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Robustness to strategic uncertainty

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Abstract. We model a player's uncertainty about other players' strategy choices as smooth probability distributions over their strategy sets. We call a strategy profile (strictly) robust to strategic uncertainty if it is the limit, as uncertainty vanishes, of some sequence (all sequences) of strategy profiles, in each of which every player's strategy is optimal under under his or her uncertainty about the others. We derive general properties of such robustness, and apply the definition to Bertrand competition games and the Nash demand game, games that admit infinitely many Nash equilibria. We show that our robustness criterion selects a unique Nash equilibrium in the Bertrand games, and that this agrees with recent experimental findings. For the Nash demand game, we show that the less uncertain party obtains the bigger share.

Keyword: Nash equilibrium, refinement, strategic uncertainty, price competition, Bertrand competition, bargaining, Nash demand game

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1. Introduction

Many games admit multiple Nash equilibria. As is well known, even in games with a unique Nash equilibrium, common knowledge of the game and of all players' rationality does not, in general, suffice to provide them with clear guidance, see e.g. Bernheim (1984), Pearce (1984) and Aumann and Brandenburger (1995). Players cannot, in general, deduce the behavior of other players, and yet beliefs about others' strategy choices are pivotal to a player's decision as to what strategy to choose. Following Harsanyi and Selten (1988) and Brandenburger (1996), it has become customary to allude to this type of uncertainty, concerning the behavior of other players in the game, as 'strategic uncertainty', as opposed to uncertainty regarding the underlying structure of the game played, which is often called 'structural uncertainty' (see e.g. Morris and Shin [2002]).

Strategic uncertainty matters because in a lot of games, many of the equilibria represent fragile situations in which players are supposed to choose a particular strategy, even though this would be optimal only if they held knife-edge beliefs about the actions taken by other players.[1] In such settings, even the slightest uncertainty about other players' choices might lead them to deviate from their equilibrium strategy. It is then arguably reasonable to require strategy profiles to be robust to small amounts of uncertainty about other players' strategies. That human beings' behavior in games admitting multiple equilibria is fairly stable and predictable in the aggregate is the finding of several experimental studies. See, e.g., Van Huyck, Battalio and Beil (1990, 1991), Abbink and Brandts (2008) or Heinmann, Nagel and Ockenfels (2009).

In this study, we formalize a notion of strategic uncertainty for games with one-dimensional continuum action spaces and propose a criterion for robustness to such uncertainty.[2] Our approach is roughly as follows. A player's uncertainty about others' strategy choices is represented by a probability distribution over others' strategy sets, scaled with a parameter $t \geq 0$. The probability distributions are assumed to be atomless and have standard regularity properties. For each value of the uncertainty parameter t, we define a t-equilibrium as a Nash equilibrium of the game in which each player strives to maximize her expected payoff under her strategic uncertainty so defined. For $t = 0$, this is nothing else than Nash equilibrium in the original game. We call a strategy profile robust to strategic uncertainty if there exists a collection of probability distributions in the admitted class, one for each player, such that some accompanying sequence of t-equilibria converges to this profile, as the uncertainty parameter t tends to zero. If convergence holds for all distributions in the admitted class, we say that the strategy profile is strictly robust to strategic uncertainty.

[1] The equilibrium strategy remains a best response if either players are certain about the actions of other players or they hold beliefs lying in a lower-dimensional subspace of probabilistic beliefs.

[2] We believe that our approach can be generalized to higher-dimensional strategy sets.

Our approach is not eductive or normative: we do not attempt at telling players what beliefs they should hold about other players' actions. Neither is our approach epistemic in the sense of being derived from assumptions about rationality, knowledge or beliefs in abstract type spaces.[3] Our aim is more modest: we simply study what play may arise when players are slightly strategically uncertain, whatever the reasons for their uncertainty may be. Contrary to the literature on global games (Carlsson and van Damme [1993], Morris and Shin [2003]), we do not derive players' uncertainty from primitive assumptions within a meta-game in which the game at hand is embedded. When compared to this literature, our approach is given in reduced form: we derive players' actions from arbitrary subjective beliefs (in an admissible class) without deriving those beliefs from a structural source of uncertainty.

We first study general properties of our robustness criterion. For games with continuous payoff functions, we establish existence results and clarify the relationship with some existing solution concepts. In particular, we show that robustness is a refinement of Nash equilibrium and that it implies, for two-player games, weak perfection in the sense of Simon and Stinchcombe (1995). The general picture is more complex for games with discontinuous payoff functions. In particular, we show by way of an example that there are games in which non-Nash equilibrium strategy profiles are robust to strategic uncertainty.

Our robustness criterion is closely related to Selten's (1975) "substitute perfection". Selten defined a Nash equilibrium in a finite game to have this property if there exists a sequence of completely mixed strategy profiles, converging to the equilibrium in question, such that each player's equilibrium strategy is a best reply to all but finitely many strategy profiles in the sequence. Substitute perfect equilibria exist in all finite games, and, as Selten (1975) shows, they coincide with (trembling-hand) perfect equilibria. However, in generic non-linear games with continuum strategy spaces, no Nash equilibrium is literally substitute perfect, the reason being that small perturbations of players' beliefs induce small changes in their best replies (while the discreteness in finite games allows best replies to remain unchanged under such perturbations).

Simon and Stinchcombe (1995) extended Selten's perfection criterion to games with compact strategy sets and continuous payoff functions. By contrast, we here analyze games with discontinuous payoff functions. Binmore (1987) and Carlsson (1991) study equilibrium selection in the Nash demand game (Nash, 1953). Both authors assume that players "tremble." By contrast, in our model players do not "tremble"; they are only uncertain about other players' action. Young (1993) takes the evolutionary route to show that the (generalized) Nash bargaining solution is the only stochastically stable outcome of a game where individuals randomly drawn

[3] See the discussion in Brandenburger (2007).

from two populations are matched to play the Nash demand game.[4] Carlsson and Ganslandt (1998) investigate "noisy equilibrium selection" in symmetric coordination games and derive results that agree with the experimental findings on minimal effort games in Van Huyck et al. (1990). While Carlsson's and Ganslandt's (1998) study is tailored to the minimal effort game, we here make general assumptions concerning players' beliefs, assumptions that permit an operational definition of robustness to strategic uncertainty for a large class of games. Our approach is related to that in Friedman and Mezzetti (2005), who introduce a notion of "robust equilibria" in finite games, as the limit of sequences of "random belief equilibria." In a random belief equilibrium, all players' beliefs are random variables, and a player's best-reply distribution, implied by her belief distribution, is required to be consistent, in terms of statistical expectation, with others' beliefs about that player's action. By contrast, we analyze continuum-action games and impose no such interpersonal consistency requirement.

We provide two main applications. The first is to Bertrand competition.[5] By way of a simple duopoly example with constant and identical marginal costs, we first show that our refinement admits the unique and weakly dominated Nash equilibrium. By contrast, when marginal costs are strictly increasing, our robustness criterion selects a unique strategy profile out of the continuum of Nash equilibria (see Dastidar [1995] for an analysis of such games).[6] Our prediction agrees with recent findings in experimental studies of (discretized versions of) the model, see Abbink and Brandts (2008) and Argenton and Müller (2009).[7] Abbink and Brandts (2008) remark that "[that] price level (...) is not predicted by any benchmark theory [they] are aware of"

[4] The literature on the non-cooperative foundations of the Nash bargaining solution and the so-called Nash program is enormous, see e.g. Serrano (2005) for a recent survey.

[5] Following Vives (1999, p.117), we take Bertrand competition to mean that (a) sellers simultaneously choose their prices and (b) each firm has to serve all its clients at the price it has chosen. As mentioned by Vives (1999), for certain utilities and auctions, provision is legally mandated, and in other markets firms have a strong incentive to serve all their clients, especially in industries in which customers have an on-going relationship with suppliers (subscription, repeat purchases, etc.) or where the costs of restricting output in real time are high.

[6] There are a number of papers focused on price competition with convex costs. Dixon (1990) studies such competition when firms are not obliged to serve all demand, but incur a cost when turning costumers down. He shows that under such circumstances there may still exist a continuum of pure-strategy Nash equilibria. Spulber (1995) assumes that firms are uncertain about rivals' costs and shows that there exists a unique symmetric Nash equilibrium in pure strategies. As the number of firms grows, equilibrium pricing strategies tend to average cost pricing. Chowdhury and Sengupta (2004) show that, in Bertrand games with convex costs, there exists a unique coalition-proof Nash equilibrium (in the sense of Bernheim, Peleg and Whinston 1987), which converges to the competitive outcome under free entry. Our criterion selects another price, which, moreover, does not depend on the number of firms.

[7] Abbink and Brandts (2008) ran experiments with fixed groups of two, three, and four identical firms. They find that duopolists are often able to collude on the joint profit-maximizing price.

(p. 3). The present refinement provides a theoretical foundation for their finding.

Heuristically, strategic uncertainty in these pricing games results in uncertainty-perturbed profit functions that are continuous, since the likelihood of serving the entire market is continuous in one's own price. The deviation incentives are asymmetric, though. For high Nash equilibrium prices, a strategically uncertain player has an incentive to cut her price, since she has a lot to lose if others' prices lie a bit below her price and little to gain if they lie a bit above it. Conversely, for low Nash equilibrium prices, each player has an incentive to raise her price, since she has a lot to loose if others' prices lie a bit above her price and little to gain if they lie a bit below. The only price that is robust to strategic uncertainty is the price at which the monopoly profit is zero. This is also the maximal Nash equilibrium price in the limit as the number of competitors tend to infinity. At that price, and no other price, the incentives to move up and down for a strategically uncertain player are of the same order of magnitude.

Our second application is the classical Nash (1953) bargaining game, in which two players simultaneously bid for their share of a given pie, and obtain their shares if and only if their bids are compatible. We show that equal division is robust to strategic uncertainty when the two parties are equally uncertain and, more generally, that pie sharing is related to the 'relative uncertainty' affecting players, thus providing some strategic foundation for the generalized Nash bargaining solution (Kalai [1977]).

The remainder of the paper is organized as follows. Section 2 provides definitions and general results. Section 3 deals with Bertrand competition and Section 4 with the Nash demand game. Section 5 concludes.

2. Robustness to strategic uncertainty

Let $G = (N, S, \pi)$ be an n-player normal-form game with player set $N = \{1, ..., n\}$, in which the pure-strategy set of each player is $S_i = \mathbb{R}$, and thus $S = \mathbb{R}^n$ is the set of pure-strategy profiles $\mathbf{s} = (s_1, ..., s_n)$, and $\pi : S \to \mathbb{R}^n$ is the combined payoff-function, with $\pi_i(\mathbf{s})$ being the payoff to player i when $\mathbf{s}$ is played.[8]

Let $\mathcal{F}$ be the class of *log-concave probability distributions* with finite mean. More exactly, by log-concavity we mean cumulative probability distribution functions $\Phi : \mathbb{R} \to [0, 1]$ with everywhere positive and differentiable density $\phi = \Phi'$, such that $\ln \phi$ is a concave function.[9] A useful feature of those distributions is that they have

However, the lowest price in the range of Nash equilibria which involves no loss in case of miscoordination (24 in their specification), a much smaller number than the collusive price, is also an attractor of play. With more than two firms in a market, it actually is the predominant market price. This outcome is also observed in the complete information, symmetric treatment in Argenton and Müller (2009).

[8] See below for how this machinery can be adapted to interval strategy sets.

[9] The log-concavity assumption is common in the economics literature and has applications in

non-decreasing hazard rates in both directions; that is, the hazard rate

$$h(x) = \frac{\phi(x)}{1 - \Phi(x)}$$

is non-decreasing (see Corollary 2 in Bagnoli and Bergstrom, 2005) and the reversed hazard rate,

$$h^{-}(x) = \frac{\phi(x)}{\Phi(x)}$$

non-increasing.[10] Examples of log-concave distributions are the normal, exponential and Gumbel distributions.

Definition 1. *For any given $t \geq 0$, a strategy profile* $\mathbf{s}$ *is a **t-equilibrium** of G if, for each player i, the strategy s_i maximizes i's expected payoff under the probabilistic belief that all other players' strategies are random variables of the form*

$$\tilde{s}_{ij} = s_j + t \cdot \varepsilon_{ij} \tag{1}$$

for some statistically independent "noise" terms $\varepsilon_{ij} \sim \Phi_{ij}$, where $\Phi_{ij} \in \mathcal{F}$ for all $j \neq i$.

Remark 1. *For $t = 0$, this definition coincides with that of Nash equilibrium.*

Remark 2. *For $t > 0$, the random variable $\tilde{s}_{ij}$ has the c.d.f. $F_{ij}^t \in \mathcal{F}$ defined by*

$$F_{ij}^t(x) = \Phi_{ij}\left(\frac{x - s_j}{t}\right) \quad \forall x \in \mathbb{R}.$$

Note that we do not require that noise terms be symmetric or have expectation zero. In particular, in a t-equilibrium players may believe that others have a systematic tendency to deviate upwards or downwards.

Example 1. *Let Φ_{ij} be a normal distribution, $N(\mu, \sigma)$, with $\mu = \sigma = 1$, and hence $E[\tilde{s}_{ij}] = s_j + t$. Then the density f_{ij}^t is skewed to the right, as shown in the diagram below for $s_j = 10$, and $t = 0.3$ (thick), $t = 0.1$ (dashed) and $t = 0.05$ (thin).*

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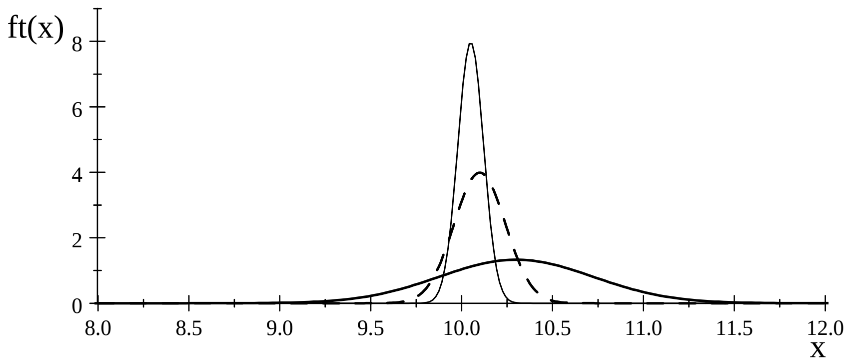
mechanism design, game theory and labor economics, see Bagnoli and Bergstrom (2005).

[10] The latter follows from the fact that if ϕ is log-concave then so is Φ (see Theorem 1 in Bagnoli and Bergstrom, 2005) and hence $(\phi(x)/\Phi(x))' = (\ln \Phi(x))'' \leq 0$.



Player i's probabilitic belief about s_j

Let $\tilde{\mathbf{s}}_{-i}$ denote the $(n-1)$-vector of random variables $(\tilde{s}_{ij})_{j\neq i}$. We note that a t-equilibrium is a Nash equilibrium of a game with perturbed payoff functions:

Remark 3. *Let $t > 0$ and $\Phi_{ij} \in \mathcal{F}$ for all $i \in N$ and $j \neq i$. A strategy profile $\mathbf{s} \in S$ is a t-equilibrium of $G = (N, S, \pi)$, with $\varepsilon_{ij} \sim \Phi_{ij}$, if and only if it is a Nash equilibrium of the* ***perturbed game*** *$G^t = (N, S, \pi^t)$, where*

$$\begin{aligned}\pi_i^t(\mathbf{s}) &= \mathbb{E}\left[\pi_i\left(s_i, \tilde{\mathbf{s}}_{-i}\right)\right] \\ &= \int_{S_1}..\int_{S_{i-1}}\int_{S_{i+1}}..\int_{S_n} \pi_i\left(s_i, \mathbf{s}_{-i}\right) dF_{i1}^t\left(s_1\right)..dF_{i,i-1}^t\left(s_{i-1}\right) dF_{i,i+1}^t\left(s_{i+1}\right)..dF_{in}^t\left(s_n\right) \\ &= \frac{1}{t^{n-1}}\int..\int..\int\left[\prod_{j\neq i}\phi_{ij}\left(\frac{x_j - s_j}{t}\right)\pi_i\left(s_i, x_{-i}\right)\right] dx_1..dx_{i-1}dx_{i+1}..dx_n\end{aligned}$$

We are now in a position to define robustness to strategic uncertainty.

Definition 2. *A strategy profile $\mathbf{s}^*$ in the game G is* ***robust to strategic uncertainty*** *if there exists a collection of c.d.f:s $\{\Phi_{ij} \in \mathcal{F} : \forall i \in N, j \neq i\}$ and an accompanying sequence of t-equilibria, $\langle \mathbf{s}^{t_k}\rangle_{k\in\mathbb{N}}$ with $t_k \downarrow 0$, such that $s^{t_k} \to s^*$ as $k \to +\infty$. The strategy profile $\mathbf{s}^*$ is* ***strictly*** *robust to strategic uncertainty if this holds for all collections of c.d.f:s $\{\Phi_{ij} \in \mathcal{F} : \forall i \in N, j \neq i\}$.*

Remark 4. *This definition can be adapted as follows to games in which the strategy set of each player j is an interval $S_j = [0, b_j]$ for some $b_j > 0$.*[11] *For any $\Phi_{ij} \in \mathcal{F}$, let*

$$F_{ij}^t(x) = \frac{\Phi_{ij}\left(\frac{x-s_j}{t}\right) - \Phi_{ij}\left(-\frac{s_j}{t}\right)}{\Phi_{ij}\left(\frac{b_j-s_j}{t}\right) - \Phi_{ij}\left(-\frac{s_j}{t}\right)}$$

[11] Without loss of generality, we normalize the left end of each interval to $a_j = 0$.

This defines a c.d.f. for $\tilde{s}_{ij}$ with support $[0, b_j]$, such that, for any $s_j, x \in [0, b_j]$:

$$\lim_{t \to 0} F_{ij}^t(x) = \begin{cases} 0 & \text{if } x < s_j \\ 1 & \text{if } x \geq s_j \end{cases}$$

Taking expectations with respect to such c.d.f:s F_{ij}^t, one obtains a perturbed game with payoff functions

$$\begin{aligned} \pi_i^t(\mathbf{s}) &= \mathbb{E}\left[\pi_i\left(s_i, \tilde{\mathbf{s}}_{-i}\right)\right] \\ &= \frac{1}{t^{n-1}} \int \dots \int \dots \int \left[\prod_{j \neq i} \phi_{ij}^t\left(\frac{x_j - s_j}{t}\right) \pi_i\left(s_i, x_{-i}\right)\right] dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n \end{aligned}$$

where

$$\phi_{ij}^t\left(\frac{x_j - s_j}{t}\right) = \frac{\phi_{ij}\left(\frac{x - s_j}{t}\right)}{\Phi_{ij}\left(\frac{b_j - s_j}{t}\right) - \Phi_{ij}\left(-\frac{s_j}{t}\right)}. \tag{2}$$

We note that for any interior strategy profile, $\mathbf{s} \in \times_{i \in N}(0, b_i)$, our robustness criterion is the same, whether or not the noise terms are truncated to the strategy sets in this way: for any $s_j \in (0, b_j)$, the denominator in (2) converges to 1 and its derivative converges to zero. If instead $S_i = \mathbb{R}_+$ for all players i, then all properties are retained by setting

$$F_{ij}^t(x) = \frac{\Phi_{ij}\left(\frac{x - s_j}{t}\right) - \Phi_{ij}\left(-\frac{s_j}{t}\right)}{1 - \Phi_{ij}\left(-\frac{s_j}{t}\right)}. \tag{3}$$

2.1. Continuous games. We here establish that, for games with continuous payoff functions, which we call *continuous games*, robustness to strategic uncertainty implies Nash equilibrium. We also show that the converse is not true, and derive some results concerning the relationship to some other Nash equilibrium refinements for such games.

Proposition 1. *If the payoff functions in G are continuous and $\mathbf{s}^*$ is robust to strategic uncertainty, then $\mathbf{s}^*$ is a Nash equilibrium.*

Proof: Suppose that the claim is false. Let $\{\Phi_{ij} : \forall i \in N, j \neq i\} \subset \mathcal{F}$ and let $\langle \mathbf{s}^{t_k} \rangle_{k \in \mathbb{N}}$ be a sequence of t_k-equilibria with $t_k \downarrow 0$, such that $\mathbf{s}^{t_k} \to \mathbf{s}^*$, where $\mathbf{s}^*$ is not a Nash equilibrium in G. Then, there exists a player i and pure strategy $s_i^0 \in S_i$ such that $\pi_i\left(s_i^0, \mathbf{s}_{-i}^*\right) - \pi_i(\mathbf{s}^*) = \delta$ for some $\delta > 0$. Now, π_i^t converges pointwise to π_i as $t \to 0$, by continuity of π_i. Hence, $\pi_i^t\left(s_i^0, \mathbf{s}_{-i}^*\right) - \pi_i^t(\mathbf{s}^*) > \delta/2$ for all $t > 0$ sufficiently close to zero, say, for all $t < \tau(\delta)$, where $\tau(\delta) > 0$. Moreover, since π_i^t is continuous, there exists a neighborhood B of $\mathbf{s}^*$ such that

$$\pi_i^t\left(s_i^0, \mathbf{s}_{-i}'\right) - \pi_i^t(\mathbf{s}') > \delta/3$$

for all $s' \in B$ and $t \in (0, \tau(\delta))$. In particular, B contains no t-equilibrium with $t < \tau(\delta)$. But this contradicts the hypothesis that all the elements of the sequence $\langle \mathbf{s}^k \rangle_{k=1}^{\infty} \to \mathbf{s}^*$ are t_k-equilibria. **End of proof.**

The following example shows that there are continuous games that have non-robust Nash equilibria.

Example 2. *Let G be a two-player game on $\mathbb{R}^2$, where $\pi_2(s_1, s_2) = -(s_2)^2$, so that 2's unique best reply is always $s_2^* = 0$. Let $\pi_1(s_1, s_2) = -(s_1)^2(s_2)^2$. Then 1's best reply to each $s_2 \neq 0$ is $s_1^* = 0$. For $s_2 = 0$, all $s_1 \in \mathbb{R}$ are best replies. However, $s_1^* = 0$ is the unique undominated best reply. This game has infinitely many Nash equilibria, $\{(s_1, s_2) \in \mathbb{R}^2 : s_2 = 0\}$, but only one of them, $(0,0)$ is robust to strategic uncertainty (indeed, it is strictly robust). For example, $(s_1, s_2) = (1, 0)$ is not robust, since for any $t > 0$, the unique t-equilibrium is $(0,0)$.*

Hence, for continuous games, our definition of robustness to strategic uncertainty is a refinement of Nash equilibrium. We proceed to show that in continuous games with (non-empty) compact strategy sets, strict robustness to strategic uncertainty implies weak perfection in the sense of Simon and Stinchcombe (1995), while mere robustness is sufficient in two-player games.

For each player $i \in N$, let $S_i = [0, b_i]$ for $b_i > 0$, and let Δ_i denote the set of Borel probability measures over S_i. For any $\mu \in \Box = \times_{i \in N} \Delta_i$, let $\beta_i(\mu) \in \Delta_i$ denote i's set of mixed best replies to the mixed-strategy profile μ. Following Simon and Stinchcombe (1995), define the *weak-metric distance* between two mixed strategies, μ_i and υ_i, as follows:

$$\rho^w(\mu_i, \upsilon_i) = \inf\{\delta > 0 : \mu_i(B) \leq \upsilon_i\left(B^\delta\right) + \delta \text{ and } \upsilon_i(B) \leq \mu_i\left(B^\delta\right) + \delta, \text{ for all Borel sets } B \subset S_i\},$$

where B^δ is the δ-neighborhood of B. Identify pure strategies with unit point masses.

Definition 3 [Simon and Stinchcombe, 1995]**.** *For any $\varepsilon > 0$, a **weak perfect ε-equilibrium** is a completely mixed-strategy profile, $\mu^\varepsilon \in int(\Box)$, such that, for every player $i \in N$, $\rho_i(\mu_i^\varepsilon, \beta_i(\mu^\varepsilon)) < \varepsilon$. A strategy profile $\mu^* \in \Box$ is **weakly perfect** if it is the limit as $\varepsilon_k \to 0$ of a sequence of weak perfect ε_k-equilibria.*

The reason why strict robustness implies weak perfection is, heuristically, that such robustness requires (inter alia) robustness to interpersonally consistent subjective beliefs, that is, subjective probability distributions about others' (pure) strategy choices, shared by all players but the one in question. Being interpersonally consistent, a shared subjective probability distribution, concerning a player's strategy choice, can be viewed as a completely mixed strategy for that player. For $t > 0$ sufficiently small, and any t-equilibrium, these distributions will place almost all probability mass in any given ε-neighborhood to that player's t-equilibrium strategy. In the two-player case, only one player has to hold a belief about a given player, so that the issue of mutual consistency of beliefs does not arise. As a result, mere robustness to strategic uncertainty implies weak perfection.[12]

Proposition 2. *If the payoff functions in G are continuous and the strategy sets compact, then every strategy profile that is strictly robust to strategic uncertainty is also weakly perfect. In the two-player case, every strategy profile that is robust to strategic uncertainty is also weakly perfect.*

Proof: Suppose that $\mathbf{s}^*$ is strictly robust to strategic uncertainty. Let $\Psi_1, ..., \Psi_n \in \mathcal{F}$ and, for each $j \in N$, let $\Phi_{ij} = \Psi_j$ for all $i \neq j$. Clearly $\Phi_{ij} \in \mathcal{F}$ for all $i \in N$ and $j \neq i$. Since $\mathbf{s}^*$ is strictly robust to strategic uncertainty, it is the limit as $t_k \to 0$ of a sequence of t_k-equilibria, $\mathbf{s}^{t_k}$. The claim is established if we can extract a subsequence of weak perfect ε_h-equilibria with $\varepsilon_h \to 0$. For each $k \in \mathbb{N}$, let

$$\tilde{F}_i^k(x) = \Psi_i\left(\frac{x - s_i^{t_k}}{t_k}\right) \quad \forall i \in N, x \in S_i.$$

For each $h \in \mathbb{N}$, let $\varepsilon_h = 1/h$, and, for each player i, let $k(i,h)$ be the minimal $k \in \mathbb{N}$ such that $\tilde{F}_i^k\left(s_i^{t_k} - 1/h, s_i^{t_k} + 1/h\right) > 1 - 1/h$. Such a k clearly exists. Let $k(h) = \max_{i \in N} k(i,h)$. Then $\mathbf{s}^{t_{k(h)}}$ is a weak perfect ε_h-equilibrium. In the two-player case: robustness implies that there exist $\Phi_{12}, \Phi_{21} \in \mathcal{F}$ such that $\mathbf{s}^*$ is the limit of a sequence of t-equilibria. Let $\Psi_2 = \Phi_{12}$ and $\Psi_1 = \Phi_{21}$ above. **End of proof.**

It is not difficult to verify that strategy profiles that are robust to strategic uncertainty exist in continuous games if each strategy set is compact and convex, and each payoff function π_i is concave in s_i (for every $\mathbf{s}_{-i}$):

Proposition 3. *Suppose that the payoff functions in G are continuous, that the strategy sets are compact and convex, and that each payoff function π_i is concave in s_i (for every $\mathbf{s}_{-i}$). Let $\Phi_{ij} \subset \mathcal{F}$, $\forall i \in N, j \neq i$. For each $t > 0$, the perturbed game G^t has at least one Nash equilibrium. Moreover, G admits at least one strategy profile that is robust to strategic uncertainty.*

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[12] By contrast, our notion of strict robustness to strategic uncertainty appears to be weaker than Simon's and Stinchcombe's (1995) notion of strong perfection. The reason is that they then work with point masses (attached to the pure best reply), while our probability distributions are continuous.

Proof: Let G be as stated in the proposition. Let $\Phi_{ij} \subset \mathcal{F}$, $\forall i \in N, j \neq i$. Consider any sequence $\langle t_k \rangle_{k=1}^{\infty} \to 0$, where each $t_k > 0$. For each $k \in N$, suppose that $\mathbf{s}^k$ is a Nash equilibrium of G^{t_k}. Then, $\mathbf{s}^k$ is a t_k-equilibrium of G. Now $\mathbf{s}^k \in S$ for all $k \in N$, where S is a non-empty and compact set, so $\langle \mathbf{s}^k \rangle_{k=1}^{\infty}$ admits a convergent subsequence, with limit $\mathbf{s}^* \in S$, by the Bolzano-Weierstrass Theorem.

It remains to establish that for each $t > 0$, the perturbed game G^t has at least one Nash equilibrium. For this purpose, it is sufficient to show that each payoff function π_i^t is continuous and concave in s_i (for every $\mathbf{s}_{-i}$). We prove this for the case of two players, but the generalization is immediate. By definition,

$$\pi_i^t(s) = \mathbb{E}\left[\pi_i(s_i, \tilde{s}_j)\right] = \int \pi_i(s_i, \tilde{s}_j)\, dF_{ij}^t$$

Continuity of π_i^t follows from the continuity of π_i and F_{ij}^t. Moreover, by concavity of π,

$$\begin{aligned} \pi_i^t\left[\lambda s_i + (1-\lambda) s_i', s_j\right] &= \int \left[\lambda \pi_i(s_i, \tilde{s}_j) + (1-\lambda)\pi_i(s_i', \tilde{s}_j)\right] dF_{ij}^t \\ &= \lambda \int \pi_i(s_i, \tilde{s}_j)\, dF_{ij}^t + (1-\lambda)\int \pi_i(s_i', \tilde{s}_j)\, dF_{ij}^t \\ &= \lambda \pi_i^t(s_i, s_j) + (1-\lambda)\pi_i^t(s_i', s_j). \end{aligned}$$

for any $\lambda \in (0,1)$ and $s_i, s_i', s_j \in \mathbb{R}$, proving that π_i^t is concave in s_i (for every $\mathbf{s}_{-i}$). **End of proof**.

The following observation follows from Propositions 3 and 1**:**

Corollary 1. *Suppose that the payoff functions in G are continuous, that the strategy sets are compact and convex, and that each payoff function π_i is concave in s_i (for every $\mathbf{s}_{-i}$). If $\mathbf{s}^*$ is the unique Nash equilibrium of G, then $\mathbf{s}^*$ is strictly robust to strategic uncertainty.*

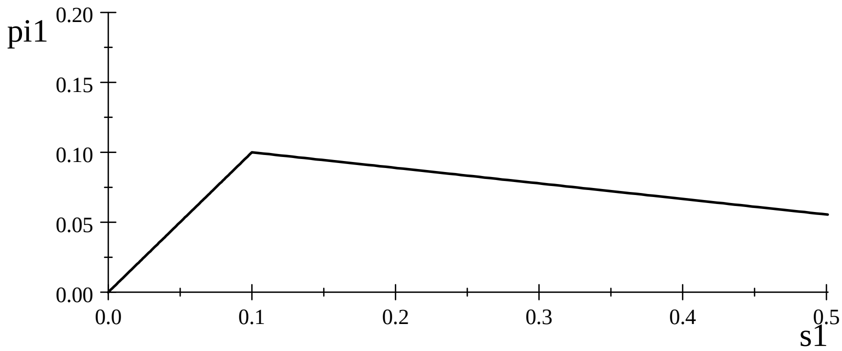
Proof: Let G be as stated. By Proposition 3, G admits a robust strategy profile. By Proposition 1 this is also a Nash equilibrium. By hypothesis, G has a unique Nash equilibrium, so this is robust to strategic uncertainty. Since the collection $\{\Phi_{ij} \subset \mathcal{F}, \forall i \in N, j \neq i\}$ in the proof of Proposition 3 was arbitrary, it follows that $\mathbf{s}^*$ is strictly robust to strategic uncertainty. **End of Proof.**

We conclude by noting that robustness to strategic uncertainty does not imply admissibility. In other words: it does not exclude all equilibria that involve the use of weakly dominated strategies. Indeed, as noted by Simon and Stinchcombe (1995), admissibility is not a property that one can generally expect from Nash equilibrium refinements in continuum-action games. The following example is taken from Simon and Stinchcombe (1995, Example 2.1).

Example 3. *Consider the symmetric two-player game with* $S_1 = S_2 = [0, 1/2]$ *and payoff functions*

$$\pi_1(s_1, s_2) = \begin{cases} s_1 & \text{if } \; s_1 \leq s_2/2 \\ s_2(1 - s_1)/\left(2 - s_2\right) & \text{otherwise} \end{cases}$$

and $\pi_2(s_1, s_2) \equiv \pi_1(s_2, s_1)$. *The figure below illustrates the graph of* $\pi_1(s_1, s_2)$ *for* $s_2 = 0.2$. *This game has a unique Nash equilibrium,* $(0, 0)$, *but for each player* i, *the strategy* $s_i = 0$ *is weakly dominated by all other strategies. Nevertheless, this is a game that meets the conditions in Corollary 1, and hence the weakly dominated Nash equilibrium* $(0, 0)$ *is strictly robust to strategic uncertainty.*



2.2. Discontinuous games. The general picture is less clear for games with discontinuous payoff functions. In particular, deriving general existence results proves challenging and we will not attempt it here. Instead, we will subsequently illustrate the workings of our robustness criteria in two well-known classes of discontinuous games. However, before embarking on that analysis, we make two general observations. First, there exist discontinuous games with strategy profiles that are robust to strategic uncertainty without being Nash equilibria. Second, there exist discontinuous games with strict Nash equilibria that are not robust to strategic uncertainty (which we also show in the next section, in the context of price competition games). We substantiate these claims by means of two examples.

Example 4. *Consider the two-player game in which each player's pure-strategy space is* $\mathbb{R}$ *and the payoff functions are:*

$$\pi_1\left(s_1, s_2\right) = \begin{cases} s_1 & \text{if } s_1 < s_2 \\ 0 & \text{otherwise} \end{cases}$$

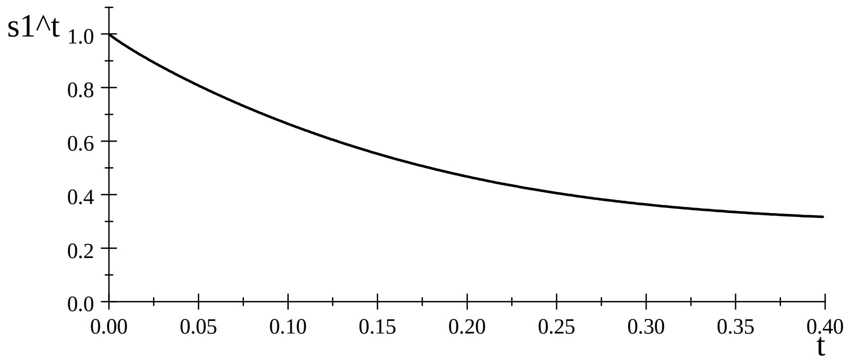
and $\pi_2(s_1, s_2) = -(s_2 - 1)^2$. *Hence, irrespective of 1's action, 2's best reply is* $s_2^* = 1$, *and 1's payoff function is discontinuous on the diagonal in* $\mathbb{R}^2$. *In any t-equilibrium,* $s_2^t = 1$, *and thus*

$$s_1^t \in \arg\max_{s_1} \left[1 - \Phi\left(\frac{s_1 - 1}{t}\right)\right] \cdot s_1 \tag{4}$$

The associated necessary first-order condition is

$$\left[1 - \Phi\left(\frac{s_1 - 1}{t}\right)\right] - \frac{s_1}{t}\phi\left(\frac{s_1 - 1}{t}\right) = 0 \tag{5}$$

Let Φ *be the standard normal distribution. As* $t \to 0$, s_1^t *tends to* 1 *from below, see diagram. Hence, the pure-strategy profile* $(1, 1)$ *is robust to strategic uncertainty. However, it is clearly not a Nash equilibrium of the unperturbed game.*



The best reply correspondence of player 1 as a function of t.

Example 5. *Consider the symmetric two-player game with* $S_1 = S_2 = [0, 1]$ *and with payoff functions*

$$\pi_i(s_1, s_2) = \left\{ \begin{array}{ll} 2 & \text{if } \; s_1 = s_2 \\ 1 - s_i & \text{otherwise} \end{array} \right\}$$

for $i = 1, 2$. *The set of pure Nash equilibria is the diagonal,* $s_1 = s_2$, *and all those equilibria are strict. By continuity of the probability distributions in* $\mathcal{F}$:

$$\pi_i^t(s_1, s_2) = 1 - s_i$$

for all $t > 0$. *Hence,* $s_i = 0$ *is a dominant strategy in every perturbed game, so the strategy profile* $(0, 0)$ *is the unique t-equilibrium, for any* $t > 0$. *It follows that* $(0, 0)$ *is the unique strategy profile that is robust to strategic uncertainty (it is even strictly robust).*

3. Price competition with convex costs

As we will show in this section, our definition of robustness selects a unique Nash equilibrium out of a continuum of equilibria in a class of price-competition games with convex costs. Before embarking on that analysis, we briefly consider the canonical Bertrand model of pure price competition with linear costs.

Example 6. *Consider two identical firms, each with constant unit cost $c > 0$, in a simultaneous-move pricing game à la Bertrand in a market for a homogeneous good. Let the demand function be linear, $D(p) = a - p$, for all $p \in [0, a]$ with $a > c$.*[13] *Then, the monopoly profit function, $\Pi(p) = (a - p)(p - c)$, is strictly concave with a unique maximum at $p^m = (a + c)/2$ and $\Pi(p^m) > 0$. By contrast, the unique duopoly Nash equilibrium, $p_1 = p_2 = c$, results in zero profits. This Nash equilibrium is weakly dominated. Nevertheless, it is robust to strategic uncertainty. For sufficiently small degrees of strategic uncertainty, both firms will set their prices a little bit above marginal cost, and less so, the less uncertain they are. To see this, suppose that $\varepsilon_{ij} \sim \Phi \in \mathcal{F}$.*[14] *For each $t > 0$ and all $p_1, p_2 \in [0, a]$,*

$$\pi_i^t(p_i, p_j) = \left[1 - \frac{\Phi\left(\frac{p_i - p_j}{t}\right) - \Phi\left(-\frac{p_j}{t}\right)}{1 - \Phi\left(-\frac{p_j}{t}\right)}\right] \cdot \Pi(p_i) \qquad i = 1, 2,\ j \neq i.$$

This can be rewritten as

$$\pi_i^t(p_i, p_j) = [1 - \Phi(-p_j/t)]^{-1} \cdot \left[1 - \Phi\left(\frac{p_i - p_j}{t}\right)\right] \cdot \Pi(p_i) \qquad i = 1, 2,\ j \neq i,$$

where the first factor is positive and independent of p_i. A necessary first-order condition for symmetric t-equilibrium[15] *is thus that*

$$t \cdot \frac{\Pi'(p_i)}{\Pi(p_i)} = \frac{\phi(0)}{[1 - \Phi(0)]} \qquad i = 1, 2,\ j \neq i. \tag{6}$$

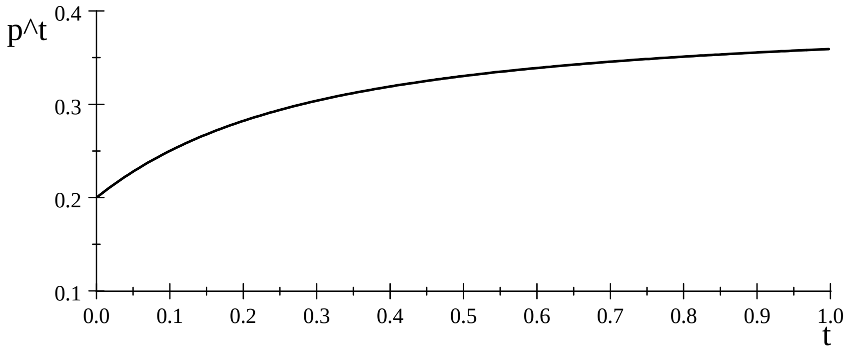
The right-hand side of (6) is a positive constant. Consequently, in the perturbed game, it is never optimal to choose $p_i \leq c$ or $p_i \geq p^m$. Hence, without loss of generality, we restrict attention to $p_i \in (c, p^m)$. On this interval, the left-hand side is a continuous and strictly decreasing function that runs from plus infinity to zero. Hence, there exists a unique symmetric t-equilibrium price, p^t, for every $t > 0$. Moreover, as $t \to 0$, the denominator of the left-hand side has to tend to zero for (6) to

[13] To keep the intuition clear, we take a simple functional form but the argument extends to general demand curves.

[14] We focus on symmetric error distributions in this example only for expositional convenience. The Nash equilibrium is robust to strategic uncertainty under asymmetric distributions as well.

[15] It is easily verified that there does not exist any asymmetric t-equilibrium.

hold. Consequently, $p^t \downarrow c$. The diagram below shows how the t-equilibrium price p^t depends on t, when Φ is the standard normal distribution, $a = 1$ and $c = 0.2$.



The $t-$equilibrium price as a function of t.

The above example shows that there are discontinuous games with weakly dominated Nash equilibria that are robust to strategic uncertainty. We now turn to Bertrand games with strictly convex costs.

3.1. Convex costs. There are $n \geq 2$ firms $i \in N = \{1, 2, ..., n\}$ in a market for a homogeneous good. Aggregate demand $D : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is twice differentiable and such that $D(0) = q^{\max} \in \mathbb{R}$ and $D(p^{\max}) = 0$ for some $p^{\max}, q^{\max} > 0$.[16] Moreover, we assume that $D'(p) < 0$ for all $p \in (0, p^{\max})$. All firms i simultaneously set their prices $p_i \in \mathbb{R}_+$. Let $\mathbf{p} = (p_1, p_2, ..., p_n)$ be the resulting strategy profile (or *price vector*). The minimal price, $p_0 := \min\{p_1, p_2, ..., p_n\}$, will be called *the (going) market price.* Let m be the number of firms that quote the going market price, $m := |\{i : p_i = p_0\}|$. Each firm i faces the demand

$$D_i(\mathbf{p}) := \begin{cases} D(p_0)/m & \text{if } p_i = p_0 \\ 0 & \text{otherwise} \end{cases}$$

All firms have the same cost function, $C : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, which is twice differentiable with $C(0) = 0$ and $C', C'' > 0$. Each firm is required to serve all demand addressed to it at its posted price. The profit to each firm i is thus

$$\pi_i(\mathbf{p}) = \begin{cases} p_0 D(p_0)/m - C[D(p_0)/m] & \text{if } p_i = p_0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

This defines a simultaneous-move n-player game G in which each player i has pure-strategy set $\mathbb{R}_+$ and payoff function $\pi_i : \mathbb{R}^n_+ \rightarrow \mathbb{R}$, defined in equation (7). A

[16] In this section, we follow closely Dastidar (1995).

strategy profile $\mathbf{p}$ will be called *symmetric* if $p_1 = ... = p_n$, and we will call a price $p \in \mathbb{R}_+$ a *symmetric Nash equilibrium price* if $\mathbf{p} = (p, p, ..., p)$ is a Nash equilibrium of G. For each positive integer $m \leq n$ and non-negative price p, let

$$v_m(p) = pD(p)/m - C[D(p)/m]$$

This defines a finite collection of twice differentiable functions, $\langle v_m \rangle_{m \in \{1,2,..,n\}}$, where $v_m(p)$ is the profit to each of m firms if they all quote the same price p and all other firms post higher prices (so that p is the going market price). In particular, v_1 defines the profit to a monopolist as a function of its price p.

We impose one more condition on C and D, namely, that the associated monopoly profit function, v_1, is strictly concave. More exactly, we assume that $v_1'' < 0$ and $v_1'(p^{mon}) = 0$ for some price $p^{mon} \in (0, p^{\max})$. Since the cost function is strictly convex by assumption, this concavity assumption on v_1 effectively requires the demand function to be "not too convex".[17] We have $v_1(p^{mon}) \geq 0$. By convexity of the cost function, there exists prices $p \in (0, p^{\max})$ at which all n firms, when quoting the same price p, make positive profits, $v_n(p) > 0$.

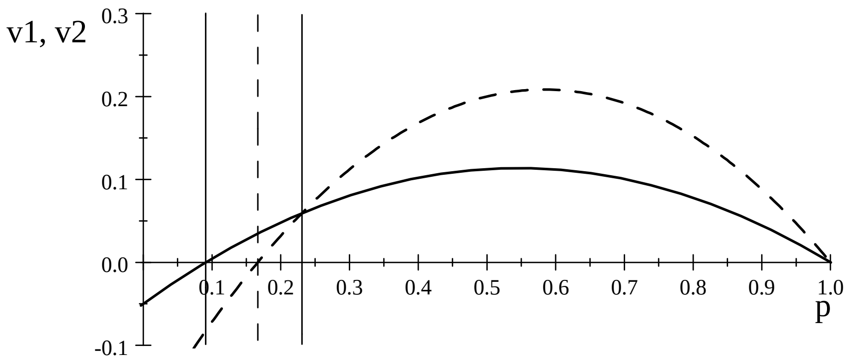
The game G has a continuum of symmetric Nash equilibria.[18] For any number of firms, $n \geq 2$, let $\check{p}_n \in (0, p^{\max})$ be the price p at which $v_n(p) = 0$ and let $\hat{p}_n \in (0, p^{\max})$ be the price p at which $v_n(p) = v_1(p)$. Dastidar (1995, Lemmas 1, 5 and 6) shows existence and uniqueness of $\check{p}_n$ and $\hat{p}_n$ and that $\check{p}_n < \hat{p}_n$. As also shown in Dastidar (1995, Proposition 1), all prices in the interval $P_n^{NE} = [\check{p}_n, \hat{p}_n]$ are symmetric Nash equilibrium prices in the game G, and no price outside this interval is a symmetric Nash equilibrium price.

As shown in Dastidar (1995, Lemmas 4 and 6), there exists a unique price $\bar{p}$ at which a monopolist makes zero profit, $v_1(\bar{p}) = 0$, and, moreover, $\bar{p} \in (\check{p}_n, \hat{p}_n)$. Dastidar (1995, Lemma 7) also shows that both $\check{p}_n$ and $\hat{p}_n$ are strictly decreasing in n. In the present setting, it is easily verified that $\check{p}_n \downarrow 0$ and $\hat{p}_n \downarrow \bar{p}$, and hence $P_n^{NE} \to (0, \bar{p}]$, as $n \to \infty$.

Example 7. *Consider a duopoly with identical firms with quadratic cost functions, $C(q) = cq^2$, where $c = 0.2$, and linear aggregate demand: $D(p) = \max\{0, 1-p\}$. The diagram below shows the graphs of v_1 (dashed curve) and v_2 (solid curve). The associated set, P_2^{NE}, is the interval $[1/11, 3/13]$, indicated by the two solid vertical lines, and $\bar{p} = 1/6$ is indicated by the dashed vertical line.*

[17] This is a more stringent assumption than the one made in Dastidar (1995), who instead assumes that there exists a unique monopoly price.

[18] Dastidar (1995) and Weibull (2006) have shown existence and multiplicity of Nash equilibria under weaker conditions.



Monopoly (dashed) and duopoly (solid) profits, as functions of a common price p.

We make two further observations. First, that $\hat{p}_n$ cannot exceed the monopoly price, and second, that the pricing game G admits no asymmetric Nash equilibrium.

Proposition 4. $\hat{p}_n \leq p^{mon}$ *for all* $n > 1$.

Proof: Dastidar (1995; Lemma 3) shows that, if $v_n(p) \geq v_1(p)$ then $v_1(p) > v_1(p - \alpha)$, $\forall \alpha > 0$ for $p - \alpha \in [0, p)$. So, if p is a Nash equilibrium, then the left-derivative of v_1 at p must be positive. The concavity of v_1 implies that $\hat{p}_n \leq p^{mon}$. **End of proof**.

Proposition 5. *Every Nash equilibrium in* G *is symmetric.*

Proof: Let $(p_1, ..., p_n)$ be a Nash equilibrium. Suppose, first, that $p_i < \min_{j \neq i} p_j$ for some i. If $p_i < \hat{p}_n$, then firm i could increase its profit by unilaterally increasing its price. Hence, $p_i \geq \hat{p}_n$. If $p_i \leq p^{mon}$, then any firm $j \neq i$ could increase its profit by unilaterally decreasing its price to p_i and earn $v_2(p_i) > 0$ instead of zero. If $p_i > p^{mon}$ then firm i can increase its profit by a unilateral deviation to p^{mon}. Hence, $p_i \geq \min_{j \neq i} p_j$ for all i. Suppose, secondly, that $p_i = \min_{j \neq i} p_j$ and that $p_k > p_i$ for some k. Either $v_{|j \in N: p_j = p_i|}(p_i) > 0$ or $v_{|j \in N: p_j = p_i|}(p_i) = 0$. (If $v_{|j \in N: p_j = p_i|}(p_i) < 0$, then i can profitably deviate to $p^{\max}$ and earn zero profit.) In any case, k can profitably deviate to p_i and make a positive profit since by strict convexity of C, if $v_l(p) \geq 0$, then $v_{l+1}(p) > 0$. Hence, $p_i = p_j$ for all $i, j \in N$. **End of proof**.

3.2. Robust price equilibrium. We proceed to apply the robustness definition from Section 2 to the pricing game described in Section 3.1. Let $t > 0$ and suppose that a firm i holds a probabilistic belief of form (1) about other firms' prices. For any price p_i that firm i might contemplate to set, its subjective probability that any other firm will choose exactly the same price is zero. Hence, with probability one, its own price will either lie above the going market price or it will be the going market price and all other firms' prices will be higher, so i will then be a monopolist at its price p_i. From equation (3), each firm i's payoff function in the perturbed game $G^t = (N, S, \pi^t)$ is, for any $t > 0$, defined by

$$\pi_i^t(\mathbf{p}) = v_1(p_i) \cdot \left(\prod_{j \neq i} \left[1 - \Phi_{ij}\left(\frac{-p_j}{t} \right) \right]^{-1} \right) \cdot \left(\prod_{j \neq i} \left[1 - \Phi_{ij}\left(\frac{p_i - p_j}{t} \right) \right] \right) \tag{8}$$

The second factor being positive and independent of p_i, a price profile $\mathbf{p}$ is a Nash equilibrium of G^t if and only if

$$p_i \in \arg \max_{p \in [\bar{p}, p^{mon}]} u_i^t(p, \mathbf{p}_{-i}) \qquad \forall i, \tag{9}$$

where

$$u_i^t(\mathbf{p}) = v_1(p_i) \cdot \prod_{j \neq i} \left[1 - \Phi_{ij}\left(\frac{p_i - p_j}{t} \right) \right]$$

and the restriction $p \in [\bar{p}, p^{mon}]$ is non-binding, since $v_1(p) < 0$ for all $p < \bar{p}$, $v_1(p) > 0$ for all $p \in (\bar{p}, p^{mon})$, and $v_1'(p) < 0$ for all $p > p^{mon}$. For any $t > 0$, let $\bar{G}^t$ be the normal-form game $(N, [\bar{p}, p^{mon}]^n, u^t)$. For any $t > 0$, a price profile p is a t-equilibrium in the pricing game G if and only if it is a Nash equilibrium of the game $\bar{G}^t$.

Proposition 6. *Let $t > 0$ and assume that $\Phi_{ij} \in \mathcal{F}$ $\forall i \in N, j \neq i$. Then $\bar{G}^t$ has at least one Nash equilibrium. Moreover, any such Nash equilibrium $\mathbf{p}^t$ is interior.*

Proof: We note that for given $t > 0$, each player's strategy set is non-empty, convex and compact and each player's payoff function is continuous. By Weierstrass's maximum theorem, each player's best reply correspondence is non-empty and compact-valued. By Berge's maximum-theorem, it is also upper hemi-continuous. Existence of a Nash equilibrium in $\bar{G}^t$ thus follows from Kakutani's fixed-point theorem if, in addition, each player's best-reply correspondence is convex-valued. In order to verify this, first note that no equilibrium price can lie on the boundary of the strategy set in $\bar{G}^t$, since $\partial u_i^t / \partial p_i$ is positive at its left boundary and negative at its right boundary. Hence, any Nash equilibrium of $\bar{G}^t$ is in $(\bar{p}, p^{mon})^n$. It remains to show that the set $\arg\max_{p_i \in (\bar{p}, p^{mon})} u_i^t(\mathbf{p})$ is convex. For this purpose, note that

$$p_i^t \in \arg \max_{p_i \in (\bar{p}, p^{mon})} u_i^t(p_i, p_{-i})$$

if and only if

$$p_i^t \in \arg \max_{p_i \in (\bar{p}, p^{mon})} \left(\ln [v_1(p_i)] + \sum_{i \neq j} \ln \left[1 - \Phi_{ij}\left(\frac{p_i - p_j}{t} \right) \right] \right).$$

Since v_1 is strictly concave by assumption, also $\ln[v_1(\cdot)]$ is strictly concave. By assumption, ϕ_{ij} is continuously differentiable and concave on an open interval, which, by Theorem 3 in Bagnoli and Bergstrom (2005), implies that also the survival function, $1-\Phi_{ij}$, is log concave. Hence, each term in the above sum is a concave function of p_i (given p_j and t). Concavity is preserved under summations, so the maximand is concave, and thus i's best-reply correspondence is convex-valued. **End of proof.**

Theorem 1. *The Nash equilibrium $(\bar{p},...,\bar{p})$ is strictly robust to strategic uncertainty. No other strategy profile of G is robust to strategic uncertainty.*

Proof: Let $\{\Phi_{ij} : \forall i \in N, j \neq i\} \subset \mathcal{F}$. Consider any sequence $\langle t_k \rangle_{k=1}^{\infty} \to 0$, where each $t_k > 0$. For each $k \in N$, let p^k be a Nash equilibrium of $\bar{G}^{t_k}$. Since all games $\bar{G}^{t_k}$ have the same strategy space, $[\bar{p}, p^{mon}]^n$, and this is non-empty and compact, the sequence $\langle \mathbf{p}^{t_k} \rangle_{k=1}^{\infty}$ contains a convergent subsequence with limit in $[\bar{p}, p^{mon}]^n$, according to the Bolzano-Weierstrass Theorem. Hence, without loss of generality we may assume that $\lim_{k\to\infty} p^k = p^* \in [\bar{p}, p^{mon}]^n$.

First, we prove that $p_i^* = p_j^*$ for all $i, j \in N$. For this purpose, first note that no price can lie on the boundary of the strategy set in $\bar{G}^t$, since $\partial u_i^t / \partial p_i$ is positive at its left boundary and negative at its right boundary.

$$\frac{\partial u_i^t(\mathbf{p})}{\partial p_i^k} = \prod_{j\neq i}\left[1-\Phi_{ij}\left(\frac{p_i^k-p_j^k}{t}\right)\right]\cdot\left[v_1'\left(p_i^k\right)-\frac{v_1\left(p_i^k\right)}{t}\sum_{j\neq i}h_{ij}\left(\frac{p_i^k-p_j^k}{t}\right)\right]$$

Hence, $\bar{p} < p_i^k < p^{mon}$ for all i and k, and thus $\bar{p} \leq p_i^* \leq p^{mon}$ for all i. The first-order condition gives,

$$t_k v_1'\left(p_i^k\right) = v_1\left(p_i^k\right)\sum_{j\neq i} h_{ij}\left(\frac{p_i^k-p_j^k}{t_k}\right) \quad \forall i,k \tag{10}$$

where h_{ij} is the hazard-rate function of Φ_{ij}. Consider a firm $i \in N$. Suppose that $p_j^* < p_i^*$ for some $j \neq i$, and let $\varepsilon = p_i^* - p_j^* > 0$. Then, there is a K such that $p_i^k - p_j^k > \varepsilon/2$ for all $k > K$. The hazard rate being non-decreasing, we thus have

$$h_{ij}\left(\frac{p_i^k-p_j^k}{t_k}\right) \geq h_{ij}\left(\frac{\varepsilon}{2t_k}\right)$$

for that $j \neq i$ and all $k > K$. Let $\delta = h_{ij}\left[\varepsilon/\left(2t_K\right)\right] > 0$. Then

$$h_{ij}\left(\frac{p_i^k-p_j^k}{t_k}\right) \geq \delta$$

for that $j \neq i$ and all $k > K$, and hence, since all hazard rates are positive:

$$t_k v_1' \left(p_i^k\right) > \delta \cdot v_1 \left(p_i^k\right)$$

for all $k > K$. However, $t_k v_1' \left(p_i^k\right) \to 0$ and $v_1 \left(p_i^k\right) \to v_1 \left(p_i^*\right)$ as $k \to \infty$, since v_1 is continuous, so $v_1 \left(p_i^*\right) = 0$. Hence, $p_i^* = \bar{p}$. But this contradicts the hypothesis $p_i^* > p_j^* \in [\bar{p}, p^{mon}]$. Hence, $p_j^* \geq p_i^*$. Since holds for all i and $j \neq i$, we conclude that $p_j^* = p_i^*$ for all $i, j \in N$.

Secondly, we prove $p_i^* = \bar{p}$ for all $i \in N$. Since $v_1 \left(p_i^k\right) > 0$ on $(\bar{p}, p^{mon})$ and all hazard rates are positive, by (10),

$$v_1 \left(p_i^k\right) \cdot h_{ij} \left(\frac{p_i^k - p_j^k}{t_k}\right) \to 0 \quad \forall i, j \neq i$$

as $k \to +\infty$. Suppose that $p_i^* > \bar{p}$. Then $v_1 \left(p_i^*\right) > 0$ and thus

$$h_{ij} \left(\frac{p_i^k - p_j^k}{t_k}\right) \to 0 \quad \forall j \neq i$$

implying that $p_i^k < p_j^k$ for all k sufficiently large. But, by the same token: since $p_j^* = p_i^*$, for all $j \neq i$, we also have $p_j^* > \bar{p}$ and $v_1 \left(p_j^*\right) > 0$ and thus

$$h_{ji} \left(\frac{p_j^k - p_i^k}{t_k}\right) \to 0$$

implying that $p_j^k < p_i^k$ for all k sufficiently large. Both strict inequalities cannot hold. Hence, $p_i^* = \bar{p}$ for all $i \in N$. In sum: the only strategy profile that is robust to strategic uncertainty is $(\bar{p}, ..., \bar{p})$. The strict robustness claim follows immediately from the fact that the collection $\{\Phi_{ij} : \forall i \in N, j \neq i\} \subset \mathcal{F}$ above was arbitrary. **End of proof**.

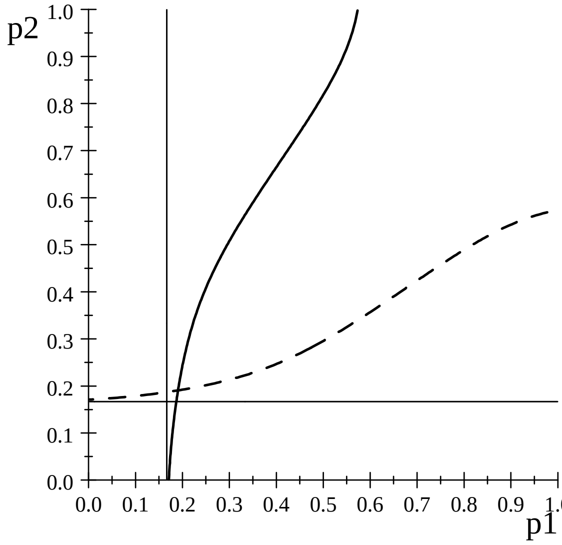
Example 8. *Consider again a duopoly with identical firms, with quadratic cost function,* $C(q) = 0.2q^2$, *and linear aggregate demand:* $D(p) = \max\{0, 1 - p\}$. *Suppose that both firms' uncertainty takes the form of normally distributed noise,* $\varepsilon_1, \varepsilon_2 \sim N(0, 1)$. *We then have* $\bar{p} = 1/6 \approx 0.167$. *The necessary first-order condition for interior* t*-equilibrium consists of the equations*

$$t v_1' (p_1) = v_1 (p_1)\, h \left(\frac{p_1 - p_2}{t}\right)$$

and

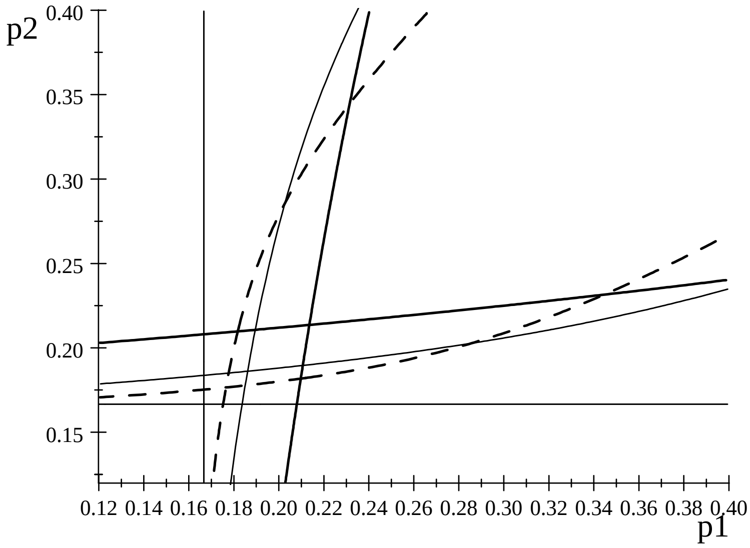
$$t v_1' (p_2) = v_1 (p_2)\, h \left(\frac{p_2 - p_1}{t}\right).$$

The diagram below shows these best-reply curves (solid and dashed, respectively) and $t = 0.1$, with $\bar{p}$ marked by thin straight lines.



The best-reply curves in the perturbed pricing game.

The next diagram displays the best-reply curves of both players for $t = 0.25$ (solid curves), $t = 0.1$ (thin curves), and $t = 0.05$ (dashed curves). As t decreases, the intersection of the associated pair of curves approaches $(\bar{p}, \bar{p}) = (1/6, 1/6)$, the intersection between the thin horizontal and vertical lines.



Convergence of t-equilibria towards $(\bar{p}, \bar{p})$

4. The Nash demand game

The Nash demand game (Nash, 1953) is a two-player game $G = (N, S, \pi)$, with strategy sets $S_1 = S_2 = [0,1]$. The players simultaneously submit "bids". The payoff to a player who bids s_1, while the other player bids s_2, is

$$\pi_i(s_1, s_2) = \begin{cases} s_i & \text{if } s_1 + s_2 \leq 1 \\ 0 & \text{otherwise} \end{cases} .$$

As is well-known, this game admits a multiplicity of Nash equilibria: for each $\lambda \in [0,1]$, $(\lambda, 1-\lambda)$ is a strict Nash equilibrium. Besides, $s_1 = s_2 = 1$ is a Nash equilibrium, though not strict.

Our robustness criterion has little general cutting power in this game. However, sharp results can be obtained in special cases. In particular, equal division is robust to strategic uncertainty when the two parties are equally uncertain, while in situations of asymmetric uncertainty, the party that is less uncertain obtains more than half the pie, in some cases even the full pie.

From equation (3), each player i's payoff function in the perturbed game, $G^t = (N, S, \pi^t)$, is, for any $t > 0$, defined by

$$\pi_i^t(s_i, s_j) = \left[\Phi_{ij}\left(\frac{1-s_j}{t}\right) - \Phi_{ij}\left(\frac{-s_j}{t}\right)\right]^{-1} \cdot \left[\Phi_{ij}\left(\frac{1-s_i-s_j}{t}\right) - \Phi_{ij}\left(\frac{-s_j}{t}\right)\right] \cdot s_i$$

for $i = 1, 2$ and $j \neq i$. Since by hypothesis Φ_{ij} is strictly increasing, the first factor is always positive. Necessary first-order conditions for an interior t-equilibrium $(s_1^t, s_2^t) \in (0,1)^2$ are thus

$$\Phi_{12}\left(\frac{1-s_1^t-s_2^t}{t}\right) - \Phi_{12}\left(-\frac{s_2^t}{t}\right) = \frac{1}{t}\phi_{12}\left(\frac{1-s_1^t-s_2^t}{t}\right)s_1^t \tag{11}$$

and

$$\Phi_{21}\left(\frac{1-s_1^t-s_2^t}{t}\right) - \Phi_{21}\left(-\frac{s_1^t}{t}\right) = \frac{1}{t}\phi_{21}\left(\frac{1-s_1^t-s_2^t}{t}\right)s_2^t. \tag{12}$$

Proposition 7. *Let $\Phi_{12}, \Phi_{21} \subset \mathcal{F}$ and $t > 0$. There exists a t-equilibrium, and every t-equilibrium is interior. A strategy pair $(s_1^t, s_2^t) \in (0,1)^2$ is a t-equilibrium if and only if conditions (11) and (12) are met.*

Proof: A strategy profile $(s_1^t, s_2^t) \in [0,1]^2$ is a Nash equilibrium of G^t if and only if

$$s_i^t \in \arg\max_{s_i \in [0,1]} \pi_i^t(s_i, s_j^t) \tag{13}$$

for $i = 1, 2$ and $j \neq i$. In order to establish existence of t-equilibrium, we note that, given $t > 0$, each player's strategy set is non-empty, convex and compact, and each player's payoff function is continuous. By Weierstrass's maximum theorem, each player's best reply correspondence is thus nonempty- and compact-valued. By Berge's maximum-theorem, it is also upper hemi-continuous. Hence, existence of t-equilibrium follows from Kakutani's fixed-point theorem if, moreover, each player's best-reply correspondence is convex-valued. In order to establish this, let $t > 0$ and $s_j^t \in [0, 1]$. First note that

$$\frac{\partial}{\partial s_i}\pi_i^t(s_i, s_j^t) = \Phi_{ij}\left(\frac{1 - s_i - s_j^t}{t}\right) - \Phi_{ij}\left(-\frac{s_j^t}{t}\right) - \frac{1}{t}\phi_{ij}\left(\frac{1 - s_i - s_j^t}{t}\right) \cdot s_i$$

In particular

$$\frac{\partial}{\partial s_i}\pi_i^t(0, s_j^t) = \Phi_{ij}\left(\frac{1 - s_j^t}{t}\right) - \Phi_{ij}\left(-\frac{s_j^t}{t}\right) > 0,$$

since Φ_{ij} is strictly increasing by assumption, and

$$\frac{\partial}{\partial s_i}\pi_i^t(1, s_j^t) = -\frac{1}{t}\phi_{ij}\left(\frac{-s_j^t}{t}\right) < 0,$$

since $\phi_{ij} > 0$ by assumption. From these observations it follows that

$$\varnothing \neq \arg\max_{s_i \in [0,1]} \pi_i^t(s_i, s_j^t) \subset (0, 1).$$

Thus, any t-equilibrium is interior. It remains to show that the set $\arg\max_{s_i \in (0,1)} \pi_i^t(s_i, s_j^t)$ is convex. For this purpose, note that

$$s_i^t \in \arg\max_{s_i \in (0,1)} \pi_i^t(s_i, s_j^t)$$

if and only if $s_i^t = 1 - s_j^t - tx^t$, where (after a change of variables and taking the logarithm):

$$x^t \in \arg\max_{x \in \left(-s_j^t/t, \left(1 - s_j^t\right)/t\right)} \left[\ln \Psi_i(x) + \ln\left(1 - s_j^t - tx\right)\right] \tag{14}$$

for

$$\Psi_i(x) = \left[\Phi_{ij}\left(\frac{1 - s_j^t}{t}\right) - \Phi_{ij}\left(\frac{-s_j^t}{t}\right)\right]^{-1} \cdot \left[\Phi_{ij}(x) - \Phi_{ij}\left(\frac{-s_j^t}{t}\right)\right]$$

By assumption, $\ln \phi_{ij}$ is concave, which, by Theorem 9 in Bagnoli and Bergstrom (2005), implies that $\ln \Psi_i$ is concave, since Ψ_i is the truncation of the c.d.f. Φ_{ij} to the (open) interval $\left(-s_j^t/t, \left(1 - s_j^t\right)/t\right)$. But this implies that the maximand (being the sum of one concave and one strictly concave function) in (14) is strictly concave.

Thus i's best-reply correspondence is singleton-valued. This establishes the first claim in the proposition. Since each player's payoff function is concave in his or her own strategy, the necessary first-order conditions (11) and (12) are also sufficient, thus establishing the second claim. **End of proof.**

Proposition 8. *If $\langle (s_1^{t_k}, s_2^{t_k}) \rangle_{k\in\mathbb{N}}$ is a sequence of t-equilibria where $t_k \downarrow 0$ as $k \to +\infty$, then*

$$(i) \qquad \lim_{k\to+\infty} \frac{1}{t_k}\left(1 - s_1^{t_k} - s_2^{t_k}\right) = +\infty$$

$$(ii) \qquad \lim_{k\to+\infty} \left(s_1^{t_k} + s_2^{t_k}\right) = 1$$

$$(iii) \qquad \lim_{k\to+\infty} \frac{s_1^{t_k}}{s_2^{t_k}} = \lim_{w\to+\infty} \frac{h_{21}^{-}(w)}{h_{12}^{-}(w)}$$

Proof: Let $t > 0$ and suppose that (s_1^t, s_2^t) solves (11) and (12). Let $w^t = (1 - s_1^t - s_2^t)/t$. The two equations imply that

$$w^t = \frac{1}{t} - \frac{\Phi_{12}(w^t) - \Phi_{12}(-s_2^t/t)}{\phi_{12}(w^t)} - \frac{\Phi_{21}(w^t) - \Phi_{21}(-s_1^t/t)}{\phi_{21}(w^t)} \tag{15}$$

First note that since $s_1^t, s_2^t \in (0,1)$, both $\Phi_{12}(-s_2^t/t)$ and $\Phi_{21}(-s_1^t/t)$ are monotonically decreasing in t with limits $\lim_{t\to 0} \Phi_{12}(-s_2^t/t) = 0$ and $\lim_{t\to 0} \Phi_{21}(-s_1^t/t) = 0$. Claim (i) is equivalent with the claim that for all subsequences $\langle w^{t_k} \rangle_{k\in\mathbb{N}}$ with $t_k \to 0$, $w^{t_k} \to +\infty$. Suppose not. Then there is a subsequence that converges to some real number w^*, or to minus infinity. In the first case,

$$w^* = \lim_{k\to\infty} 1/t_k - \lim_{k\to\infty} \Phi_{12}\left(w^{t_k}\right)/\phi_{12}\left(w^{t_k}\right) - \lim_{k\to\infty} \Phi_{21}\left(w^{t_k}\right)/\phi_{21}\left(w^{t_k}\right) \tag{16}$$

$$= \lim_{k\to\infty} 1/t_k - \lim_{k\to\infty} 1/h_{12}^{-}\left(w^{t_k}\right) - \lim_{k\to\infty} 1/h_{21}^{-}\left(w^{t_k}\right) \tag{17}$$

However, this is impossible, since the right-hand side tends to plus infinity.[19] Suppose, thus, that there is a subsequence that tends to minus infinity. By assumption, $h_{12}^{-}(w^{t_k})$ and $h_{21}^{-}(w^{t_k})$ will then tend to some nonnegative limit or to plus infinity. Again, this is impossible, since the right-hand side of (15) will then tend to plus infinity while the left-hand side tends to minus infinity. This establishes claim (i).

In order to establish claim (ii), first note that by (i), for all $t > 0$ sufficiently small, we must have $z^t > 0$, where $z^t = 1 - s_1^t - s_2^t$. We may thus presume, without loss of generality, that $z^t \in (0,1)$ for all t. It remains to show that $z^t \to 0$ as $t \to 0$. Suppose not. Since $(0,1)$ is non-empty and bounded, there then exists, by

[19] By continuity, $1/h_{12}^{-}(w^{t_k}) \to 1/h_{12}^{-}(w^*)$ and $1/h_{21}^{-}(w^{t_k}) \to 1/h_{21}^{-}(w^*)$.

the Bolzano-Weierstrass Theorem, a subsequence $\langle z^{t_k}\rangle_{k\in\mathbb{N}}$ with $t_k \to 0$ such that $z^{t_k} \to z^* \in (0,1]$. However, for each $t > 0$ we also have

$$z^t = 1 - t \cdot \frac{\Phi_{12}(z^t/t) - \Phi_{12}(-s_2^t/t)}{\phi_{12}(z^t/t)} - t \cdot \frac{\Phi_{21}(z^t/t) - \Phi_{21}(-s_1^t/t)}{\phi_{21}(z^t/t)}$$

so we necessarily then have

$$z^* = 1 - \lim_{k\to\infty} t_k/\phi_{12}(z^*/t_k) - \lim_{k\to\infty} t_k/\phi_{21}(z^*/t_k)$$

If the right-hand side is to be a number in $(0,1]$, which is necessary, we cannot have

$$\lim_{k\to\infty} \frac{1}{t_k}\phi_{12}(z^*/t_k) \to 0$$

But the latter is necessary for the c.d.f. Φ_{12} to have finite expectation, which we have assumed. This establishes claim (ii).

To establish claim (iii), note that division of the two first-order conditions gives

$$\frac{s_1^t}{s_2^t} = \frac{\phi_{21}(w^t) \cdot [\Phi_{12}(w^t) - \Phi_{12}(-s_2^t/t)]}{\phi_{12}(w^t) \cdot [\Phi_{21}(w^t) - \Phi_{21}(-s_1^t/t)]}$$

Since $s_1^t, s_2^t \in (0,1)$

$$\begin{aligned}\lim_{t\to 0} \frac{\phi_{21}(w^t) \cdot [\Phi_{12}(w^t) - \Phi_{12}(-s_2^t/t)]}{\phi_{12}(w^t) \cdot [\Phi_{21}(w^t) - \Phi_{21}(-s_1^t/t)]} &= \lim_{t\to 0} \frac{\phi_{21}(w^t)\,\Phi_{12}(w^t)}{\phi_{12}(w^t)\,\Phi_{21}(w^t)}. \\ &= \lim_{t\to 0} \frac{h_{21}^-(w^t)}{h_{12}^-(w^t)}\end{aligned}$$

By claim (i), $w^t \to +\infty$, from which claim (iii) follows immediately. **End of proof.**

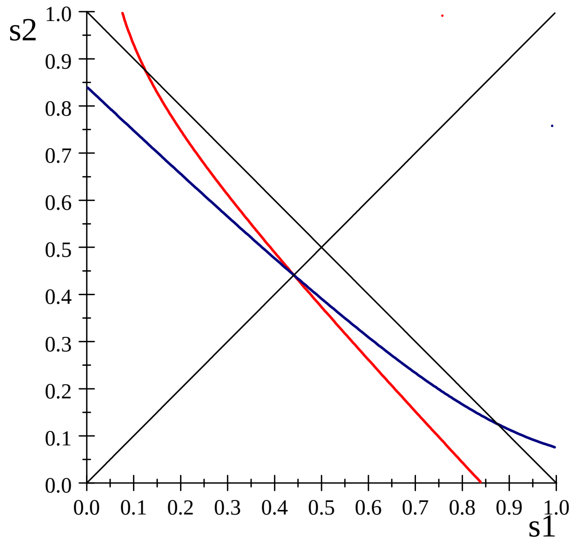
From this last proposition we obtain three corollaries. The first makes the point that if the two parties are equally uncertain about each other's bid, then robustness to strategic uncertainty implies equal division:

Corollary 2. *The unique strategy profile that is robust to symmetric uncertainty ($\Phi_{12} = \Phi_{21}$) is $s_1 = s_2 = 1/2$.*

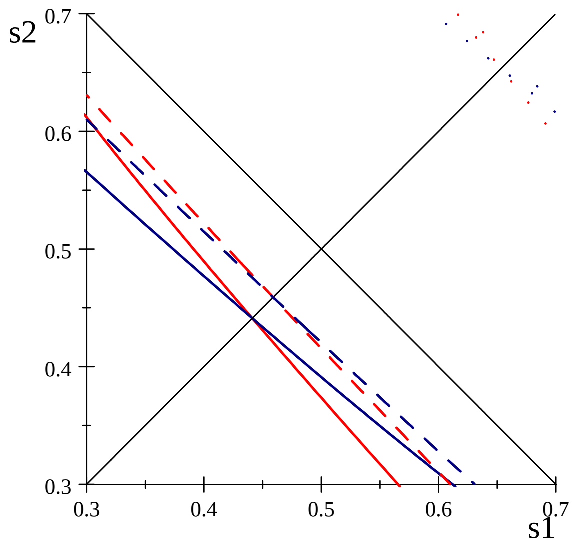
Proof: If $\Phi_{12} = \Phi_{21}$ and $(s_1^t, s_2^t) \in (0,1)^2$ is a t-equilibrium for each $t > 0$, then $s_1^t + s_2^t \to 1$ and $s_1^t/s_2^t \to 1/2$, by claims (ii) and (iii) in Proposition 8. Hence $s_i^t \to 1/2$ for $i = 1, 2$. **End of proof.**

The following example illustrates this result.

Example 9. *Let $\Phi_{12} = \Phi_{21} = \Phi$ be the normal distribution with mean zero and unit variance. The first diagram shows the two best-reply curves for $t = 0.1$.*



A close-up, also inserting the curves for $t = 0.05$ (dashed):

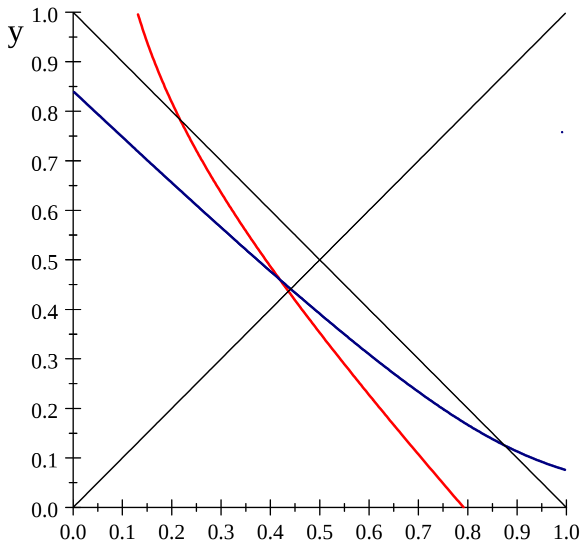


By hypothesis, both inverse hazard rates, $h_{ij}^{-}(w) = \phi_{ij}(w)/\Phi_{ij}(w)$ are non-increasing in w, for $i = 1, 2$ and $j \neq i$. Arguably, if player i is much more uncertain about j's bid (than player j is about i), then $h_{ji}^{-}(w)/h_{ij}^{-}(w) \to 0$ as $w \to +\infty$. It follows immediately from Proposition 8 that if one player is much more uncertain than the other, then robustness to strategic uncertainty requires that the latter obtains the whole pie:

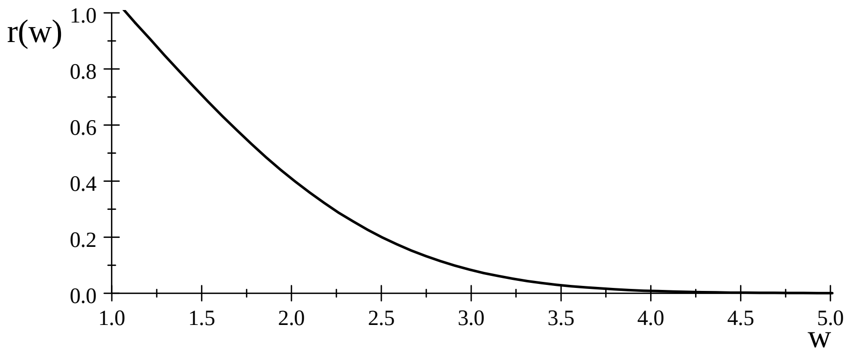
Corollary 3. *Let $\Phi_{ij} \in \mathcal{F}$, $\forall i \in N, j \neq i$ and suppose that $h_{21}^{-}(w)/h_{12}^{-}(w) \to 0$ as $w \to +\infty$. If $\langle (s_1^{t_k}, s_2^{t_k}) \rangle_{k\in\mathbb{N}} \in (0,1)^2$ is a sequence of t_k-equilibria, with all $t_k > 0$ and $t_k \to 0$, then $s_1^{t_k}/s_2^{t_k} \to 0$.*

To see that distributions do not have to be too different, consider the following variation of the above example:

Example 10. *Consider the setup from the previous example with the exception that player 1's belief has higher variance ($\sigma = 3$), that is, player 1 is more uncertain about 2 than 2 is uncertain about 1. The following diagram shows that, for $t = 0.1$, player 1 obtains less than half:*



From part (iii) of Proposition 7 we have that $\lim_{t\to 0} s_1^t/s_2^t = \lim_{w\to+\infty} h_{21}^{-}(w)/h_{12}^{-}(w)$ The next diagram shows the ratio $r(w) = h_{21}^{-}(w)/h_{12}^{-}(w)$ as a function of w. In the limit, the player with the less noisy belief, in this example player 2, obtains the whole pie.



Our third and final corollary establishes that (a) every division of the pie is robust to some strategic uncertainty, (b) none of these divisions is strictly robust, and (c) no other strategy pairs are robust to strategic uncertainty:

Corollary 4. *A strategy profile* (s_1, s_2) *is robust to strategic uncertainty if and only if* $s_1 + s_2 = 1$. *No strategy profile is strictly robust to strategic uncertainty.*

Proof: For the first claim, the proof is constructive, and uses claim (iii) in Proposition 8. Consider first $(s_1, s_2) = (1, 0)$. Let Φ_{12} be the Gumbel distribution with mean 0 and variance 1, and let Φ_{21} be the Gumbel distribution with mean 0 and variance 2. Then

$$\frac{h_{21}^{-}(w)}{h_{12}^{-}(w)} = \frac{e^{-w}}{\frac{1}{2}e^{-w/2}} = 2e^{-w/2},$$

so $\lim_{w\to+\infty} h_{21}^{-}(w)/h_{12}^{-}(w) = 0$. Hence, any sequence of t-equilibria (with respect to Φ_{12} and Φ_{21}) converges towards $(1, 0)$ as $t \to 0$, so $(1, 0)$ is robust to strategic uncertainty. By reversing the variances, $(0, 1)$ is robust to strategic uncertainty.

Consider now any (s_1, s_2) such that $0 < s_1 \leq 1/2$ and $s_2 = 1 - s_1$. Let $\mu = \ln(s_2/s_1)$. Let Φ_{21} be the Gumbel distribution with mean 0 and variance 1 and let Φ_{12} be the Gumbel distribution with mean μ and variance 1. Then

$$\frac{h_{21}^{-}(w)}{h_{12}^{-}(w)} = \frac{e^{-(w-\mu)}}{e^{-w}} = e^{\mu} = s_2/s_1.$$

Any sequence of t-equilibria (with respect to Φ_{12} and Φ_{21}) converges to the strategy profile (s_1, s_2), so (s_1, s_2) is robust to strategic uncertainty. A similar argument applies to any (s_1, s_2) such that $0 < s_2 \leq 1/2$ and $s_1 = 1 - s_2$. Finally, consider (s_1, s_2) such that $s_1 + s_2 \neq 1$. By claim (ii) in Proposition 8, no sequence of t-equilibra converges to (s_1, s_2). So no such strategy pair is robust to strategic uncertainty. This establishes the first claim in the corollary. The second claim follows immediately from the above constructive proof, since for any strategy profile $(s_1, s_2) \in [0,1]^2$ we identified $\Phi_{12}, \Phi_{21} \in \mathcal{F}$ for which any sequence of t-equilibria converges to another strategy profile. **End of proof.**

5. Conclusion

In this paper, we propose a way to model strategic uncertainty in a straightforward fashion. After defining a robustness criterion and deriving a handful of general results, including its relations to Nash equilibrium in continuous and discontinuous games, we have investigated in detail two well-known games with discontinuous payoffs and continuum action spaces, each game admitting infinitely many Nash equilibria. Arguably, strategic uncertainty is considerable in those games, due to the richness of the strategy spaces and the large number of equilibria. In the Bertrand competition game with convex costs, we showed that our notion of robustness to strategic uncertainty selects a unique Nash equilibrium, that, moreover, figured prominently in recent laboratory experiments. In the Nash demand game, we showed that robustness to symmetric strategic uncertainty singles out the Nash bargaining solution. Contrary to Carlsson (1991), where players "tremble" when making their bids, our predictions are not distribution independent. We find that the party who is least uncertain about the other party's bid obtains the bigger share.

We believe that our concept of robustness to strategic uncertainty has a wide domain of application and that it comes at a relatively low analytical cost when used for predictions in simultaneous-move games.

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