Matrix Constructions of Divisible Designs

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ABSTRACT

We present two new constructions of group divisible designs. We use skew-symmetric Hadamard matrices and certain strongly regular graphs together with (v, k, λ)-designs. We include many examples, in particular several new series of divisible difference sets.

1. INTRODUCTION

A *(group) divisible design (GDD)* with parameters $(m, n, k, \lambda_1, \lambda_2)$ is a finite incidence structure $\mathcal{D}$ with the following properties: $\mathcal{D}$ has mn points

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and (not necessarily the same number of) blocks of size k. The points are divided into m point classes (sometimes called *groups*) with n points each. Any two distinct points in the same (respectively, different) point classes are joined by exactly λ_1 (respectively, λ_2) blocks. We denote the number of blocks by b and the number of blocks through a point by r. It follows by easy counting arguments that r is indeed constant and that

$$bk = vr.$$

A GDD is called *square* if $b = v$. If $\lambda_1 = \lambda_2$ $(= \lambda)$ (or if m or $n = 1$), the definition of a GDD becomes just the definition of a (v, k, λ)-*design* with $v = mn$ points. We also call a (v, k, λ)-design a (v, b, r, k, λ)-design, where again b denotes the number of blocks and r the number of blocks through a point.

Let $p_1^{(1)}, \cdots, p_n^{(1)}, p_1^{(2)}, \cdots, p_n^{(2)}, \cdots, p_1^{(m)}, \cdots, p_n^{(m)}$ be the points of $\mathscr{D}$ ordered in an obvious way according to the point classes. The *incidence matrix* of $\mathscr{D}$ is a $v \times b$ matrix N with 0-1 entries: We label the rows of N with the points of the GDD in the above order, and the columns by the blocks. The (p, B) entry of N is 1 if p lies on B, and 0 otherwise. Then it follows from the definition that

$$NN^T = \begin{pmatrix} A & B & \cdot & \cdot & \cdot & B \\ B & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & B \\ B & \cdot & \cdot & \cdot & B & A \end{pmatrix} \in \mathbb{N}^{(mn,\, mn)} \tag{1}$$

with $A, B \in \mathbb{N}^{(n,\, n)}$ and $A = (r - \lambda_1) I_n + \lambda_1 J_{n,\, n}$, $B = \lambda_2 J_{n,\, n}$. Here I_n is the $n \times n$ identity matrix and $J_{n,\, n}$ is the $n \times n$ matrix whose entries are all 1. We keep this notation throughout this paper, where, in general, $J_{s,\, t}$ is the $s \times t$ matrix whose entries are all 1. A matrix with entries 0 and 1 and constant column sum satisfying (1) has to be the incidence matrix of a GDD. The incidence matrix N of a (v, b, r, k, λ)-design satisfies $NN^T = (r - \lambda) I_v + J_{v,\, v}$, the column sum is k, and the row sum is r.

In this paper we will construct incidence matrices N of GDDs using Kronecker products of incidence matrices of some (v, k, λ)-designs with adjacency matrices of strongly regular graphs and with skew-symmetric Hadamard matrices. We denote the Kronecker product between two matrices A and B

by $A \otimes B$; it is defined by

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n_2}B \\ \vdots & & \vdots \\ a_{n_1 1}B & \cdots & a_{n_1 n_2}B \end{pmatrix} \in K^{(n_1 m_1,\, n_2 m_2)},$$

where $A = (a_{ij}) \in K^{(n_1, n_2)}$ and $B \in K^{(m_1, m_2)}$.

A *strongly regular graph* (SRG) with parameters (t, h, α, β) is an h-regular graph without loops on t vertices, for which any two adjacent vertices have exactly α common neighbors and any two distinct nonadjacent vertices have exactly β common neighbors. We can define the adjacency matrix of a graph similarly to the incidence matrix of a design. The adjacency matrix A of an SRG satisfies the equation

$$AA^T = hI_t + \alpha A + \beta(J_{t,t} - I_t - A),$$

and conversely, every symmetric 0-1 matrix whose diagonal entries are 0 that satisfies this equation is the matrix of an SRG. In this paper we consider SRGs with $\alpha = \beta + 1$.

Now we define skew-symmetric Hadamard matrices. A *Hadamard matrix* is an $n \times n$ matrix H with entries $\pm 1 \in \mathbb{Z}$ satisfying

$$HH^T = nI_n.$$

If $H = I_n + S$ with $S^T = -S$, we call H *skew-symmetric*. Multiplying the ith row and ith column by -1 results in a matrix that is still a skew-symmetric Hadamard matrix; thus we may assume

$$H = \begin{pmatrix} +1 & \cdots & +1 \\ -1 & & \\ -1 & & \\ \vdots & H' & \\ -1 & & \end{pmatrix}, \tag{2}$$

where $H' = I_{n-1} + S'$ for a skew-symmetric matrix S'. It is well known that the size of H is 1, 2, or divisible by 4. Many skew-symmetric Hadamard matrices are known to exist; we mention just one series:

Result 1.1. *There exist skew-symmetric Hadamard matrices of size n whenever*

$$n = 2^t \prod_i (q_i + 1),$$

where $t \geqq 1$ and the q_i's are prime powers $\equiv 3 \bmod 4$.

In concluding this introduction we refer the reader for more on designs, SRGs, and Hadamard matrices to [2], [5], and [13] respectively. We also mention that a very general construction of partially balanced incomplete block designs (PBIBDs) with many association classes is given in [12]. GDDs are a special type of PBIBDs with just two association classes. Hence there is a chance that some of our examples can be constructed also by the method in [12]; however, it is not at all clear how this might work. So our construction is in any case preferable for producing GDDs and divisible difference sets.

We note that Theorems 2.1 and 3.1 generalize two constructions in [3].

2. DIVISIBLE DESIGNS AND HADAMARD MATRICES

There are constructions of divisible designs using Hadamard matrices in the literature; see for instance [7]. The following construction is new.

THEOREM 2.1. *Assume the existence of a skew-symmetric Hadamard matrix of size $4s$ and the existence of a (v, b, r, k, λ)-design. Then there exists a divisible design with parameters*

$$m = 4s - 1, \qquad n = v, \qquad k' = v(2s - 1) + k,$$

$$\lambda_1 = b(2s - 1) + \lambda, \qquad \lambda_2 = b(s - 1) + r, \qquad b' = b(4s - 1).$$

Proof. We assume that H is in "normal form" (2) and that S' is the skew-symmetric matrix appearing in (2) and $H' = I_{4s-1} + S'$. Let A denote the incidence matrix of the (v, k, λ)-design. We claim that

$$N := I_{4s-1} \otimes A + \tfrac{1}{2}\left(S' - I_{4s-1} + J_{4s-1,4s-1}\right) \otimes J_{v,b}$$

is the incidence matrix of a GDD with the desired parameters. In other words, we replace the diagonal of S' by the matrix A, the 1's in S' by $J_{v,b}$, and -1 by the $v \times b$ zero matrix. Obviously the number of points is $(4s-1)v$, the number of blocks is $b(4s-1)$, and the points come naturally in $4s-1$ classes with v points each. By the orthogonality of H every column of H' has $2s-1$ entries $+1$ off the diagonal; thus every column of S' has $2s-1$ entries $+1$. This shows $k' = v(2s-1)+k$.

Now we have to check that the inner product of two rows of N is λ_1 or λ_2 according as the points are in the same or in distinct point classes. If they are in the same point class, the product is $b(2s-1)+\lambda\ (=\lambda_1)$, since the inner product of two rows of the incidence matrix of a (v, b, r, k, λ)-design is λ. Now assume the points are in distinct classes. It is well known that two distinct rows $h = (h_1, \cdots, h_{4s})$ and $h' = (h'_1, \cdots, h'_{4s})$ of H have exactly s positions j with $(h_j, h'_j) = (1, 1)$. From the skew symmetry of H it follows that exactly one of these s positions involves a diagonal element, so that either

$$\begin{array}{c} \begin{array}{cc} \begin{array}{c} h \\ h' \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & +1 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & +1 & \cdots & +1 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array} \\ \qquad h \qquad\quad h' \end{array}$$

or

$$\begin{array}{c} \begin{array}{cc} \begin{array}{c} h \\ h' \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & +1 & \cdots & +1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & -1 & \cdots & +1 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array} \\ \qquad h \qquad\quad h' \end{array}$$

occurs. This shows $\lambda_2 = b(s-1)+r$ and finishes the proof. ■

Note that the constructed GDD is square if and only if $b = v$ in the (v, b, r, k, λ)-design. The existence of a Hadamard matrix of size $4s$ implies the existence of a $(4s-1, 4s-1, 2s-1, 2s-1, s-1)$-design [i.e. a symmetric $(4s-1, 2s-1, s-1)$-design having the same number of points and blocks]. The design is specified by its incidence matrix, which is the matrix H' in (2) where the -1's are replaced by 1's and 1's by 0. (It is more common to replace -1 by 0 after multiplying rows 2 through n by -1.) Denote this design by $\mathscr{H}$, and the (v, b, r, k, λ)-design used in Theorem 2.1 by $\mathscr{D}$. If $\mathscr{H} = \mathscr{D}$, our construction yields for instance the following interesting series of GDDs with $m = n$:

PROPOSITION 2.2. *The existence of a skew-symmetric Hadamard matrix of size* $4s$ *implies the existence of a*

$$\left(4s - 1, 4s - 1, 8s^2 - 4s, 8s^2 - 5s, 4s^2 - 3s\right) - \text{GDD};$$

hence (3, 3, 4, 3, 1)- *and* (7, 7, 24, 22, 10)-*GDDs exist, since Result* 1.1 *shows the existence of skew-symmetric Hadamard matrices of size* 4 *and* 8.

EXAMPLE 2.3. We take $\mathscr{D}$ to be the design $AG_d(n, q)$ having the parameters

$$\left(q^n, q^{n-d}\begin{bmatrix} n \\ d \end{bmatrix}_q, \begin{bmatrix} n \\ d \end{bmatrix}_q, q^d, \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right),$$

where

$$\begin{bmatrix} n \\ d \end{bmatrix}_q$$

is the number of d-dimensional subspaces of $GF(q)^n$. We obtain for instance the following nonsquare GDDs:

(3, 9, 12, 13, 4)-GDD	$(q = 3,\ n = 2,\ d = 1,\ s = 1)$,
(7, 9, 30, 37, 16)-GDD	$(q = 3,\ n = 2,\ d = 1,\ s = 2)$,
(3, 16, 20, 21, 5)-GDD	$(q = 4,\ n = 2,\ d = 1,\ s = 1)$,
(3, 8, 12, 17, 7)-GDD	$(q = 2,\ n = 3,\ d = 2,\ s = 1)$.

It is obvious from our construction that the automorphism group of our new design contains in the direct product Aut $\mathscr{H} \times$ Aut $\mathscr{D}$; in particular, the new design admits a sharply transitive automorphism group (on points) if $\mathscr{H}$ and $\mathscr{D}$ have such groups. It is well known that a square GDD with a group acting sharply transitively on points gives rise to a divisible difference set and symmetric designs with a sharply transitive group admit difference sets in the usual sense. For difference sets we refer the reader to [2], and for divisible difference sets to [6].

If $q \equiv 3 \bmod 4$ is a prime power, there exists a skew-symmetric Hadamard matrix such that the corresponding design $\mathscr{H}$ has a sharply transitive group (Paley difference sets). The corresponding difference set is constructed, for instance, as the set of nonzero squares of GF(q). Using the terminology of difference sets, we can rephrase Theorem 2.1 as follows [where EA(q) denotes the elementary abelian group of order q]:

THEOREM 2.4. *Let D_1 be the Paley difference set of nonzero squares in* GF(q), *where $q = 4s - 1$ is a prime power. If D_2 is any (v, k, λ)-difference set in G, then we get divisible difference sets with parameters*

$$m = 4s - 1, \qquad n = v, \qquad k' = v(2s - 1) + k,$$

$$\lambda_1 = v(2s - 1) + \lambda, \qquad \lambda_2 = v(s - 1) + k, \qquad b' = v(4s - 1)$$

by taking $D = (D_1 \times G) \cup (\{0\} \times D_2) \subseteq \mathrm{EA}(q) \times G$.

Note that the special case that D_2 is a $(v, 1, 0)$-difference set is contained in [1, Lemma 4.2].

COROLLARY 2.5. *There exist divisible difference sets with parameters*

$$(4s - 1, 4s - 1, 8s^2 - 4s, 8s^2 - 5s, 4s^2 - 3s)$$

in EA(q^2) *whenever $q = 4s - 1$ is a prime power.*

We now combine the Paley difference sets with some other series of difference sets. We start with the difference sets corresponding to $PG_{n-1}(n, q)$ (see [9]) and obtain the following triply infinite series of divisible difference sets:

PROPOSITION 2.6. *Let q' be a prime power $\equiv 3 \bmod 4$, and q any prime power. Then there exist*

$$\left(q', \frac{q^{n+1} - 1}{q - 1}, \frac{q' - 1}{2}\frac{q^{n+1} - 1}{q - 1} + \frac{q^n - 1}{q - 1},\right.$$

$$\left.\frac{q' - 1}{2}\frac{q^{n+1} - 1}{q - 1} + \frac{q^{n-1} - 1}{q - 1}, \frac{q' - 3}{4}\frac{q^{n+1} - 1}{q - 1} + \frac{q^n - 1}{q - 1}\right) - \text{GDDs}$$

for each positive integer $n \geqq 2$ admitting a sharply transitive automorphism group. Thus divisible difference sets with the above parameters always exist.

PROPOSITION 2.7. *There exist divisible difference sets with parameters*

$$\left(4s-1, q(q+2), \frac{4s-1}{2}q(q+2)-\frac{1}{2},\right.$$

$$\left.\frac{8s-3}{4}q(q+2)-\frac{3}{4}, \frac{1}{2}(2s-1)q(q+2)-\frac{1}{2}\right)$$

in EA($4s-1$) $\times$ EA(q) $\times$ EA($q+2$) *whenever* q, $q+2$ *and* $4s-1$ *are odd prime powers.*

Proof. We use the existence of the well-known difference sets in EA(q) $\times$ EA($q+2$) whenever q and $q+2$ are odd prime powers; see [11]. ∎

One can of course use any other series of difference sets to produce divisible difference sets. In the following we list a few examples that were obtained using some small members of the series of difference sets due to McFarland [8] and Spence [10]:

EXAMPLE 2.8. Divisible difference sets with the following parameters exist:

(3, 45, 57, 48, 12)	(McFarland),
(3, 96, 116, 100, 20)	(McFarland),
(3, 36, 51, 42, 15)	(Spence),
(7, 36, 123, 114, 51)	(Spence).

3. DIVISIBLE DESIGNS AND STRONGLY REGULAR GRAPHS

The following construction involves adjacency matrices of SRGs with $\beta = \alpha + 1$.

THEOREM 3.1. *Assume the existence of a* $(t, h, \alpha, \alpha+1)$-SRG *and the existence of a* (v, b, r, k, λ)-*design with* $b = 2r$ (*equivalently* $v = 2k$). *Then we can construct a* GDD *with parameters*

$$m = t, \qquad n = v, \qquad k' = k + hv, \qquad \lambda_1 = \lambda + hb,$$

$$\lambda_2 = (\alpha + 1)b, \qquad b' = tb.$$

Proof. Let T be the adjacency matrix of the graph, and A the incidence matrix of the design. We consider the matrix

$$N := I_t \otimes A + T \otimes J_{v,b},$$

i.e., we replace the diagonal of T by A and every entry $+1$ of T by $J_{v,b}$. We show that N is the incidence matrix of the desired GDD. The column sum of N (which is the block size) is $k + hv$. The points (or rows of N) come (as before) naturally in t point classes with v points each. We have to calculate the inner product of two rows of N. If the rows correspond to points in the same class, their inner product is obviously $\lambda + hb$. To compute the inner product of rows that belong to points in distinct point classes we must distinguish the cases that the corresponding vertices in the SRG are adjacent or not. If they are adjacent, the inner product is $2r + \alpha b$; if they are nonadjacent, it is $(\alpha + 1)b$. So N is the incidence matrix of a GDD iff $2r + \alpha b = (\alpha + 1)b$ $(= \lambda_2)$, which gives the desired parameters. ■

We note that our proof shows that the construction cannot produce interesting designs if the difference between the parameters α and β of the SRG is larger than 1. We remark that Theorem 3.1 generalizes Theorem 3.1 in [1], which is the special case of an "extension" using a symmetric $(2, 2, 1, 1, 0)$-design. This is the only symmetric design having $v = 2k$; thus the only new examples beyond [1] are GDDs with more blocks than points.

We now mention a few applications of our construction.

EXAMPLE 3.2. There are three known SRGs with $\alpha = 0$ and $\beta = 1$; they have parameters $(5, 2, 0, 1)$, $(10, 3, 0, 1)$, and $(50, 7, 0, 1)$ (see [5], for instance). Combining them with a $(6, 10, 5, 3, 2)$-design gives GDDs with parameters $(5, 6, 15, 22, 10)$, $(10, 6, 21, 32, 10)$, and $(50, 6, 45, 72, 10)$.

Let S be an SRG of Paley type, i.e. with parameters $(4\mu + 1, 2\mu, \mu - 1, \mu)$. Several infinite series of such graphs are known to exist; see [4] for instance. One series occurs if $4\mu + 1$ is a prime power. Combining this with the designs $AG_{n-1}(n, 2)$, we obtain a doubly infinite series of GDDs.

PROPOSITION 3.3. *Let q be a prime power $\equiv 1 \bmod 4$. Then there exists a* GDD *with parameters*

$$\left(q, 2^n, 2^{n-1}q, (2^n - 1)q - 2^{n-1}, \frac{q-1}{2}(2^n - 1)\right).$$

Using Proposition 3.3, we obtain GDDs with parameters $(5, 2, 5, 4, 2)$, $(5, 4, 10, 13, 6)$, and $(9, 2, 9, 8, 4)$.

More generally, every symmetric $(4n-1, 2n-1, n-1)$-design can be extended to a $(4n, 8n-2, 4n-1, 2n, 2n-1)$-design, and the existence of such designs is equivalent to the existence of Hadamard matrices of size $4n$. Using these designs, we obtain the following proposition (which includes the examples of Proposition 3.3):

PROPOSITION 3.4. *Let $q = 4\mu + 1$ be a prime power, and let $4n$ be the size of a Hadamard matrix. Then there exists a* GDD *with parameters*

$$\big(4\mu + 1, 4n, 2n(4\mu + 1), (2n-1) + 2\mu(8n-2), \mu(8n-2)\big).$$

It is conjectured that Hadamard matrices of order $4n$ exists for every $n \geqq 1$. Many series of Hadamard matrices are known (see [13]), and every series can be used in Proposition 3.4 to produce a doubly infinite series of GDDs as in Proposition 3.3. We obtain, for instance, a (5, 12, 30, 49, 22)-GDD ($\mu = 1$, $n = 3$). It is worth noting that Theorem 3.1 gives rise to GDDs where the line size is half the number of points if Paley type SRGs are used in the construction.

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