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Non-smooth Dynamics and Multiple Equilibria in a Cournot-Ramsey Model with Endogenous Markups[☆]

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Abstract

We consider a Ramsey model with a continuum of Cournotian industries where free entry generates an endogenous markup. The model produces two different regimes, monopoly and oligopoly, resulting in non-smooth dynamics. We analyze the global dynamics of the model, demonstrating the model may exhibit heteroclinic orbits connecting multiple equilibria. Small transitory changes in parameters can lead to large permanent effects and there can be a Rostovian poverty trap separating a low-capital and high-markup equilibrium from a high-capital low-markup equilibrium. The paper applies recent results from applied mathematics for non-smooth dynamic systems.

Keywords: Endogenous markups, Non-smooth dynamics, Discontinuity induced bifurcations, Heteroclinic orbits.

JEL codes: C62, D43, E32

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1. Introduction

Traditionally, the study of differential equations has been based on the assumption of continuous differentiability of at least the first and second order, resulting in smooth dynamics. Furthermore, in economics most dynamic systems are restricted to a unique stable steady-state equilibrium. In this paper, we take the analysis of dynamic systems in economics beyond both of these boundaries in developing a model of entry in Cournot product markets (resulting in an endogenous markup) in a dynamic general equilibrium continuous time Ramsey model. Cournot competition has the attraction that increases in economic activity are associated with more firms and hence lower markups, which captures the empirical feature of counter-cyclical markups - see *inter alia* Martins and Scarpetta (2002). We embed this entry process into an otherwise fairly standard intertemporal representative-household macromodel. The Ramsey household consumes and accumulates capital. Free entry drives profits in each instant to zero, leading to an endogenous markup and hence a wedge between the marginal product and the marginal revenue os capital. We apply recent advances in applied mathematics (Leine (2006), di Bernardo et al. (2008)) which extend traditional analysis to allow for non-smooth piecewise continuous dynamic systems. Furthermore, we develop a comprehensive analysis of the local and global dynamics of an economic system which possesses up to three steady-state equilibria, including up to two saddle-path stable equilibria, and perform the corresponding bifurcation analysis.

Whilst there are several papers that have explored the relationship between entry and the resultant "endogenous markups" using the Cournot model, most have done so in the context of discrete-time frameworks either in an OLG environment as in Chatterjee et al. (1993), D'Aspremont et al. (1995), dos Santos Ferreira and Lloyd-Braga (2005), or Kaas and Madden (2005), or in a unique steady-state Real Business Cycle (RBC) setting as in dos Santos Ferreira and Dufourt (2006), Portier (1995), Costa (2001) and Costa (2006). Continuous-time models of endogenous growth have also employed the Cournot mechanism for endogenizing markups - Zilibotti (1994), Galí and Zilibotti (1995). Related to the

Cournotian approach of this paper is the Linneman (2001) model of entry in monopolistic competition as used in Jaimovich (2007) and Jaimovich and Floetotto (2008), and Bilbiie et al. (2007). Entry reduces the market share of firms, and hence reduces the "own price effect" of the monopolist on the aggregate price index, which increases the elasticity of demand (see Yang and Heidra (1993))[1].

Whilst we allow for entry and exit, we place a *lower bound* on the measure of firms at unity. This eliminates the undesirable property of Cournot entry that the markup can go to unity and hence the marginal revenue product of capital to zero as the "number" of firms gets below unity and falls towards zero[2]. In a symmetric Cournot equilibrium, the firm's elasticity of demand is the product of the number of firms n and the industry elasticity σ, $n\sigma$ hence when $n = 1/\sigma$, the firm's elasticity is unity so that the equilibrium output is zero and the economy ceases to produce output or accumulate capital[3]. Whilst placing a lower bound on the number of firms in an industry avoids this absurd outcome, it creates a *discontinuity* in the dynamic system, since when the number of firms falls to one, the markup remains constant at σ, and the number of industries shrinks. The typical industry in the economy can thus be in one of two states: *monopoly*, where there is only one firm producing each good and charging the monopoly markup, or *oligopoly* where there is more than one firm and the markup is below its monopoly ceiling. These two states result in two dynamic regimes for the economy: for low levels of capital the dynamics is in the monopoly regime, for high levels it is in the oligopoly regime, and in-between there is a *switching boundary* and resultant non-smooth dynamics.

We find that there can be one, two or three steady-state equilibria in this economy. There are two types of stable equilibria (*saddles*): one is a low-output high-markup monopoly; the

[1]Other papers that consider a variety of aggregate feedback mechanisms are D'Aspremont et al. (1989), Wu and Zhang (2000).

[2]Recall, that in most economic analysis, the industry elasticitiy of demand σ is assumed to be *elastic*, so that $\sigma > 1$

[3]See for example Galí and Zilibotti (1995) who define this case as "autarky".

other is a high-output low-markup oligopoly. All other types of equilibria are locally unstable. We analyze the global dynamics of the model, which is non-trivial in a multiple-equilibrium environment, demonstrating the model exhibits robust heteroclinic orbits, i.e. orbits that connect the different equilibria together, and we do it in both the smooth and non-smooth cases (depending on whether the orbit passes through the switching boundary). Furthermore, we show how two fundamentally similar economies may behave very differently, as they may be in two different regimes with distinct dynamic behavior, especially in terms of markups. Even for the same economy, there is the possibility of regime change along the convergence to a stable long-run equilibrium if the switching boundary is passed. From the bifurcation analysis, the "deep" parameters associated with the dominant market structure (fixed costs and the elasticity of demand) play a crucial role in this model and a change in their values may alter the dynamics in a radical way, either by inducing a discontinuous transition or a discontinuous hysteresis. A *transitory* (and possibly small) technology shock can give rise to a *large permanent* shift in the equilibrium. If the shock takes the economy from the initial equilibrium across the switching boundary, the economy can amplify the initial shock and lead the economy (through capital accumulation) to the alternative stable equilibrium.

This economy can display a "Rostovian" threshold effect, or "poverty trap" (Easterly (2006)). Unless the economy starts off with a high enough capital stock, it will be trapped in the low-output high-markup monopoly. If, however, the capital stock is high enough, the economy will be attracted to the high-output low-markup equilibrium. The dividing line is (for a range of parameters) a totally unstable equilibrium, i.e. an unstable focus. The implication of this threshold effect is that an economy may be stuck in a monopoly with a high markup, which reduces the marginal revenue product of capital below its marginal product, and discourages saving so that only a low steady-state capital emerges. This would make a good argument for the government to intervene in some way to enable a *great leap forward* to achieve the critical capital stock so that it can then leave the outcome to the market. This intervention could take the form of regulation (reducing the gap between the marginal product of capital and the return to savings), the encouragement of savings and the accumulation of capital[4].

This paper is organized as follows. In section 2, we introduce the basic Ramsey model with capital accumulation and entry *à la* Cournot. In section 3, we explore in detail the switching boundary and the dynamics in the two regions: monopolistic and Cournot. In section 4 we characterize the steady-state equilibrium, perform the bifurcation analysis, and determine the local dynamics of the equilibria. In section 5, we characterize the global dynamics of our general-equilibrium system. Section 6 analyses the long-run effects of technology shocks leading to permanent regime changes, section 7 concludes.

2. A Ramsey Model with Endogenous Markups

We assume there is a single infinitely living household that consumes a basket of goods and supplies one unit of labor and K units of capital to firms. Total population is constant, it was normalized to unity, and we assume that the rate of technical progress is zero[5]. Thus, quantity variables may be interpreted as expressed in units of efficient labor. The household is assumed to maximize an intertemporal utility function in the absence of uncertainty:

$$\max_{C(t)} U = \int_0^\infty e^{-\rho t} \ln C\left(t\right) dt,$$

where $\rho > 0$ represents the rate of time preference and C stands for consumption. For sake of simplicity we assume a logarithmic felicity function, but most results hold with a general isoelastic function.

The final good can be used either for consumption or for capital accumulation and its price is normalized to unity, i.e. the final good is used as *numéraire*. Therefore, the instantaneous

[4]More radical alternatives would be forced saving or the nationalization of the means of production in the initial stages of development.

[5]This is for simplicity. Exogenous population growth or exogenous technical progress do not change the main message of the model, but it complicates notation substantially.

budget constraint is given by

$$\dot{K}(t) = w(t) + R(t)K(t) + \Pi(t) - C(t) - \delta K(t), \tag{1}$$

where w is the wage rate, R stands for the rental price of capital, Π represents pure profits, and $\delta > 0$ is the capital depreciation rate.

Optimal consumption and labor supply paths verify the Euler and the transversality conditions:

$$\frac{\dot{C}(t)}{C(t)} = R(t) - (\rho + \delta), \tag{2}$$

$$\lim_{t\to\infty} e^{-\rho t}\frac{K(t)}{C(t)} = 0. \tag{3}$$

2.1. The final-good sector

The final good sector is perfectly competitive, with the representative firm maximizing its profits given by

$$\max_{Y(t),[y(v,t)]_{v=0}^{z(t)}} Y(t) - \int_0^{z(t)} p(v,t)\, y(v,t)\, dv,$$

where $Y(t)$ is the production of final good, $y(v,t)$ and $p(v,t)$ stand for intermediate consumption of variety $v \in [0, z(t)]$ at the moment t, and for its relative price, and $z(t)$ is the mass of the continuum of intermediate goods. Profit maximization is subject to a constant returns to specialization[6] CES technology that transforms a continuum of intermediate goods into a homogenous final good:

$$Y(t) = z(t)^{\frac{1}{1-\sigma}} \left[\int_0^{z(t)} y(v,t)^{\frac{\sigma-1}{\sigma}}\, dv\right]^{\frac{\sigma}{\sigma-1}},$$

where $\sigma > 1$ represents the elasticity of substitution between inputs.

The first-order conditions lead to the demand function for each input and also to

$$y(v,t) = p(v,t)^{-\sigma}\frac{Y(t)}{z(t)}, \tag{4}$$

[6] That is, there is no "love of variety" and the range of intermediate goods has no effect per se on unit cost.

and

$$1 = \left(\frac{1}{z(t)} \int_0^{z(t)} p(v,t)^{1-\sigma} \, dv \right)^{\frac{1}{1-\sigma}}, \tag{5}$$

where the left-hand side represents the price and the right-hand side is the marginal cost of producing it.

The market-clearing condition for this sector is given by $Y(t) = C(t) + I(t)$ where $I(t) = \dot{K}(t) + \delta K(t)$ is the gross investment defined as $I(t) = \dot{K}(t) + \delta K(t)$.

2.2. The intermediate goods sector

First, we assume $0 < z(t) \leq 1$, i.e. the mass of the continuum of intermediate goods, is bounded above by one, a fixed technological frontier. Industry V that produces good $v \in [0, z(t)]$ is composed of $n(v,t) \geq 1$ producers at moment t[7]. Each industry can be in one of two states: *monopoly*, when only one firm is operative, or *oligopoly*, when more than one is operative. New firms prefer to be monopolists than to share existing industries. When monopoly profit opportunities are still available, i.e. $0 < z(t) < 1$, new firms will set up as monopolies. Only when all industries have at least one firm operating $z(t) = 1$, will new firms be forced to enter existing oligopolistic industries. Using the terminology in D'Aspremont et al. (1997), we assume *Cournotian Monopolistic Competition* (CMC), i.e. firms choose quantity to influence their own market price (as in Cournot), but treat the aggregate price level as given (as in monopolistic competition).

The representative firm $i \in \{1, ..., n(v,t)\}$, in industry V, maximizes its real profits given by

$$\max_{y_i(v,t), L_i(v,t), K_i(v,t)} \Pi_i(v,t) = p(v,t) \, y_i(v,t) - w(t) \, L_i(v,t) - R(t) \, K_i(v,t), \tag{6}$$

where y_i represents the output of firm i, K_i and L_i represent its capital and labor inputs. The production technology is given by the following expression valid for $y_i > 0$:

$$y_i(v,t) + \phi = A(t) \, K_i(v,t)^{\alpha} \, L(v,t)^{1-\alpha}, \tag{7}$$

[7]Of course this number is an integer. However, we will treat it as a real number, for simplicity. We can think about it as the average number of firms in each industry.

where $A(t) > 0$ stands for total factor productivity, $0 < \alpha < 1$, and $\phi > 0$ induces increasing returns to scale. Firm i also faces the following inverse residual demand for its variety, given the outputs of the other firms in industry V $(k \neq i \in V)$ and given the prices of firms producing goods that are an imperfect substitute to good v,

$$p(v,t) = \left(z(t) \frac{y_i(v,t) + \sum_{k \neq i \in V} y_k(v,t)}{Y(t)} \right)^{-\frac{1}{\sigma}}. \tag{8}$$

Notice this is a static problem, as the firm does not accumulate capital. The first-order conditions are given by

$$(1 - \mu_i(v,t))(1-\alpha) A(t) \left(\frac{K_i(v,t)}{L_i(v,t)} \right)^{\alpha} = \frac{w(t)}{p(v,t)}, \tag{9}$$

$$(1 - \mu_i(v,t)) \alpha A(t) \left(\frac{K_i(v,t)}{L_i(v,t)} \right)^{\alpha-1} = \frac{R(t)}{p(v,t)}, \tag{10}$$

where $\mu_i(v,t) = y_i(v,t) / \left(\sigma \sum_{s \in V} y_s(v,t) \right) \in (0,1)$ is the Lerner index for firm i in industry V. Henceforth we will call this market-power measure the "markup". Observe that if industry V is a monopoly, i.e. $n(v,t) = 1$, then the markup is $\mu(v,t) = 1/\sigma$, and if industry V is a oligopoly, i.e. $n(v,t) > 1$, then $0 < \mu(v,t) < 1/\sigma$.

2.3. Symmetric equilibrium and aggregation

Considering an inter-industrial symmetric equilibrium, we have $n(v,t) = n(t)$; and within-industry symmetry implies[8] $\mu_i(v,t) = \mu(t) = 1/(\sigma n(t))$ and $p(v,t) = 1$.

The market-clearing condition for the labor market equates aggregate labour demand to aggregate labour supply. The market-clearing condition in the final good market, $Y(t) = D(t)$, allows us to derive an aggregate production function for final output

$$Y = F(K) - zn\phi, \tag{11}$$

[8]Note that since $\sigma > 1$ and $n \geq 1$, $\mu(t) < 1$.

where $F(K) = AK^{\alpha}$ is a reduced-form production function for "gross" output. Capital market clearing generates the rate of return of capital

$$R = (1-\mu) F'(K). \tag{12}$$

Profit income is obtained by aggregating profits across all firms in the economy, i.e. $\Pi = Y - w - RK$. Considering the equilibrium factor prices and the aggregate production function, total profits can be expressed as

$$\Pi = \mu F(K) - zn\phi. \tag{13}$$

2.4. Entry, endogenous markups, and regime switching

In the absence of sunk entry costs there will be instantaneous free entry, i.e. wither the continuum of intermediate goods (z) or the number of firms in each industry (n) adjust in order to keep pure profits of all firms firm equal to zero, i.e. $\Pi = 0$.

Together, equations (11), (13), and the zero-profit condition imply that the reduced-form production function becomes[9]

$$Y = (1-\mu)F(K), \tag{14}$$

and the aggregate resource constraint of the economy, at the equilibrium is

$$\dot{K} = (1-\mu)F(K) - C - \delta K. \tag{15}$$

Up to this point the type of entry is undefined. Given the assumptions made, the incentive scheme in the economy separates entry into two regimes and a regime-switch transition:

1. there is a Monopolistic-Competition (MC) regime when $n = 1$ and the best possibility for an entrant is to start a new industry, i.e. entry determines the equilibrium value of $z \in (0,1)$, as vertical entry can only take place for $z < 1$;

[9]Equation (14) does not depend on the specific functional form assumed in equation (7). In fact, the same result is obtained for a general linear-homogeneous production function for "gross" output: $y_i(v,t) + \phi = f(K_i(v,t), L_i(v,t))$.

2. there is a CMC regime when $z = 1$ and entry can only take place into existing industries, i.e. horizontal entry determines the number of firms in the industry n, such that $n > 1$;
3. there is a regime-switch frontier if and only if $z = n = 1$, that is the case in which new industries cannot be created and there is place for exactly one producer in each industry.

Although we treat n as a continuous variable, we still want to impose a lower bound of $n \geq 1$, so that the markup cannot take a value greater than $1/\sigma$, the level attained in the MC regime. Without this restriction, the economy would possess a "black hole" where as n falls towards $1/\sigma < 1$ the markup rises to unity and wages and the return on capital go to zero along with output and consumption. Setting the lower bound at 1 seems to us a reasonable way of avoiding this exotic and implausible phenomenon (any lower bound above $1/\sigma$ would do).

It turns out that both the markup and the regime we are in becomes endogenous and depend on the (current) capital stock. Taking into account that $n = 1/(\sigma\mu)$, we obtain the rule governing the endogenous mark-up: the actual markup $\mu = \mu(K)$ is the smaller of the free-entry MC ($n = 1$) markup given by $1/\sigma$ and its CMC ($z = 1$) counterpart given by $m(K) = (\sigma F(K)/\phi)^{-1/2}$,[10]

$$\mu = \mu(K) = \min\{1/\sigma, m(K)\}. \tag{16}$$

Then, there is a critical value for capital for which a regime switch occurs: the capital stock for which entry reduces the number of firms per industry to exactly one in the CMC

[10] See dos Santos Ferreira and Lloyd-Braga (2005), equation (15), p. 852 for a similar equation to $\mu = m(K)$. Galí and Zilibotti (1995), equation (8), p. 201 is equivalent to $\mu = m(K)$ when the marginal product of labour is constant (they assume an AK model). dos Santos Ferreira and Dufourt (2006), equation (7), p. 316 is a partial equilibrium version. Costa (2004), equation (25), p. 65 is equivalent to the one presented here.

regime and induces a mass of the continuum of varieties equal to its maximum (i.e. one) in the MC regime. Equation (16) allow us to define it as $\tilde{K} \equiv \{ K : m(K) = 1/\sigma \}$. Given the Cobb-Douglas production function in equation (7), the critical level for the capital stock is given by

$$\tilde{K} = \left(\frac{\phi\sigma}{A} \right)^{1/\alpha} > 0. \tag{17}$$

If $0 < K < \tilde{K}$ then the economy operates in a MC regime and if $K > \tilde{K}$ then the economy operates in a CMC regime[11]. Poor economies tend to operate at the MC regime and rich economies in the CMC regime. Regime switches may occur in two different ways. First, if the capital stock is far away from its steady state and along the transition it passes through $\tilde{K}$, triggering an endogenous change in the regime. Second, if the economy is in a steady state and a shock to parameters α, ϕ, σ, or A occurs, moving it to a different regime.

2.5. Global production and return functions

Given that the critical value $\tilde{K}$ defines a switching boundary, the endogenous markup function given by equation (16) has two contiguous branches. Thus, the aggregate production function in (14) and the rate of return on capital in (12) also exhibit two contiguous branches:

$$Y(K) = \begin{cases} Y_1(K) \equiv (1 - \frac{1}{\sigma})F(K) & \text{if } 0 \leq K \leq \tilde{K}, \\ Y_2(K) \equiv (1 - m(K))F(K) & \text{if } K \geq \tilde{K}, \end{cases} \tag{18}$$

and

$$R(K) = \begin{cases} R_1(K) \equiv (1 - \frac{1}{\sigma})F'(K) & \text{if } 0 \leq K \leq \tilde{K}, \\ R_2(K) \equiv (1 - m(K))F'(K) & \text{if } K \geq \tilde{K}. \end{cases} \tag{19}$$

[11] Given that $m(K)$ is positive, monotonous and has $\lim_{K\to 0} m(K) = +\infty$, $\lim_{K\to\infty} m(K) = 0$, then $\tilde{K} \equiv \{K : m(K) = 1/\sigma\}$ is unique. Therefore: (1) for all K in the interval $(0, \tilde{K})$ we have $m(K) > 1/\sigma$ which implies $\mu = \min\{m(K), 1/\sigma\} = 1/\sigma$, and (2) for all K in the interval in the interval $(\tilde{K}, +\infty)$ we have $m(K) < 1/\sigma$ which implies $\mu = \min\{m(K), 1/\sigma\} = m(K)$.

The first branch in both functions corresponds to the MC regime and the second corresponds to the CMC regime. Next we will characterize the properties of the aggregate technology and return on capital arising from those functions.

At $K = \tilde{K}$ both production and return functions are continuous, because

$$\begin{aligned} R_1(\tilde{K}) &= R_2(\tilde{K}) = \tilde{R} = \alpha(\sigma - 1)(A/\sigma)\tilde{K}^{\alpha-1} \\ Y_1(\tilde{K}) &= Y_2(\tilde{K}) = \tilde{Y} = (\sigma - 1)(A/\sigma)\tilde{K}^{\alpha}, \end{aligned}$$

but classic derivatives do not exist. However, we can determine generalized gradients for the production function as

$$\partial Y(K) = \begin{cases} Y_1'(K) & \text{if } 0 \leq K < \tilde{K}, \\ \partial Y(\tilde{K}) & \text{if } K = \tilde{K}, \\ Y_2'(K) & \text{if } K > \tilde{K}. \end{cases} \tag{20}$$

where $\partial Y(\tilde{K})$ is the convex hull of $Y_1'(K)$ and $Y_2'(K)$ at $K = \tilde{K}$, $\partial Y(\tilde{K}) = \{\ (1-q)Y_1'(\tilde{K}) + qY_2'(\tilde{K}) : 0 \leq q \leq 1\}$, and for the return function

$$\partial R(K) = \begin{cases} R_1'(K) & \text{if } 0 \leq K < \tilde{K}, \\ \partial R(\tilde{K}) & \text{if } K = \tilde{K}, \\ R_2'(K) & \text{if } K > \tilde{K}, \end{cases} \tag{21}$$

where $\partial R(\tilde{K})$ is the convex hull of $R_1'(K)$ and $R_2'(K)$ at $K = \tilde{K}$, $\partial R(\tilde{K}) = \{\ (1-q)R_1'(\tilde{K}) + qR_2'(\tilde{K}) : 0 \leq q \leq 1\}$.

Lemma 1. *Functions $Y(K)$ and $R(K)$ are continuous and piecewise smooth (PWS).*

All proofs are in the appendix. Next we observe that the concavity of both functions depend critically on the elasticity of intermediate input substitution, σ. Let us define

$$\bar{\sigma} = \bar{\sigma}(\alpha) = \frac{2-\alpha}{2(1-\alpha)} > 1, \tag{22}$$

an admissible value for σ considering $\bar{\sigma} > 1$ for all $0 < \alpha < 1$.

Lemma 2. *Let* $\bar{K} \equiv (\bar{\sigma}/\sigma)^{2/\alpha}\,\tilde{K}$. *(a) If the elasticity of substitution between inputs is large, i.e. for* $\sigma > \bar{\sigma}$, *then the technology is concave in a generalized way, i.e.* $\partial^2 Y(K) \in \mathbb{R}_{--}$, *and there are decreasing returns, i.e.* $\partial R(K) \in \mathbb{R}_{--}$. *(b) If* $\max\{1, \bar{\sigma}/2\} < \sigma < \bar{\sigma}$ *then the production function is concave and the return function is non-monotonous:* $R_2(K)$ *is increasing for* $K \in [\tilde{K}, \bar{K})$, *reaches a local maximum at* $K = \bar{K}$, *and it is decreasing for* $K > \bar{K}$. *(c) If* $1 < \sigma < \max\{1, \bar{\sigma}/2\}$ *then the return function has the same properties as in (b) but the production function becomes concave-convex:* $Y_2(K)$ *is locally convex for* $K \in [\tilde{K}, \bar{K})$, *reaches a local maximum at* $K = \bar{K}$, *and it is locally concave for* $K > \bar{K}$. [12]

Figures A.1 and A.2 below depict the first two (more plausible) cases in which the production function is concave, i.e. for $\sigma > \bar{\sigma}$, and the case in which the return function may be locally increasing, at the onset of the CMC regime, and the production function is concave.

`Figure A.1 around here`

`Figure A.2 around here`

The existence of two regimes has interesting economic implications. First, the firm-level production technology is one corresponding to "natural monopoly" since there is a globally decreasing average cost when $\phi > 0$ with constant marginal cost. The degree of inefficiency can be proxied by the gap between *average cost (ac)* and *marginal cost (mc)*. In both MC and CMC regimes free entry implies that price equals average cost $p = ac$ (zero profits) and industry equilibrium implies marginal revenue, $p(1 - 1/(n\sigma))$, equals marginal cost: hence

$$\frac{ac}{mc} = \left(1 - \frac{1}{n\sigma}\right).$$

In the MC regime where $n = 1$, this gap is fixed and hence output per-firm is tied down. In the CMC regime, however, matters are different because n increases with K, $n =$

[12] The third case, is somewhat uninteresting, as we can only obtain $1 < \bar{\sigma}/2$ for $\alpha > 2/3$.

$(\sigma\phi/F(K))^{-1/2}$. More firms means a lower markup, which narrows the gap between ac and mc, as the zero-profit condition requires output per firm to increase. In the limit as K (and hence n) tend to infinity, ac tends to mc, and output per firm also tends to infinity, the condition for efficient production.

The possibility of a concave-convex technology relates to an existing literature: Skiba (1978) and Dechert and Nishimura (1983). However, these contributions were made for centralized economies in which the externalities are fully internalized. Therefore, the return and the marginal product of capital are permanently equalized. In our model we retain some of the dynamic properties of that type of model (e.g. the existence of multiple stationary equilibria) while circumventing criticisms that those functions are ad-hoc. In addition, when utility is logarithmic, the general equilibrium is Markovian in contrast to those models - see Santos (2002) (see below).

3. General Equilibrium

Definition 1. ***General Equilibrium*** *The general equilibrium (GE) is defined by the paths for the mass of industries and for the number firms per industry,* $[(z(t), n(t))]_{t=0}^{\infty}$*, by the allocations and prices* $[([y(v,t)]_{v\in(0,z(t)]}, [p(v,t)]_{v\in(0,z(t)]})]_{t=0}^{\infty}$*, and by the aggregate capital stock and consumption trajectories* $[(K(t), C(t))]_{t=0}^{\infty}$ *such that both final- and intermediate-goods firms and consumers optimize, there is symmetric equilibria in all markets for intermediate goods, and the equilibrium conditions for both factor and final-good markets hold.*

In the previous section, we proved that in a symmetric equilibrium, the paths of z, n, and of the distributions $[y(v)]$ and $[p(v)]$ depend on the paths of the aggregate capital stock K, and the last one is jointly determined with the trajectory of aggregate consumption C at the GE level. As $R(K)$ and $Y(K)$ are defined by the PWS continuous functions (18) and (19) the GE paths are generated by a dynamic system that also displays discontinuities.

Let us define the following subsets of $(K, C) \in \mathbb{R}_+^2$:

$$\mathcal{S}_1 = \{\ (K, C) \in \mathbb{R}_+^2 : m(K) > 1/\sigma\} = \{\ (K, C) \in \mathbb{R}_+^2 : 0 \leq K < \tilde{K}\}, \tag{23}$$

$$\Sigma = \{\ (K, C) \in \mathbb{R}^2_+ : m(K) = 1/\sigma\} = \{\ (K, C) \in \mathbb{R}^2_+ : K = \tilde{K}\}, \tag{24}$$

$$\mathcal{S}_2 = \{\ (K, C) \in \mathbb{R}^2_+ : m(K) < 1/\sigma\} = \{\ (K, C) \in \mathbb{R}^2_+ : K > \tilde{K}\}, \tag{25}$$

where $\mathcal{S}_1$ ($\mathcal{S}_2$) corresponds to states of the economy where a MC (CMC) regime is in place. If the economy is in state Σ we say it is on the switching boundary between the previous two regimes. We also know that $\bar{\mathcal{S}}_1 \cap \bar{\mathcal{S}}_2 = \Sigma$.

The equilibrium trajectories $[(K(t), C(t))]_{t \in \mathbb{R}_+}$ are the solutions to the system

$$\dot{C} = \begin{cases} (R_1(K) - (\rho + \delta))\, C & \text{if } (K, C) \in \mathcal{S}_1, \\ (R_2(K) - (\rho + \delta))\, C & \text{if } (K, C) \in \mathcal{S}_2, \end{cases} \tag{26}$$

$$\dot{K} = \begin{cases} Y_1(K) - \delta K - C & \text{if } (K, C) \in \mathcal{S}_1, \\ Y_2(K) - \delta K - C & \text{if } (K, C) \in \mathcal{S}_2. \end{cases} \tag{27}$$

together with the initial condition $K(0) = K_0$ and the transversality condition in (3).

Equations (26) and (27) define a PWS continuous dynamic system since: (1) both functions are smooth in the two branches; (2) they are continuous at the boundary Σ, as $R_1(\tilde{K}) = R_2(\tilde{K})$ and $Y_1(\tilde{K}) = Y_2(\tilde{K})$; and (3) nevertheless, their derivatives differ at $K = \tilde{K}$, i.e. $R_1'(\tilde{K}) < R_2'(\tilde{K})$ and $Y_1'(\tilde{K}) < Y_2'(\tilde{K})$ [13]. In order to understand the nature of the dynamics of our PWS system, we have distinguish between *candidate* and GE paths. We denote them as $\Phi^c(t) = (\Phi^c_K(t, K(0)), \Phi^c_C(t, K(0)))$ and $\Phi(t) = (\Phi_K(t, K(0)), \Phi_C(t, K(0)))$ respectively. Candidate trajectories, are solutions to the system (26)-(27) for a given initial capital stock $K(0)$, belonging to one of the two possible states of the economy: MC when $0 < K(0) < \tilde{K}$ (branch $\mathcal{S}_1$) or CMC when $K(0) > \tilde{K}$ (branch $\mathcal{S}_2$). We denote

[13] Were the functions discontinuous at the boundary Σ, we would have a Filippov system, from Filippov (1988). See di Bernardo et al. (2008), and Leine and Nijmeier (2004) for the state of the art on the analysis of both types of dynamic systems. Fillipov systems have been subject to more attention. However, recent advances in dealing with PWS continuous systems are presented in the above-mentioned references and also in Freire et al. (1998) and Leine (2006). We will use the approach in di Bernardo et al. (2008) and di Bernardo et al. (2008) from now on.

$\Phi_j^c(t) = (\Phi_{j,K}^c(t), \Phi_{j,C}^c(t))$ and $\Phi_j(t) = (\Phi_{j,K}(t), \Phi_{j,C}(t))$ the candidate and the GE trajectories belonging to branch $\mathcal{S}_j$ for $j = 1, 2$.

The following types of behavior are possible, starting from any initial capital stock[14]: if $(K(0), C(0)) \in \mathcal{S}_j$ for $j = 1, 2$, with an arbitrary $C(0)$, the solution $\Phi^c(t) = \Phi_j^c(t)$, for $t > 0$, has one the following alternative types of trajectory. First, it may stay inside the same area $\mathcal{S}_j$ converging to a steady state $(K_j^*, C_j^*) \in \mathcal{S}_j$, or it may converge to zero or unbounded values for one or both variables. Alternatively, it may contact the boundary Σ at time $t = t_\Sigma > 0$ where $\Phi_{j,K}^c(t_\Sigma, K(0)) = \tilde{K}$. It was proved that for PWS continuous systems[15] four types of behavior may unfold upon contact with the switching boundary: (1) there is a steady state located at the boundary $(\tilde{K}, \tilde{C}) \in \Sigma$, which is reached in infinite time and we have $\Phi_j^c(\infty, K(0)) = (C_\Sigma^*, K_\Sigma^*)$; (2) the trajectory crosses the boundary and proceeds to $\mathcal{S}_{-j}$ ($\mathcal{S}_{-j} = \mathcal{S}_1$ for $j = 2$ and $\mathcal{S}_{-j} = \mathcal{S}_2$ for $j = 1$), taking the value $\Phi_{-j}^c(t, \Phi_j^c(t_\Sigma, K(0))$ at time $t > t_\Sigma$ and converges to a steady state $(K_{-j}^*, C_{-j}^*) \in \mathcal{S}_{-j}$ to zero or unbounded values of one or both variables; (3) the trajectory *grazes* the boundary and turns back inside area $\mathcal{S}_j$, converging to a steady state $(K_j^*, C_j^*) \in \mathcal{S}_j$ to zero or to unbounded values of one or both variables; (4) the trajectory crosses Σ at time t_Σ^j, penetrates $\mathcal{S}_{-j}$, curls back to the switching boundary Σ at time $t_\Sigma^{-j} > t_\Sigma^j$, and re-enters the initial branch $\mathcal{S}_j$ - in this case, if $t > t_\Sigma^{-j}$ the candidate path takes the value $\Phi_j^c(t, \Phi_{-j}^c(t_\Sigma^{-j}, \Phi_j^c(t_\Sigma^j, K(0)))$.

GE trajectories, $\Phi(t)$, are candidate trajectories starting from a given $(K(0), C(0)) \in \mathcal{S}_j$ such that the transversality condition holds. If the equilibrium is determinate, there is an unique value $\Phi_{j,C}(0, K(0))$ such that the transversality condition holds, whether that value is not unique when indeterminacy exists. In our case this means that the trajectories should converge to a steady state in $\mathcal{S}_j$ (with or without grazing the boundary Σ), in Σ or in $\mathcal{S}_{-j}$, after traversing the boundary. If the last case occurs, the GE paths concatenate

[14] It is well known in the literature that solutions exist when the vector fields are continuous, as in our model.

[15] See di Bernardo et al. (2008).

the solutions belonging to both branches. If $(K(0), C(0)) \in \mathcal{S}_j$ then $\Phi(t) = \Phi_j(t, K(0))$ for $t < t_\Sigma$, $\Phi(t_\Sigma) = \Phi_j(t_\Sigma, K(0))$ for $t = t_\Sigma$, and $\Phi(t) = \Phi_{-j}(t, \Phi_j(t_\Sigma, K(0))$, for $t > t_\Sigma$. Intuitively, this means that there is an endogenous and transient change in the regime from MC to CMC, or vice-versa.

Thus, two possible changes in regime may occur: (1) a *transient* change in regime takes place if the initial and the steady-state levels for the capital stock are located in different sides of the switching boundary Σ and the adjustment dynamics involves traversing the switching boundary; (2) a *long-run* regime shift occurs if there is a parameter change that justifies it.

In the next two sections, we use global-bifurcation analysis to determine what types of GE dynamics features our model can display. We do this by characterizing the local behavior at equilibria, the behavior at the switching boundary, and finally global dynamics.

4. Local Dynamics

We will first examine local dynamics around the steady-states. In the last section we presented two types of steady states that can exist in system (26)-(27): *regular steady states* if they belong to the interior of $\mathcal{S}_j$ with $j = 1, 2$, $(K_j^*, C_j^*) \equiv \{(K, C) \in \mathcal{S}_j : \dot{C} = \dot{K} = 0\}$, and *boundary steady states* if they belong to Σ, $(K_\Sigma^*, C_\Sigma^*) \equiv \{(K, C) \in \Sigma : \dot{C} = \dot{K} = 0\}$. For given values of the parameters, steady states may be isolated or multiple, and in the latter they can belong either to the same or to different regimes.

From the properties of functions (18) and (19) we obtain the following relationship between steady-state consumption and capital stock:

$$C_j^* = \beta K_j^*, \ \beta \equiv \frac{\rho + (1-\alpha)\delta}{\alpha} > 0, \text{for } j = 1, 2, \Sigma, \tag{28}$$

which implies that each steady state is completely characterized using the steady-state stock of capital, $K_j^* \equiv \{K \in \mathcal{S}_j : \ R_j(K) = \rho + \delta\}$, $j = 1, 2$, or $K_\Sigma^* = \{K \in \Sigma : R_1(K) = R_2(K) = \rho + \delta\}$, respectively for regular or boundary steady states.

Let us define the following critical values for parameter ϕ:

$$\tilde{\phi} \equiv \frac{1}{\sigma}\left(A\left(\frac{\alpha}{\rho+\delta}\left(1-\frac{1}{\sigma}\right)\right)^{\alpha}\right)^{1/(1-\alpha)}. \tag{29}$$

and

$$\bar{\phi} \equiv \frac{\sigma}{\bar{\sigma}^2}\left(A\left(\frac{\alpha}{\rho+\delta}\left(1-\frac{1}{\bar{\sigma}}\right)\right)^{\alpha}\right)^{1/(1-\alpha)} \text{ for } 1<\sigma\leq\bar{\sigma}. \tag{30}$$

The next proposition and the bifurcation diagrams in Figure A.3 summarize the local dynamics, in the neighborhood of steady states.

Proposition 1. ***Local dynamics and bifurcations at steady-state equilibria.***

1. *If $1 < \sigma < \bar{\sigma}$ either a unique or multiple equilibria may exist, depending on ϕ. For $\phi > \bar{\phi}$ there is a unique saddle-point stable regular MC equilibrium. For $\tilde{\phi} < \sigma < \bar{\sigma}$ there are three regular equilibria, a MC saddle-point stable and a pair of CMC equilibria: a high-markup unstable and a low-markup saddle-point stable. For $\phi < \tilde{\phi}$ there is a unique saddle-point stable CMC equilibrium. For $\phi = \bar{\phi}$ there are two steady states: a saddle-point stable regular MC and a regular CMC that is locally a smooth-fold bifurcation. For $\phi = \tilde{\phi}$ there are two equilibria: a saddle-point stable CMC equilibrium and a boundary equilibrium displaying a non-smooth fold bifurcation.*
2. *If $\sigma \geq \bar{\sigma}$ there is a unique equilibrium whose type depends on ϕ. For $\phi > \tilde{\phi}$ it is a saddle-point stable MC equilibrium. For $\phi < \tilde{\phi}$ it is a saddle-point stable CMC equilibrium. For $\phi = \tilde{\phi}$ it is a boundary-equilibrium bifurcation displaying saddle-point stable persistence.*

We will derive Proposition 1 step by step in the next sub-section 4.1. However, first we will interpret its meaning and implications.

`Figure A.3 around here`

Figure A.3 presents a bifurcation diagram in (K, ϕ) space. The left-hand-side (LHS) panel represents the $1 < \sigma < \bar{\sigma}$ case and the right-hand-side (RHS) panel the $\sigma \geq \bar{\sigma}$ case. For different values of the fixed cost it shows the number of equilibria and the local stability properties of each steady-state equilibrium, including the location of the discontinuity-induced bifurcations (persistence P on the RHS panel and the non-smooth fold NF on the LHS panel) and the smooth fold bifurcation (F on the LHS panel).

If we start with high values of ϕ, on the RHS panel, the initial equilibrium is saddle-point stable (C_M^*, K_M^*), it is a MC equilibrium, and further reductions of the fixed cost keep the local stability properties, but force it to pass through the switching boundary (C_Σ^*, K_Σ^*) and further evolve into a CMC equilibrium (C_L^*, K_L^*). We associate the later to a low markup as the return function is locally decreasing.

If we do the same reasoning for the left-hand-panel panel, we should observe that there is structural instability, i.e. there is a change on the number of equilibria and on their local-dynamics properties for variations of ϕ close to its bifurcation levels: $\tilde{\phi}$ associated to a discontinuity-induced bifurcation of the non-smooth fold type[16], and $\bar{\phi}$ associated to a smooth fold bifurcation. If we start with high values of ϕ the initial equilibrium (C_M^*, K_M^*) is also saddle-point stable and it is a MC equilibrium. If we reduce ϕ from $\bar{\phi}$, then three equilibria emerge: (C_M^*, K_M^*), an unstable high-markup CMC equilibrium (C_H^*, K_H^*), and a saddle-point stable low-markup equilibrium (C_L^*, K_L^*). Further reductions of ϕ from $\tilde{\phi}$, would result in the collision of the first two equilibria at the discontinuous boundary equilibrium bifurcation point (C_Σ^*, K_Σ^*), followed by their disappearance. For lower values of ϕ the steady-state equilibrium is unique and again it is a saddle-point stable low-markup CMC (C_L^*, K_L^*).

Observe that the MC steady-state level for the capital stock does not depend on ϕ. However, when the fixed cost varies, the distance to the switching boundary changes: if the fixed cost increases, the "competitive distance" increases. If the economy is in the CMC regime, not only the distance of the steady state to the switching boundary varies (because $\tilde{K}$ depends on ϕ), but the steady-state stock of capital also varies. If the elasticity σ is low and there are multiple CMC equilibria the relative "competitive distance" of the two

[16]See subsection 4.1 below.

equilibria changes in a symmetric way. When the fixed cost is reduced, the unstable high-markup equilibrium moves towards the switching barrier and the low-markup saddle-point stable equilibrium moves away from it.

We can also observe the same by defining a partition over the domain of (ϕ, σ), i.e. $\mathbb{R}_{++} \times (1, +\infty)$ - see Figure A.4:

$$\begin{aligned} \mathcal{A} &= \{\ (\sigma, \phi) : \sigma > 1,\ \phi > \max\{\overline{\phi}, \tilde{\phi}\ \}\}, & \mathcal{D}^F &= \{\ (\sigma, \phi) : 1 < \sigma < \bar{\sigma},\ \phi = \overline{\phi}\}, \\ \mathcal{B} &= \{\ (\sigma, \phi) : 1 < \sigma < \bar{\sigma},\ \tilde{\phi} < \phi < \overline{\phi}\}, & \mathcal{D}^{NF} &= \{\ (\sigma, \phi) : 1 < \sigma < \bar{\sigma},\ \phi = \tilde{\phi}\}, \\ \mathcal{C} &= \{\ (\sigma, \phi) : \sigma > 1,\ 0 < \phi < \tilde{\phi}\}, & \mathcal{D}^P &= \{\ (\sigma, \phi) : \sigma > 1,\ \phi = \tilde{\phi}\}. \end{aligned}$$

`Figure A.4 around here`

This pair of crucial parameters allow us to identify the region where steady-state equilibria lie. If $(\sigma, \phi) \in \mathcal{A}$, given by $\mathcal{A}_1 \cup \mathcal{A}_2$ in Figure A.4 , there is a single saddle-point stable stationary MC equilibrium $(K_M^*, C_M^*) \in \mathcal{S}_1$. If $(\sigma, \phi) \in \mathcal{C}$, given by $\mathcal{C}_1 \cup \mathcal{C}_2$ in the same picture, there is a unique saddle-point stable stationary CMC equilibrium $(K_L^*, C_L^*) \in \mathcal{S}_2$. If $(\sigma, \phi) \in \mathcal{B}$, three stationary equilibria exist: the saddle-point stable stationary MC equilibrium $(K_M^*, C_M^*) \in \mathcal{S}_1$ and the two CMC equilibria in the $\mathcal{S}_2$ region, the saddle-point stable (K_M^*, C_M^*) and the unstable (K_L^*, C_L^*). When we observe the limits between two of the previous regions bifurcations occur. If $(\sigma, \phi) \in \mathcal{D}^F \cup \mathcal{D}^{NF}$ two stationary equilibria exist with a fold bifurcation associated to one of them: on the $\mathcal{D}^F$ frontier the bifurcation is a regular smooth one and on the $\mathcal{D}^{NF}$ border it is a non-smooth one. Finally, if $(\sigma, \phi) \in \mathcal{D}^P$ persistence is the discontinuity-induced bifurcation that emerges.

We can find a single saddle-point stable MC equilibrium like (K_M^*, C_M^*) in dynamic GE models with a Dixit and Stiglitz (1977) MC market structure - see Rotemberg and Woodford (1995). Galí and Zilibotti (1995) present a CMC model that produces a pair of equilibria that correspond to (K_L^*, C_L^*) and (K_H^*, C_H^*), with the possibility of complex eigenvalues for the latter. Galí (1995) produced a different type of market structure producing endogenous markups and also leading to multiple equilibria and non-saddle dynamics. However, the existence of endogenous markups is not a sufficient condition to generate multiple equilibria as we can see in Goodfriend and King (1997), Jaimovich (2007) or even in the CMC model of Portier (1995).

4.1. Derivation of 1

Our first step is to establish the existence and number of steady states:

Lemma 3. *Let $\bar{\sigma}$, $\tilde{\phi}$, and $\bar{\phi}$ be given by equations (22), (29), and (30), respectively. Then: (1) if $\phi > \max\{\tilde{\phi}, \bar{\phi}\}$, for any $\sigma > 1$, there is a unique regular MC steady state; (2) if $0 < \phi < \min\{\tilde{\phi}, \bar{\phi}\}$, for any $\sigma > 1$, there is a unique regular CMC steady state; (3) if $\sigma \geq \bar{\sigma}$ and $\phi = \tilde{\phi}$ then there is a unique boundary steady state; (4) if $1 < \sigma < \bar{\sigma}$ and $\tilde{\phi} < \phi < \bar{\phi}$, there are three regular steady states, a MC and two CMC steady states; (5) if $1 < \sigma < \bar{\sigma}$ and $\phi = \bar{\phi}$, there are two regular steady states, one MC and the other CMC; (6) if $1 < \sigma < \tilde{\sigma}$ and $\phi = \tilde{\phi}$, there are two steady states, a regular CMC and a boundary one.*

If the elasticity of substitution between varieties is large, $\sigma \geq \bar{\sigma}$, then the steady state is unique, but it may be associated to any regime or belong to the switching boundary. The type of steady-state regime depends on the cost of entry ϕ relative to the other parameters. First, if the fixed cost is high (or A is low) such that $\phi > \tilde{\phi}$ then there is a unique regular MC steady state that we can determine explicitly

$$K_1^* = K_M^* \equiv \left(\frac{\alpha A}{\rho + \delta} \left(1 - \frac{1}{\sigma} \right) \right)^{1/(1-\alpha)} < \tilde{K}.$$

Second, if the fixed cost is low (or A is high), such that $\phi < \tilde{\phi}$, then there is a regular CMC steady state which is unique from Lemma 3, with capital stock defined by

$$K_2^* = \{\ K_C^* \} \ = \{\ K \in \mathcal{S}_2 : (1 - m(K))F^{'}(K) = \rho + \delta \}.$$

In the transition between those two cases, i.e. if $\phi = \tilde{\phi}$, then the steady-state capital stock is given by

$$K_\Sigma^* = \tilde{K}(\tilde{\phi}) = \left(\frac{\sigma \tilde{\phi}}{\alpha A} \right)^{1/\alpha} = \left(\frac{\alpha^2 A}{\rho + \delta} \left(1 - \frac{1}{\sigma} \right) \right)^{1/(1-\alpha)}.$$

If the elasticity is low, verifying $1 < \sigma < \bar{\sigma}$ several cases may occur. In this case we have $\tilde{\phi} < \bar{\phi}$ and both similar and different possibilities are produced, as in the case of high σ. If $\phi > \bar{\phi}$ or $\phi < \tilde{\phi}$ we have two cases that are analogous to the previous high-elasticity ones: we observe unique steady states associated with either K^*_M or K^*_C. However, if $\bar{\phi} > \phi > \tilde{\phi}$ multiple steady states exist: a regular MC, K^*_M, and two regular CMC steady states, $K^*_2 = \{K^*_H, K^*_L\}$ such that

$$\tilde{K}^*_\Sigma < K^*_H < \bar{K}^* < K^*_L,$$

where

$$\bar{K}^* = \bar{K}(\bar{\phi}) = \left(\frac{\alpha A}{\rho + \delta} \left(1 - \frac{1}{\bar{\sigma}} \right) \right)^{1/(1-\alpha)}.$$

As $m(K^*_H) > m(K^*_L)$, we call them *high-* and *low-markup* CMC steady states, respectively. At last, in the boundary cases, associated to either $\phi = \bar{\phi}$ or $\phi = \tilde{\phi}$, there are two steady states. In the first case there is regular MC, $K^*_1 = K^*_M$, and a CMC steady state, $K^*_2 = \bar{K}^*$, and, in the second case, there is a boundary, K^*_Σ, and a regular low-markup CMC steady state, $K^*_2 = K^*_L$. As we will see below, both cases are associated to local bifurcations, a smooth or classic one, in the first case, and a non-smooth, in the second. The steady state $\bar{K}^*$ occurs at the local maximum of the $R_2(K)$ function if the return function is globally non-decreasing.

In order to study the local stability and bifurcation properties, we build the generalized Jacobian in the sense of Clarke (see Clarke (1990) and Leine (2006))

$$J(K^*) = \begin{cases} J_1(K^*_1), & \text{if } (K^*_1, C^*_1) \in \mathcal{S}_1, \\ J(K^*_\Sigma), & \text{if } (K^*_\Sigma, C^*_\Sigma) \in \Sigma, \\ J_2(K^*_2), & \text{if } (K^*_2, C^*_2) \in \mathcal{S}_2. \end{cases}$$

If we have a regular steady state belonging to the interior of subset $\mathcal{S}_j$, $j = 1, 2$ the classic Jacobian is

$$J_j(K^*_j) = \begin{pmatrix} 0 & C_j(K^*_j) R'_j(K^*_j) \\ -1 & C'_j(K^*_j) \end{pmatrix}, K^*_j \in S_j,\ j = 1, 2. \tag{31}$$

where $C_j(K_j^*) \equiv Y_j(K_j^*) - \delta K_j^*$ for $\left(K_j^*, C_j^*\right) \in \mathcal{S}_j$, $j = 1, 2$. If we have a boundary steady state, $K_\Sigma^* = \tilde{K}$, the classic Jacobian does not exist, but we can determine the generalized differential of Clarke evaluated at that point as the convex hull of the derivatives at that point, which in our case becomes

$$J(K_\Sigma^*) = \{(1-q)J_1(K) + qJ_2(K) : 0 \leq q \leq 1,\ K = K_\Sigma^*\},$$

as the boundary steady state is completely characterized by the capital stock level K_Σ^*.

The eigenvalues of the Jacobians associated to regular steady states are

$$\lambda_j^- = \tfrac{C_j'(K_j^*)}{2} - \Delta(J_j(K_j^*))^{1/2}, \quad \lambda_j^+ = \tfrac{C_j'(K_j^*)}{2} + \Delta(J_j(K_j^*))^{1/2}, \text{ for } j = 1, 2,$$

where the discriminant is

$$\Delta(J_j) = \left(\frac{C_j'(K_j^*)}{2}\right)^2 - C_j(K_j^*)R_j'(K_j^*),\ j = 1, 2.$$

The eigenvalues are continuous functions of the parameters, in particular of ϕ and σ. If, as a result of a continuous change in a parameter, at least one eigenvalue associated to a regular steady state crosses the imaginary axis we say the equilibrium point (K^*, C^*) undergoes a *smooth bifurcation*[17].

Lemma 4. 1. *Assume there is a regular MC steady state* $(K_1^*, C_1^*) \in \mathcal{S}_1$. *Then it is saddle-point stable.*

2. *Assume there is a regular CMC steady state* $(K_2^*, C_2^*) \in \mathcal{S}_2$. *Then, if* $R_2'(K_2^*) < 0$ *it is saddle-point stable, if* $R_2'(K_2^*) > 0$ *it is a non-oscilatory unstable node, and if* $R_2'(K_2^*) = 0$ *it is a smooth fold bifurcation point.*

[17]For the existence of classic or smooth bifurcations in PWS dynamic systems see di Bernardo et al. (2008). Leine (2006) calls them continuous bifurcations.

Local dynamics and bifurcations in the neighborhood of boundary steady states for PWS continuous systems have been studied in the applied mathematics literature, though there are different approaches and nomenclatures[18]. di Bernardo et al. (2008) use the expression *discontinuity-induced bifurcations* for a generic change in structural stability in a PWS dynamic system, upon a change in a parameter. There is a *boundary-equilibrium bifurcation* at a boundary steady state, K^*_{Σ}, if for $\phi = \tilde{\phi}$ the Jacobians at the boundaries of the two contiguous branches, $\mathcal{S}_1$ and $\mathcal{S}_2$, verify $\det\left(J_j(K^*_{\Sigma}, \tilde{\phi})\right) \neq 0$, for $j = 1, 2$. Two types of boundary-equilibrium bifurcations may exist: *persistence*, if at the bifurcation point $(K^*_{\Sigma}, C^*_{\Sigma})$ a regular equilibrium in branch $\mathcal{S}_j$ is turned into a regular equilibrium in branch $\mathcal{S}_{-j}$ after a small variation in the bifurcating parameter, ϕ; or *non-smooth fold*, if there is a collision on the boundary of two regular steady states with different stability properties from both branches, and we observe their disappearance after a small change in the bifurcating parameter.

Lemma 5. *Let $(K^*_{\Sigma}, C^*_{\Sigma})$ be a boundary steady state. Then a boundary-equilibrium bifurcation occurs at $\phi = \tilde{\phi}$ if $\sigma \neq \bar{\sigma}$[19]. If $1 < \sigma < \bar{\sigma}$, there is a non-smooth fold bifurcation and if $\sigma > \bar{\sigma}$, there is persistence.*

We need to determine the eigenvalues of the Jacobian in order to have a specific characterization of the local dynamics at the boundary steady state. As the Jacobian $J(K^*_{\Sigma})$ is set-valued, so are its generalized eigenvalues,

$$\Lambda^{\pm}(K^*_{\Sigma}) = \left\{ \lambda^{\pm}_{\Sigma}(q) : 0 \leq q \leq 1 \right\},$$

where $\lambda^{\pm}_{\Sigma}(q)$ are the eigenvalues of the generalized Jacobian $J_{\Sigma}(q) \equiv J(K^*_{\Sigma}, q) = (1 - q)J_1(K^*_{\Sigma}) + qJ_2(K^*_{\Sigma})$. Observe that, although $\lambda^{\pm}_{\Sigma}(0) = \lambda^{\pm}_1(K^*_{\Sigma})$ and $\lambda^{\pm}_{\Sigma}(1) = \lambda^{\pm}_2(K^*_{\Sigma})$, we

[18]For two alternative approaches see Leine and Nijmeier (2004) or di Bernardo et al. (2008). On local-bifurcation analysis see Freire et al. (1998), Leine (2006), and the previous references.

[19]This is a co-dimension-one bifurcation. If $\phi = \tilde{\phi}$ and $\sigma = \bar{\sigma}$ a co-dimension-two type of bifurcation may occur, which seems to be new in the literature. It is a kind of non-smooth fold-fold bifurcation.

have $\lambda^{\pm}_{\Sigma}(q) \neq (1-q)\lambda^{\pm}_1(K^*_{\Sigma}) + q\lambda^{\pm}_2(K^*_{\Sigma})$.

Lemma 6. *The generalized eigenvalues $\Lambda^{\pm}(K^*_{\Sigma})$ are real. If $\sigma > \bar{\sigma}$ then $\Lambda^- \subset \mathbb{R}_{--}$ and $\Lambda^+ \subset \mathbb{R}_{++}$. If $1 < \sigma < \bar{\sigma}$ then $\Lambda^- \subset \mathbb{R}$ and $\Lambda^+ \subset \mathbb{R}_{++}$. In this case there is a value for q, $q_0 \equiv (\sigma - 1)/(\bar{\sigma} - 1)$ such that $\lambda^-_{\Sigma}(q^0)) = 0$.*

If we consider Lemmas 5 and 6 together we can conclude the following. If the elasticity σ is large and the bifurcation parameter ϕ is reduced continuously towards $\tilde{\phi}$, in the neighborhood of a boundary steady state, a saddle-point regular MC steady state is continued as a saddle point at the boundary-equilibrium bifurcation, and for further reductions, into a saddle-point regular CMC steady state. This is the meaning of persistence in our case. On the other hand, if the elasticity σ is low and ϕ is slightly larger than $\tilde{\phi}$ there will be two regular steady states, a saddle-point stable MC and an unstable CMC, which, upon reduction of the fixed cost will both collide at the boundary steady state and will both disappear for further reductions. This is the particular instance of a non-smooth fold bifurcation in our model. This type of behavior is generated by a discontinuity in the first derivatives which does not occur in smooth dynamic systems.

4.2. Local dynamics at the switching boundary

Candidate trajectories, including equilibrium trajectories presented in section 3, may contact the switching boundary in infinite time when there is a boundary steady-state equilibrium. If $\sigma > \bar{\sigma}$, equilibrium trajectories converge to it along the saddle paths passing through $(K^*_{\Sigma}, C^*_{\Sigma})$. Piecewise smoothness of the vector field implies that the slopes of the saddle paths are not collinear in both branches separated by Σ. If $\sigma < \bar{\sigma}$ equilibrium trajectories converge to the boundary equilibrium if $K(0) < K^*_{\Sigma}$ and they diverge from it, moving towards the low-markup CMC equilibrium if $K(0) > K^*_{\Sigma}$. This is the consequence of the existence of a non-smooth fold bifurcation.

Candidate trajectories, including equilibrium trajectories, may contact the switching boundary in finite time if a boundary steady-state equilibrium does not exist. Advances in the study of this behavior use the Filippov convex method, as we do below (see Filippov (1988), and Leine and Nijmeier (2004)).

Let us write compactly the vector fields in the two sub-domains as $f_j(K,C)$ for $(K,C) \in \mathcal{S}_j$ and $j=1,2$. Accordingly we will use the vector-field notation $\dot{C} = f_{j,C}(K,C)$, i.e. the RHS of equation (26), and $\dot{K} = f_{j,K}(K,C)$, i.e. the RHS of equation (27), again for $(K,C) \in \mathcal{S}_j$ with $j=1,2$. Consider a trajectory that reaches the switching boundary Σ at time t_Σ, $\Phi_j^c(t_\Sigma) \in \Sigma$, coming from branch $\mathcal{S}_j$ ($j=1,2$). Recalling the definition of Σ in equation (24), we know that $\Phi_{j,K}^c(t_\Sigma) = \tilde{K}$. Let us define the function $h(K,C) \equiv m(K) - 1/\sigma$. The behavior of the vector field $f_j(\cdot)$, as regards the normal to Σ, can be obtained from the following two functions $v_j(K,C)$ and $a_j(K,C)$:

$$v_j(K,C) = \frac{\partial}{\partial t} h\left(\Phi_j^c(0)\right) \; = \mathcal{L}_{f_j} h(K,C), \; j=1,2, \tag{32}$$

and

$$a_j(K,C) = \frac{\partial^2}{\partial t^2} h\left(\Phi_j^c(0)\right) = \mathcal{L}_{f_j}^2 h(K,C), \; j=1,2, \tag{33}$$

where $\mathcal{L}_{f_j} h$ and $\mathcal{L}_{f_j}^2 h$ are the first and second Lie derivatives of function $h(K,C)$ evaluated at $(K,C) \in \Sigma$, along the projections of vector field $f_j(\cdot)$ into the normal of $h(K,C)=0$:

$$\mathcal{L}_{f_j} h(K,C) = \frac{\partial h}{\partial C} f_{j,C} + \frac{\partial h}{\partial K} f_{j,K}, \tag{34}$$

and

$$\mathcal{L}_{f_j}^2 h(K,C) = \frac{\partial}{\partial C} \left(\frac{\partial h}{\partial C} f_{j,C} + \frac{\partial h}{\partial K} f_{j,K} \right) f_{j,C} + \frac{\partial}{\partial K} \left(\frac{\partial h}{\partial C} f_{j,C} + \frac{\partial h}{\partial K} f_{j,K} \right) f_{j,K}. \tag{35}$$

The literature on PWS continuous systems tells us that there are two main types of contact of trajectories originated in branch $\mathcal{S}_j$ with the switching boundary, when no boundary equilibrium points exist: traversing or grazing trajectories. This allows a partition of the switching boundary into a *traverse subset*,

$$\Sigma^t \equiv \{ \; (K,C) \in \Sigma : v_1(K,C) v_2(K,C) > 0 \},$$

and a subset in which grazing is observed inside $\mathcal{S}_j$

$$\Sigma_j^g \equiv \{\ (K,C) \in \Sigma : v_j(K,C) = 0,\ a_j(K,C) \neq 0\}\,.$$

The full *grazing subset* is defined as $\Sigma^g = \Sigma_1^g \cup \Sigma_2^g$.

Given our definitions of sets $\mathcal{S}_j$ in equations (23) and (25), and the fact that $h(K,C) > 0$ ($h(K,C) < 0$) for $K < \tilde{K}$ ($K > \tilde{K}$), if $\mathcal{L}_{f_j} h(K,C) > 0$ the trajectory crosses Σ in the direction of a decreasing K and if $\mathcal{L}_{f_j} h(K,C) < 0$ it crosses the boundary in the direction of an increasing K. Also, if a grazing point exists, the structure of $h(\cdot)$ implies that, for $a_j(K,C) > 0$, grazing takes place from side $\mathcal{S}_1$.

Lemma 7. *Let $\tilde{C} = Y_1(\tilde{K}) - \delta\tilde{K} = Y_2(\tilde{K}) - \delta\tilde{K}$. Then:*

1. $\Sigma^t = \{(\tilde{K}, C) : C \neq \tilde{C}\}$*;*
2. *For any values of the parameters, if $\Phi_C^c(\tilde{K}, t_\Sigma) > \tilde{C}$ then candidate trajectories traverse Σ by decreasing K, i.e. $\partial\Phi_{j,K}^c(\tilde{K}, t_\Sigma)/\partial t < 0$, for $j = 1, 2$. If $\Phi_C^c(\tilde{K}, t_\Sigma) < \tilde{C}$ then candidate trajectories traverse Σ by increasing K, i.e. $\partial\Phi_{j,K}^c(\tilde{K}, t_\Sigma)/\partial t > 0$, for $j =$ $1, 2$.*
3. *If $\phi < \tilde{\phi}$ then $\Sigma^g = \Sigma_1^g = \{(\tilde{K}, \tilde{C})\}$ and Σ_2^g is empty. If $\phi > \tilde{\phi}$ then $\Sigma^g = \Sigma_2^g = \{(\tilde{K}, \tilde{C})\}$ and Σ_1^g is empty.*

From Lemma 7 we conclude that grazing is unilateral. For $\phi < \tilde{\phi}$, there is a candidate trajectory which grazes Σ within $\mathcal{S}_1$, i.e. $\Phi_{1,C}^c(\tilde{K}, t_\Sigma) = \tilde{C}$. For $\phi > \tilde{\phi}$, there is a candidate trajectory which grazes Σ within $\mathcal{S}_2$, i.e. $\Phi_{2,C}^c(\tilde{K}, t_\Sigma) = \tilde{C}$. di Bernardo et al. (2008) call a point like $(\tilde{K}, \tilde{C})$ a *regular grazing point.*

We call *grazing trajectory* to a candidate trajectory that is tangent to Σ. If grazing happens to trajectories inside $\mathcal{S}_j$, then a grazing trajectory separates $\mathcal{S}_j$ into a subset in which trajectories departing from inside of it approach Σ, but do not collide with the boundary and curl back into $\mathcal{S}_j$, from a subset in which trajectories starting within traverse the boundary Σ to branch $\mathcal{S}_{-j}$. Traversing candidate trajectories close to the grazing trajectory may exhibit two types of behavior: either they curl back to Σ and traverse it returning to $\mathcal{S}_j$, or they continue only at branch $\mathcal{S}_{-j}$. Then there is a separating trajectory, that tends to converge to an equilibrium point in $\mathcal{S}_j$. In fact, this trajectory is the GE trajectory and belongs to the stable manifold associated to a stationary equilibrium point. From this, we conclude that grazing trajectories are candidate trajectories but are not GE trajectories.

5. Global General-Equilibrium Dynamics

Apparently, explicit solutions for equilibrium (and candidate) trajectories do not exist, so we have to resort to qualitative analysis of the equilibrium dynamics by drawing on the results from the two previous sections.

There are two main results in terms of global GE dynamics. First, there is the possibility of transient or long-run change in regime from MC to CMC, depending on the relationship between the fixed cost and the initial level of the capital stock to the remaining parameters in the model. Second, there is the dependence of the structural dynamics on the curvature properties of the return function, which is related to the elasticity of substitution amongst varieties.

Next we present the most representative GE trajectories using phase diagrams for the cases where there is either a unique steady-state equilibrium or multiple long-run equilibria.

Proposition 2. *Let $\phi > \max\{\bar{\phi}, \tilde{\phi}\}$, for any $\sigma > 1$, and assume that $K(0) > 0$. Then there is a unique saddle-point stable MC stationary equilibrium (K^*_M, C^*_M). The GE path $[\Phi(t, K(0))]_{t\geq 0}$, where $\Phi(t) = (\Phi_K(t), \Phi_C(t))^\top$, converges asymptotically to (K^*_M, C^*_M), independently of the regime associated to $K(0)$ at time $t = 0$. In particular, if $K(0) > \tilde{K}$ then both capital and consumption adjust downwards, the convergence is PWS continuous, the equilibrium path verifies $\Phi_K(t_\Sigma, K(0)) = \tilde{K}$, where $t_\Sigma > 0$ is the time of collision with the boundary Σ, and $\Phi^g_C(t, K(0)) < \Phi_C(t, K(0)) < \beta\Phi_K(t, K(0))$ for $0 < t \leq t_\Sigma$, where $[\Phi^g(t)]$ is the grazing trajectory, which grazes Σ from $\mathcal{S}_2$ at point $(\tilde{K}, \tilde{C})$.*

Figure A.5 depicts the complete phase diagram associated to Proposition 2. If the initial capital stock satisfies $0 < K(0) < \tilde{K}$ then both the initial and the stationary state of the economy exhibit a MC regime and the markup is constant, i.e. it is independent of the transitional dynamics. The dynamics is smooth in this region ($\mathcal{S}_1$) and it depends on the relative position of $K(0)$ regarding K_M^*. If $K(0)$ is greater (smaller) than K_M^*, then there is a downward (upward) adjustment in both variables. If the initial capital stock is above the switching boundary, i.e. if $K(0) > \tilde{K}$, then the economy starts from a CMC regime and it shifts to the MC regime along the transition path at time $t = t_\Sigma$. In the beginning of the adjustment the markup is endogenous and counter-cyclical: both output and consumption decrease while the markup adjusts upwards. At time t_Σ there is exactly one firm per industry, as the economy switches from a CMC to a MC regime. At this point, the markup becomes exogenous and constant while the mass of the continuum of varieties shrinks continuously along the convergence to the stationary equilibrium (K_M^*, C_M^*). The GE equilibrium path lies along the stable manifold associated with MC equilibrium, represented by $W_M^s = W^s(K_M^*, C_M^*)$, which belongs to both branches $\mathcal{S}_1$ and $\mathcal{S}_2$, and is therefore non-smooth. Notice that for $K(0) > K_M^*$ the stable manifold lies between the schedule $C = \beta K$ and the PWS isocline $\dot{K} = 0$.

`Figure A.5 around here`

Proposition 3. *Let $\phi < \min\{\bar{\phi}, \tilde{\phi}\}$, for any $\sigma > 1$, and assume that $K(0) > 0$. Then there is a unique saddle-point stable CMC stationary equilibrium (K_L^*, C_L^*). The GE path $[\Phi(t, K(0))]_{t \geq 0}$, where $\Phi(t)$ converges asymptotically to (K_L^*, C_L^*), independently of the regime associated to $K(0)$ at time $t = 0$. In particular, if $K(0) < \tilde{K}$ then both capital and consumption adjust upwards, the convergence is PWS continuous, the equilibrium path verifies $\Phi_K(t_\Sigma, K(0)) = \tilde{K}$, where $t_\Sigma > 0$ is the time of collision with the boundary Σ, and $\Phi_C^g(t, K(0)) > \Phi_C(t, K(0)) > \beta\Phi_K(t, K(0))$ for $0 < t \leq t_\Sigma$, where $[\Phi^g(t)]$ is the grazing trajectory, which grazes Σ from $\mathcal{S}_1$ at point $(\tilde{K}, \tilde{C})$.*

Figure A.6 depicts the complete phase diagram associated to Proposition 3. The interpretation is similar to that of Figure A.5. The GE equilibrium path lies along the stable manifold associated with the low-markup CMC equilibrium, $W_L^s = W^s(K_L^*, C_L^*)$, which also belongs to both branches $\mathcal{S}_1$ and $\mathcal{S}_2$, and is non-smooth. In this case, if the initial state of the economy is in the MC regime and the initial capital stock lies in the interval $\left(0, \tilde{K}\right)$, then it increases in the transition and there is a switch in the regime along the way. In the beginning, the markup is uncorrelated with economic activity, but then intra-industrial entry occurs as soon as the capital stock reaches $\tilde{K}$. From that point onwards competition will drive the markup counter-cyclically down.

`Figure A.6 around here`

Multiple steady-state equilibria exist in the remaining cases. Here we deal with the generic case in which $1 < \sigma < \bar{\sigma}$ and $\tilde{\phi} < \phi < \bar{\phi}$. From Proposition 1, we already know that there are three steady-state equilibria, (K_M^*, C_M^*), (K_H^*, C_H^*), and (K_L^*, C_L^*), which are collinear, i.e. $C_j^* = \beta K_j^*$, $j = M, H, L$, with $K_M^* < K_H^* < K_L^*$.

As we will show next, two heteroclinic orbits exist. First, there is a smooth heteroclinic orbit joining the two CMC stationary equilibria (K_H^*, C_H^*) and (K_L^*, C_L^*) denoted by Γ_{HL}. This heteroclinic orbit lives in subset $\mathcal{S}_2$ and coincides with the intersection between the unstable manifold associated to (K_H^*, C_H^*), W_H^u, and the stable manifold associated to (K_L^*, C_L^*), W_L^s, i.e. $\Gamma_{HL} = W_H^u \cap W_L^s$. There is also a second, PWS heteroclinic orbit, Γ_{HM}, joining the CMC equilibrium $(K_H^*, C_H^*) \in \mathcal{S}_2$ to the MC equilibrium $(K_M^*, C_M^*) \in \mathcal{S}_1$. Again, the heteroclinic orbit is defined as the intersection between the unstable manifold associated to (K_H^*, C_H^*), W_H^u, and the stable manifold associated to (K_M^*, C_M^*), W_M^s. Therefore, $\Gamma_{HM} = W_H^u \cap W_M^s$ is continuous, but PWS since it has a discontinuity in its derivatives at Σ.

Proposition 4. *Assume that $1 < \sigma < \bar{\sigma}$ and $\tilde{\phi} < \phi < \bar{\phi}$, and also that $K(0) > 0$. Then there are three stationary equilibria: a MC saddle-point stable equilibrium, (K_M^*, C_M^*), a* *high-markup CMC unstable equilibrium,* (K_H^*, C_H^*)*, and a low-markup saddle-point stable CMC equilibrium,* (K_L^*, C_L^*)*, such that* $K_L^* > K_H^* > K_M^*$ *and* $C_j^* = \beta K_j^*$*, with* $j = L, H, M$*. If* $K(0) > K_L^*$ *then the equilibrium path* $[\Phi(t, K(0)]_{t \geq 0}$ *adjusts downwards towards the low-markup equilibrium* (K_L^*, C_L^*)*. If* $K(0) < K_M^*$ *then the equilibrium path converges upwards to the MC equilibrium* (K_M^*, C_M^*)*. If* $K_H^* < K(0) < K_L^*$ *then the equilibrium path converges upwards to* (K_L^*, C_L^*) *along the smooth heteroclinic trajectory* Γ_{HL}*. If* $K_M^* < K(0) < K_H^*$ *then the economy converges downwards to* (K_M^*, C_M^*) *along the PWS heteroclinic trajectory* Γ_{HM}*.*

`Figure A.7 around here`

Figure A.7 illustrates Proposition 4. The equilibrium trajectory belongs to the stable manifolds associated to (K_M^*, C_M^*) and (K_L^*, C_L^*), W_M^s and W_L^s. Both these manifolds are obtained as the union of two branches and exhibit a common point at the unstable steady-state equilibrium (K_H^*, C_H^*). The branches that start in this long-run equilibrium feature heteroclinic trajectories Γ_{HM} and Γ_{HL}. The necessity of the existence of heteroclinic orbits, which have a global nature, is obvious: the basins of attraction of the two saddle-point stable equilibria should be bounded.

Equilibrium paths are "trapped" by the two manifolds defined by curves $C = \beta K$ and $C = C(K)$, i.e. the isocline $\dot{K} = 0$. Geometrically, if the latter is above (below) the former then both capital and consumption increase (decrease).

Depending on the initial level of the capital stock, rational agents "choose" a MC equilibrium (K_M^*, C_M^*) or a CMC equilibrium with a low markup (K_L^*, C_L^*). The dividing barrier is given by the CMC equilibrium with a high markup (K_H^*, C_H^*). Off course, the economy may be trapped in the high-markup CMC equilibrium, which is totally unstable. A small shift in any parameter forces the economy to converge to one of the two saddle-point stable equilibria. In some sense there is a kind of juxtaposition of the two previous cases with unique stationary equilibria if the economy is located in one of the sides of the barrier. Nonetheless, there is an important difference: there are restrictions on the possibility of regime shifts along transition paths. If the economy starts from a MC equilibrium it never converges to a CMC stationary equilibrium. However, the converse is not true: if the initial stock of capital associated with a CMC dynamics is such that $\tilde{K} < K(0) < K_H^*$ then the economy converges to a MC equilibrium. This asymmetry is related to the fact that in this case there is a small elasticity of substitution in the demand for intermediate goods, which can be interpreted as a case where there is an overall low level of flexibility in the economy.

In all the cases in Propositions 2, 3, and 4 the equilibrium is determinate, and it is Markovian (in the sense of Santos (2002)), even for the cases where the function is non-monotonic, similarly to Skiba (1978) and Dechert and Nishimura (1983). However, to our knowledge, this type of global dynamics arising from the change in regime associated to PWS dynamics, has not been analyzed in the macroeconomics literature. The regime switch along the path, linked to the non-smooth dynamics, offers a novel vision on the interaction between the industrial structure of economies and their macroeconomic equilibrium. Reasonably competitive economies may end up in a bad equilibrium if the fixed costs faced by its firms are high. Conversely, economies showing a poor competitive performance may end up in a good (second-best) equilibrium if increasing returns are not substantial.

6. Technology and regime switching

One interesting possibility in this model is the effect of a permanent technological shock, i.e. a permanent increase in A, on the long-run regime of the model. Let us start with the simplest case and assume $(\sigma, \phi) \in \mathcal{A}_2$. The initial long-run equilibrium is depicted by point E_0 in Figure A.8. Now, assume there is a permanent technology shock that sets a new value $A = A_1 > A_0$. Considering equation (17), then we know that the value for $\tilde{K}$ decreases to $\tilde{K}(A_1)$.

Assume also that $K_0^* > \tilde{K}(A_1)$, i.e. the shock is large enough to make the economy switch from a MC to a CMC regime. Since we know that, for the CMC region, we have $R(K) = (1 - m(K)) F'(K)$, an increase in A will lead to a upward shift in the $R(\cdot)$ schedule, as $\partial m/\partial A < 0$ and $\partial F'/\partial A > 0$. Following the shock, the economy moves to point A and eventually to a new long-run equilibrium like point E_1 in Figure A.8 [20].

`Figure A.8 around here`

Thus, we can conclude that a small permanent technology shock hitting an economy in a unique stable MC equilibrium close to the switching boundary may have substantial long-run effects since the economy shifts to a CMC regime with a lower equilibrium markup. Of course a negative shock can have the opposite effect. Notice this means the economy is no longer in region $\mathcal{A}_2$, as it moved to $\mathcal{C}_2$.

The situation is even more interesting when $(\sigma, \phi) \in \mathcal{B}$ and we have three equilibria. With a similar shock we may have the case depicted in Figure A.9.

`Figure A.9 around here`

In this case the economy crosses the bifurcation to the $\mathcal{A}_1$ region and the new short-run equilibrium would be given by point A. Considering a permanent shock, there would be new low-markup CMC equilibrium for the long run given by point E_P.

Interestingly, even if the shock is temporary, the economy may not go back to E_0. Assume the shock vanishes at $t = \tau > 0$, when the economy is at point B. As long as it accumulated enough capital in order to have $K(\tau) > K_H^*$ then it jumps to point C at time τ and it converges to the new steady state E_1.

[20] The increase in the value of K^* is unambiguous since $R(\cdot)$ is always non-increasing in K.

In the three-equilibria case, starting from the unstable equilibrium, any shock - no matter how small or temporary - will be sufficient to move the economy towards one of the two stable equilibria. Thus, even temporary shocks can have permanent effects in this economy with multiple equilibria.

7. Conclusion

When we analyze dynamic systems, we may encounter structural changes associated to non-smoothness, such as discontinuities in derivatives. We find exactly such non-smooth dynamics in a simple dynamic general equilibrium model with Cournotian Monopolistic Competition and instantaneous free entry. The model endogenously generates two regimes with different economic and dynamic features: i) a stable monopoly regime associated with very high markup levels and low welfare and ii) an oligopoly regime that may produce one or two equilibria, a stable low-markup and an unstable high-markup one, where the latter works as a threshold between regimes. The two regimes have different dynamics and there is a switching boundary between them. We provide a rigorous study of the non-smooth dynamics and we also analyze the global dynamics of the model, demonstrating that it may exhibit robust heteroclinic orbits, either of the smooth or the non-smooth type. We show that two economies exhibiting the same fundamental parameters may behave very differently, as they may be in two different regimes. Additionally, it is possible for one economy to change regime along its convergence to a stable long-run equilibrium. Fixed costs and elasticities of demand play a crucial role in this model: changing their values may alter the dynamics in a radical way, either by inducing a discontinuous transition or a discontinuous hysteresis. In economic terms, we have the possibility of one, two or three steady-state equilibria, connected by heteroclinic orbits. When there is a unique equilibrium, we have two possible types (both saddle-point stable): one monopolistic with a high and constant markup, and the other oligopolistic with a low and endogenous countercyclical markup. When there are three equilibria, there can be a "threshold" effect, with low initial levels of capital stock leading to the low-output monopolistic equilibrium and with high levels leading to the high-output oligopolistic equilibrium.

In this paper we have assumed that there is an exogenous labor supply: we leave it to further work to see how the dynamics become even richer with an endogenous labor supply. We would also like to relax the assumption of instantaneous free entry, allowing for the flow of entry to be determined by an endogenous cost of entry as in Brito and Dixon (2009): there will be two state variables (the number of firms and the capital stock) driving the markup. This would not affect the steady-state equilibria (since in steady state the cost of entry is zero), but would influence the dynamics out of steady state.

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Appendix A. Proofs

Proof of Lemma 1. Considering $F(K)$ is continuous and smooth, and taking into account equations (18) and (19), then $Y(K)$ and $R(K)$ are continuous, but are only smooth in the interior the subsets $(0, \tilde{K})$ or $(\tilde{K}, \infty)$. Both functions are also continuous at the boundary value $\tilde{K}$, as $Y_1(\tilde{K}) = Y_2(\tilde{K})$ and $R_1(\tilde{K}) = R_2(\tilde{K})$. However, they are not differentiable in the classical sense since $Y_1'(\tilde{K}) \neq Y_2'(\tilde{K})$ and $R_1'(\tilde{K}) \neq R_2'(\tilde{K})$ as we demonstrate below. Following Leine and Nijmeier (2004) we use the generalized derivative of Clarke (1990), evaluated at $K = \tilde{K}$, denoted here as $\partial Y(\tilde{K})$ and $\partial R(\tilde{K})$. Therefore the generalized gradients $\partial Y(K)$ and $\partial R(K)$, as in equations (20) and (21) are set-valued functions. We obtain $Y_1'(K) = (1 - \frac{1}{\sigma})F'(K)$ and $Y_2'(K) = (1 - m(K)/2)\, F'(K)$ for non-boundary K and, for $K = \tilde{K}$ we have

$$0 < Y_1'(\tilde{K}) = \left(1 - \frac{1}{\sigma}\right) F'(\tilde{K}) < Y_2'(\tilde{K}) = \left(1 - \frac{1}{2\sigma}\right) F'(\tilde{K})$$

implying that the generalized derivative at $\tilde{K}$ is $\partial Y(\tilde{K}) = \left[\, Y_1'(\tilde{K}), Y_2'(\tilde{K})\right] \subset \mathbb{R}_{++}$. We derive $R_1'(K) = (1 - \frac{1}{\sigma})F''(K)$ and $R_2'(K) = (1 - m(K))\, F''(K) - m'(K)F'(K)$ where $m'(K) = -m(K)F'(K)/(2F(K)) < 0$. Then, $R_2'(K) = (1 - m(K))\, F''(K) + m(K)\frac{(F'(K))^2}{2F(K)}$, in general and for the Cobb-Douglas production function we have

$$R_2'(K) = -\frac{(2-\alpha)}{2} \left(\frac{2(1-\alpha)}{2-\alpha} - m(K)\right) \alpha A K^{\alpha-2}. \tag{A.1}$$

For boundary $K = \tilde{K}$ we obtain

$$R_1'(\tilde{K}) = \left(1 - \frac{1}{\sigma}\right) F''(\tilde{K}) < R_2'(\tilde{K}) = \left(1 - \frac{1}{\sigma}\right) F''(\tilde{K}) + \frac{(F'(\tilde{K}))^2}{2\sigma F(\tilde{K})},$$

in general, and for Cobb-Douglas function we have

$$R_1'(\tilde{K}) = \left(1 - \frac{1}{\sigma}\right) \alpha(\alpha-1) A\tilde{K}^{\alpha-2} < 0,\ R_2'(\tilde{K}) = -\frac{(2-\alpha)}{2} \left(\frac{2(1-\alpha)}{2-\alpha} - \frac{1}{\sigma}\right) \alpha A \tilde{K}^{\alpha-2},$$

which has an ambiguous sign. Then the generalized gradient is $\partial R(\tilde{K}) = \left[\, R_1'(\tilde{K}),\ R_2'(\tilde{K})\right] \subset \mathbb{R}$ has a positive measure for any $\tilde{K}$. $\square$

Proof of Lemma 2. The generalized second derivative for the production function (18) is

$$\partial^2 Y(K) = \begin{cases} Y_1^{''}(K) & \text{if } 0 \leq K < \tilde{K}, \\ \partial^2 Y(\tilde{K}) & \text{if } K = \tilde{K}, \\ Y_2^{''}(K) & \text{if } K > \tilde{K}, \end{cases} \tag{A.2}$$

where $\partial^2 Y(\tilde{K})$ is the convex hull of $Y_1^{''}(K)$ and $Y_2^{''}(K)$ at $K = \tilde{K}$, $\partial^2 Y(\tilde{K}) = \{ (1 - q)Y_1^{''}(\tilde{K}) + qY_2^{''}(\tilde{K}) : 0 \leq q \leq 1\}$. For the Cobb-Douglas function we have $Y_1^{''}(K) = -(1-\alpha)(1-1/\sigma)F^{'}(K)/K < 0$ and

$$Y_2^{'}(K) = -\frac{(2-\alpha)}{2}\left(\frac{2(1-\alpha)}{2-\alpha} - \frac{m(K)}{2}\right)\frac{F^{'}(K)}{K}.$$

where $F^{'}(K)/K = \alpha A K^{\alpha-2}$. Now, recall the (generalized) differential of $R(K)$ in equation (21) and its properties from Lemma 1. Let us consider separately the two sub-domains divided by $\tilde{K}$. For the first sub-domain, where $K \in (0, \tilde{K})$, we have $R_1^{'}(K) = Y_1^{''}(K) < 0$ for all possible values for the parameters. In the second sub-domain, where $K > \tilde{K}$, we already know that $m(K) < 1/\sigma$. If we define $\bar{m} = 1/\bar{\sigma}$, we conclude from equation (A.1) that $R_2^{'}(K)$ has the same sign as $m(K) - \bar{m}$ and from equation (A.2) that $Y_2^{''}(K)$ has the same sign as $m(K) - 2\bar{m}$. Therefore, several cases can occur: First, if $\sigma > \bar{\sigma}$ then $m(K) < 1/\sigma < 1/\bar{\sigma} < 2/\bar{\sigma}$ which implies $R_2^{'}(K) < 0$ and $Y_2^{''}(K) < 0$ for all $K > \tilde{K}$. Second, if $\sigma < \bar{\sigma}$ then $\bar{m} = 1/\bar{\sigma} < 1/\sigma$ and the relationship between $m(K)$ and $\bar{m}$ is ambiguous. However, the critical level for the capital stock $\bar{K}$ is such that $m(\bar{K}) = 1/\sigma$. Therefore $R_2^{'}(K) \gtreqless 0$ if and only if $m(K) \lesseqgtr \bar{m}$, but $m(K) \lesseqgtr \bar{m}$ if and only if $\tilde{K} < K \lesseqgtr \bar{K}$. In the same sub-domain, we also have $Y_2^{''}(K) \gtreqless 0$ if and only if $m(K) \lesseqgtr 2\bar{m}$. But $m(K) \lesseqgtr 2\bar{m}$ if and only if $\tilde{K} < K \lesseqgtr \left(\frac{2\bar{\sigma}}{\sigma}\right)^{2/\alpha}\tilde{K}$. However, the latter holds only if $1 < \sigma < \bar{\sigma}/2 = (2-\alpha)/4(1-\alpha)$, which is only possible for $\alpha > 3/4$. □

Proof of Lemma 3. Equation (28) implies that steady states of system (26)-(27) are fully characterized by steady states of the capital stock, formally determined as $K^* = \{ K \in \mathbb{R}_+ : R(K) = \rho + \delta\}$. From the definition and the properties of the return function, in equation

(19) and Lemma 2, and from its depiction in Figures A.1 and A.2, we can readily see that the following cases may occur:

1. If $\sigma \geq \bar{\sigma}$ the return function $R(K)$ is monotonically decreasing and PWS concave (see Lemma 1), and we have three possible cases:
   (a) if $\rho + \delta < R_1(\tilde{K}) = R_2(\tilde{K})$, there is a unique regular equilibrium $(K_1^*, C_1^*) \in \mathcal{S}_1$ given by the equilibrium condition $R_1(K) = \rho + \delta$. This case occurs if and only if $\phi > \tilde{\phi}$ where $\tilde{\phi}$ is defined in equation (29);
   (b) if $\rho + \delta = R_1(\tilde{K}) = R_2(\tilde{K}) = R(\tilde{K})$, there is a unique boundary equilibrium $(K_\Sigma^*, C_\Sigma^*) \in \Sigma$. This holds if and only if $\phi = \tilde{\phi}$;
   (c) if $\rho + \delta > R_1(\tilde{K}) = R_2(\tilde{K})$, there is a unique regular equilibrium $(K_2^*, C_2^*) \in \mathcal{S}_2$ given by the equilibrium condition $R_2(K) = \rho + \delta$. This case occurs if and only if $\phi < \tilde{\phi}$.

2. If $1 < \sigma < \bar{\sigma}$ we proved in Lemma 1 that the return function is neither monotonically decreasing nor globally concave. Additionally, we proved that it is decreasing for $K \in (0, \tilde{K}) \cup (\bar{K}, \infty)$, it is increasing for $K \in (\tilde{K}, \bar{K})$, and it reaches a local maximum at $K = \bar{K}$. At point $\tilde{K}$, in addition to the generalized derivative, there is the convex hull of slopes ranging to negative (at the left) to positive (at the right). Those properties imply that $R_2(\bar{K}) > R_1(\tilde{K}) = R_2(\tilde{K})$. Then, five cases are possible:
   (a) for $\rho + \delta > R_2(\bar{K})$, there is one regular steady state $(K_1^*, C_1^*) \in \mathcal{S}_1$ given by the equilibrium condition $R_1(K) = \rho + \delta$. This case occurs if and only if $\phi > \bar{\phi}$;
   (b) for $\rho + \delta = R_2(\bar{K})$, there are two regular steady states: $(K_1^*, C_1^*) \in \mathcal{S}_1$, such that $R_1(K_1^*) = \rho + \delta$, and $\left(\bar{K}^*, \overline{C}^*\right) \in \mathcal{S}_2$, such that $R_2(K) = \rho + \delta$ and $R_2^{'}(K) = 0$. This case occurs if and only if $\phi = \bar{\phi}$;
   (c) for $R(\tilde{K}) < \rho + \delta < R_2(\bar{K})$, there are three regular steady states: $(K_1^*, C_1^*) \in \mathcal{S}_1$, such that $R_1(K_1^*) = \rho + \delta$ and a pair of $(K_2^*, C_2^*) \in \mathcal{S}_2$ such that $K_2^* = \{\ K_H^*, K_L^*\}$ with $\tilde{K} < K_M^* < \bar{K} < K_L^*$, considering the condition $R_2(K) = \rho + \delta$ holds for two values of $K \in \left(\bar{K}, +\infty\right)$. This case occurs if and only if $\bar{\phi} < \phi < \tilde{\phi}$;

(d) for $\rho+\delta = R(\tilde{K})$, once again two steady states exist: a boundary steady state $(K^*_\Sigma, C^*_\Sigma) \in \Sigma$ and a regular steady state $(K^*_2, C^*_2) \in \mathcal{S}_2$, such that $K^*_2 > \bar{K}$. This case holds if and only if $\phi = \tilde{\phi}$;

(e) for $\rho+\delta < R(\tilde{K})$, there is again an isolated regular steady state $(K^*_2, C^*_2) \in \mathcal{S}_2$ and the equilibrium condition is given by $R_2(K) = \rho+\delta$. This case occurs if and only if $\phi < \tilde{\phi}$.

□

Proof of Lemma 4. The classic Jacobian in equation (31), associated to a regular steady state, has trace and determinant given by

$$\begin{aligned}
\mathrm{tr}(J_j(K^*_j)) &= C'_j(K^*_j),\ \left(K^*_j, C^*_j\right) \in \mathcal{S}_j,\ j=1,2,\\
\det(J_j(K^*_j)) &= C_j(K^*_j)R'_j(K^*_j),\ \left(K^*_j, C^*_j\right) \in \mathcal{S}_j,\ j=1,2.
\end{aligned}$$

If the steady-state equilibrium point (K^*_1, C^*_1) belongs to branch $\mathcal{S}_1$, then we obtain $\mathrm{tr}(J_1(K^*_1)) = C'_1(K^*_1) = Y'_1(K^*_1) - \delta = \rho > 0$ and $\det(J_1(K^*_1)) = C_1(K^*_1)R'_1(K^*_1) = -(1-\alpha)(\rho+\delta)\beta < 0$. Therefore, the eigenvalues of $J_1(K^*_1)$ are both real and given by $\lambda^-_1 < 0 < \lambda^+_1$. If the steady-state equilibrium point (K^*_2, C^*_2) belongs to branch $\mathcal{S}_2$ then

$$\begin{aligned}
C_2(K^*_2) &= Y_2(K^*_2) - \delta K^*_2 = (1-m(K^*_2))F(K^*_2) - \delta K^*_2 =\\
&= \left((1-m(K^*_2))\frac{F'(K^*_2)}{\alpha} - \delta\right) K^*_2 = \beta K^*_2,
\end{aligned}$$

and

$$\begin{aligned}
R'_2(K^*_2)K^*_2 &= -\left[\ 1-\alpha-(2-\alpha)\frac{m(K^*_2)}{2}\ \right]\ F'(K^*_2) =\\
&= -(1-m(K^*_2))F'(K^*_2) + \alpha\left(1 - \frac{m(K^*_2)}{2}\right) F'(K^*_2) =\\
&= -R_2(K^*_2) + \alpha\delta + \alpha C'_2(K^*_2) =\\
&= \alpha\left(C'_2(K^*_2) - \frac{C_2(K^*_2)}{K^*_2}\ \right) =\\
&= \alpha\left(C'_2(K^*_2) - \beta\right).
\end{aligned}$$

Thus

$$\mathrm{tr}(J_2(K_2^*)) = C_2'(K_2^*) = \beta + R_2'(K_2^*)K_2^*/\alpha,$$

and

$$\det(J_2(K_2^*)) = C_2(K_2^*)R_2'(K_2^*) = \beta R_2'(K_2^*)K_2^*.$$

If $R_2'(K_2^*) < 0$ then $\det(J_2(K_2^*)) < 0$ and the equilibrium point is saddle-point stable, i.e. $\lambda_2^- < 0 < \lambda_2^+$. If $R_2'(K_2^*) = 0$ then $\det(J_2(K_2^*)) = 0$ and $\lambda_2^- = 0 < \lambda_2^+ = \beta$. If $R_2'(K_2^*) > 0$ then $\mathrm{tr}(J_2(K_2^*)) > \beta > 0$ and $\det(J_2(K_2^*)) > 0$, and since

$$0 < \left(\frac{\beta - R_2'(K_2^*)K_2^*/\alpha}{2}\right)^2 < \Delta(J_2) < \left(\frac{\beta + R_2'(K_2^*)K_2^*/\alpha}{2}\right)^2,$$

then the eigenvalues are real and verify $\lambda_2^+ > \lambda_2^- > 0$. Considering that $1 < \sigma < \overline{\sigma}$ is a necessary condition to obtain $R_2'(K_2^*) = 0$ and it is verified if ϕ crosses continuously the value of $\overline{\phi}$, then the point K_2^* is a continuous bifurcation point - see Leine (2006). □

Proof of Lemma 5. If $K^* = \tilde{K}(\tilde{\phi}) = K_\Sigma^*$, from the expressions in the proof of Lemma 4 we obtain the determinants of the Jacobians evaluated at the closures of branches $\mathcal{S}_1$ and $\mathcal{S}_2$, $\det(J_1(K_\Sigma^*)) = -(1-\alpha)(\rho+\delta)\beta < 0$, and $\det(J_2(K_\Sigma^*)) = -\beta(1-\alpha)(\rho+\delta)(\sigma-\bar{\sigma})/(\sigma-1)$, which is different from zero for $\sigma \neq \bar{\sigma}$. If this condition holds then an equilibrium such that $K^* = \tilde{K}$, occurring for $\phi = \tilde{\phi}$ in which case $\tilde{K}(\tilde{\phi}) = K_\Sigma^*$, is a boundary-equilibrium bifurcation. To characterize the type of bifurcation we use the method in di Bernardo et al. (2008). Let us consider the system (26)-(27) and study a perturbation of the parameter ϕ about the value $\tilde{\phi}$ of the equilibrium away from K_Σ^* by linearizing it locally. If we write the system (26)-(27) compactly as

$$\dot{\mathbf{x}} = \begin{cases} f_1(\mathbf{x}, \phi) & \text{if } h(\mathbf{x}, \phi) > 0, \\ f_2(\mathbf{x}, \phi) & \text{if } h(\mathbf{x}, \phi) < 0, \end{cases}$$

where $\mathbf{x} = (C, K)^\top$. Let $\mathbf{x}^*$ be the boundary equilibrium for $\phi = \tilde{\phi}$. For any pair $(\mathbf{x}, \phi)$ we have $f_2(\mathbf{x}, \phi) - f_1(\mathbf{x}, \phi) = g(\mathbf{x}, \phi)h(\mathbf{x}, \phi)$. Those authors prove that, under the non-degeneracy conditions $\det(\frac{\partial}{\partial \mathbf{x}} f_1(\mathbf{x}^*, \tilde{\phi})) \neq \mathbf{0}$, $D_1 \equiv \frac{\partial}{\partial \phi} h(\mathbf{x}^*, \tilde{\phi}) - \frac{\partial}{\partial \phi} h(\mathbf{x}^*, \tilde{\phi})^\top \frac{\partial}{\partial \mathbf{x}} f_1(\mathbf{x}^*, \tilde{\phi})^{-1} \frac{\partial}{\partial \mu} f_1(\mathbf{x}^*, \tilde{\phi}) \neq 0$ and $D_2 \equiv 1 + \frac{\partial}{\partial \phi} h(\mathbf{x}^*, \tilde{\phi})^\top \frac{\partial}{\partial \mathbf{x}} f_1(\mathbf{x}^*, \tilde{\phi})^{-1} g(\mathbf{x}^*, \tilde{\phi})$, there is persistence at the boundary equilibrium bifurcation if $D_2 > 0$ and there is a non-smooth fold bifurcation if $D_2 < 0$. In our case, the non-degeneracy conditions hold at a boundary equilibrium bifurcation as $\det(J_2(K^*_\Sigma) \neq 0$, due to $D_1 = (2\sigma\tilde{\phi})^{-1} > 0$ and $D_2 = (\sigma - \bar{\sigma})/(\sigma - 1)$. Then $D_2 > 0$ ($D_2 < 0$) if $\sigma > \bar{\sigma}$ ($1 < \sigma < \bar{\sigma}$) and $D_2 \neq 0$ for a boundary-equilibrium bifurcation (of co-dimension-one). □

Proof of Lemma 6. At the boundary equilibrium K^*_Σ we have $0 < \mathrm{tr}(J_1(K^*_\Sigma) = \rho < \mathrm{tr}(J_2(K^*_\Sigma)) = \rho + (\rho + \delta)/(2(\sigma - 1))$ and $\det(J_2(K^*_\Sigma)) = \det(J_1(K^*_\Sigma))(\sigma - \bar{\sigma})(\sigma - 1)$ and $\det(J_1(K^*_\Sigma)) = -(1 - \alpha)(\rho + \delta)\beta < 0$. Therefore, $\mathrm{sign}\{\det(J_2(K^*_\Sigma))\} = \mathrm{sign}\{\bar{\sigma} - \sigma\}$. Then

$$\mathrm{tr}(J_\Sigma(q)) = \rho + \frac{q(\rho + \delta)}{2(\sigma - 1)},$$

and

$$\det(J_\Sigma(q)) = -\beta(1 - \alpha)(\rho + \delta)\left(1 - q\left(\frac{\bar{\sigma} - 1}{\sigma - 1}\right)\right).$$

Thus, $\mathrm{tr}(J_\Sigma(q)) > 0$ for any $0 \leq q \leq 1$ and $\det(J_\Sigma(q)) < 0$ if $\sigma > \bar{\sigma}$ for any q, and $\det(J_\Sigma(q)) \leq 0$ if $1 < \sigma < \bar{\sigma}$ and $0 \leq q \leq q_0$ where $q_0 \equiv (\sigma - 1)/(\bar{\sigma} - 1)$, and $\det(J_\Sigma(q)) > 0$ if $1 < \sigma < \bar{\sigma}$ and $q_0 < q \leq 1$. Then the trace and the determinant of the generalized Jacobian are the sets

$$\mathrm{tr}(J(K^*_\Sigma) = \left[\rho, \rho + \frac{\rho + \delta}{2(\sigma - 1)}\right] \subset \mathbb{R}_{++},$$

and

$$\det(J(K^*_\Sigma) = \left[\det(J_1(K^*_\Sigma)), \det(J_1(K^*_\Sigma)))\left(\frac{\sigma - \bar{\sigma}}{\sigma - 1}\right)\right],$$

is a subset of $\mathbb{R}_{--}$ if $\sigma > \bar{\sigma}$ and of $\mathbb{R}$ if $1 < \sigma < \bar{\sigma}$. The generalized eigenvalues are

$$\lambda^{\mp}_\Sigma(q) = \frac{\mathrm{tr}(J_\Sigma(q))}{2} \mp \Delta(J_\Sigma(q))^{1/2}, \ 0 \leq q \leq 1,$$

where the discriminant is

$$\Delta(J_\Sigma(q)) = \frac{1}{4}\left\{\left[\frac{q}{2}\left(\frac{\rho + \delta}{\sigma - 1}\right) + \rho - 2\alpha\beta\right]^2 + 4(1 - \alpha)\alpha\beta^2\right\},$$

is positive for any q. Therefore, all the generalized eigenvalues are real and Λ^+ is a subset of $\mathbb{R}_{++}$ and Λ^- is a subset of $\mathbb{R}_{--}$ if $\sigma > \bar{\sigma}$ and of $\mathbb{R}$ if $\sigma < \bar{\sigma}$. □

Proof of Lemma 7. If we evaluate equations (32) and (34) we obtain

$$v_1(\tilde{K}, C) = v_2(\tilde{K}, C) = -\frac{\alpha\left(\tilde{C} - C\right)}{\sigma\tilde{K}},$$

where $\frac{\partial}{\partial K} f_1(\tilde{K}) = \frac{\partial}{\partial K} f_2(\tilde{K}) = (1 - 1/\sigma)F(\tilde{K}) - \delta\tilde{K} - C = \tilde{C} - C$. Then $v_1(\tilde{K})v_2(\tilde{K}) = \left(\alpha\left(\tilde{C} - C\right)/(\sigma\tilde{K})\right)^2$ and $v_1(\tilde{K}, C)v_2(\tilde{K}, C) > 0$ if and only if $C \neq \tilde{C}$. We also obtain the projection of the two vector fields to the normal of h, $v_1(\tilde{K}, C) > 0$ ($v_1(\tilde{K}, C) < 0$) if and only if $C > \tilde{C}$ ($C < \tilde{C}$). The grazing subset of Σ possesses a unique element, a grazing point, at $C = \tilde{C}$. At this point we have $v_1(\tilde{C}, \tilde{K}) = v_2(\tilde{C}, \tilde{K}) = 0$. If we evaluate the second Lie derivative in (35), we obtain

$$a_1(\tilde{K}, \tilde{C}) = a_2(\tilde{K}, \tilde{C}) = \frac{\alpha(\rho + \delta)\tilde{C}}{2\sigma\tilde{K}} \left(\left(\frac{\tilde{\phi}}{\phi}\right)^{\frac{1-\alpha}{\alpha}} - 1\right).$$

Therefore, if $\phi < \tilde{\phi}$ ($\phi > \tilde{\phi}$) then $a_1(\tilde{K}, \tilde{C}) > 0$ ($a_1(\tilde{K}, \tilde{C}) < 0$) and grazing occurs from the side $\mathcal{S}_1$ ($\mathcal{S}_2$). □

Proof of Proposition 2. From Lemmas 3 and 4 we already know there is a unique equilibrium steady state $(K_M^*, C_M^*) \in \mathcal{S}_1$ that is saddle-point stable. The local dynamics for initial points belonging to set $\mathcal{S}_1$ is similar to the Ramsey model: the isocline $\dot{C} = 0$ is vertical, the isocline $\dot{K} = 0$, has slope $\frac{dC}{dK}\big|_{\dot{K}=0} = C_1'(K_M^*) = \mathrm{tr}(J_1(K_M^*)) = \rho$, and the steady-state equilibrium points are located along the equilibrium radius $C^* = \beta K^*$. The transversality condition (3) is met if the GE trajectories $\Phi(t)$ are tangent to the stable manifold passing through (K_M^*, C_M^*), $W_M^s = W^s(K_M^*, C_M^*)$. Locally, W_M^s is tangent to the linear subspace E_M^s, with slope $\frac{dC}{dK}\big|_{E_M^s} = \lambda_M^+ > 0$. It can be proved that $\rho < \lambda_M^+ < \beta$, which implies that the stable manifold W_M^s has a slope asymptotically verifying the following inequalities

$$0 < \left.\frac{dC(K_M^*)}{dK}\right|_{\dot{K}=0} < \left.\frac{dC(K_M^*)}{dK}\right|_{E_M^s} < \left.\frac{dC(K_M^*)}{dK}\right|_{C=\beta K}.$$

Then, equilibrium trajectories are unique, positively slopped, and are located between the equilibrium radius $C = \beta K$ and the isocline $\dot{K} > 0$. One implication from this results is that if $K(0) = \tilde{K}$ then there is a unique equilibrium value for consumption $\Phi_C(0, \tilde{K})$. It belongs to stable manifold, and therefore, should verify $\beta\tilde{K} > \Phi_C(0, \tilde{K}) > (1 - 1/\sigma)F(\tilde{K}) - \delta\tilde{K}$. If the initial point belongs to subset $\mathcal{S}_2$, $K(0) > \tilde{K}$, and the proof involves non-smooth-dynamics analysis. As the steady state equilibrium is unique, the GE trajectory starting in branch $\mathcal{S}_2$ should reach the switching boundary Σ at time $t = t_\Sigma > 0$. From Lemma 7 and section 4.2 we know that if there is continuation to subset $\mathcal{S}_1$, then the equilibrium trajectory should cross Σ with a value for consumption larger than $\tilde{C} = Y_2(\tilde{K}) - \delta\tilde{K}$, with both variables decreasing in time. Therefore the equilibrium trajectory should verify $\Phi_C(t, K(0)) > \Phi^g_C(t, K(0))$, where $[\Phi^g(t)]$ is the candidate grazing trajectory that grazes Σ from side $\mathcal{S}_2$ and it is tangent to Σ at the intersection with the isocline $\dot{K} = 0$. The equilibrium trajectory has also an upper bound on $\mathcal{S}_2$ given by schedule $C = \beta K$. To prove this observe that

$$\left.\frac{d\Phi^c_C}{d\Phi^c_K}\right|_{C=\beta K} = \frac{\beta K(R_2(K) - (\rho + \delta)}{Y_2(K) - \delta K - \beta K} = \alpha\beta < \left.\frac{dC}{dK}\right|_{C=\beta K} = \beta. \tag{A.3}$$

As $\dot{K} < 0$ and $\dot{C} < 0$ for candidate trajectories verifying $\Phi^c_C(t) > \Phi^g_C(t)$, then candidate trajectories intercept the curve $C = \beta K$ from below and they diverge from the steady state. The equilibrium trajectory, converging to the steady state, is the unique candidate trajectory which cuts that schedule at the equilibrium point. Therefore, the saddle path should lie below $C = \beta K$ if $K^*_M \leq \Phi_K(t, K(0)) \leq K(0)$ at both branches. □

Proof of Proposition 3. The proof is similar to the proof of Proposition 2, *mutatis mutandis*. In this case, observe that the unique steady-state equilibrium point is $(K^*_L, C^*_L) \in \mathcal{S}_2$ and the transversality condition is met if the trajectories, for every admissible initial capital stock $K(0)$, belong to the stable manifold $W^s_L = W^s(K^*_L, C^*_L)$, which is also non-smooth and extends to branches $\mathcal{S}_1$ and $\mathcal{S}_2$. Asymptotically the GE trajectory converges to the subspace tangent to the stable manifold in the neighborhood of $(K^*_L, C^*_L) \in \mathcal{S}_2$, E^s_L, which has again a steeper slope than the isocline $\dot{K} = 0$, but is less steep than the ray $C = \beta K$, since

$$0 < \left.\frac{dC(K^*_L)}{dK}\right|_{\dot{K}=0} = C_2'(K^*_L) = \mathrm{tr}(J_2(K^*_L)) < \left.\frac{dC(K^*_L)}{dK}\right|_{E^s_L} = \lambda^+_L < \beta,$$

as we will prove next. An implication is that the equilibrium trajectory traverses Σ below the isocline $\dot{K}=0$ and both variables are increasing in Σ. As in the last case, equation (A.3) is still valid, meaning that candidate trajectories cross the line $C=\beta K$ with a slope smaller than β. Again, equilibrium trajectories are the only ones that intercept the line $C=\beta K$ at the equilibrium point. This means that, if $K(0)\leq\tilde{K}$ then $\beta\Phi_K(t_\Sigma,K(0))<\Phi_C(t_\Sigma,K(0))<Y_2(\Phi_K(t_\Sigma,K(0)))-\delta\Phi_K(t_\Sigma,K(0))$. Thus, equilibrium trajectories starting in $\mathcal{S}_1$ should pass below the grazing trajectory, $\Phi^g(t)$, which in this case grazes Σ from side $\mathcal{S}_1$. $\square$

Proof of Proposition 4. Proposition 1 presents the number of steady states and the local dynamics for the case in which $1<\sigma<\bar{\sigma}$ and $\overline{\phi}<\phi<\tilde{\phi}$. There are three equilibria (K^*_M,C^*_M), (K^*_H,C^*_H), and (K^*_L,C^*_L). Geometrically, these equilibria are located in the intersection of the equilibrium ray $C=\beta K$ and the isocline $\dot{K}=0$, which is a PWS one-dimensional manifold in (C,K). We prove below that the equilibrium trajectories, for any initial value of the capital stock belong to the stable manifolds associated to the two saddle point stable steady states (K^*_M,C^*_M) and (K^*_L,C^*_L). Those trajectories are located between the two schedules mentioned above, they are split by steady state (K^*_H,C^*_H), and they are connected to this steady state by two heteroclinic trajectories, Γ_{HL} and Γ_{HM}. We prove their existence in that order.

First, we prove there is a smooth heteroclinic orbit connecting stationary equilibria (K^*_H,C^*_H) and (K^*_L,C^*_L). We first address local dynamics and the prove there is a connecting orbit.

Steady state (K^*_L,C^*_L) is saddle-point stable and its Jacobian exhibits eigenvalues $\lambda^-_L<0<\lambda^+_L$. In addition, $E^s_L=(\lambda^+_L,1)^\top$ is the eigenvector associated to λ^-_L and $E^u_L=(\lambda^-_L,1)^\top$ is the eigenvector associated to λ^+_L. The stable subspace E^s_L is tangent to the stable manifold $W^s_L=W^s(K^*_L,C^*_L)$. Locally, the eigenspaces verify

$$\left.\frac{dC(K^*_L)}{dK}\right|_{C=\beta K}=\beta>\left.\frac{dC(K^*_L)}{dK}\right|_{E^s_L}=\lambda^+_L>\left.\frac{dC(K^*_L)}{dK}\right|_{\dot{K}=0}>0>\left.\frac{dC(K^*_L)}{dK}\right|_{E^u_L}=\lambda^-_L,$$

since $0<\left.\frac{dC(K^*_L)}{dK}\right|_{\dot{K}=0}=C_2'(K^*_L)=\mathrm{tr}(J_2(K^*_L))<\lambda^+_L<\beta$. Then the eigenspace tangent to the stable manifold is delimited by lines $\dot{K}=0$ and $C=\beta K$. Steady state (K_H^*, C_H^*) is unstable and its Jacobian has eigenvalues $0<\lambda_H^- < \lambda_H^+$. Furthermore, $E_H^{u,+}=(\lambda_H^-, 1)^\top$ is the eigenvector associated to λ_H^+ and $E_H^{u,-}=(\lambda_H^+, 1)^\top$ is the eigenvector associated to λ_H^-. They span the subspace tangent to the unstable manifold $W_H^u = W^u(K_H^*, C_H^*)$. In the neighborhood of (K_H^*, C_H^*) we have

$$\left.\frac{dC(K_H^*)}{dK}\right|_{\dot{K}=0} > \left.\frac{dC(K_H^*)}{dK}\right|_{E_H^{u,-}} = \lambda_H^+ > \left.\frac{dC(K_H^*)}{dK}\right|_{C=\beta K} = \beta > \left.\frac{dC(K_H^*)}{dK}\right|_{E_H^{u,+}} = \lambda_H^-,$$

as $0<\lambda_H^- < \beta < \lambda_L^+ < \left.\frac{dC(K_H^*)}{dK}\right|_{\dot{K}=0} = \mathrm{tr}(J_2(K_H^*))$. Then the eigenspace associated to the smaller eigenvalue in absolute value is also delimited by lines $\dot{K}=0$ and $C=\beta K$.

Now consider the global dynamics in the trapping area for $K_H^* \le K \le K_L^*$ and delimited by schedules $C=\beta K$ and $C=C_2(K)=Y_2(K)-\delta K$ (i.e. $\dot{K}=0$):

$$T_1 \equiv \{(K,C)\in \mathcal{S}_2 : C_2(K) \ge C \ge \beta K,\ K_H^* \le K \le K_L^*\}.$$

The behavior of candidate trajectories in the boundaries of the trapping area are described next. First, on the first manifold we have $\left.\frac{d\Phi_C^c(t,K)}{dt}\right|_{C=C_2(K)} = (Y_2(K)-\delta K)(R_2(K_H^*)-(\rho+\delta)) > 0$ and $\left.\frac{d\Phi_K^c(t,K)}{dt}\right|_{C=C_2(K)} = 0$. Therefore the slope of the candidate flows in the (K,C) space is given by

$$\left.\frac{d\Phi_C(t,K)}{d\Phi_K(t,K)}\right|_{C=C_2(K)} = \infty.$$

Thus, the dynamic paths point outwards at side $\dot{K}=0$. Second, the candidate flows at $C=\beta K$ verify $\left.\frac{d\Phi_C^c(t,K)}{dt}\right|_{C=\beta K} = \beta K(R_2(K)-(\rho+\delta)) > 0$ and $\left.\frac{d\Phi_K^c(t,K)}{dt}\right|_{C=\beta K} = Y_2(K) - (\beta+\delta)K > 0$. This implies that the slope of the candidate paths verify

$$\left.\frac{d\Phi_C^c(t,K)}{d\Phi_K^c(t,K)}\right|_{C=\beta K} = \frac{\beta K(R_2(K)-(\rho+\delta))}{(R_2(K)-\alpha(\beta+\delta))K/\alpha} = \alpha\beta < \beta = \left.\frac{dC}{dK}\right|_{C=\beta K}.$$

Again, as in Proposition 3, candidate trajectories approach the schedule $C=\beta K$ with a smaller slope, which means that they move outwards from the trapping area T_1. Then, with the exception of one path that converges to the steady state (K_L^*, C_L^*), all the other candidate paths starting within the trapping area T_1 eventually move outwards. The convergent path is the equilibrium path, $\Phi(t)$. Then the stable manifold W_L^s has a branch inside the trapping area T_1 and the equilibrium trajectory belonging to it lay along the heteroclinic trajectory, Γ_{HL}, joining K_H^* and K_L^*. The heteroclinic trajectory is tangent to $E_H^{u,-}$ in the neighborhood of (K_H^*, C_H^*) and to E_L^s in the neighborhood of (K_L^*, C_L^*), which lies within T_1, as we have seen. Geometrically, it separates the paths that start on side $C = C_2(K)$ from those leaving from side $C = \beta K$.

Next, we prove that there is a unique PWS heteroclinic orbit connecting both stationary equilibria, (K_H^*, C_H^*) and (K_M^*, C_M^*), and this is an equilibrium trajectory. In order to do that, we apply the same method used above, but taking into account the non-smoothness properties of the dynamic system, as two branches ($\mathcal{S}_1$ and $\mathcal{S}_2$) are crossed. We also define a trapping area, joining two contiguous subsets belonging to each branch, in which the dynamics is governed by different equations. For $K_M^* \leq K \leq \tilde{K}$, then the dynamics is driven by equations $\dot{C} = C(R_1(K) - (\rho + \delta))$ and $\dot{K} = Y_1(K) - C - \delta K$, and for $\tilde{K} \leq K \leq K_M^*$, it is given by $\dot{C} = C(R_2(K) - (\rho + \delta))$ and $\dot{K} = Y_2(K) - C - \delta K$. The trapping area is delimited by two manifolds, a smooth one $C = \beta K$ and the non-smooth manifold $C = C(K)$ where $C(K) = Y_j(K) - \delta K$ for $(K, C) \in \bar{\mathcal{S}}_j$, $j = 1, 2$,

$$T_2 \equiv \{\ (K, C) \in \bar{\mathcal{S}}_1 \cup \bar{\mathcal{S}}_2 : \beta K \geq C \geq C(K),\ K_H^* \geq K \geq K_M^*\}.$$

In Proposition 2 we showed that equilibria (K_M^*, C_M^*) is saddle-point stable and (K_H^*, C_H^*) is a source. The stable manifold associated to (K_M^*, C_M^*), W_M^s, has a tangent, E_M^s, that lies inside the trapping area due to

$$0 < \left.\frac{dC(K_M^*)}{dK}\right|_{C=C_1(K)} = tr(J_1(K_M^*)) < \left.\frac{dC(K_M^*)}{dK}\right|_{E_M^s} = \lambda_M^+ < \left.\frac{dC(K_M^*)}{dK}\right|_{C=\beta K} = \beta.$$

On the other hand, we showed that the eigenspace associated to the smaller eigenvalue of the Jacobian J_2 evaluated at (K_H^*, C_H^*), also lies inside the trapping area, as

$$\left.\frac{dC\ (K_H^*)}{dK}\right|_{C=C_2(K)} = \mathrm{tr}(J_2(K_H^*)) > \left.\frac{dC\ (K_H^*)}{dK}\right|_{E_H^{u,-}} = \lambda_H^+ > \left.\frac{dC\ (K_H^*)}{dK}\right|_{C=\beta K} = \beta > 0.$$

This means that the heteroclinic orbit Γ_{HM} should depart from inside the trapping area in the neighborhood of the MC equilibrium. For $K_M^* < K < K_H^*$, all candidate trajectories crossing $C = \beta K$ point outwards T_2, as their slope is smaller in the (K, C) space:

$$\left.\frac{d\Phi_C^c}{d\Phi_K^c}\right|_{C=\beta K} = \frac{\beta K(R_1(K) - (\rho+\delta))}{Y_1(K) - (\delta+\beta)K} = \frac{\beta K(R_2(K) - (\rho+\delta))}{Y_2(K) - (\delta+\beta)K} = \alpha\beta < \beta,$$

and

$$\beta\tilde{K}(R_i(\tilde{K}) - (\rho+\delta)) < \left.\frac{d\Phi_C^c(t,K)}{dt}\right|_{C=\beta K} = \beta K(R_i(K) - (\rho+\delta)) < 0,\ i = 1, 2,$$

and

$$\left.\frac{d\Phi_K^c(t,K)}{dt}\right|_{C=\beta K} = \left(R_i(K)/\alpha - (\delta+\beta)\right) K < 0,\ i = 1, 2.$$

For $K_M^* < K < K_H^*$, all candidate trajectories crossing $C = C(K)$ also point outwards T_2, as

$$\begin{aligned} C_1(K_M^*)(R_1(K_M^*) - (\rho+\delta)) = 0 \quad &> \quad \left.\frac{d\Phi_C^c(t,K)}{dt}\right|_{C=C_1(K)} = C_1(K)(R_1(K) - (\rho+\delta)) > \\ &> \quad C_1(\tilde{K})(R_1(\tilde{K}) - (\rho+\delta)), \end{aligned}$$

$$\begin{aligned} C_2(\tilde{K})(R_2(\tilde{K}) - (\rho+\delta)) \quad &< \quad \left.\frac{d\Phi_C^c(t,K)}{dt}\right|_{C=C_2(K)} = C_2(K)(R_2(K) - (\rho+\delta)) < \\ &< \quad C_2(K_H^*)(R_2(K_H^*) - (\rho+\delta)) = 0, \end{aligned}$$

and

$$\left.\frac{d\Phi_K^c(t,K)}{dt}\right|_{C=C_1(K)} = \left.\frac{d\Phi_K^c(t,K)}{dt}\right|_{C=C_2(K)} = 0.$$

The later implies that candidate trajectories, not converging to steady-state equilibria, cross the PWS continuous manifold $C = C(K)$ and head outwards of T_2 with slope

$$\left.\frac{d\Phi_C^c(t,K)}{d\Phi_K(t,K)}\right|_{C=C(K)} = \infty.$$

This includes the grazing trajectory, which is tangent to $C = C(K)$ at point $K = \tilde{K}$. This means that both the heteroclinic Γ_{HM} and the equilibrium trajectories departing from

$\tilde{K} < K(0) < K_H^*$ are located between a grazing trajectory departing from (K_H^*, C_H^*), which grazes Σ from the side $\mathcal{S}_2$, and the line $C = \beta K$ cross Σ within the interval $(C(\tilde{K}, \beta\tilde{K})$. This also implies that they asymptotically converge to the equilibrium point (K_M^*, C_M^*). It can also be proved that the heteroclinic trajectory is located between the loci $E_H^{u,-}$ and $C = \beta K$ ($C = C(K)$ and $E_H^{u,-}$) if $\frac{K_H^* - \tilde{K}}{\tilde{K} - K_M^*} - \frac{\beta - \lambda_M^+}{\lambda_H^+ - \beta} > 0$ (< 0). $\square$

Figures

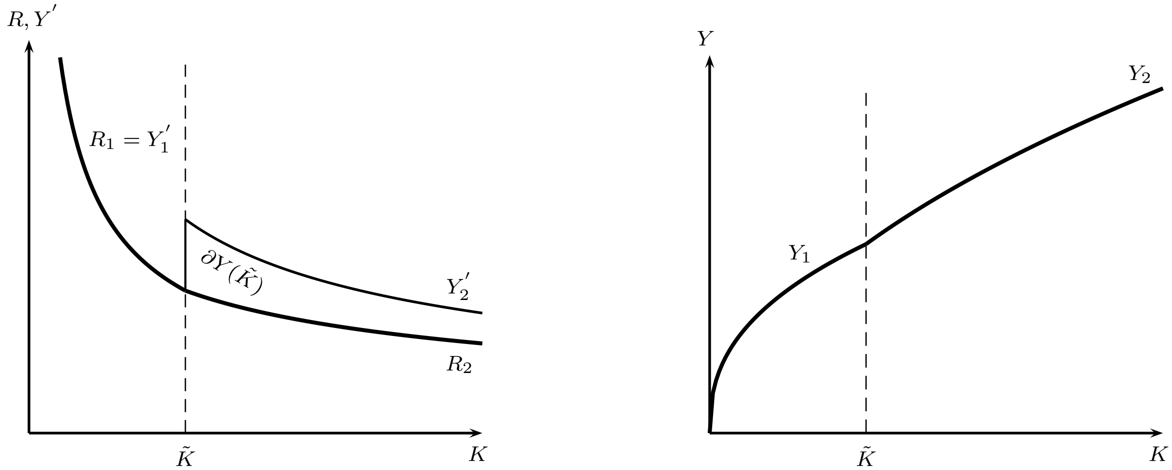

Figure A.1: Aggregate return and marginal product of capital, and production functions for $\sigma > \tilde{\sigma}$.

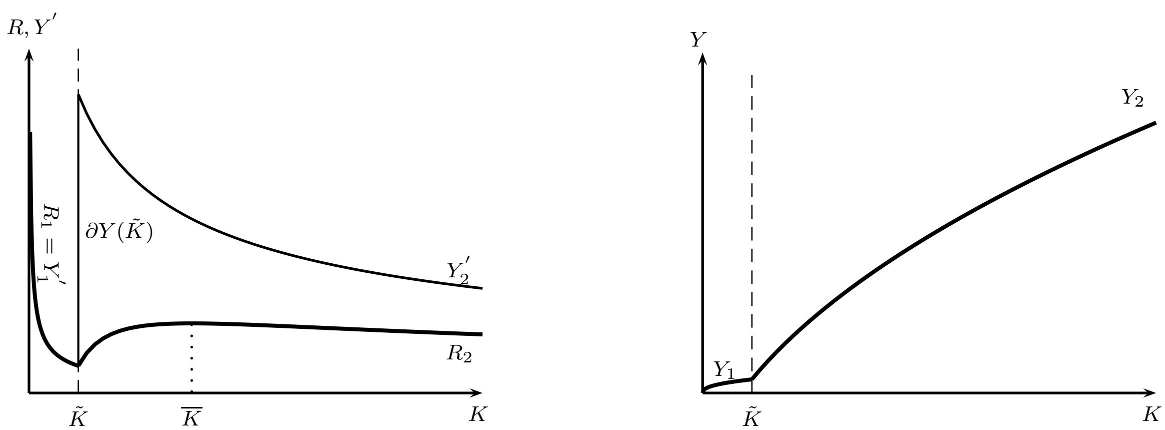

Figure A.2: Aggregate return and marginal product of capital, and production functions for $\max\{\tilde{\sigma}/2, 1\} < \sigma < \tilde{\sigma}$.

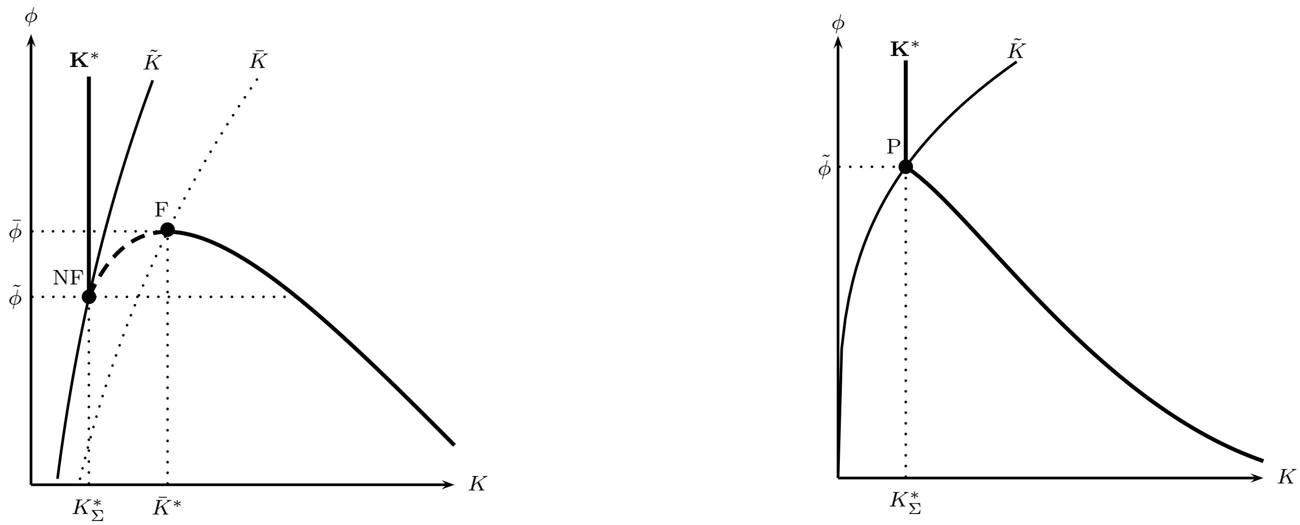

Figure A.3: Bifurcation diagram for $1 < \sigma < \bar{\sigma}$ (left panel) and for $\sigma > \bar{\sigma}$ (right panel) with ϕ as the bifurcation parameter. Legends: persistence (P), smooth fold bifurcation (F), non-smooth fold bifurcation (NF). Steady states: saddle point (filled line), unstable steady state (dashed line).

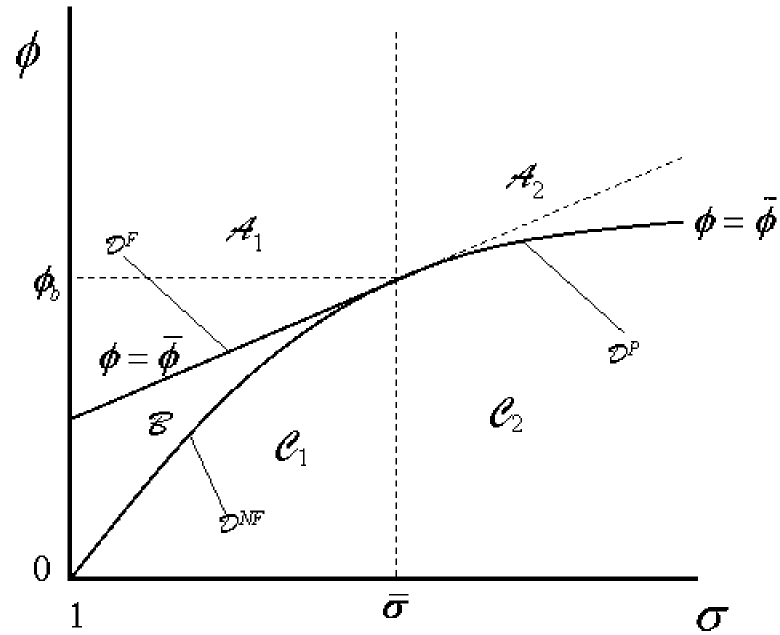

Figure A.4: Bifurcation diagram in the (σ, ϕ)-space.

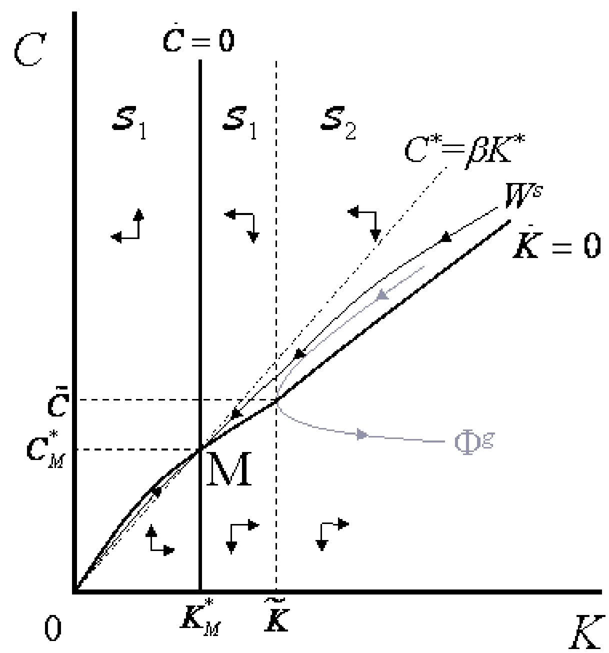

Figure A.5: Phase diagram 1 - $\phi > \max\{\tilde{\phi}, \bar{\phi}\}$ Unique MC stationary equilibrium.

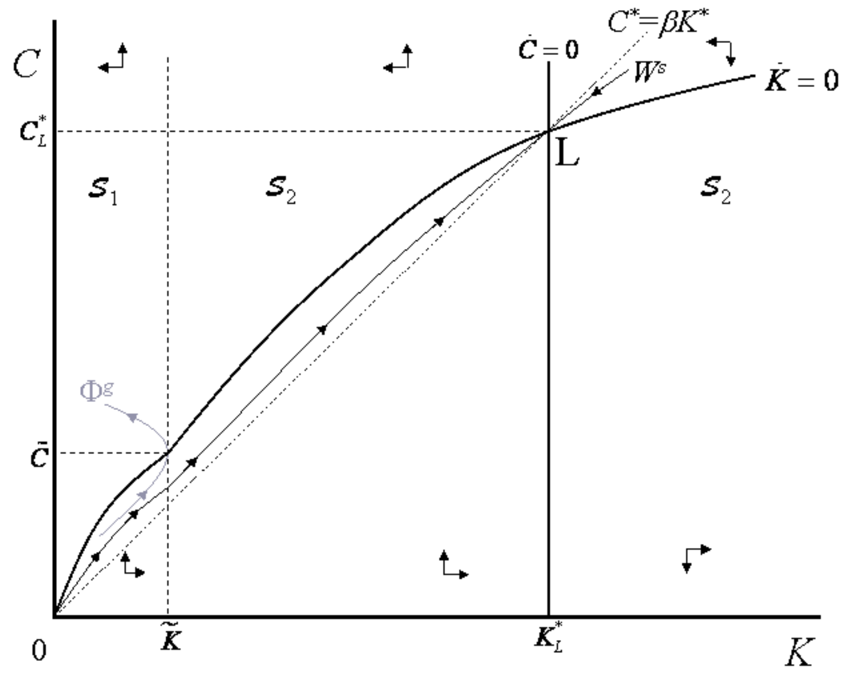

Figure A.6: Phase diagram 2 - $\phi < \tilde{\phi}$ Unique CMC stationary equilibrium.

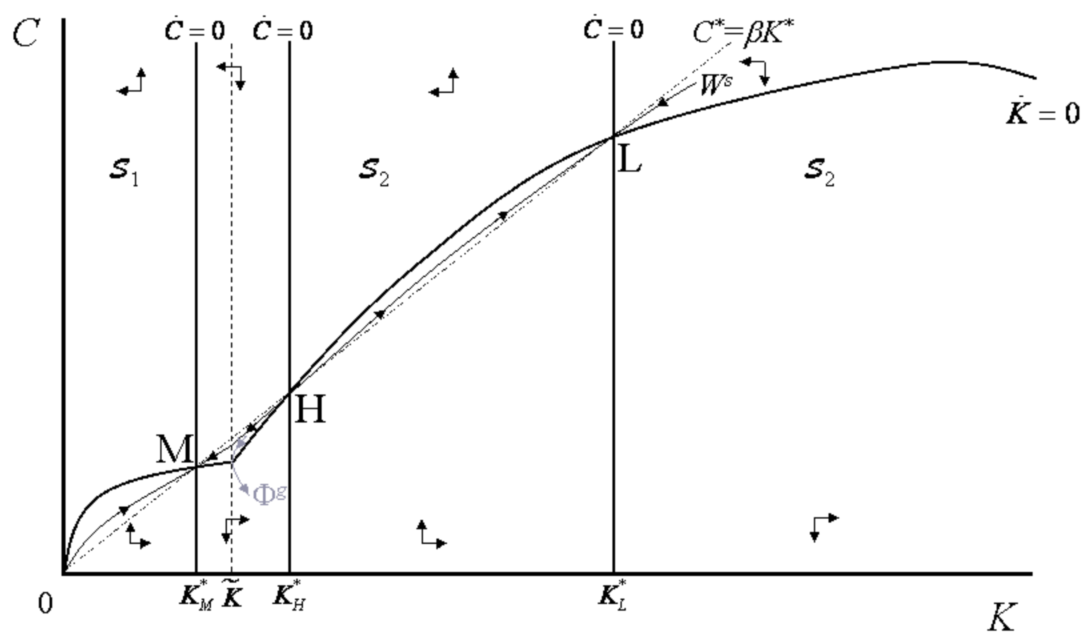

Figure A.7: Phase diagram 3 - $\tilde{\phi} < \phi < \bar{\phi}$ and $1 < \sigma < \bar{\sigma}$ Multiplicity of stationary equilibrium.

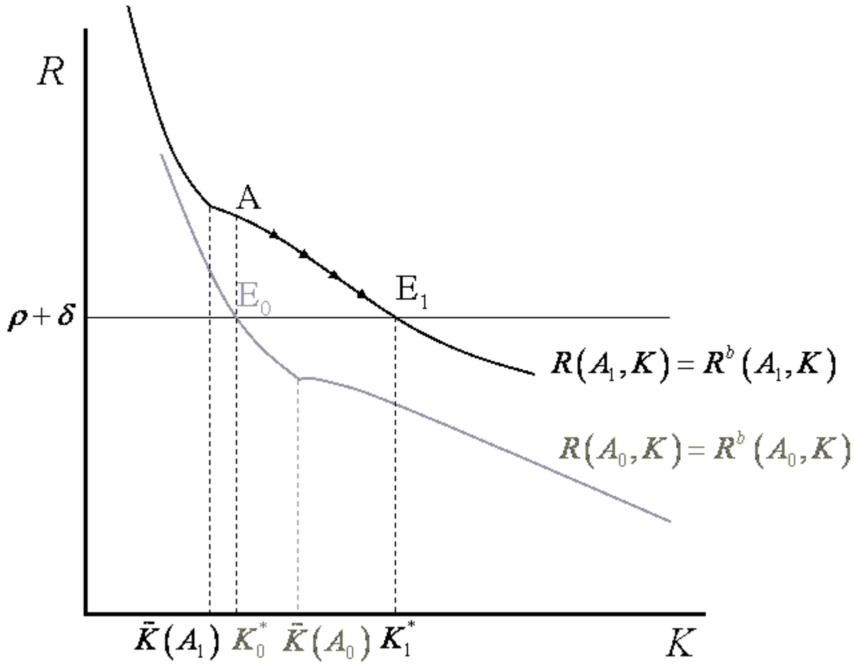

Figure A.8: Permanent increase in productivity for $\sigma > \bar{\sigma}$.

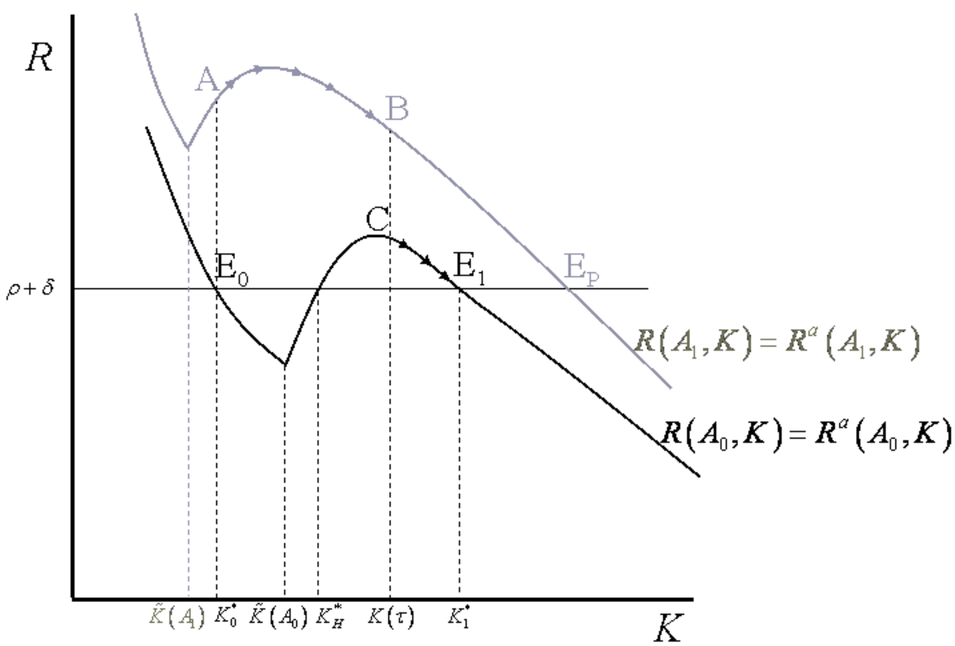

Figure A.9: Temporary increase in productivity for $1 < \sigma < \bar{\sigma}$.