Discrete Mathematics 106/107 (1992) 83–92
North-Holland

Some 2-ranks

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Received 22 January 1992
Revised 16 March 1992

Abstract

Brouwer, A.E., H.A. Wilbrink and W.H. Haemers, Some 2-ranks, Discrete Mathematics 106/107 (1992) 83–92.

We compute the dimension of some of the binary codes associated with the unital $U(3, q)$, the generalized quadrangle $U(4, q)$ and its dual $O_6^-(q)$ for odd q.

Introduction

Suppose $(X, \mathscr{B})$ is some kind of combinatorial structure, i.e., X is a set and $\mathscr{B}$ is a collection of subsets of X. The F-code of $(X, \mathscr{B})$, where F is a field, is the subspace of F^X generated by the (characteristic vectors of) elements of $\mathscr{B}$ or in other words, it is the row space of the incidence matrix of $(X, \mathscr{B})$. There is an increasing amount of literature in which combinatorial structures are being studied through their codes. In this note we examine binary codes related to the unitary geometries $U(n, q)$ for $n = 3, 4$ and the orthogonal geometry $O_6^-(q)$ for odd q. In particular we are interested in the dimension of these codes (which is the same as the 2-rank of the incidence matrices).

1. Codes spanned by lines in an affine space

Proposition 1.1. *Let q be a prime power, and let p be a prime not dividing q. Let B be a blocking set for hyperplanes in* $\mathrm{PG}(d-1, q)$, *where $d \geqslant 2$. Then the*

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F_p-code $\mathfrak{C}_B$ spanned by the lines in AG(d, q) *that have a direction meeting the hyperplane at infinity in a point of B, is the full code (of dimension q^d). Conversely, if* $\dim \mathfrak{C}_B = q^d$, *then B is a blocking set for hyperplanes.*

Proof (Blokhuis and Brouwer). The all-1 vector **1** is a code word, since $B \neq \emptyset$. Each affine $(d-1)$-space is a code word, since every $(d-2)$-space in the hyperplane at infinity meets B. But if each affine r-space is a code word for $r > m$, and S is an affine m-space contained in an $(m+2)$-space T, then summing the $q+1$ (characteristic vectors of the) $(m+1)$-spaces on S contained in T, and subtracting T, we find a nonzero multiple of S. Thus each affine m-space is a code word, too, and by induction single points are code words. Conversely, if H is a hyperplane with $H \cap B = \emptyset$, then taking linear combinations of translates of H shows that $\dim \mathfrak{C}_B^{\perp} \geq q-1$. □

2. The generalized quadrangle $O_6^-(q)$ for odd q

Let $(X, \mathscr{L})$ be the generalized quadrangle of order (q, q^2) formed by the isotropic points and totally isotropic lines in an O_6^- geometry. Note that the collinearity graph Γ of $(X, \mathscr{L})$ is strongly regular with parameters $v = (q+1)(q^3+1)$, $k = q(q^2+1)$, $\lambda = q-1$, $\mu = q^2+1$ and eigenvalues k, $r = q-1$ and $s = -q^2-1$ with multiplicities 1, $f = q^2(q^2+1)$ and $g = q(q^2-q+1)$, respectively. Assume that q is odd, and let $\mathfrak{P}$ be the binary code spanned by the point neighbourhoods in Γ, i.e., generated by the rows of the adjacency matrix A of Γ. It was shown in Bagchi et al. [1] that $\dim(\mathfrak{P}) = q^3 - q^2 + q + 1$ (indeed, this holds for an arbitrary generalized quadrangle of order (q, q^2)).

Proposition 2.1. *The binary code $\mathfrak{C}$ spanned by the lines of $(X, \mathscr{L})$ has dimension $q^4 + q^2 + 1$.*

This will be obtained as a corollary of the following proposition.

Fix $x \in X$ and let $(Y, \mathscr{M})$ be the subgeometry obtained by removing all points of $x^{\perp}$ from X and from each line in $\mathscr{L}$ (and discarding empty lines), so that in $\mathscr{M}$ all lines have size q. Note that the collinearity graph of $(Y, \mathscr{M})$ is strongly regular with parameters $v = q^4$, $k = (q^2+1)(q-1)$, $\lambda = q-2$, $\mu = q(q-1)$) and eigenvalues k, $r = q-1$ and $s = -(q^2-q+1)$ with multiplicities 1, $f = q(q-1)(q^2+1)$ and $g = (q-1)(q^2+1)$, respectively.

Proposition 2.2. *The binary code spanned by the lines of $(Y, \mathscr{M})$ has dimension q^4.*

Proof. This is the code spanned by the lines in AG$(4, q)$ of one of q^2+1 directions, where these directions are points of an elliptic quadric in the hyperplane at infinity. Since each plane in PG$(3, q)$ meets any elliptic quadric, the result follows from Proposition 1.1. □

Remark. Each word $C \in \mathfrak{C}$ that meets Y in a single point must be of the form $\{x\} + x^{\perp} \cap y^{\perp} + \{y\}$ (by [1], since $\pi_C \in \langle \mathbf{1} \rangle$), and by this proposition $\mathfrak{C}$ does indeed contain all such words. Maybe it is possible to see this more directly.

Let $\mathfrak{P}^+$ and $\mathfrak{C}^+$ be the binary codes generated by the sums of two point neighbourhoods or the sums of two lines, respectively. Since each line meets each point neighbourhood oddly, we have $\dim(\mathfrak{P}/\mathfrak{P}^+) = \dim(\mathfrak{C}/\mathfrak{C}^+) = 1$ and $\mathfrak{P}^+ \subseteq \mathfrak{C}^{\perp}$. The following theorem shows the inclusions between all codes mentioned. In fact, we conjecture that this is the full lattice of $O_6^-(q)$-invariant submodules of 2^X.

For $O_6^-(q)$, q odd:

$$\begin{tikzcd}[row sep=small]
0^{\perp} \arrow[d, dash] & (q^4 + q^3 + q + 1) \\
1^{\perp} \arrow[d, dash] & (q^4 + q^3 + q) \\
\mathfrak{C} = \mathfrak{P}^{+\perp} \arrow[d, dash] & (q^4 + q^2 + 1) \\
\mathfrak{C}^+ = \mathfrak{P}^{\perp} \arrow[d, dash] & (q^4 + q^2) \\
\mathfrak{C}^{+\perp} = \mathfrak{P} \arrow[d, dash] & (q^3 - q^2 + q + 1) \\
\mathfrak{C}^{\perp} = \mathfrak{P}^+ \arrow[d, dash] & (q^3 - q^2 + q) \\
1 \arrow[d, dash] & (1) \\
0 & (0).
\end{tikzcd}$$

For $U_4(q)$, q odd:

$$\begin{tikzcd}[row sep=small]
 & & 0^{\perp} \arrow[d, dash] & (q^5 + q^3 + q^2 + 1) \\
 & & 1^{\perp} \arrow[d, dash] & (q^5 + q^3 + q^2) \\
 & & {*}\mathfrak{P}^{+\perp} \arrow[d, dash] & (q^5 - q^4 + 2q^3 - q^2 + q + 1) \\
(q^4 + q^2 + 1) & {*}\mathfrak{C} \arrow[ur, dash] \arrow[d, dash] & {*}\mathfrak{P}^{\perp} \arrow[d, dash] & (q^5 - q^4 + 2q^3 - q^2 + q) \\
(q^4 + q^2) & {*}\mathfrak{C}^+ \arrow[ur, dash] \arrow[d, dash] & {*}\mathfrak{C}^{+\perp} \arrow[d, dash] & (q^5 - q^4 + q^3 + 1) \\
(q^4 - q^3 + 2q^2 - q + 1) & {*}\mathfrak{P} \arrow[ur, dash] \arrow[d, dash] & {*}\mathfrak{C}^{\perp} & (q^5 - q^4 + q^3) \\
(q^4 - q^3 + 2q^2 - q) & {*}\mathfrak{P}^+ \arrow[ur, dash] \arrow[d, dash] & & \\
(1) & 1 \arrow[d, dash] & & \\
(0) & 0. & &
\end{tikzcd}$$

Landazuri and Seitz [4] show that any nontrivial representation of $U_4(q) \cong O_6^-(q)$ over a field of characteristic not dividing q, has degree at least $(q-1)(q^2+1)$. The module $\mathfrak{P}^+/\langle \mathbf{1} \rangle$ is an example of a module achieving this lower bound, and hence is irreducible. Can this be seen directly?

Theorem 2.3. *All statements about dimensions, inclusions and identities made in the above diagram for $O_6^-(q)$ hold. Any additional $O_6^-(q)$-invariant submodule of 2^X contains $\mathfrak{P}$ and is contained in $\mathfrak{P}^\perp$.*

Proof. The map $C \mapsto C \cap Y$ for $C \in \mathfrak{C}$ has the q^2+1 lines on x in its kernel, and has an image of dimension q^4, so $\dim \mathfrak{C} \geq q^4+q^2+1$ and $\dim \mathfrak{C}^\perp \leq q^3-q^2+q$. But $\mathfrak{P}^+ \subseteq \mathfrak{C}^\perp$ and by [1] $\dim \mathfrak{P}^+ = q^3-q^2+q$, so equality holds everywhere. (This proves Proposition 2.1.) For the last sentence, apply (the proof of) Theorem 10.3 in [1], noting that $\mathfrak{P}^+/\langle \mathbf{1} \rangle$ must be irreducible by [4]. □

Remains to prove that also $\mathfrak{P}^\perp/\mathfrak{P}$ is irreducible.

3. The generalized quadrangle $U_4(q)$ for odd q

Let $(Z, \mathcal{N})$ be the generalized quadrangle of order (q^2, q) formed by the isotropic points and the totally isotropic lines in a $U_4(q)$ geometry. Then $(Z, \mathcal{N})$ is (isomorphic to) the dual of the generalized quadrangle $(X, \mathcal{L})$ considered above. The collinearity graph of this (or any) $\mathrm{GQ}(q^2, q)$ has parameters $v = (q^2+1)(q^3+1)$, $k = q^2(q+1)$, $\lambda = q^2-1$, $\mu = q+1$ and spectrum k, $r = q^2-1$, $s = -q-1$ with multiplicities 1, $f = q^4+q^2$ and $g = q^3(q^2-q+1)$, respectively. Let $^*\mathfrak{P}$ and $^*\mathfrak{C}$ be the binary codes generated by the point neighbourhoods and lines in $(Z, \mathcal{N})$.

Proposition 3.1. *We have* $\dim {}^*\mathfrak{C} = q^4+q^2+1$ *and* $\dim {}^*\mathfrak{P} = (q^2+1)(q^2-q+1)$.

Proof. Clearly, $\dim {}^*\mathfrak{C} = \dim \mathfrak{C}$ because both are the 2-rank of the point-line incidence matrix. And $^*\mathfrak{P} \cong (\mathfrak{C} \cap \mathfrak{C}^\perp)$ (by the map sending $C \in \mathfrak{C}$ to the set of lines meeting C oddly). □

Theorem 3.2. *All statements about dimensions and inclusions made in the above diagram for $U_4(q)$ hold. This diagram is the Hasse diagram for a sublattice of the lattice of all $U_4(q)$-invariant submodules of 2^Z.*

Proof. By the foregoing proposition, all dimensions are as shown. All inclusions are obvious. No other inclusions exist—e.g., $^*\mathfrak{C}^+$ is not contained in $^*\mathfrak{C}^{+\perp}$. All intersections are as shown, e.g., $^*\mathfrak{C}^+ \cap {}^*\mathfrak{C}^{+\perp}$ must coincide with $^*\mathfrak{P}$, since this intersection contains $^*\mathfrak{P}$ and $^*\mathfrak{C}^+/{}^*\mathfrak{P} \cong \mathfrak{P}^+/\langle \mathbf{1} \rangle$. □

Of course we conjecture that in fact the diagram above is the full lattice of submodules. In order to show this, we have to prove that $^*\mathfrak{P}^+/\langle \mathbf{1}\rangle$ and $^*\mathfrak{C}^\perp/^*\mathfrak{P}^+$ are irreducible. (For $q=3$, this is indeed the case.)

Other natural submodules that one might consider can no doubt be identified with those found already. It looks like the code spanned by the hyperbolic lines is $\langle \mathbf{1}^\perp\rangle$ (and indeed, as the referee remarks, this is easy to see: adding all hyperbolic lines in a tangent plane $z^\perp$ that meet a given t.i. line L on z in one of two given points x and y we get the set $\{x, y\}$), that spanned by the pairs of orthogonal hyperbolic lines is $^*\mathfrak{C}^\perp$, and that spanned by the unitals (perps of non-isotropic points) is $^*\mathfrak{P}$.

4. The unital $U(3, q)$ for odd q

Let X be the set of q^3+1 isotropic points in $\mathrm{PG}(2, q^2)$ with respect to a Hermitian form. On X we shall define several structures with X as the underlying set of *points*. Each of these structures will be represented as a collection of subsets of X. The first structure is the unital $\mathscr{L}$, i.e., $\mathscr{L}$ is the set of all intersections of X with hyperbolic (non-absolute) lines of $\mathrm{PG}(2, q^2)$. The elements of $\mathscr{L}$ are called *lines*. There are $q^2(q^2-q+1)$ lines in $\mathscr{L}$, each line contains $q+1$ points and any two distinct points are on a unique line, i.e., $\mathscr{L}$ is a 2-$(q^3+1,\ q+1,\ 1)$ design (cf. [3, 2]).

The binary code $\mathfrak{L}$ of $\mathscr{L}$ is defined as the subspace in $\mathfrak{X} := \boldsymbol{F}_2^X$ spanned by the characteristic vectors of the lines. In other words, $\mathscr{L}$ is the column space (over $\boldsymbol{F}_2$) of the point–line incidence matrix of $\mathscr{L}$. We shall determine the dimension of $\mathfrak{L}$ if $q \equiv 1\ (4)$ but for $q \equiv 3\ (4)$ we only have an upper bound. For this we need a second structure on X, the unitary 2-graph. The *unitary* 2-graph on X is the collection $\mathscr{T}$ of 3-element subsets of X defined by

$$\{\langle x\rangle, \langle y\rangle, \langle z\rangle\} \in \mathscr{T} :\Leftrightarrow (x, y)(y, z)(z, x) \text{ is a nonsquare in } \boldsymbol{F}_{q^2},$$

where (x, y) is the inner product of x and y (elements of $\boldsymbol{F}_{q^2}^3$) with respect to the Hermitian form. See, e.g., Seidel [5] or Taylor [7] for a general introduction to 2-graphs; Taylor [6] discusses the unitary 2-graph. Notice that our definition of the unitary 2-graph is different from the one given by Taylor in [6]: for $q \equiv 3\ (4)$ our definition yields the 2-graph which is the complement of the one in Taylor [6].

5. The code of the unitary 2-graph

First notice that, since $U_3(q)$ is a 2-transitive automorphism group of $\mathscr{T}$, every 2-element subset of X is in the same number a of triples from $\mathscr{T}$, i.e., $\mathscr{T}$ is a *regular* 2-graph. It turns out that

$$a = \begin{cases} \frac{1}{2}(q-1)(q^2+1) & \text{if } q \equiv 1\ (4), \\ \frac{1}{2}(q-1)(q+1)^2 & \text{if } q \equiv 3\ (4). \end{cases}$$

(See the proof of Proposition 6.2 for such a computation, cf. also Taylor [6]). Assume for the moment that $q \equiv 1$ (4). The eigenvalues of $\mathcal{T}$ as defined in Taylor [7] are $\rho_1 = q^2$ and $\rho_2 = -q$ respectively.

For a point $\infty \in X$ we define the graph $\mathcal{T}^\infty$ on X to be the collection of all 2-element subsets $\{x, y\}$ of X for which $\{\infty, x, y\} \in \mathcal{T}$. Notice that ∞ is an isolated point of $\mathcal{T}^\infty$. If we throw away ∞ the resulting graph is a strongly regular graph on $v = q^3$ points of valency $k = a = \frac{1}{2}(q-1)(q^2+1)$. The other parameters of this graph (as defined in Taylor [7]) are $\lambda = \frac{1}{4}(q^3 - 3q^2 + 3q - 5)$, $\mu = \frac{1}{4}(q-1)(q^2+1)$ and the nontrivial eigenvalues (of the 0–1-adjacency matrix) are $r = \frac{1}{2}(q-1)$ with multiplicity $f = (q-1)(q^2+1)$ and $s = -\frac{1}{2}(q^2+1)$ with multiplicity $g = q(q-1)$. Thus the matrix equation for the 0–1-adjacency matrix A of this graph is

$$A^2 + \tfrac{1}{2}(q^2 - q + 2)A - \tfrac{1}{4}(q-1)(q^2+1)I = \tfrac{1}{4}(q-1)(q^2+1)J. \qquad (1)$$

Let $\mathscr{E}$ be any graph on X and let F be any field. We define the *F-code* of $\mathscr{E}$ to be the subspace of F^X spanned by the characteristic vectors of the subsets $\mathscr{E}(x) := \{y \in X \mid \{x, y\} \in \mathscr{E}\}$, $x \in X$ (i.e., it is the subspace spanned by the rows (or columns) of the adjacency matrix of $\mathscr{E}$). If $q \equiv 1$ (4) we can compute the dimension of the binary code of the graphs $\mathcal{T}^\infty$: it is the 2-rank of the matrix A. Modulo 2 equation (1) reduces to $A^2 + A = O$. Hence, it follows that $\mathrm{rk}_2(A) + \mathrm{rk}_2(A+I) \leq v$. But clearly $v = \mathrm{rk}_2(I) = \mathrm{rk}_2(I + A + A) \leq \mathrm{rk}_2(I+A) + \mathrm{rk}_2(A)$ and so it follows that $\mathrm{rk}_2(A) + \mathrm{rk}_2(A+I) = v$. Since the $\boldsymbol{Q}$-rank of

$$A - sI - (k-s)/vJ = A + (q^2+1)/2I - (q^2+1)/(2q^2)J$$

is f, it follows that $\mathrm{rk}_2(A + I + J) \leq f$ and therefore

$$\mathrm{rk}_2(A+I) \leq \mathrm{rk}_2(A+I+J) + \mathrm{rk}_2(J) \leq f + 1.$$

Similarly, since the $\boldsymbol{Q}$-rank of

$$A - rI - (k-r)/vJ = A - (q-1)/2I - (q-1)/(2q)J$$

is g, it follows that $\mathrm{rk}_2(A) \leq g$. From $v = f + g + 1$ it now follows that $\mathrm{rk}_2(A + I) = f + 1$, $\mathrm{rk}_2(A + I + J) = f$ and $\mathrm{rk}_2(A) = g = q^2 - q$, i.e., the dimension of the binary code of $\mathcal{T}^\infty$ is $q^2 - q$.

We define the binary *code* $\mathfrak{X}$ of the 2-graph $\mathcal{T}$ to be the subspace of the vector space $\mathfrak{X}$ spanned by the all-one vector $\mathbf{1}$ and the $\boldsymbol{F}_2$-code of $\mathcal{T}^\infty$. This definition is independent of the choice of the point ∞. To see this suppose $\mathscr{E}$ is a graph in the switching class of $\mathcal{T}$ and suppose $\mathscr{E}'$ is the graph we get from $\mathscr{E}$ if we switch w.r.t. a subset Y of X. Then

$$\mathscr{E}'(x) = \begin{cases} \mathscr{E}(x) \div Y & \text{if } x \notin Y, \\ \mathscr{E}(x) \div (X \backslash Y) & \text{if } x \in Y \end{cases}$$

(here $\div$ denotes the symmetric difference). Since $\mathcal{T}^{\infty_1}$ can be obtained from $\mathcal{T}^{\infty_2}$ by switching w.r.t. the neighbours of ∞_2 in $\mathcal{T}^{\infty_1}$ (a set whose characteristic vector is in the code of $\mathcal{T}^{\infty_1}$), it follows that the code obtained by using ∞_2 is contained in the code obtained by using ∞_1, hence these codes are the same.

Proposition 5.1. *If $q \equiv 1$ (4), then the binary code $\mathfrak{X}$ of the unitary 2-graph satisfies*

(a) $\dim(\mathfrak{X}) = q^2 - q + 1$,

(b) $\langle \mathbf{1} \rangle = \mathfrak{X} \cap \mathfrak{X}^{\perp}$,

(c) $\mathfrak{X}^{\perp}$ *is generated by* $\mathbf{1}$ *and the rows of $A + I + J$ (extended with a zero in the ∞-coordinate).*

Proof. Clearly, the dimension of $\mathfrak{X}$ is one more then the dimension of the code of the graph $\mathcal{T}^{\infty}$. In our case the dimension of $\mathfrak{X}$ is therefore $q^2 - q + 1$.

To prove (b), notice that by definition $\mathbf{1} \in \mathfrak{X}$ and since both $v + 1$ and k are even we also have $\mathbf{1} \in \mathfrak{X}^{\perp}$. Let $x \in \mathfrak{X} \cap \mathfrak{X}^{\perp}$. Since $x \in \mathfrak{X}$ there exists $y \in \mathcal{X}$ such that

$$x = y\left(\begin{array}{c|c} 1 & \mathbf{1} \\ \hline \mathbf{0}^{t} & A \end{array}\right),$$

where A is the adjacency matrix of the strongly regular graph on $X \setminus \{\infty\}$. If we multiply this equation by

$$\left(\begin{array}{c|c} 1 & \mathbf{1} \\ \hline \mathbf{0}^{t} & A + I + J \end{array}\right) = I + \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0}^{t} & A \end{array}\right) + \left(\begin{array}{c|c} 0 & \mathbf{1} \\ \hline \mathbf{0}^{t} & J \end{array}\right)$$

then the LHS is just x since $x \in \mathfrak{X}^{\perp}$. Using $A(I + A) = O$ and $AJ = O$, the RHS is easily seen to be $y_{\infty}\mathbf{1}$ (where y_{∞} is the first component of y).

For (c), notice that $\mathbf{1}$ and the rows of $A + I + J$ are contained in $\mathfrak{X}^{\perp}$. Since $\mathrm{rk}_2(A + I + J) = f$, these vectors span a code of dimension $f + 1 = q^3 - q^2 + q = q^3 + 1 - \dim(\mathfrak{X}) = \dim(\mathfrak{X}^{\perp})$. $\square$

For $q \equiv 3$ (4) the code of the 2-graph turns out to be self-orthogonal and we again have to use a result from [4] to compute its dimension.

Proposition 5.2. *If $q \equiv 3$ (4), then the binary code $\mathfrak{X}$ of the unitary 2-graph satisfies*

(a) $\dim(\mathfrak{X}) = q^2 - q + 1$,

(b) $\langle \mathbf{1} \rangle \subseteq \mathfrak{X} \subseteq \mathfrak{X}^{\perp}$.

Proof. Again let A denote the adjacency matrix of the strongly regular graph on q^3 points. The same reasoning as in the case of $q \equiv 1$ (4) now gives that $A + (q + 1)/2I - (q + 1)/(2q)J$ has $\mathbb{Q}$-rank $q^2 - q$. Hence, $\dim(\mathfrak{X}) \leqslant q^2 - q + 1$. In [4] it is shown that a nontrivial representation of $U(3, q)$ over a field of characteristic not dividing q has degree at least $q^2 - q$. Therefore $\dim(\mathfrak{X}/\langle \mathbf{1} \rangle) \geqslant q^2 - q$. This proves (a). The rows of A have even weight and modulo 2 the matrix equation for A reduces to $A^2 = O$ proving (b). Note that in this case $(A + I)^2 \equiv I$ (mod 2) so that $\mathrm{rk}_2(A + I) = v$ and we have no description of $\mathfrak{T}^\perp$ similar to 5.1(c). $\square$

6. The code of the unital

The discussion of the 2-graph already indicates that the cases $q \equiv 1$ (4) and $q \equiv 3$ (4) behave differently. Our aim is to show that $\mathfrak{L} = \mathfrak{T}^\perp$ but we can do this only for $q \equiv 1$ (4) and only have some partial results for $q \equiv 3$ (4). Our first result is valid for all odd q.

Proposition 6.1. $\dim(\mathfrak{L}) \leq q(q^2 - q + 1)$.

Proof. Let N be the $(q^3 + 1) \times q^2(q^2 - q + 1)$ incidence matrix of the unitial. Fix a point ∞ of the unital. The coherent subsets of $\mathfrak{T}$ are precisely the induced subgraphs of $\mathfrak{T}^\infty$ which are the disjoint union of at most 2 cliques and the incoherent subsets are the cocliques and complete bipartite subgraphs of $\mathfrak{T}^\infty$ (this is true for any 2-graph). In particular, if $q \equiv 3$ (4), every line of the unital is a $K_{a,q+1-a}$ or $\bar{K}_{q+1}$ (one gets of course the complements of these graphs for $q \equiv 1$ (4)). For each column of N, arbitrarily choose one part of the corresponding bipartite graph and replace the ones in this column corresponding to this part by minus ones. One then obtains a matrix $\tilde{N}$ which satisfies

$$\tilde{N}\tilde{N}^t = B + q^2 I,$$

where B is the ± 1-adjacency matrix of $\mathfrak{T}^\infty$. Since $-q^2$ is an eigenvalue of B with multiplicity $g + 1 = q^2 - q + 1$ it follows that the $\mathbf{Q}$-rank of $\tilde{N}\tilde{N}^t$ is $q(q^2 - q + 1)$. $\square$

We shall now relate the code $\mathfrak{T}$ with the code $\mathfrak{L}$. We start with the case $q \equiv 1$ (4) which we can settle completely.

Proposition 6.2. *If* $q \equiv 1$ (4) *then* $\mathfrak{L} = \mathfrak{T}^\perp$.

Proof. Let L be a line of the unital and let $\boldsymbol{l} \in \mathfrak{L}$ be its characteristic vector. We shall show that $\boldsymbol{l} \in \mathfrak{T}^\perp$. Since $|L| = q + 1 \equiv 0$ (2) it is clear that $\boldsymbol{l} \in \mathbf{1}^\perp$ and it remains to show that a point x is joined to an even number of points on L (in the graph $\mathcal{T}^\infty$ with ∞ suitably chosen). It is well known that L is a coherent set of maximal size $q + 1$ in the 2-graph $\mathcal{T}$ (see Taylor [7]). Such a set C has the property that for every point $x \notin C$ there is a partition of C into two subsets C' and C'' of equal size $\frac{1}{2}(q + 1)$ such that for all $y, z \in C$ we have $\{x, y, z\} \in \mathcal{T}$ if and only if y and z are both in C' or both in C'' (Proposition 5.2. in Taylor [7]). Thus, if we fix the point ∞ on L, then in the graph T^∞, a point $x \in L \backslash \{\infty\}$ is joined to the $q-1$ points of $L\setminus\{x,\infty\}$, a point $x\notin L$ is joined to $\frac{1}{2}(q-1)$ points of L, and ∞ is joined to 0 points of L. Since $q\equiv 1$ (4) it follows that $\mathfrak{L}\subseteq\mathfrak{T}^{\perp}$.

It remains to show that $\mathfrak{T}^{\perp}\subseteq\mathfrak{L}$. We know from Proposition 5.1 that $\mathfrak{T}^{\perp}$ is spanned by $\mathbf{1}$ and the rows of $A+I+J$. It suffices to prove that these generating vectors are in $\mathfrak{L}$. For $\mathbf{1}$ this is easy: the q^2 lines on a given point sum up to $\mathbf{1}$. Remains to show that the set of nonneighbours of a point x in the graph $\mathcal{T}^{\infty}$ is in $\mathfrak{L}$. By 2-transitivity of $U_3(q)$ it suffices to do this for two suitably chosen points x and ∞. We shall use the Hermitian form

$$(x,y)=x_1y_1^q+x_2y_3^q+x_3y_2^q$$

and choose $x=\langle(0,0,1)\rangle$ and $\infty=\langle(0,1,0)\rangle$. The nonneighbours of x in $\mathcal{T}^{\infty}$ are the points $\langle(\xi,\eta,\zeta)\rangle$ for which $\xi\xi^q+\eta\zeta^q+\zeta\eta^q=0$ and $1\cdot\eta^q\cdot\zeta$ is a (nonzero) square in $\boldsymbol{F}_{q^2}$. There are $\frac{1}{2}(q-1)(q+1)^2$ such points and these are on $\frac{1}{2}(q^2-1)$ lines through the non-isotropic point $\langle(1,0,0)\rangle$ (the polar of the line through x and ∞). Indeed, normalize by taking $\eta=1$. Choose a square ζ ($\frac{1}{2}(q^2-1)$ possibilities). Then $\zeta+\zeta^q\neq 0$ since $q\equiv 1$ (4) and so there are $q+1$ values ξ with $\xi\xi^q+\zeta+\zeta^q=0$ and these $q+1$ points are precisely the isotropic points on the polar of $\langle(0,1,-\zeta^q)\rangle$. □

Remark. For $q\equiv 3$ (4) the first half of the above proof carries over without many problems: we now get that a point x is *not* joined to an even number of points on any given line L. Since $|L|=q+1$ is even x is also *joined* to an even number of points on L and so the same conclusion follows: $\mathfrak{L}\subseteq\mathfrak{T}^{\perp}$. But redoing the second half of the proof now only shows that $\mathfrak{T}\subseteq\mathfrak{L}$.

Proposition 6.3. *If $q\equiv 3$ (4), then, $\mathfrak{T}\subseteq\mathfrak{L}\subseteq\mathfrak{T}^{\perp}$.*

We conjecture that $\mathfrak{L}=\mathfrak{T}^{\perp}$ also for $q\equiv 3$ (4). We conclude with a result that shows $\mathfrak{L}/(\mathfrak{L}\cap\mathfrak{L}^{\perp})$ is irreducible as a PGU$(3,q^2)$-module.

Proposition 6.4. *Let $\mathfrak{M}$ be a* PGU$(3,q^2)$*-submodule of $\mathfrak{X}$ containing* $\mathbf{1}$ *and contained in $\mathbf{1}^{\perp}$. Then either $\mathfrak{L}\subseteq\mathfrak{M}$ or $\mathfrak{M}\subseteq\mathfrak{L}^{\perp}$.*

Proof. Let $\boldsymbol{m}\in\mathfrak{M}\setminus\mathfrak{L}^{\perp}$. Then there is a line $L\in\mathcal{L}$ such that $\boldsymbol{m}\cdot\boldsymbol{l}\neq 0$ where $\boldsymbol{l}$ is the characteristic vector of L. Fix a point $x\in\operatorname{supp}(\boldsymbol{m})\cap L$ and let G be the stabilizer of x and L in PGU$(3,q^2)$. Notice that G has order q^3-q and acts regularly on $X\setminus L$ (and is transitive on $L\setminus\{x\}$). From this it follows that $\sum_{g\in G}g(\boldsymbol{m})$ is a word in $\mathfrak{M}$ which is zero on every coordinate position in L (since q^2-1 is even) and is $\boldsymbol{m}\cdot(\mathbf{1}-\boldsymbol{l})\neq 0$ on every coordinate position in $X\setminus L$. Hence $\mathbf{1}-\boldsymbol{l}\in\mathfrak{M}$ and since also $\mathbf{1}\in\mathfrak{M}$ we have $\boldsymbol{l}\in\mathfrak{M}$. □

References

[1] B. Bagchi, A.E. Brouwer and H.A. Wilbrink, Notes on binary codes related to the $O(5, q)$ generalized quadrangle for odd q, Geom. Dedicata 39 (1991) 339–355.
[1] Th. Beth, D. Jungnickel and H. Lenz, Design Theory (Bibliographisches Institut, Zürich, 1985).
[3] D.R. Hughes and F.C. Piper, Design Theory (Cambridge Univ. Press, Cambridge, 1985).
[4] V. Landazuri and G.M. Seitz, On the minimal degree of projective representations of the finite Chevalley groups, J. Algebra 32 (1974) 418–443.
[5] J.J. Seidel, A survey of two-graphs, Proc. Intern. Colloq. Theorie Combinatorie (Roma 1973), Accad. Naz. Lincei (1976) 481–511.
[6] D.E. Taylor, Graphs and block designs associated with the three-dimensional unitary groups, Combinatorial Mathematics, Proc. 2nd Austral. Conf., A. Dold and B. Eckmann, eds., Lecture Notes in Math., Vol. 403 (Springer Berlin, 1974) 128–131.
[7] D.E. Taylor, Regular 2-graphs, Proc. London Math. Soc. 35 (3) (1977) 257–274.