.$$

Since $\xi'(b|N) = 1/B'(v|N)$ for $v = \xi(b|N)$, (15) follows from the differential equation (2) for the bidding strategy . ■

Proof of Proposition 4. The proof of the results in (a) is omitted because it parallels the proofs of Theorems 1 and 4 in GPV.

For (b), suppose the data are generated by an AME with some $k > 0$ and $F(\cdot|\cdot)$ such that $F(r) \in (0, 1)$. Consider any standard model ($k = 0$) with the distribution of valuations given by

$$F_0(v) = F_0(v|v < r) F_0(r) + F^*(v|N)[1 - F_0(r)]$$

where $F_0(r) = 1 - p(N)$ and $F_0(\cdot|v < r)$ is an arbitrary absolutely continuous CDF with support $[\underline{v}_0, r]$ for some $\underline{v}_0 \in (0, r)$. One can check that the differential equation (2) for the bidding strategy remains the same as for the original AME, so the inverse bidding strategy is also $\xi(\cdot|N)$. Since the distribution of active bidders' valuations $F^*(v|N)$ is the same by construction, it follows that this standard model induces, in the Bayesian-Nash equilibrium, the same distribution of observable bids $G^*(\cdot|N)$. Furthermore, since each bidder becomes active with probability $1 - F(r) = p(N)$, the distribution of the number of active bidders is also the same.

Going in the other direction, suppose the data $(G^*(\cdot|N), p(N))$ are generated by some standard model with the distribution of valuations F_0 above. Since LS model is included in the AME class, it is sufficient to construct an observationally equivalent LS model. Consider an LS model with the distribution of valuations

$$F_{LS}(v) = F_{LS}(v|v < r) F_{LS}(r) + F^*(v|N) [1 - F_{LS}(r)]$$

where $F_{LS}(\cdot|v < r)$ is an arbitrary absolutely continuous CDF with support $[\underline{v}_{LS}, r]$ for some $\underline{v}_{LS} \in (0, r)$, and $F_{LS}(r)$ is chosen to assure that the probability of bidding is still $p(N)$:

$$\alpha [1 - F_{LS}(r)] = p(N)$$

for some *entry probability* $\alpha \in (0, 1)$. To complete the construction, let

$$k = \int_r^{\bar{v}} (1 - F_{LS}(v) (\alpha + (1 - \alpha) F_{LS}(v))^{N-1} dv.$$

Again, it is straightforward to show that this LS model's differential equation for the bidding strategy is the same as in the original standard model. Since $F_{LS}(v|v \geq r) = F_0(v \geq r)$, the induced distribution of bids is the same. Also, according to (4), bidders are indifferent between entering and not entering provided the rivals enter with probability $\alpha \in (0, 1)$, while the probability of submitting a bid is equal to $p(N)$ as in the original standard model.

For part (c), if $\xi(r|N) > r$, similar arguments imply that any S model with $k = (\xi(r|N) - r)(1 - p(N))^{N-1}$ and

$$F_S(v) = F_S(v|v < \xi(r|N)) [1 - p(N)] + F^*(v|N) p(N)$$

where $F_S(\cdot|v < \xi(r|N))$ is any absolutely continuous CDF with support $[\underline{v}_{LS}, r]$ for some $\underline{v}_{LS} \in (0, r)$, rationalizes the data. ■

Proof of Proposition 5. In the S model, $F(v)$ is identified for $v \geq \bar{s}(N) > r$, using, for example, (23):

$$F(v) = p(N) F^*(v|N) + 1 - p(N) \tag{35}$$

Consequently, $F(v)$ is identified for $v \geq \min\{\bar{s}(N) : N \in \mathcal{N}\} = \bar{s}(\underline{N})$. The entry cost is also identified:

$$k = (\xi(r|N) - r)(1 - p(N))^{N-1}. \tag{36}$$

In the LS model, k is smaller than the expected profit from bidding even under full entry for $N = \underline{N}, ..., N_*$, and is strictly greater than the expected profit for $N = N_* + 1, ..., \bar{N}$. Equivalently,

$$\int_r^{\bar{v}} (1 - F(v))(F(r) + [1 - F(r)] F(v|V_i \geq r))^{N_*-1} dv > k$$
$$> \int_r^{\bar{v}} (1 - F(v))(F(r) + [1 - F(r)] F(v|V_i \geq r))^{N_*} dv. \tag{37}$$

For $N = \underline{N}, ..., N_*$, non-participation in the auction is caused by the truncating effect of the reserve price. In other words, $p(N) = 1 - F(r)$ for $N = \underline{N}, ..., N_*$, and $F(r) = 1 - p(N)$ is identifiable. This implies the identification of both $F(v) = [1 - F(r)] F^*(v|N) + F(r)$ and k according to (19). ■

Remark 7. Note that (37) is a condition on the primitives of the model sufficient for the pattern (17) to arise.

B Additional details of the DGP

We make the following assumptions concerning the DGP.

Assumption 3. **(a)** *$\{(N_l, x_l) : l = 1, \ldots, L\}$ are i.i.d.*

(b) *Let $\varphi(x)$ denote the marginal PDF of x_l. We assume that $\varphi(\cdot) > 0$ on its compact support $\mathcal{X} \subset \mathbb{R}^d$ and admits up to $R \geq 2$ continuous partial derivatives on the interior of $\mathcal{X}$.*

(c) *Let $\pi(N|x)$ denote the distribution of N_l conditional on x_l. We assume that $\pi(\cdot|x)$ has support $\mathcal{N} = \{\underline{N}, ..., \bar{N}\}$ for all $x \in \mathcal{X}, \underline{N} \geq 2$, and $\pi(N|\cdot)$ admits R continuous bounded derivatives on the interior of $\mathcal{X}$ for all $N \in \mathcal{N}$.*

(d) *(V_{il}, S_{il}) and N_l are independent conditional on x_l.*

(e) *$\{(V_{il}, S_{il}) : i = 1, \ldots N_l; l = 1, \ldots, L\}$ are i.i.d. conditional on (N_l, x_l).*

(f) *In the AME, suppose that the density $f(v, s|x)$ is strictly positive and bounded away from zero on its support, a compact interval $[\underline{v}(x), \overline{v}(x)]^2 \subset \mathbb{R}_+^2$ for all $x \in \mathcal{X}$; $f(\cdot, \cdot|x)$ admits up to $R - 1$ continuous partial derivatives on $[\underline{v}(x), \overline{v}(x)]^2$ for all $x \in \mathcal{X}$, and $f(v, s|\cdot)$ admits up to R continuous partial derivatives on the interior of $\mathcal{X}$ for all $v, s \in [\underline{v}(x), \overline{v}(x)]^2$. In the S model, suppose that the density $f(v|x)$*

is strictly positive and bounded away from zero on its support, a compact interval $[\underline{v}(x), \overline{v}(x)] \subset \mathbb{R}_+$ *for all* $x \in \mathcal{X}$*;* $f(\cdot|x)$ *admits up to* $R-1$ *continuous derivatives on* $[\underline{v}(x), \overline{v}(x)]$ *for all* $x \in \mathcal{X}$*,* $f(v|\cdot)$ *admits up to* R *continuous partial derivatives on the interior of* $\mathcal{X}$ *for all* $v \in [\underline{v}(x), \overline{v}(x)]$*.*

(g) *The reserve price* $r(x)$ *is binding for all* $x \in \mathcal{X}, N \in \mathcal{N}$*, i.e.* $\underline{v}(x) < r(x)$*, and* $r(\cdot)$ *admits up to* R *continuous partial derivatives on the interior of* $\mathcal{X}$*.*

(h) *The entry cost* $k(x)$ *admits up to* R *continuous partial derivatives on the interior of* $\mathcal{X}$*.*

Remark 8. Assumption 3(a) is the usual i.i.d. assumption on the DGP for the covariates. The smoothness conditions in Assumptions 3(b), (c), (f), (g) and (h) are standard in the nonparametric literature and used for deriving the asymptotic properties of the kernel estimators. Assumption 3(c) also defines the support of the distribution of N_l conditional on the covariates. Assumption 3(d) repeats the condition in Assumption 2 for completeness. Assumption 3(e) is the IPV assumption.

Lemma 1. *The following results hold for the AME and the S model.*

(a) *Let* $f^*(v|N,x)$ *denote the conditional PDF of valuations given* $N_l = N, x_l = x$ *and conditional on bidding. Then* $f^*(\cdot|N,x)$ *is strictly positive and bounded away from zero on its support* $[\underline{v}(x), \overline{v}(x)]$*, admits up to* $R-1$ *continuous derivatives on* $[\underline{v}(x), \overline{v}(x)]$ *for all* $x \in \mathcal{X}, N \in \mathcal{N}$*, and* $f^*(v|N,\cdot)$ *admits up to* R *continuous partial derivatives on the interior of* $\mathcal{X}$ *for all* $v \in [\underline{v}(N,x), \overline{v}(N,x)], N \in \mathcal{N}$*.*

(b) *The conditional probability of entry* $p(N,x)$ *admits up to* R *continuous partial derivatives with respect to* x *on the interior of* $\mathcal{X}$ *for all* $N \in \mathcal{N}$*.*

Proof of Lemma 1. Consider the AME first. Note that $\bar{s}(N,x)$ is determined by

$$\int_{r(x)}^{\bar{v}(x)} (1 - F(v|s,x))\, \lambda(v,\bar{s},x)^{N-1}\, dv - k(x) = 0, \text{ where} \tag{38}$$

$$\lambda(v,\bar{s},x) = F(\bar{s}|x) + \int_{\bar{s}}^{\bar{v}(x)} F(v|s,x)\, f(s|x)\, ds.$$

By Lemma A1(i) in GPV, $\bar{v}(x)$ admits up to R continuous partial derivatives on the interior of $\mathcal{X}$. Together with our Assumptions 3(f)-(h), this implies that the left-hand side in (38) is smooth up to order R in x. Moreover, its partial derivative with respect to $\bar{s}$ is

$$\int_{r(x)}^{\bar{v}(x)} (1 - F(v|x))\, (f(\bar{s}|x) - F(v|\bar{s},x)\, f(\bar{s}|x))^{N-1}\, dv > 0.$$

The Implicit Function Theorem then implies that $\bar{s}(N,x)$ admits up to R continuous partial derivatives with respect to x on the interior of $\mathcal{X}$ for all $N \in \mathcal{N}$. The result follows then from the conditions of the lemma, and the definitions of $f^*(v|N,x)$ and $p(N,x)$:

$$\begin{aligned} f^*(v|N,x) &= \frac{\int_{s\geq\bar{s}(N,x)}^{\bar{v}(x)} f(v,s|x)\,ds}{\int_{v\geq r(x)}^{\bar{v}(x)} \int_{s\geq\bar{s}(N,x)}^{\bar{v}(x)} f(v,s|x)\,dsdv}, \\ p(N,x) &= \int_{r(x)}^{\bar{v}(x)} \int_{\bar{s}(N,x)}^{\bar{v}(x)} F(v,s|N,x)\, f(s|N,x)\,dsdv. \end{aligned}$$

In the S model, the cutoff $\bar{s}(N,x)$ is determined as an implicit function from the equation $(\bar{s}-r(x))\,F(\bar{s}|x)^{N-1} - k(x) = 0$. The derivative with respect to $\bar{s}$ of the expression on the left-hand side is $F(\bar{s}|x)^{N-1} + (\bar{s}-r(x))(N-1)\,f(\bar{s}|x)\,F(\bar{s}|x)^{N-2} > 0$. Assumptions of the lemma also imply that the left-hand side of equation determining $\bar{s}(N,x)$ has continuous partial derivatives up to order R with respect to $\bar{s}$ and x. The Implicit Function Theorem then implies that the solution $\bar{s}(N,x)$ has continuous derivatives up to order R in x on the interior of $\mathcal{X}$. The result follows since in this case for $v \geq \bar{s}(N,x)$, $f^*(v|N,x) = f(v|x)/p(N,x)$, and $p(N,x) = 1 - F(\bar{s}(N,x)|x)$. ■

We can now prove the following result about the order of smoothness of $g(b|N,x)$.[35]

Lemma 2. *Suppose that Assumptions 3(f) and (g) hold. Then for all $N \in \mathcal{N}$, the conditional PDF of bids $g^*(b|N,x)$ is strictly positive and bounded away from zero on its support $[\underline{b}(N,x), \bar{b}(N,x)]$, and $g^*(\cdot|N,\cdot)$ admits up to R continuous partial derivatives on the interior of the set $\{(b,x) : x \in \mathcal{X}, b \in [\underline{b}(N,x), \bar{b}(N,x)]\}$.*

Proof of Lemma 2. First, we establish the order of smoothness of inverse bidding strategy $\xi(v|N,x)$ is R in both v and x. It is straightforward to show that differential equation (2) can be re-written in terms of $\xi(b|N,x)$ for $b \in (\underline{b}(N,x), \bar{b}(N,x))$ as

$$\begin{aligned} \frac{\partial \xi(b|N,x)}{\partial b} &= \frac{1}{N-1}\frac{1}{\xi(b|N,x)-b}\frac{p(N,x)\,f^*(\xi(b|N,x)|N,x)}{1-p(N,x)+p(N,x)\,F^*(\xi(b|N,x)|N,x)} \\ &\equiv \Phi_N(\xi(b|N,x),x). \end{aligned} \tag{39}$$

By 3(f), $f^*(v|N,x)$ admits $R-1$ derivatives with respect to v and R derivatives with respect to x, while $p(N,x)$ admits R derivatives with respect to x. Therefore $\Phi_N(v,x)$ also admits $R-1$ derivatives with respect to its first argument and R derivatives with respect to second. A fundamental results in the theory of differential equations (see, for example, Theorem 2.6 in Anosov, Aranson, Arnold, Bronshtein, Grines, and Il'Yashenko (1997)) implies that $\xi(\cdot|N,x)$ admits R derivatives on $(\underline{b}(N,x), \bar{b}(N,x))$ as a solution of this differential equation. Also, $\xi(b|N,\cdot)$ admits R partial derivatives on the interior of $\mathcal{X}$. Next, since

[35] This result parallels Proposition 1(iii) in GPV.

$G^*(b|N,x) = F^*(\xi(b|N,x)|N,x)$, we have $g(b|N,x) = f^*(\xi(b|N,x)|N,x)\,\partial\xi(b|N,x)/\partial b$. Substituting $\partial\xi(b|N,x)/\partial b$ from (39) yields (note that $f^*(\xi(b)|N,x)$ cancels out):

$$g^*(b|N,x) = \frac{1}{N-1}\frac{1}{\xi(b|N,x)-b}\frac{p(N,x)}{1-p(N,x)+p(N,x)F^*(\xi(b|N,x)|N,x)}.$$

The result follows from the just established order of smoothness of $\xi(\cdot|N,\cdot)$ and Assumptions 3(f) and (g). ■

C Details of the estimation method

For kernel estimation, we use kernel functions K satisfying the following standard assumption (see, for example, Newey (1994)).

Assumption 4. *The kernel K has at least $R \geq 2$ continuous and bounded derivatives on $\mathbb{R}$, compactly supported on $[-1,1]$ and is of order R: $\int K(u)\,du = 1$, $\int u^j K(u)\,du = 0$ for $j = 1, ..., R-1$.*

The standard nonparametric regression arguments imply that the estimator of entry probabilities $\hat{p}(N,x)$ is asymptotically normal (see, for example, Pagan and Ullah (1999), Theorem 3.5, page 110):

Proposition 6. *Assume that the bandwidth h satisfies $Lh^d \to \infty$ and $\sqrt{Lh^d}h^R \to 0$ as $L \to \infty$. Then, for x in the interior of $\mathcal{X}$ and under Assumptions 3 and 4, $\sqrt{Lh^d}(\hat{p}(N,x) - p(N,x))$ is asymptotically normal with mean zero and variance*

$$V_p(N,x) = \frac{p(N,x)(1-p(N,x))}{N\pi(N|x)\varphi(x)}\left(\int K(u)^2\,du\right)^d.$$

Moreover, the estimators $\hat{p}(N,x)$ are asymptotically independent for any distinct $N, N' \in \left\{\underline{N}, ...\bar{N}\right\}$ and x, x' in the interior of $\mathcal{X}$.

Since the distribution of values and, consequently, the distribution bids have compact supports, the estimator of the PDF $g^*(b|N,x)$ in Section 4.3 is asymptotically biased near the boundaries of the bids' support. Our quantile approach allows one to avoid the problem by considering only inner intervals of the supports. Specifically given $N \in \mathcal{N}$ and x in the interior of $\mathcal{X}$, let $0 < \tau_1(N,x) < \tau_2(N,x) < 1$. In our approach, we use quantiles $Q^*(\tau|N,x)$ with $\tau \in [\tau_1(N,x), \tau_2(N,x)]$. Due to the assumptions on $f^*(v|N,x)$ and since the bidding function is monotone, there are $b_1(N,x)$ and $b_2(N,x)$ such that

$$[b_1(N,x), b_2(N,x)] \subset \left(\underline{b}(N,x), \bar{b}(N,x)\right), \text{ and} \quad (40)$$

$$[q^*(\tau_1(N,x)|N,x), q^*(\tau_2(N,x)|N,x)] \subset (b_1(N,x), b_2(N,x)). \quad (41)$$

By (40), the estimator $\hat{g}^*(b|N,x)$ in Section 4.3 consistently estimates $g^*(b|N,x)$ on the interval $[b_1(N,x), b_2(N,x)]$ as we show in the lemma below. Condition (41) is used for establishing consistency of $\hat{Q}^*(\hat{q}^*(\tau|N,x)|N,x)$.

In practice, $\tau_1(N,x)$ and $\tau_2(N,x)$ can be selected as follows. Following the discussion on page 531 of GPV, in the case with no covariates one can choose $\tau_1(N)$ and $\tau_2(N)$ such that

$$[\hat{q}^*(\tau_1(N)|N), \hat{q}^*(\tau_2(N)|N)] \subset (b_{\min}(N)+h, b_{\max}(N)-h),$$

where $b_{\min}(N)$ and $b_{\max}(N)$ denote the minimum and maximum bids respectively in the auctions with $N_l = N$. When there are covariates available, one can replace $b_{\min}(N)$ and $b_{\max}(N)$ with the corresponding minimum and maximum bids in the neighborhood of x as defined on page 541 of GPV.

Lemma 3. *Under Assumptions 3 and 4, for all x in the interior of $\mathcal{X}$ and $N \in \mathcal{N}$,*

(a) $\hat{\varphi}(x) - \varphi(x) = O_p((Lh^d/\log L)^{-1/2} + h^R)$.

(b) $\hat{\pi}(N|x) - \pi(N|x) = O_p((Lh^d/\log L)^{-1/2} + h^R)$.

(c) $\hat{p}(N,x) - p(N,x) = O_p((Lh^d/\log L)^{-1/2} + h^R)$.

(d) $\sup_{b\in[\underline{b}(N,x),\bar{b}(N,x)]} |\hat{G}^*(b|N,x) - G^*(b|N,x)| = O_p((Lh^d/\log L)^{-1/2} + h^R)$.

(e) $\sup_{\tau\in[\varepsilon,1-\varepsilon]} |\hat{q}^*(\tau|N,x) - q^*(\tau|N,x)| = O_p((Lh^d/\log L)^{-1/2} + h^R)$*, for any* $0 < \varepsilon < 1/2$.

(f) $\sup_{b\in[b_1(N,x),b_2(N,x)]} |\hat{g}^*(b|N,x) - g^*(b|N,x)| = O_p((Lh^{d+1}/\log L)^{-1/2} + h^R)$*, where* $b_1(N,x)$ *and* $b_2(N,x)$ *are defined in (40) and (41).*

(g) $\sup_{\tau\in[\tau_1(N,x),\tau_2(N,x)]} |\hat{Q}^*(\tau|N,x) - Q^*(\tau|N,x)| = O_p((Lh^{d+1}/\log L)^{-1/2} + h^R)$.

(h) $\hat{Q}^*(\hat{\beta}(\tau,N,x)|N,x) = Q^*(\beta(\tau,N,x)|N,x) + O_p((Lh^{d+1}/\log L)^{-1/2} + h^R)$ *uniformly in* τ *such that* $\beta(\tau,N,x) \in [\tau_1(N,x)+\varepsilon, \tau_2(N,x)-\varepsilon]$*, for any* $0 < \varepsilon < (\tau_2(N,x) - \tau_1(N,x))/2$.

Proof of Lemma 3. Parts (a)-(c) of the lemma follow from Lemma B.3 of Newey (1994).

For part (d), define

$$G_0^*(b,N,x) = Np(N,x)\,\pi(N|x)\,G^*(b|N,x)\,\varphi(x),$$

and its estimator

$$\hat{G}_0^*(b,N,x) = \frac{1}{L}\sum_{l=1}^{L}\sum_{i=1}^{N_l} y_{il} 1\{N_l = N\}\, 1(b_{il} \le b)\, K_{*h}(x_l - x),$$

where

$$K_{*h}(x_l - x) = \frac{1}{h^d} K_d\left(\frac{x_l - x}{h}\right), \text{ and}$$
$$K_d\left(\frac{x_l - x}{h}\right) = \prod_{k=1}^{d} K\left(\frac{x_{kl} - x_k}{h}\right). \tag{42}$$

Similarly to Lemma B.2 of Newey (1994), by Lemma 2 and Assumptions 3(b), (c), and (g),

$$\sup_{b \in \left[\underline{b}(N,x), \bar{b}(N,x)\right]} \left| G_0^*(b, N, x) - E\hat{G}_0^*(b, N, x) \right| = O\left(h^R\right). \tag{43}$$

Next, we show that

$$\sup_{b \in \left[\underline{b}(N,x), \bar{b}(N,x)\right]} |\hat{G}_0^*(b, N, x) - E\hat{G}_0^*(b, N, x)| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2}\right). \tag{44}$$

We follow the approach of Pollard (1984). Consider, for given $N \in \mathcal{N}$ and x in the interior of $\mathcal{X}$, a class of functions $\mathcal{Z}$ indexed by h and b, with a representative function

$$z_l(b, N, x) = \sum_{i=1}^{N_l} y_{il} 1\{N_l = N\} 1(b_{il} \leq b) h^d K_{*h}(x_l - x).$$

By the result in Pollard (1984) (Problem 28), the class $\mathcal{Z}$ has polynomial discrimination. Theorem 37 in Pollard (1984) (see also Example 38) implies that for any sequences δ_L, α_L such that $L\delta_L^2\alpha_L^2/\log L \to \infty$, $Ez_l^2(b) \leq \delta_L^2$,

$$\alpha_L^{-1}\delta_L^{-2} \sup_{b \in \left[\underline{b}(N,x), \bar{b}(N,x)\right]} \left|\frac{1}{L}\sum_{l=1}^{L} z_l(b, N, x) - Ez_l(b, N, x)\right| \to 0 \tag{45}$$

almost surely. We claim that this implies that

$$\left(\frac{Lh^d}{\log L}\right)^{1/2} \sup_{b \in \left[\underline{b}(N,x), \bar{b}(N,x)\right]} |\hat{G}_0^*(b, N, x) - E\hat{G}_0^*(b, N, x)|$$

is bounded as $L \to \infty$ almost surely, which in turn implies the result in (44). The proof is by contradiction. Suppose not. Then there exist a sequence $\gamma_L \to \infty$ and a subsequence of L such that along this subsequence

$$\sup_{b \in \left[\underline{b}(N,x), \bar{b}(N,x)\right]} |\hat{G}_0^*(b, N, x) - E\hat{G}_0^*(b, N, x)| \geq \gamma_L \left(\frac{Lh^d}{\log L}\right)^{-1/2}. \tag{46}$$

on a set of events $\Omega' \subset \Omega$ with a positive probability measure. Now if we let $\delta_L^2 = h^d$ and $\alpha_L = (\frac{Lh^d}{\log L})^{-1/2}\gamma_L^{1/2}$, then the definition of z implies that, along the subsequence on a set of events Ω',

$$
\begin{aligned}
& \alpha_L^{-1}\delta_L^{-2} \sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]} |\frac{1}{L}\sum_{l=1}^{L} z_l\left(b,N,x\right) - Ez_l\left(b,N,x\right)| \\
= & \left(\frac{Lh^d}{\log L}\right)^{1/2} \gamma_L^{-1/2} h^{-d} \sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]} |\frac{1}{L}\sum_{l=1}^{L} z_l\left(b,N,x\right) - Ez_l\left(b,N,x\right)| \\
= & \left(\frac{Lh^d}{\log L}\right)^{1/2} \gamma_L^{-1/2} \sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]} |\hat{G}_0^*\left(b,N,x\right) - E\hat{G}_0^*\left(b,N,x\right)| \\
\geq & \left(\frac{Lh^d}{\log L}\right)^{1/2} \gamma_L^{-1/2}\gamma_L \left(\frac{Lh^d}{\log L}\right)^{-1/2} \\
= & \gamma_L^{1/2} \to \infty,
\end{aligned}
$$

where the inequality follows by (46), a contradiction to (45). This establishes (44), so that (43), (44) and the triangle inequality together imply that

$$
\sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]} |\hat{G}_0^*\left(b,N,x\right) - G_0^*\left(b,N,x\right)| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right). \tag{47}
$$

To complete the proof, recall that from the definitions of $G_0^*\left(b,N,x\right)$ and $\hat{G}_0^*\left(b,N,x\right)$,

$$
G^*\left(b|N,x\right) = \frac{G_0^*\left(b,N,x\right)}{p\left(N,x\right)\pi\left(N|x\right)\varphi\left(x\right)} \text{ and } \hat{G}^*\left(b|n,x\right) = \frac{\hat{G}_0^*\left(b,N,x\right)}{\hat{p}\left(N,x\right)\hat{\pi}\left(N|x\right)\hat{\varphi}\left(x\right)},
$$

so that by the mean-value theorem,

$$
\left|\hat{G}^*\left(b|N,x\right) - G^*\left(b|N,x\right)\right| \leq \tilde{C}\left(b,N,x\right)\left\|\begin{pmatrix} \hat{G}_0^*\left(b,N,x\right) - G_0^*\left(b,N,x\right) \\ \hat{p}\left(N,x\right) - p\left(N,x\right) \\ \hat{\pi}\left(N|x\right) - \pi\left(N|x\right) \\ \hat{\varphi}\left(x\right) - \varphi\left(x\right) \end{pmatrix}\right\|, \tag{48}
$$

where $\|\cdot\|$ denotes the Euclidean norm, $\tilde{C}\left(b,N,x\right)$ is given by

$$
\left|\frac{1}{\tilde{p}\left(N,x\right)\tilde{\pi}\left(N,x\right)\tilde{\varphi}\left(x\right)}\right| \left\|\left(1, \frac{\tilde{G}_0^*\left(b,N,x\right)}{\tilde{p}\left(N,x\right)}, \frac{\tilde{G}_0^*\left(b,N,x\right)}{\tilde{\pi}\left(N,x\right)}, \frac{\tilde{G}_0^*\left(b,N,x\right)}{\tilde{\varphi}\left(x\right)}\right)\right\|,
$$

and $\|(\tilde{G}_0^* - G_0^*, \tilde{p} - p, \tilde{\pi} - \pi, \tilde{\varphi} - \varphi)\| \leq \|(\hat{G}_0^* - G_0^*, \hat{p} - p, \hat{\pi} - \pi, \hat{\varphi} - \varphi)\|$. Further by Assumption 3(b), (c), and (g), and the results in parts (a)-(c) of the lemma, with probability approaching one $\tilde{\varphi}$, $\tilde{\pi}$, and $\tilde{p}$ are bounded away from zero. The desired result follows from (47), (48) and parts (a)-(c) of the lemma.

For part (e) of the lemma, since $\hat{G}^*(\cdot|N,x)$ is monotone by construction,

$$\begin{aligned} P\left(\hat{q}^*(\varepsilon|N,x) \leq \underline{b}(N,x)\right) &= P\left(\inf_b \left\{b : \hat{G}^*(b|N,x) \geq \varepsilon\right\} \leq \underline{b}(N,x)\right) \\ &= P\left(\hat{G}^*\left(\underline{b}(N,x)|N,x\right) \geq \varepsilon\right) \\ &= o(1), \end{aligned}$$

where the last equality is by the result in part (d). Similarly,

$$\begin{aligned} P\left(\hat{q}^*(1-\varepsilon|N,x) \geq \overline{b}(N,x)\right) &= P\left(\hat{G}^*\left(\overline{b}(N,x)|N,x\right) \leq 1-\varepsilon\right) \\ &= o(1). \end{aligned}$$

Hence, for all x in the interior of $\mathcal{X}$ and $N \in \mathcal{N}$, $\underline{b}(N,x) < \hat{q}^*(\varepsilon|N,x) < \hat{q}^*(1-\varepsilon|N,x) < \overline{b}(N,x)$ with probability approaching one. Since the distribution $G^*(b|N,x)$ is continuous in b, $G^*(q^*(\tau|N,x)|N,x) = \tau$, and for $\tau \in [\varepsilon, 1-\varepsilon]$, we can write the identity

$$G^*(\hat{q}^*(\tau|N,x)|N,x) - G^*(q^*(\tau|N,x)|N,x) = G^*(\hat{q}^*(\tau|N,x)|N,x) - \tau. \tag{49}$$

Next,

$$0 \leq \hat{G}^*(\hat{q}^*(\tau|N,x)|N,x) - \tau \leq \frac{(\sup_u K(u))^d}{\hat{p}(N,x)\,\hat{\pi}(N|x)\,\hat{\varphi}(x)\,NLh^d}, \tag{50}$$

where the first inequality is by Lemma 21.1(ii) in van der Vaart (1998), and the second inequality holds with probability one since $\hat{G}^*(\cdot|N,x)$ is an empirical CDF and the distribution of bids is continuous so that ties occur with probability zero. By (50) and the results in (a)-(c),

$$\hat{G}^*(\hat{q}^*(\tau|N,x)|N,x) = \tau + O_p\left(\left(Lh^d\right)^{-1}\right) \tag{51}$$

uniformly over $\tau \in [\varepsilon, 1-\varepsilon]$. Combining (49) and (51), and applying the mean-value theorem to the left-hand side of (49), we obtain

$$\begin{aligned} &\hat{q}^*(\tau|N,x) - q^*(\tau|N,x) \\ &= \frac{G^*(\hat{q}^*(\tau|N,x)|N,x) - \hat{G}^*(\hat{q}^*(\tau|N,x)|N,x)}{g^*(\widetilde{q}^*(\tau|N,x)|N,x)} + O_p\left(\left(Lh^d\right)^{-1}\right), \end{aligned} \tag{52}$$

where $\widetilde{q}^*$ lies between $\hat{q}^*$ and q^* for all (τ, N, x). Now, by Lemma 2, $g^*(b|N,x)$ is bounded away from zero, and the result in part (e) follows from (52) and part (d) of the lemma.

To prove part (f), by Lemma 2, $g^*(\cdot|N,\cdot)$ admits up to R continuous bounded partial derivatives. Let

$$g_0^*(b,N,x) = p(N,x)\,\pi(N|x)\,\varphi(x)\,g^*(b|N,x), \text{ and} \tag{53}$$

$$\hat{g}_0^*(b,N,x) = \hat{p}(N,x)\,\hat{\pi}(N|x)\,\hat{\varphi}(x)\,\hat{g}^*(b|N,x). \tag{54}$$

By Lemma B.3 of Newey (1994), $\hat{g}_0^*(b, N, x)$ is uniformly consistent in b over the interval $[b_1(N,x), b_2(N,x)]$. By the results in parts (a)-(c), the estimators $\hat{p}(N,x)$, $\hat{\pi}(N|x)$, and $\hat{\varphi}(x)$ converge at a faster rate than that of $\hat{g}_0^*(b, N, x)$. The desired result follows by the same argument as in the proof of part (d), equation (48).

Next, we prove part (g). By Lemma 2, $g^*(b|N,x) > c_g > 0$ for some constant c_g. Then,

$$\begin{aligned}
& \left|\hat{Q}^*(\tau|N,x) - Q^*(\tau|N,x)\right| \\
\leq\ & |\hat{q}^*(\tau|N,x) - q^*(\tau|N,x)| + 2\frac{|\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x) - g^*(q^*(\tau|N,x)|N,x)|}{p(N,x)\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)c_g} \\
& + \frac{|\hat{p}(N,x) - p(N,x)|}{\hat{p}(N,x)p(N,x)\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)} \\
\leq\ & \left(1 + \frac{2\sup_{b\in[b_1(N,x),b_2(N,x)]}|\partial g^*(b|N,x)/\partial b|}{p(N,x)\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)c_g}\right)|\hat{q}^*(\tau|n,x) - q^*(\tau|n,x)| \\
& + 2\frac{|\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x) - g^*(\hat{q}^*(\tau|N,x)|N,x)|}{p(N,x)\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)c_g} \\
& + \frac{|\hat{p}(N,x) - p(N,x)|}{\hat{p}(N,x)p(N,x)\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)}. \qquad (55)
\end{aligned}$$

Define the event

$$E_L(N,x) = \{\hat{q}^*(\tau_1(N,x)|N,x) \geq b_1(N,x), \hat{q}^*(\tau_2(N,x)|N,x) \leq b_2(N,x)\},$$

and let $\beta_L = \left(Lh^{d+1}/\log L\right)^{-1/2} + h^R$. By the result in part (e), $P(E_L^c(N,x)) = o(1)$. Hence, it follows from part (e) of the lemma that the estimator $\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)$ is bounded away from zero with probability approaching one, and the first term on the right-hand side of (55) is $O_p(\beta_L)$ uniformly over $[\tau_1(N,x), \tau_2(N,x)]$. Next,

$$\begin{aligned}
& P\left(\sup_{\tau\in[\tau_1(N,x),\tau_2(N,x)]}\beta_L^{-1}|\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x) - g^*(\hat{q}^*(\tau|N,x)|N,x)| > M\right) \\
\leq\ & P\left(\sup_{\tau\in[\tau_1(N,x),\tau_2(N,x)]}\beta_L^{-1}|\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x) - g^*(\hat{q}^*(\tau|N,x)|N,x)| > M, E_L(x)\right) \\
& + P(E_L^c(x)) \\
\leq\ & P\left(\sup_{b\in[b_1(N,x),b_2(N,x)]}\beta_L^{-1}|\hat{g}^*(b|N,x) - g^*(b|N,x)| > M\right) + o(1). \qquad (56)
\end{aligned}$$

Part (g) follows from (55) and (56) by the results in parts (c) and (f) of the lemma.

For part (h), first for all $\tau \in [0,1]$,

$$\left|\hat{\beta}(\tau, N, x) - \beta(\tau, N, x)\right| \leq \left|\frac{\hat{p}(\bar{N}, x)}{\hat{p}(N,x)} - \frac{p(\bar{N}, x)}{p(N,x)}\right|,$$

and therefore by the result in part (c) of the lemma, $\sup_{\tau\in[0,1]}|\hat{\beta}(\tau,N,x)-\beta(\tau,N,x)|=O_p((Lh^d/\log L)^{-1/2}+h^R)$ for all $N\in\mathcal{N}$ and x in the interior of $\mathcal{X}$. The desired result then follows by the triangular inequality, uniform consistency of $\hat{\beta}(\tau,N,x)$, the result in part (g) of the lemma, differentiability of $Q^*(\cdot|N,x)$, the mean-value theorem, and since $f^*(\cdot|N,x)$ is bounded away from zero by Assumption 3(f). ■

Lemma 4. *Suppose that Assumptions 3 and 4 hold, and that the bandwidth h is such that $Lh^{d+1}\to\infty$, $\sqrt{Lh^{d+1}}h^R\to 0$. Then,*

$$\sqrt{Lh^{d+1}}\left(\hat{g}^*(b|N,x)-g^*(b|N,x)\right)\to_d N\left(0,V_g(b,N,x)\right)$$

for $b\in[b_1(N,x),b_2(N,x)]$, x in the interior of $\mathcal{X}$, $N\in\mathcal{N}$, where $b_1(N,x)$ and $b_2(N,x)$ are defined in (40) and (41), and $V_g(b,N,x)$ is given by

$$V_g(N,b,x)=\frac{g^*(b|N,x)}{Np(N,x)\pi(N|x)\varphi(x)}\left(\int K(u)^2\,du\right)^{d+1}.$$

Furthermore, $\hat{g}^*(b|N_1,x)$ and $\hat{g}^*(b|N_2,x)$ are asymptotically independent for all $N_1\neq N_2$, $N_1,N_2\in\mathcal{N}$.

Proof of Lemma 4. Consider $g_0^*(b,n,x)$ and $\hat{g}_0^*(b,n,x)$ defined in (53) and (54) respectively. It follows from parts (a)-(c) of Lemma 3,

$$\begin{aligned}&\sqrt{Lh^{d+1}}\left(\hat{g}^*(b|N,x)-g^*(b|N,x)\right)\\&\quad=\frac{1}{p(N,x)\pi(N|x)\varphi(x)}\sqrt{Lh^{d+1}}\left(\hat{g}_0^*(b,N,x)-g_0^*(b,N,x)\right)+o_p(1).\end{aligned}\tag{57}$$

Furthermore, as in Lemma B2 of Newey (1994), $E\hat{g}_0^*(b,N,x)-g_0^*(b,N,x)=O\left(h^R\right)$ uniformly in $b\in[b_1(N,x),b_2(N,x)]$ for all x in the interior of $\mathcal{X}$ and $N\in\mathcal{N}$. Thus, it remains to establish asymptotic normality of $\sqrt{Lh^{d+1}}\left(\hat{g}_0^*(b,N,x)-E\hat{g}_0^*(b,N,x)\right)$.

Define

$$\begin{aligned}w_{il,N}&=\sqrt{\frac{1}{h^{d+1}}}y_{il}1\{N_l=N\}K\left(\frac{b_{il}-b}{h}\right)K_d\left(\frac{x_l-x}{h}\right),\\\overline{w}_{L,N}&=\frac{1}{NL}\sum_{l=1}^{L}\sum_{i=l}^{N_l}w_{il,N},\end{aligned}$$

where K_d is defined in (42). With the above definitions we have that

$$\sqrt{NLh^{d+1}}\left(\hat{g}_0^*(b,N,x)-E\hat{g}_0^*(b,N,x)\right)=\sqrt{NL}\left(\overline{w}_{L,N}-E\overline{w}_{L,N}\right).\tag{58}$$

Then, by the Liapunov CLT (see, for example, Corollary 11.2.1 on page 427 of Lehman and Romano (2005)),

$$\sqrt{NL}\left(\overline{w}_{L,N}-E\overline{w}_{L,N}\right)/\sqrt{NLVar\left(\overline{w}_{L,N}\right)}\to_d N(0,1),\tag{59}$$

provided that $Ew_{il,N}^2 < \infty$, and for some $\delta > 0$,

$$\lim_{L\to\infty} \frac{1}{L^{\delta/2}} E\left|w_{il,N} - Ew_{il,N}\right|^{2+\delta} = 0.$$

The last condition follows from the Liapunov's condition (equation (11.12) on page 427 of Lehman and Romano (2005)) and because $w_{il,N}$ are i.i.d. Next, $Ew_{il,N}$ is given by

$$\begin{aligned}
& \sqrt{\frac{1}{h^{d+1}}} E\left(p\left(N, x_l\right) \pi\left(N | x_l\right) \int K\left(\frac{u-b}{h}\right) g^*\left(u | N, x_l\right) du K_d\left(\frac{x_l - x}{h}\right)\right) \\
= & \sqrt{\frac{1}{h^{d+1}}} \int\int p\left(N, y\right) \pi\left(N | y\right) K\left(\frac{u-b}{h}\right) g^*\left(u | N, y\right) K_d\left(\frac{y-x}{h}\right) \varphi\left(y\right) dudy \\
= & \sqrt{h^{d+1}} \\
& \times \int\int p\left(N, x+hy\right) \pi\left(N | x+hy\right) K\left(u\right) g^*\left(b+hu | N, x+hy\right) K_d\left(y\right) \varphi\left(x+hy\right) dudy \\
\to & \ 0.
\end{aligned}$$

Further, $Ew_{il,N}^2$ is given by

$$\begin{aligned}
& \frac{1}{h^{d+1}} \int\int p\left(N, y\right) \pi\left(N | y\right) K^2\left(\frac{u-b}{h}\right) g^*\left(u | N, y\right) K_d^2\left(\frac{y-x}{h}\right) \varphi\left(y\right) dudy \\
= & \int\int p\left(N, x+hy\right) \pi\left(N | x+hy\right) K^2\left(u\right) g^*\left(b+hu | N, x+hy\right) K_d^2\left(y\right) \varphi\left(x+hy\right) dudy \\
< & \ \infty.
\end{aligned}$$

Hence,

$$NLVar\left(\overline{w}_{L,N}\right) \to p\left(N, x\right) \pi\left(N | x\right) g^*\left(b | N, x\right) \varphi\left(x\right) \left(\int K^2\left(u\right) du\right)^{d+1} du. \tag{60}$$

Next, $E\left|w_{il,N}\right|^{2+\delta}$ is bounded by

$$\begin{aligned}
& \frac{1}{h^{(d+1)(1+\delta/2)}} \int\int \left|K\left(\frac{u-b}{h}\right)\right|^{2+\delta} g^*\left(u | N, y\right) \left|K_d\left(\frac{y-x}{h}\right)\right|^{2+\delta} \varphi\left(y\right) dudy \\
= & \frac{1}{h^{(d+1)\delta/2}} \int\int \left|K\left(u\right)\right|^{2+\delta} g^*\left(b+hu | N, x+hy\right) \left|K_d\left(y\right)\right|^{2+\delta} \varphi\left(x+hy\right) dudy \\
\leq & \frac{1}{h^{(d+1)\delta/2}} \sup_{u\in[-1,1]} \left|K\left(u\right)\right|^{(d+1)(2+\delta)} \sup_{x\in\mathcal{X}} \varphi\left(x\right) \sup_{b\in[b_1(N,x), b_2(N,x)]} g^*\left(b | N, x\right) \\
= & \frac{C}{h^{(d+1)\delta/2}}.
\end{aligned}$$

Lastly,

$$\begin{aligned}
\frac{1}{L^{\delta/2}} E\left|w_{il,N} - Ew_{il,N}\right|^{2+\delta} &\leq \frac{2^{1+\delta}}{L^{\delta/2}} E\left|w_{il,N}\right|^{2+\delta} \\
&\leq \frac{2^{1+\delta} C}{\left(Lh^{d+1}\right)^{\delta/2}} \\
&\to 0, \qquad (61)
\end{aligned}$$

since $Lh^{d+1} \to \infty$ by the assumption. The first result of the lemma follows now from (57)-(61).

Next, note that the asymptotic covariance of $\overline{w}_{L,N_1}$ and $\overline{w}_{L,N_2}$ involves a product of the two indicator functions, $1\{N_l = N_1\} 1\{N_l = N_2\}$, which is zero for all $N_1 \neq N_2$. The joint asymptotic normality and asymptotic independence of $\hat{g}^*(b|N_1, x)$ and $\hat{g}^*(b|N_2, x)$ follows then by the Cramér-Wold device. ■

Proposition 7. *Assume that the bandwidth h satisfies $Lh^{d+1} \to \infty$ and $\sqrt{Lh^{d+1}} h^R \to 0$. Then, for $\tau \in (0,1)$, x in the interior of $\mathcal{X}$, and under Assumptions 3 and 4,*

$$\sqrt{Lh^{d+1}} \left(\hat{Q}^*(\tau|N, x) - Q^*(\tau|N, x)\right) \to_d N\left(0, V_Q(N, \tau, x)\right),$$

$$\sqrt{Lh^{d+1}} \left(\hat{Q}^*\left(\hat{\beta}(\tau, N, x)|N, x\right) - Q^*(\beta(\tau, N, x)|N, x)\right) \to_d N\left(0, V_Q(N, \beta(\tau, N, x), x)\right),$$

where

$$V_Q(N, \tau, x) = \left(\frac{1 - p(N, x)(1-\tau)}{(N-1)p(N, x)\, g^{*2}(q^*(\tau|N, x)|N, x)}\right)^2 V_g(N, q^*(\tau|N, x), x),$$

and $V_g(N, \tau, x)$ is defined in Lemma 4. Moreover, for any distinct $N, N' \in \left\{\underline{N}, ...\bar{N}\right\}$, $\tau, \tau' \in \Upsilon$, and x, x' in the interior of $\mathcal{X}$, the estimators $\hat{Q}^*(\tau|N, x)$ are asymptotically independent, as well as the estimators $\hat{Q}^*\left(\hat{\beta}(\tau, N, x)|N, x\right)$.

Proof of Proposition 7. First, by Lemma 3 (c), (e) and (f), and the mean-value theorem,

$$\begin{aligned}
\hat{Q}^*(\tau|N, x) = Q^*(\tau|N, x) &- \frac{1 - p(N, x)(1-\tau)}{(N-1)p(N, x)\, \widetilde{g}^{*2}(q^*(\tau|N, x)|N, x)} \\
&\times \left(\hat{g}^*(q^*(\tau|N, x)) - g^*(q^*(\tau|N, x))\right) + o_p\left(\frac{1}{\sqrt{Lh^{d+1}}}\right), \qquad (62)
\end{aligned}$$

where $\widetilde{g}^*$ is a mean-value between g^* and $\hat{g}^*$ for $b = q^*(\tau|N, x)$. The result follows then by Lemma 4. ■

D Asymptotic variance for the binomial restriction test

By the delta-method, the asymptotic variance of $\hat{\Delta}_{Binomial}(n, N, x)$ is given by

$$V_{Binomial}(n, N, x) = V_\rho(n, N, x) + D^2(n, N, x)\, V_p(N, x) - 2D(n, N, x)\, \sigma_{\rho,p}(n, N, x),$$

where V_ρ and V_p are the asymptotic variances of $\hat{\rho}$ and $\hat{p}$ respectively:

$$\begin{aligned} V_\rho(n, N, x) &= \frac{\rho(n|N, x)(1 - \rho(n|N, x))\left(\int K^2(u)\, du\right)^d}{N\pi(N|x)\varphi(x)}, \\ V_p(N, x) &= \frac{p(N, x)(1 - p(N, x))\left(\int K^2(u)\, du\right)^d}{N\pi(N|x)\varphi(x)}, \end{aligned}$$

$D(n, N, x)$ is the derivative of $\begin{pmatrix} N \\ n \end{pmatrix} p(N, x)^n (1 - p(N, x))^{N-n}$ with respect to $p(N, x)$:

$$\begin{aligned} &D(n, N, x) = \\ &\quad \begin{pmatrix} N \\ n \end{pmatrix} \left(np(N, x)^{n-1}(1 - p(N, x))^{N-n} - (N - n)\, p(N, x)^n (1 - p(N, x))^{N-n-1}\right), \end{aligned}$$

and $\sigma_{\rho,p}(n, N, x)$ is the asymptotic covariance between $\hat{\rho}$ and $\hat{p}$:

$$\sigma_{\rho,p}(n, N, x) = \frac{E\left\{(1(n_l = n) - \rho(n|N, x))(n_l/N - p(N, x))|N, x\right\}\left(\int K^2(u)\, du\right)^d}{N\pi(N|x)\varphi(x)}.$$

In the above expression, the term $E\left\{(I(n_l = n) - \rho(n|N, x))(n_l/N - p(N, x))|N, x\right\}$ can be estimated as

$$\frac{\sum_{l=1}^L (1(n_l = n) - \hat{\rho}(n|N, x))(n_l/N - \hat{p}(N, x))\, 1\{N_l = N\} \prod_{k=1}^d K\left(\frac{x_{kl} - x_k}{h}\right)}{\sum_{l=1}^L 1\{N_l = N\} \prod_{k=1}^d K\left(\frac{x_{kl} - x_k}{h}\right)}.$$

E Validity of the bootstrap

We establish the bootstrap validity for the AME test (32). For the other tests, the proof is analogous and therefore omitted. As is standard in the literature on statistical testing, see for example Chapter 21.4 in Gourieroux and Monfort (1995), we control the maximum asymptotic rejection probability under our composite null hypothesis, and replace the statistic $T^{AME}(x)$ with its infeasible re-centered version $\bar{T}^{AME}(x)$,

$$\begin{aligned} \bar{T}^{AME}(x) &= \sup_{\tau \in \Upsilon} \sqrt{Lh^{d+1}} \sum_{N=\underline{N}}^{\bar{N}} \sum_{N'=N}^{\bar{N}} \frac{\left[\hat{\Delta}(\tau, N, N', x) - \Delta(\tau, N, N', x)\right]_+}{\hat{\sigma}(\tau, N, N', x)}, \text{ where} \\ \Delta\left(\tau, N, N', x\right) &= Q^*(\tau|N, x) - Q^*\left(\tau|N', x\right). \end{aligned}$$

Since under the null hypothesis, $\Delta(\tau, N, N', x) \leq 0$, it follows that $P(T^{AME}(x) > u) \leq P(\bar{T}^{AME}(x) > u)$ for all $u \in \mathbb{R}$. Therefore, when the test based on $\bar{T}^{AME}(x)$ has asymptotic size α, the asymptotic size of the test $T^{AME}(x)$ is less or equal to α. Thus, it suffices to show that

$$\sup_u \left| P(\bar{T}^{AME}(x) > u) - P^\dagger(T_m^{\dagger, AME}(x) > u) \right| \to_p 0, \tag{63}$$

where $P^\dagger$ denotes the bootstrap distribution conditional on the original data.

To show (63), we proceed as follows. First, the results in Lemma 3 imply the following delta-method expansion.

$$\begin{aligned}
&\sqrt{Lh^{d+1}} \frac{\left[\hat{\Delta}(\tau, N, N', x) - \Delta(\tau, N, N', x)\right]_+}{\hat{\sigma}(\tau, N, N', x)} = \frac{\kappa(\tau, N, N', x)}{\sigma(\tau, N, N', x)} \\
&\times \sqrt{Lh^{d+1}} \left[\hat{g}^*(q^*(\tau|N,x)) - g^*(q^*(\tau|N,x)) - \hat{g}^*\left(q^*\left(\tau|N',x\right)\right) + g^*\left(q^*\left(\tau|N',x\right)\right)\right]_+ \\
&+ o_p(1),
\end{aligned} \tag{64}$$

where $\kappa(\tau, N, N', x)$ is determined by the term in the front of $\hat{g}^*(q^*(\tau|N,x)) - g^*(q^*(\tau|N,x))$ in (62), and the $o_p(1)$ term is uniform in $\tau \in [\tau_1(N,x), \tau_2(N,x)]$.

Let $\hat{g}_m^{*,\dagger}(\cdot|N,x)$ be the bootstrap analogue of $\hat{g}^*(\cdot|N,x)$ in bootstrap sample m. We say that a bootstrap statistic $\zeta_L^\dagger$ is $o_p^\dagger(1)$ if for all $\varepsilon > 0$, $P^\dagger(|\zeta_L^\dagger| > \varepsilon) \to_p 0$. By repeating the steps of Lemma 3 for the bootstrap analogues of the original sample statistics, one can show that the bootstrap version of (64) holds as well:

$$\begin{aligned}
&\sqrt{Lh^{d+1}} \frac{\left[\hat{\Delta}_m^\dagger(\tau, N, N', x) - \hat{\Delta}(\tau, N, N', x)\right]_+}{\hat{\sigma}^\dagger(\tau, N, N', x)} = \frac{\kappa(\tau, N, N', x)}{\sigma(\tau, N, N', x)} \\
&\times \sqrt{Lh^{d+1}} \left[\hat{g}_m^{*,\dagger}(q^*(\tau|N,x)) - \hat{g}^*(q^*(\tau|N,x)) - \hat{g}_m^{*,\dagger}\left(q^*\left(\tau|N',x\right)\right) + \hat{g}^*\left(q^*\left(\tau|N',x\right)\right)\right]_+ \\
&+ o_p^\dagger(1),
\end{aligned} \tag{65}$$

where the $o_p^\dagger(1)$ term is again uniform in $\tau \in [\tau_1(N,x), \tau_2(N,x)]$.[36]

Next, we use the uniform strong approximation for the bootstrap of Chen and Lo (1997). Provided that $Lh^{d+1} \to \infty$ and by Proposition 3.2 in Chen and Lo (1997), one can construct $\tilde{g}^*(\cdot|N,x)$ independent of the original data, such that $\tilde{g}^*(\cdot|N,x) =^d \hat{g}^*(\cdot|N,x)$, and for almost all sample paths,

$$\begin{aligned}
\sup_{x \in \mathcal{X}^0} \sup_{b \in [b_1(N,x), b_2(N,x)]} \sqrt{Lh^{d+1}} \left| \hat{g}_m^{*,\dagger}(b|N,x) - \hat{g}^*(b|N,x) - (\tilde{g}^*(b|N,x) - g^*(b|N,x)) \right| \\
= O(\delta_L),
\end{aligned} \tag{66}$$

[36]The proof of (65) is given in the supplement Marmer, Shneyerov, and Xu (2011), which is available from the UBC working papers series and the authors' web-pages.

where $\delta_L \to 0$ is a sequence of constants.[37] In the above result, $[b_1(N,x), b_2(N,x)]$ and $\mathcal{X}^0$ are the inner compact subsets of the support of bids and x respectively.

Define

$$\frac{\left[\tilde{\Delta}(\tau,N,N',x)-\Delta(\tau,N,N',x)\right]_+}{\sigma(\tau,N,N',x)} = \frac{\kappa(\tau,N,N',x)}{\sigma(\tau,N,N',x)}$$
$$\times\left[\tilde{g}^*\left(q^*(\tau|N,x)\right)-g^*\left(q^*(\tau|N,x)\right)-\tilde{g}^*\left(q^*\left(\tau|N',x\right)\right)+g^*\left(q^*\left(\tau|N',x\right)\right)\right]_+,$$

and

$$\tilde{T}^{AME}(x)=\sup_{\tau\in\Upsilon}\sqrt{Lh^{d+1}}\sum_{N=\underline{N}}^{\bar{N}}\sum_{N'=N}^{\bar{N}}\frac{\left[\tilde{\Delta}(\tau,N,N',x)-\Delta(\tau,N,N',x)\right]_+}{\sigma(\tau,N,N',x)}.$$

By the results in Lemma 4, (64), and the Continuous Mapping Theorem,

$$\bar{T}^{AME}(x)\to_d \mathcal{T}, \tag{67}$$

where $\mathcal{T}$ is a random variable with a continuous CDF. Since $\tilde{g}^*(\cdot|N,x) =^d \hat{g}^*(\cdot|N,x)$ by construction, we have as well that

$$\tilde{T}^{AME}(x)\to_d \mathcal{T}. \tag{68}$$

Next, by (65) and (66),

$$T_m^{\dagger,AME}(x)-\tilde{T}^{AME}(x)=o_p^{\dagger}(1). \tag{69}$$

Since $\tilde{g}^*(\cdot|N,x)$ is independent of the original data by construction, $P^{\dagger}(\tilde{T}^{AME}(x)\le u) = P(\tilde{T}^{AME}(x)\le u)$ for all $u\in\mathbb{R}$. This, together with (69) and the fact that the CDF of $\mathcal{T}$ is continuous, implies that

$$P(\tilde{T}^{AME}(x)>u)-P^{\dagger}(T_m^{\dagger,AME}(x)>u)\to_p 0 \tag{70}$$

for all $u\in\mathbb{R}$.[38] Lastly, by (67), (68), and (70), we have that for all $u\in\mathbb{R}$,

$$P(\bar{T}^{AME}(x)>u)-P^{\dagger}(T_m^{\dagger,AME}(x)>u)\to_p 0. \tag{71}$$

The result in (63) now follows by the pointwise convergence in (71) and Pólya's Theorem (Shao and Tu, 1995, page 447).

[37] See (14) in Chen and Lo (1997) for the precise definition of δ_L.

[38] This can be shown similarly to Theorem 22.4 on page 349 in Davidson (1994).

F Monte-Carlo experiment

In this section we present a Monte-Carlo study of the small sample properties of the tests. In particular, we are interested in seeing how a dense grid of quantiles τ affects size and power of the tests.

We simulate S and V using a Gaussian copula. Let (Z_1, Z_2) be bivariate normal with zero means, variances equal to one, and correlation coefficient $\rho \in [0, 1)$. Informative signals corresponds to $\rho > 0$, while the LS model corresponds to $\rho = 0$. Let Φ denote the standard normal CDF. A pair (S, V) is generated as $S = \Phi(Z_1)$ and $V = \Phi(Z_2)$.

Details of the computation of the distributions $F(v|S)$ and $F^*(v|N)$ that are needed in order to solve for the equilibrium of the auction are as follows. First, $Z_2|Z_1 \sim N\left(\rho z_1, 1 - \rho^2\right)$, and therefore the conditional distribution of V given S is given by

$$\begin{aligned} F(v|S) &= P(V \leq v|S) \\ &= P\left(Z_2 \leq \Phi^{-1}(v) | \Phi^{-1}(S)\right) \\ &= \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(S)}{\sqrt{1-\rho^2}}\right). \end{aligned}$$

Next, note that the marginal distribution of S is uniform on the $[0, 1]$ interval, therefore

$$\begin{aligned} F^*(v|N) &= F(v|S \geq \bar{s}(N)) \\ &= \frac{1}{1-\bar{s}} \int_{\bar{s}(N)}^{1} \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(s)}{\sqrt{1-\rho^2}}\right) ds, \end{aligned}$$

where the cutoff signal $\bar{s}(N)$ can be found, given the value of N, as a solution to equation (4). Lastly, for $S \geq \bar{s}(N)$, the bids are computed according as a solution to (2).

In our simulations, we set $L = 250$, $\mathcal{N} = \{2, 3, 4, 5\}$, $\pi(N) = 1/4$ for all $N \in \mathcal{N}$, and $k = 0.17$. The number of Monte Carlo replications is 1,000; in each replication, the critical values for the tests are obtained using 999 replications. We use the tri-weight kernel function $K(u) = (35/32)\left(1 - u^2\right)^3 1\{|u| \leq 1\}$ and $1.978 \times 1.06 \times (\text{std.err.}) \times (\text{sample size})^{-(1/5)}$ for the bandwidth. Our grid consists of 200 quantiles equally spaced between 0.05 and 0.95.

We generate three random samples. The first one is $\rho = 0$, which corresponds to LS model. The second is an AME sample with $\rho = 0.5$. The last sample is the S sample. Table 4 reports the results of size simulations for the tests. T^{AME} test performs reasonably well in the finite samples. Some under rejection is observed for the LS sample ($\rho = 0$). For nominal sizes of 0.1, 0.05, and 0.01, the simulated rejection rates are approximately 0.057, 0.025 and 0.005 respectively. For the AME ($\rho = 0.5$), the simulated rejection probabilities are very close to their nominal values: for the same nominal sizes, they are approximately 0.088, 0.047, and 0.014. Finally, the rejection probabilities in the S experiment are also close to their nominal values.

Table 4: Simulated size and size-corrected power of models' tests

Test	Nominal size		
	0.1	0.05	0.01
		Size	
LS	0.0570	0.0250	0.0050
AME	0.0880	0.0470	0.0140
S	0.0850	0.0510	0.0120
		Size-corrected power	
LS	1.0000	1.0000	1.0000
AME	0.9930	0.9700	0.8550
S	0.9950	0.9810	0.9300

Table 1 also reports the *size corrected* power results (the critical values are computed from the simulated distribution of the test statistic under the null). To address the power issue for the AME test, it is necessary to come up with an alternative. We reverse the order of quantiles, making them *decreasing* in N. To do this in the simplest fashion possible, we multiply each quantile by minus one and then add a constant to all quantiles to assure that they are positive.

The results in Table 4 show that the power properties are good. For $\rho = 0.5$ and for nominal sizes of 10%, 5%, and 1%, the simulated rejection rates are 99.3%, 97% and 85.5% respectively. Similar results hold for LS ($\rho = 0$) and S models. Taken together, the simulation results indicate that our bootstrap method has both good size and power properties, even in samples of moderate size.

G Circumventing the curse of dimensionality: A single index approach

Consider a single index model

$$\begin{aligned} F(v, s|x) &= F\left(v, s|x'\beta\right), \\ r &= r\left(x'\beta\right). \end{aligned}$$

(In this section, we abuse the notation slightly by often keeping it the same for the distribution conditional on the single index, as well as for other relevant objects, r etc.) Here $x'\beta$ is a *single index* that captures the dependence of signals and valuations on covariates, and $\beta \in \mathbb{R}^d$ is a vector of coefficients, identifiable up to a common scale normalization. For simplicity, assume that the entry cost k does not vary with x.[39] Equation (21) implies that the signal cutoffs $\bar{s}$ are also functions of the single index $x'\beta$, say $\bar{s}(N, x'\beta)$, and therefore the distribution of active bidders' valuations $F^*(v|x'\beta)$ also depends on x only through $x'\beta$. Equation (20) implies that the bidding strategy $B(\cdot|N, x'\beta)$ also depends on x only through the single index $x'\beta$. Moreover, since the bidding strategy is monotone increasing, the quantiles of bids are equal to

$$\begin{aligned} q^*(\tau|N,x) &= B\left(Q^*\left(\tau|N,x'\beta\right)|N,x'\beta\right) \\ &\equiv \tilde{q}^*\left(\tau|N,x'\beta\right), \end{aligned}$$

i.e. also depend on x only through the single index $x'\beta$. Consequently, we can estimate β from the bids data by any of the methods proposed in the literature on single index quantile regression.

In particular, we can use an average derivative estimator. For $u = x'\beta$, we have

$$\frac{\partial q^*(\tau|N,x)}{\partial x} = \beta \frac{\partial \tilde{q}^*(\tau|N,u)}{\partial u}, \tag{72}$$

and β can be estimated as an average quantile derivative. (Recall that β is only identifiable up to a scale normalization). Equation (72) implies that β is *proportional* to the average derivative

$$\int \frac{\partial q^*(\tau|N,x)}{\partial x} w(x)\varphi(x)\,dx, \tag{73}$$

where $w(\cdot)$ is a nonnegative, smooth weighting function with compact support within $\mathcal{X}$. Since β is only identifiable up to a scalar multiple, we can normalize β by setting it *equal* to (73). Taking into account this compact support assumption for $w(\cdot)$ and using integration by parts in (73),

$$\begin{aligned} \beta &= -\int q^*(\tau|N,x)\frac{\partial\left[w(x)\varphi(x)\right]}{\partial x}dx \\ &= -\int q^*(\tau|N,x)\left(\varphi(x)\frac{\partial w(x)}{\partial x} + w(x)\frac{\partial \varphi(x)}{\partial x}\right)dx. \end{aligned} \tag{74}$$

Chaudhuri, Doksum, and Samarov (1997) propose an estimator of β based on a finite sample analogue to the average derivative (74):

$$\hat{\beta} = -\frac{1}{L}\sum_{l=1}^{L}\frac{\hat{q}^*(\tau|N,x)}{\hat{\varphi}(x_l)}\left(\hat{\varphi}(x_l)\frac{\partial w(x_l)}{\partial x} + w(x_l)\frac{\partial \hat{\varphi}(x_l)}{\partial x}\right), \tag{75}$$

[39] This assumption may be plausible in some applications. For example, Bajari, Hong, and Ryan (2010) in their empirical study of highway procurement auctions assume that entry costs do not depend on auction characteristics.

and provide conditions under which this estimator is root-L consistent, $\hat{\beta} = \beta + O_p\left(L^{-1/2}\right)$. Since the convergence of the estimator $\hat{\beta}$ is root-L, which is faster than nonparametric, the asymptotics of our estimators $\hat{Q}^*(\tau|N,x)$ etc. will remain unaffected if we use the single index $x'\hat{\beta}$ in place of x in the implementation of the nonparametric estimators described in the main text.

H Econometric details of the quantile-matching GMM estimator

Here, we provide the detailed expressions for Ω_0 and D_0 that appear in the asymptotic distribution of the GMM estimator $\hat{\theta}$ in Section 5.1. For generic random vectors Y and Z, we define $E_Z Y \equiv E[Y|Z]$. First,

$$\begin{aligned}
& E_{N_l,x_l}\left[y_{il}\left(\tau_1 - 1\left(b_{il} \leq q^*(\tau_1|N_l,x_l;\theta_0)\right)\right)\left(\tau_2 - 1\left(b_{il} \leq q^*(\tau_2|N_l,x_l;\theta_0)\right)\right)\right] \\
= \; & E_{N_l,x_l}\left[y_{il}E_{N_l,x_l,y_l}\left[\left(\tau_1 - 1\left(b_{il} \leq q^*(\tau_1|N_l,x_l;\theta_0)\right)\right)\left(\tau_2 - 1\left(b_{il} \leq q^*(\tau_2|N_l,x_l;\theta_0)\right)\right)\right]\right] \\
= \; & E_{N_l,x_l}\left[y_{il}\left(\tau_1 \wedge \tau_2 - \tau_1\tau_2\right)\right], \text{ where } \tau_1 \wedge \tau_2 = \min\{\tau_1,\tau_2\} \\
= \; & \left(\tau_1 \wedge \tau_2 - \tau_1\tau_2\right) p\left(N_l,x_l;\theta_0\right).
\end{aligned}$$

Next,

$$\begin{aligned}
& E_{N_l,x_l}\left[y_{il}\left(\tau - 1\left(b_{il} \leq q^*(\tau|N_l,x_l;\theta_0)\right)\right)\left(y_{il} - p(N_l,x_l;\theta_0)\right)\right] \\
= \; & E_{N_l,x_l}\left[y_{il}\left(y_{il} - p(N_l,x_l;\theta_0)\right)\left(\tau - E_{N_l,x_l,y_{il}}\left[1\left(b_{il} \leq q^*(\tau|N_l,x_l;\theta_0)\right)\right]\right)\right] \\
= \; & 0,
\end{aligned}$$

and

$$\begin{aligned}
E_{N_l,x_l}\left(y_{il} - p(N_l,x_l;\theta_0)\right)^2 &= E_{N_l,x_l}y_{il} - p\left(N_l,x_l;\theta_0\right)^2 \\
&= p\left(N_l,x_l;\theta_0\right)\left(1 - p\left(N_l,x_l;\theta_0\right)\right)
\end{aligned}$$

Therefore,

$$E_{N_l,x_l} m_l(\theta_0) m_l(\theta_0)' = \frac{1}{N_l}\begin{pmatrix} \Upsilon_{M\times M} \cdot p\left(N_l,x_l;\theta_0\right) & 0_{M\times 1} \\ 0_{1\times M} & Ep\left(N_l,x_l;\theta_0\right)\left(1 - p\left(N_l,x_l;\theta_0\right)\right) \end{pmatrix},$$

where

$$\Upsilon_{k\times k} = \begin{pmatrix} \tau_1(1-\tau_1) & \tau_1 \wedge \tau_2 - \tau_1\tau_2 & \cdots & \tau_1 \wedge \tau_M - \tau_1\tau_M \\ \tau_1 \wedge \tau_2 - \tau_1\tau_2 & \tau_2(1-\tau_2) & \cdots & \tau_2 \wedge \tau_M - \tau_2\tau_M \\ \vdots & \vdots & \ddots & \vdots \\ \tau_1 \wedge \tau_M - \tau_1\tau_M & \tau_2 \wedge \tau_M - \tau_2\tau_M & \cdots & \tau_M(1-\tau_M) \end{pmatrix}.$$

For example, if $\tau_1 < \tau_2 < \ldots < \tau_M$, then

$$\Upsilon_{k\times k} = \begin{pmatrix} \tau_1\left(1-\tau_1\right) & \tau_1(1-\tau_2) & \cdots & \tau_1(1-\tau_M) \\ \tau_1(1-\tau_2) & \tau_2\left(1-\tau_2\right) & \cdots & \tau_2(1-\tau_M) \\ \vdots & \vdots & \ddots & \vdots \\ \tau_1(1-\tau_M) & \tau_2(1-\tau_M) & \cdots & \tau_M\left(1-\tau_M\right) \end{pmatrix}.$$

We have

$$\Omega_0 = E\left[\frac{1}{N_l}\begin{pmatrix} \Upsilon_{M\times M}\cdot p\left(N_l, x_l; \theta_0\right) & 0_{M\times 1} \\ 0_{1\times M} & Ep\left(N_l, x_l; \theta_0\right)\left(1-p\left(N_l, x_l; \theta_0\right)\right) \end{pmatrix}\right].$$

Note that Ω_0 can be estimated without estimating θ_0 first. This is because $\hat{p}\left(N,x\right) \to_p p\left(N,x;\theta_0\right)$, where $\hat{p}\left(N,x\right)$ is the kernel-based nonparametric estimator of the conditional entry probabilities. Thus, a consistent estimator of Ω_0 is

$$\hat{\Omega} = \frac{1}{L}\sum_{l=1}^{L}\frac{1}{N_l}\begin{pmatrix} \Upsilon_{M\times M}\cdot \hat{p}\left(N_l, x_l\right) & 0_{M\times 1} \\ 0_{1\times M} & \hat{p}\left(N_l, x_l\right)\left(1-\hat{p}\left(N_l, x_l\right)\right) \end{pmatrix},$$

and the optimal weight matrix can be computed as $\hat{\Omega}^{-1}$.

For D_0, we have

$$\begin{aligned} & \frac{\partial}{\partial\theta}E_{N_l,x_l,y_{il}}\left[y_{il}\left(\tau - 1\left(b_{il} \le q^*(\tau|N_l, x_l; \theta)\right)\right)\right] \\ = & \frac{\partial}{\partial\theta}\left[y_{il}\left(\tau - G^*\left(q^*(\tau|N_l, x_l;\theta)|N_l, x_l\right)\right)\right] \text{ if } y_{il}=1 \text{ and zero otherwise} \\ = & -y_{il}g^*\left(q^*(\tau|N_l, x_l;\theta)|N_l, x_l\right)\frac{\partial q^*(\tau|N_l, x_l;\theta)}{\partial\theta}. \end{aligned}$$

Therefore,

$$D_0 = -E\begin{bmatrix} y_{il}g^*\left(q^*(\tau_1|N_l, x_l;\theta_0)|N_l, x_l\right)\frac{\partial}{\partial\theta}q^*(\tau_1|N_l, x_l;\theta_0) \\ \vdots \\ y_{il}g^*\left(q^*(\tau_M|N_l, x_l;\theta_0)|N_l, x_l\right)\frac{\partial}{\partial\theta}q^*(\tau_M|N_l, x_l;\theta_0) \\ \frac{\partial}{\partial\theta}p(N_l, x_l;\theta_0) \end{bmatrix}.$$

Given $\hat{\theta}$ and nonparametric kernel-based estimator $\hat{g}^*\left(\cdot|N,x\right)$, the matrix D_0 can be consistently estimated as

$$\hat{D} = -\frac{1}{L}\sum_{l=1}^{L}\frac{1}{N_l}\sum_{i=1}^{N_i}\begin{bmatrix} y_{il}\hat{g}^*\left(q^*(\tau_1|N_l, x_l;\hat{\theta})|N_l, x_l\right)\frac{\partial}{\partial\theta}q^*(\tau_1|N_l, x_l;\hat{\theta}) \\ \vdots \\ y_{il}\hat{g}^*\left(q^*(\tau_M|N_l, x_l;\hat{\theta})|N_l, x_l\right)\frac{\partial}{\partial\theta}q^*(\tau_M|N_l, x_l;\hat{\theta}) \\ \frac{\partial}{\partial\theta}p(N_l, x_l;\hat{\theta}) \end{bmatrix}.$$

References

Anosov, D. V., S. K. Aranson, V. I. Arnold, I. U. Bronshtein, V. Z. Grines, and Y. S. Il'Yashenko (1997): *Ordinary Differential Equations and Smooth Dynamical Systems*. Springer Verlag, Berlin.

Athey, S., and P. A. Haile (2002): "Identification of Standard Auction Models," *Econometrica*, 70(6), 2107–2140.

——— (2007): "Nonparametric Approaches to Auctions," in *Handbook of Econometrics*, ed. by J. J. Heckman, and E. E. Leamer, vol. 6, part 1, chap. 60, pp. 3847–3965. North-Holland, Amsterdam.

Athey, S., J. Levin, and E. Seira (2011): "Comparing Open and Sealed Bid Auctions: Evidence from Timber Auctions," *Quarterly Journal of Economics*, 126(1).

Avery, C. (1998): "Strategic Jump Bidding in English Auctions," *Review of Economic Studies*, 65(2), 185–210.

Bajari, P. (2001): "Comparing Competition and Collusion: A Numerical Approach," *Economic Theory*, 18(1), 187–205.

Bajari, P., H. Hong, and S. Ryan (2010): "Identification and Estimation of a Discrete Game of Complete Information," *Econometrica*, 78(5), 1529–1568.

Bajari, P., and A. Hortacsu (2003): "Winner's Curse, Reserve Prices and Endogenous Entry: Empirical Insights from eBay Auctions," *RAND Journal of Economics*, 34(2), 329–355.

Bergemann, D., and J. Valimaki (2002): "Information Acquisition and Efficient Mechanism Design," *Econometrica*, 70(3), 1007–1033.

Campo, S., E. Guerre, I. Perrigne, and Q. Vuong (forthcoming): "Semiparametric Estimation of First-Price Auctions with Risk Averse Bidders," *Review of Economic Studies*.

Chaudhuri, P., K. Doksum, and A. Samarov (1997): "On Average Derivative Quantile Regression," *Annals of Statistics*, 25(2), 715–744.

Chen, K., and S. H. Lo (1997): "On a Mapping Approach to Investigating the Bootstrap Accuracy," *Probability Theory and Related Fields*, 107(2), 197–217.

Coviello, D., and M. Mariniello (2010): "The Role of Publicity Requirements on Entry and Auctions Outcomes," Working Paper, Harvard University.

Davidson, J. (1994): *Stochastic Limit Theory*. Oxford University Press, New York.

Einav, L., and I. Esponda (2008): "Endogenous Participation and Local Market Power in Highway Procurement," Working Paper, Stanford University.

Gourieroux, C., and A. Monfort (1995): *Statistics and Econometric Models*. Cambridge University Press, Cambridge.

Guerre, E., I. Perrigne, and Q. Vuong (2000): "Optimal Nonparametric Estimation of First-Price Auctions," *Econometrica*, 68(3), 525–74.

——— (2009): "Nonparametric Identification of Risk Aversion in First-Price Auctions under Exclusion Restrictions," *Econometrica*, 77(4), 1193–1227.

Haile, P. A., H. Hong, and M. Shum (2003): "Nonparametric Tests for Common Values in First-Price Sealed Bid Auctions," NBER Working Paper 10105.

Harstad, R. M., J. Kagel, and D. Levin (1990): "Equilibrium Bid Functions for Auctions with an Uncertain Number of Bidders," *Economic Letters*, 33(1), 35–40.

Hendricks, K., J. Pinkse, and R. H. Porter (2003): "Empirical Implications of Equilibrium Bidding in First-Price, Symmetric, Common Value Auctions," *Review of Economic Studies*, 70(1), 115–145.

Hong, H., and M. Shum (2002): "Increasing Competition and the Winner's Curse: Evidence from Procurement," *Review of Economic Studies*, 69(4), 871–898.

Jofre-Bonet, M., and M. Pesendorfer (2003): "Estimation of a Dynamic Auction Game," *Econometrica*, 71(5), 1443–1489.

Krasnokutskaya, E. (forthcoming): "Identification and Estimation in Procurement Auctions under Unobserved Auction Heterogeneity," *Review of Economic Studies*.

Krasnokutskaya, E., and K. Seim (forthcoming): "Bid Preference Programs and Participation in Highway Procurement Auctions," *American Economic Review*.

Laffont, J. J., and Q. Vuong (1996): "Structural Analysis of Auction Data," *American Economic Review*, 86(2), 414–420.

Lehman, E. L., and J. P. Romano (2005): *Testing Statistical Hypotheses*. Springer, New York.

Levin, D., and J. L. Smith (1994): "Equilibrium in Auctions with Entry," *American Economic Review*, 84(3), 585–599.

Li, T. (2005): "Econometrics of First-Price Auctions with Entry and Binding Reservation Prices," *Joural of Econometrics*, 126(1), 173–200.

Li, T., and I. Perrigne (2003): "Timber Sale Auctions With Random Reserve Prices," *Review of Economics and Statistics*, 85(1), 189–200.

Li, T., and X. Zheng (2009): "Entry and Competition Effects in First-Price Auctions: Theory and Evidence from Procurement Auctions," *Review of Economic Studies*, 76(4), 1397–1429.

Marmer, V., and A. Shneyerov (forthcoming): "Quantile-Based Nonparametric Inference for First-Price Auctions," *Journal of Econometrics*.

Marmer, V., A. Shneyerov, and P. Xu (2011): "Supplement to What Model for Entry in First-Price Auctions? A Nonparametric Approach," Working Paper, University of British Columbia.

Milgrom, P., and R. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50(5), 1089–1122.

Newey, W. K. (1994): "Kernel Estimation of Partial Means and a General Variance Estimator," *Econometric Theory*, 10(2), 233–253.

Paarsch, H. J. (1997): "Deriving an Estimate of the Optimal Reserve Price: An application to British Columbian Timber Sales," *Journal of Econometrics*, 78(2), 333–357.

Paarsch, H. J., and H. Hong (2006): *An Introduction to the Structural Econometrics of Auction Data*. The MIT Press, Cambridge, Massachusetts.

Pagan, A., and A. Ullah (1999): *Nonparametric Econometrics*. Cambridge University Press, New York.

Persico, N. (2000): "Information Acquisition in Auctions," *Econometrica*, 68(1), 135–148.

Pollard, D. (1984): *Convergence of Stochastic Processes*. Springer-Verlag, New York.

Roberts, J. W. (2009): "Unobserved Heterogeneity and Reserve Prices in Auctions," Working Paper, Duke University.

Roberts, J. W., and A. Sweeting (2010a): "Entry and Selection in Auctions," Working Paper, Duke University.

——— (2010b): "When Should Sellers Use Auctions?," Working Paper, Duke University.

Samuelson, W. F. (1985): "Competitive Bidding with Entry Costs," *Economics Letters*, 17(1-2), 53–57.

Shao, J., and D. Tu (1995): *The Jackknife and Bootstrap*. Springer-Verlag, New York.

TAN, G., AND O. YILANKAYA (2006): "Equilibria in Second Price Auctions with Participation Costs," *Journal of Economic Theory*, 130(1), 205–219.

VAN DER VAART, A. W. (1998): *Asymptotic Statistics.* Cambridge University Press, Cambridge.

XU, P. (2009): "Nonparametric Estimation of the Entry Cost in First-Price Procurement Auctions," Working Paper, University of Hong Kong.

YE, L. (2007): "Indicative Bidding and a Theory of Two-Stage Auctions," *Games and Economic Behavior*, 58(1), 181–207.