Discrete Mathematics 106/107 (1992) 77–82
North-Holland

Structure and uniqueness of the (81, 20, 1, 6) strongly regular graph

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Received 12 November 1991
Revised 4 March 1992

Abstract

Brouwer, A.E, W.H. Haemers, Structure and uniqueness of the (81, 20, 1, 6) strongly regular graph, Discrete Mathematics 106/107 (1992) 77–82.

We prove that there is a unique graph (on 81 vertices) with spectrum $20^1 2^{60} (-7)^{20}$. We give several descriptions of this graph, and study its structure.

Let $\Gamma = (X, E)$ be a strongly regular graph with parameters $(v, k, \lambda, \mu) = (81, 20, 1, 6)$. Then Γ (that is, its 0–1 adjacency matrix A) has spectrum $20^1 2^{60} (-7)^{20}$, where the exponents denote multiplicities. We will show that up to isomorphism there is a unique such graph Γ. More generally we give a short proof for the fact (due to Ivanov and Shpectorov [9]) that a strongly regular graph with parameters $(v, k, \lambda, \mu) = (q^4, (q^2+1)(q-1), \ q-2, \ q(q-1))$ that is the collinearity graph of a partial quadrangle (that is, in which all maximal cliques have size q) is the second subconstituent of the collinearity graph of a generalized quadrangle $\mathrm{GQ}(q, q^2)$. In the special case $q = 3$ this will imply our previous claim, since $\lambda = 1$ implies that all maximal cliques have size 3, and it is known (see Cameron et al. [5]) that there is a unique generalized quadrangle GQ(3, 9) (and this generalized quadrangle has an automorphism group transitive on the points). The proof will use spectral techniques very much like those found in

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Haemers [7] and Brouwer and Haemers [3]. For completeness let us explicitly formulate the tools we use.

Tool 1. Let A and B be real symmetric matrices of orders n and m (where $m \leqslant n$) and with eigenvalues $\theta_1 \geqslant \cdots \geqslant \theta_n$ and $\eta_1 \geqslant \cdots \geqslant \eta_m$, respectively. We say that the eigenvalues of B *interlace* those of A when $\theta_j \geqslant \eta_j \geqslant \theta_{n-m+j}$ for all j ($1 \leqslant j \leqslant m$). We say that the interlacing is *tight* when for some integer l we have $\eta_j = \theta_j$ for $1 \leqslant j \leqslant l$ and $\eta_j = \theta_{n-m+j}$ for $l+1 \leqslant j \leqslant m$. If B is a principal submatrix of A then the eigenvalues of B interlace those of A. Another case of interlacing is the following result: *Given a symmetric partition of the rows and columns of a symmetric matrix A, let B be the matrix with as entries the average row sums of the parts of A. Then the eigenvalues of B interlace those of A, and when the interlacing is tight, the parts of A have constant row sums.*

Tool 2. Given a symmetric partition of a symmetric matrix A with two eigenvalues into four submatrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

the eigenvalues of A_{22} can be computed from those of A_{11}: *If A has eigenvalues α and β (where $\alpha > \beta$) with multiplicities f and $n-f$, respectively, and A_{11} (of order m) has eigenvalues $\theta_1 \geqslant \cdots \geqslant \theta_m$, then A_{22} (of order $n-m$) has eigenvalues $\eta_1 \geqslant \cdots \geqslant \eta_{n-m}$, where*

$$\eta_i = \begin{cases} \alpha & \text{if } 1 \leqslant i \leqslant f-m, \\ \beta & \text{if } f+1 \leqslant i \leqslant n-m, \\ \alpha + \beta - \theta_{f-i+1} & \text{otherwise.} \end{cases}$$

For undefined concepts and notation, see Brouwer et al. [2]. For surveys on strongly regular graphs, see Hubaut [8] and Brouwer and van Lint [4].

Let us first give a few descriptions of our graph on 81 vertices.

(A) Let X be the point set of AG(4, 3), the 4-dimensional affine space over $\mathbb{F}_3$, and join two points when the line connecting them hits the hyperplane at infinity (a PG(3, 3)) in a fixed elliptic quadric Q. This description shows immediately that $v = 81$ and $k = 20$ (since $|Q| = 10$). Also $\lambda = 1$ since no line meets Q in more than two points, so that the affine lines are the only triangles. Finally $\mu = 6$, since a point outside Q in PG(3, 3) lies on 4 tangents, 3 secants and 6 exterior lines with respect to Q, and each secant contributes 2 to μ. We find that the group of automorphisms contains $G = 3^4 \cdot \mathrm{PGO}_4^-(3) \cdot 2$, where the last factor 2 accounts for the linear transformations that do not preserve the quadratic form Q, but multiply it by a constant. In fact this is the full group, as will be clear from the uniqueness proof.

(B) A more symmetric form of this construction is found by starting with $X = \mathbf{1}^{\perp}/\langle \mathbf{1} \rangle$ in $\boldsymbol{F}_3^6$ provided with the standard bilinear form. The corresponding quadratic form ($Q(x) = \mathrm{wt}(x)$, the number of nonzero coordinates of x) is elliptic, and if we join two vertices $x + \langle \mathbf{1} \rangle$, $y + \langle \mathbf{1} \rangle$ of X when $Q(x - y) = 0$, i.e., when their difference has weight 3, we find the same graph as under A. This construction shows that the automorphism group contains $G = 3^4 \cdot (2 \times \mathrm{Sym}(6)) \cdot 2$, and again this is the full group.

(C) There is a unique strongly regular graph Σ with parameters $(v, k, \lambda, \mu) = (112, 30, 2, 10)$, the collinearity graph of the unique generalized quadrangle with parameters GQ(3, 9). Its second subconstituent is strongly regular (since Σ is a Smith graph), and hence is isomorphic to our graph Γ. (See Cameron et al. [5].) We find that Aut Γ contains (and in fact it equals) the point stabilizer in $U_4(3) \cdot D_8$ acting on GQ(3, 9).

(D) In the McLaughlin graph Λ (the unique strongly regular graph with parameters $(v, k, \lambda, \mu) = (275, 112, 30, 56)$) let x, y be two adjacent vertices. The subgraph of Λ induced by the neighbours of y is isomorphic to Σ; the subgraph T induced by the nonneighbours of y is the unique strongly regular graph with parameters $(v, k, \lambda, \mu) = (162, 56, 10, 24)$. (Again, see Cameron et al [5].) Thus, by (C) above, we may identify Γ with the subgraph of Λ induced by the vertices adjacent to y but not to x. Let Γ' be the subgraph induced by the vertices nonadjacent to both x and y, so that T is partitioned by the vertex sets of Γ and Γ'. Then also Γ' is a strongly regular graph with parameters $(v, k, \lambda, \mu) = (81, 20, 1, 6)$ (its spectrum can be computed from that of T and that of Γ). We find that Aut Γ' contains the edge stabilizer in Aut $\Lambda = \mathrm{McL}{\cdot}2$—in fact as an index 2 subgroup.

(E) The graph Γ is the coset graph of the truncated ternary Golay code C: take the 3^4 cosets of C and join two cosets when they contain vectors differing in only one place.

(F) The graph Γ is the Hermitean forms graph on $\boldsymbol{F}_9^2$; more generally, take the q^4 matrices M over $\boldsymbol{F}_{q^2}$ satisfying $M^{\mathrm{T}} = \bar{M}$, where $\bar{}$ denotes the field automorphism $x \mapsto x^q$ (applied entrywise), and join two matrices when their difference has rank 1. This will give us a strongly regular graph with parameters $(v, k, \lambda, \mu) = (q^4, (q^2 + 1)(q - 1), q - 2, q(q - 1))$.

(G) The graph Γ is the graph with vertex set $\boldsymbol{F}_{81}$, where two vertices are joined when their difference is a fourth power. (This construction was given by Van Lint and Schrijver [10].)

Now let us embark upon the uniqueness proof. Let $\Gamma = (X, E)$ be a strongly regular graph with parameters $(v, k, \lambda, \mu) = (q^4, (q^2 + 1)(q - 1), q - 2, q(q - 1))$ and assume that all maximal cliques (we shall just call them lines) of Γ have size q.

Let Γ have adjacency matrix A. Using the spectrum of A—it is $k^1(q - 1)^f(q - 1 - q^2)^g$, where $f = q(q - 1)(q^2 + 1)$ and $g = (q - 1)(q^2 + 1)$—we can obtain some structure information. Let $\boldsymbol{T}$ be the collection of subsets of X of cardinality q^3 inducing a subgraph that is regular of degree $q - 1$.

Step 1. *If $T \in \mathbf{T}$, then each point of $X \setminus T$ is adjacent to q^2 points of T.*

Look at the matrix B of average row sums of A, with sets of rows and columns partitioned according to $\{T, X\setminus T\}$. We have

$$B = \begin{pmatrix} q-1 & q^2(q-1) \\ q^2 & k-q^2 \end{pmatrix}$$

with eigenvalues k, $q-1-q^2$, so interlacing is tight, and by Tool 1 it follows that the row sums are constant in each block of A.

Step 2. *Given a line L, there is a unique $T_L \in \boldsymbol{T}$ containing L.*

Let Z be the set of vertices in $X\setminus L$ without a neighbour in L. Then $|Z| = q^4 - q - q(k-q+1) = q^3 - q$. Let $T = L \cup Z$. Each vertex of Z is adjacent to $q\mu = q^2(q-1)$ vertices with a neighbour in L, so T induces a subgraph that is regular of degree $q-1$.

Step 3. *If $T \in \boldsymbol{T}$ and $x \in X\setminus T$, then x is on at least one line L disjoint from T, and T_L is disjoint from T for any such line L.*

The point x is on q^2+1 lines, but has only q^2 neighbours in T. Each point of L has q^2 neighbours in T, so each point of T has a neighbour on L and hence is not in T_L.

Step 4. *Any $T \in \boldsymbol{T}$ induces a subgraph Δ isomorphic to q^2K_q.*

It suffices to show that the multiplicity m of the eigenvalue $q-1$ of Δ is (at least) q^2 (it cannot be more). By interlacing we find $m \geq q^2 - q$, so we need some additional work. Let $M := A - (q-1/q^2)J$. Then M has spectrum $(q-1)^{f+1}(q-1-q^2)^g$, and we want that M_T, the submatrix of M with rows and columns indexed by T, has eigenvalue $q-1$ with multiplicity (at least) q^2-1, or, equivalently (by Tool 2), that $M_{X\setminus T}$ has eigenvalue $q-1-q^2$ with multiplicity (at least) $q-2$. But for each $U \in \boldsymbol{T}$ with $U \cap T = \emptyset$ we find an eigenvector $x_U := (2-q)\chi_U + \chi_{X\setminus(T\cup U)}$ of $M_{X\setminus T}$ with eigenvalue $q-1-q^2$. A collection $\{x_U \mid U \in \boldsymbol{U}\}$ of such eigenvectors cannot be linearly dependent when $\boldsymbol{U} = \{U_1, U_2, \ldots\}$ can be ordered such that $U_i \nsubseteq \bigcup_{j<i} U_j$ and $\bigcup \boldsymbol{U} \neq X\setminus T$, so we can find (using Step 3) at least $q-2$ linearly independent such eigenvectors, and we are done.

Step 5. *Any $T \in \boldsymbol{T}$ determines a unique partition of X into members of $\boldsymbol{T}$.*

Indeed, we saw this in the proof of the previous step.

Let Π be the collection of partitions of X into members of $\boldsymbol{T}$. We have $|\boldsymbol{T}| = q(q^2+1)$ and $|\Pi| = q^2+1$. Construct a generalized quadrangle GQ(q, q^2) with point set $\{\infty\} \cup \boldsymbol{T} \cup X$ as follows: The q^2+1 lines on ∞ are $\{\infty\} \cup \pi$ for $\pi \in \Pi$. The q^2 remaining lines on each $T \in \boldsymbol{T}$ are $\{T\} \cup L$ for $L \subseteq T$. It is completely straightforward to check that we really have a generalized quadrangle GQ(q, q^2).

Other graphs. Some of our arguments can be generalized a little. Given a strongly regular graph $\Gamma = (X, E)$ with parameters (v, k, λ, μ) and spectrum $k^1 r^f s^g$, suppose that there is a subset L of X inducing a strongly regular subgraph of Γ with parameters (u, r, λ, μ). Then $k = r + u\mu$ and $v = u(k - s)$. Each point outside L has at most one neighbour in L. Let Z be the set of points in $X \setminus L$ without neighbour in L. Each point of Z has $u\mu$ neighbours outside Z, and hence Z, and also $T := L \cup Z$, is regular of valency r. In a few cases one can show using multiplicity arguments that T must consist of a number of copies of L. For example:

(a) Starting with a single point in a complete multipartite graph $K_{m \times n}$ (with spectrum $(m-1)n^1 0^{m(n-1)} (-n)^{m-1}$) we find a coclique of size n.

(b) Starting with an edge in the Petersen graph, we find a subgraph $3K_2$. (Likewise, an edge in the complement of the Clebsch graph is contained in a unique $4K_2$, but this is the special case $q = 2$ of our result above.)

(c) Starting with a pentagon in the Hoffman–Singleton graph, we find a subgraph $5C_5$.

(d) Starting with a quadrangle in the Gewirtz graph, we find a subgraph $6C_4$. (This was the starting point of Brouwer and Haemers [3]; also the uniqueness of the $(162, 56, 10, 24)$ strongly regular graph (Cameron et al. [5]) relies on this fact.)

(e) Starting with a grid 3×3 in the Berlekamp–van Lint–Seidel graph (Berlekamp et al. [1]), we find a subgraph $9\ (3 \times 3)$. Maybe one could prove uniqueness (for strongly regular graphs with parameters $(v, k, \lambda, \mu) = (243, 22, 1, 2)$) using this?

(f) Starting with a triangle in a $(57, 14, 1, 4)$ graph Γ, we find (under the assumption that Γ does not contain a 15-coclique) a subgraph $7K_3$. This implies that Γ is embeddable in a non-existing GQ(3, 6) (see Dixmier and Zara [6], or Payne and Thas [11]). Thus, the non-existence proof for Γ in Wilbrink and Brouwer [12] can be shortened considerably.

Our strongly regular graph on 81 vertices might have distance-regular antipodal 2-, 3- and 6-covers of diameter 4. Maybe one can prove non-existence for the 2- and 6-covers and uniqueness for the 3-cover (e.g., by proving that a grid 3×3 must lift to a grid again)?

p-Rank and Smith normal form. Writing $S(M)$ for the Smith normal form of a matrix M, we find for the adjacency matrix A of our 81-point graph: $S(A) = \mathrm{diag}(1^{20}, 2^{41}, 14^{19}, 140^1)$ and $S(A - 2I) = \mathrm{diag}(1^{19}, 3^1, 6^1, 0^{60})$ and $S(A + 7I) = \mathrm{diag}(1^{19}, 3^2, 9^{39}, 27^1, 0^{20})$. In particular, $A + I$ has 3-rank 19.

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