Characterizing Stability Properties in Games with Strategic Substitutes*

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Abstract

In games with strategic substitutes (GSS), convergence of the best response dynamic starting from the inf (or sup) of the strategy space is equivalent to global stability (convergence of every adaptive dynamic to the same pure strategy Nash equilibrium). Consequently, in GSS, global stability can be analyzed using a single best response dynamic. Moreover, in GSS, global stability is equivalent to dominance solvability, showing that in this class of games, two different foundations for robustness of predicted outcomes are equivalent, and both can be checked using a single best response dynamic. These equivalences are useful to study stability of equilibria in a variety of applications. Furthermore, in parameterized GSS, under natural conditions, dynamically stable equilibrium selections can be viewed in terms of monotone selections of equilibria. Several examples are provided.

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1 Introduction

Games with strategic substitutes (GSS) and games with strategic complements (GSC) formalize two basic economic interactions and have widespread applications. Bulow, Geanakoplos, and Klemperer (1985) develop these ideas and show that models of strategic investment, entry deterrence, technological innovation, dumping in international trade, natural resource extraction, business portfolio selection, and others can be viewed in a more unifying framework according as the variables under consideration are strategic complements or strategic substitutes.[1]

GSS arise more naturally in situations where there is competition for a shared resource: for example, firms producing the same good and competing for the same market, fisheries competing to fish in the same pond, versions of tournaments, games with congestion effects, managing teams with substitutable members, and so on. They also arise naturally in situations where there is conflict: for example, dove-hawk-type situations, which include mutually assured destruction games, wars of conflict, and games of political posturing. GSS have the characteristic that the best-response of each player is weakly decreasing in the action of each of the other players.

In contrast, GSC arise more naturally when there are network or coordination benefits: for example, using a particular network system, such as Facebook, or public transport; or games of coordination, such as Battle of the sexes, bank runs, managing complementary teams, production with complementary inputs, and so on. In GSC, best-response of each player is weakly increasing in actions of the other players.[2]

For some recent research on GSS, confer Amir (1996), Villas-Boas (1997), Amir and Lambson (2000), Schipper (2003), Zimper (2007), Roy and Sabarwal (2008), Acemoglu and Jensen (2009), Acemoglu and Jensen (2010), Roy and Sabarwal (2010), and Jensen (2010), among others. Moreover, GSS have recently received renewed attention in the global games literature, because of the importance of congestion effects; see, for example, Harrison (2005), Karp, Lee, and Mason (2007), and Morris and Shin (2009).[3]

In this paper, we focus on stability (or robustness) properties of predicted outcomes in GSS. In particular, we study stability of equilibrium, and stability of parameterized equilibrium selections.

With regard to stability of equilibrium, we first show that in GSS, convergence of the best response dynamic starting from the inf (or sup) of the strategy space is equivalent to

[1]Earlier developments are provided in Topkis (1978) and Topkis (1979).

[2]There is a long literature developing the theory of GSC. Some of this work can be seen in Topkis (1978), Topkis (1979), Bulow, Geanakoplos, and Klemperer (1985), Lippman, Mamer, and McCardle (1987), Sobel (1988), Milgrom and Roberts (1990), Vives (1990), Milgrom and Shannon (1994), Milgrom and Roberts (1994), Zhou (1994), Shannon (1995), Villas-Boas (1997), Edlin and Shannon (1998), Echenique (2002), Echenique (2004), Quah (2007), and Quah and Strulovici (2009), among others. Extensive bibliographies are available in Topkis (1998), in Vives (1999), and in Vives (2005).

[3]Developments in the theory of GSS show that these games behave differently from GSC along several dimensions: for example, the structure of their equilibrium set is very different, and conditions for comparative statics of equilibria are different as well. The results here show additional differences.

convergence of every adaptive dynamic[4] to the same (pure strategy) Nash equilibrium. In other words, *in GSS, convergence of the best response dynamic from inf (or sup) of the strategy space is equivalent to global stability*, where global stability is defined as convergence of every adaptive dynamic to the same (pure strategy) Nash equilibrium. Recall that adaptive dynamics allow for strategic behavior and learning processes based on past play, but with otherwise few restrictions. Intuitively, the only requirement in an adaptive dynamic is that eventually, future play should be an undominated response to the order interval determined by past play, (or at least be in the order interval determined by such undominated responses.)

This result provides a new perspective on global stability. In GSS, knowledge of convergence of a single best response dynamic yields convergence of all adaptive dynamics to the same outcome. Consequently, whether players actually play best response dynamics or not, convergence of a single best response dynamic is sufficient to conclude convergence under all adaptive behavior. This provides an alternative to checking for convergence of each and every dynamic in a class of dynamics: for example, as in Milgrom and Roberts (1991), or using the traditional eigen-value approach applied to each dynamic, as in, Al-Nowaihi and Levine (1985) and Okuguchi and Yamazaki (2008).

Next, we investigate dominance solvability in GSS. Zimper (2007) shows that in GSS in which best responses are order continuous functions, there exist smallest and largest serially undominated strategies, and a GSS is dominance solvable iff the second iterate of the (joint) best response function has a unique fixed point. We use a more general model than Zimper (2007), and show that his results go through for our generalizations. We show that *in GSS, convergence of the best response dynamic from inf (or sup) of the strategy space is equivalent to dominance solvability.*

Connecting these two results, we conclude that *in GSS, global stability is equivalent to dominance solvability.* This brings together two different foundations for robustness of predicted outcomes in games. Dominance solvability (and rationalizability) assume fully informed players, using infinite iterations of rationalizing about potential (future) responses by competitors to predict a solution to a one-shot game. Global stability uses dynamic learning and strategic processes in a series of game-play over time, using past play by typically myopic players to determine present moves, and relying on limits of such learning and strategic behavior to predict an outcome robust to the dynamics. In GSS, both approaches are equivalent, and moreover, both global stability and dominance solvability can be checked using a single best response dynamic.[5]

Recall that Moulin (1984) has shown that in all strategic games where strategy spaces are compact metric spaces and best responses are continuous functions, dominance solvability implies Cournot stability (convergence of all best response dynamics). Zimper (2007)'s results imply that in GSS, with order continuous best response functions, Cournot stability is equivalent to dominance solvability. Our results extend these to show that in our more general model of GSS, global stability (convergence of every adaptive dynamic, not just best

[4]As defined in Milgrom and Roberts (1990).

[5]Using Milgrom and Roberts (1990), it is easy to deduce that global stability is equivalent to dominance solvability in GSC as well. What is not true in GSC is the equivalence of convergence of the best response dynamic from inf (or sup) of strategy space to global stability, or to dominance solvability.

response dynamics) is equivalent to dominance solvability.[6]

In addition to showing the theoretical equivalence of these approaches, we show that these results can be used profitably in a variety of applications in diverse areas. In particular, we present general results for games with linear best responses (including versions of common pool resource games, and private provision of public goods), analyze the Cournot model in some detail, extend the analysis of two-player GSS and two-player GSC, and apply our results to R&D games, games of tournaments, and to managing team projects with substitutable tasks. In each case, knowledge of a particular aspect of the game allows for powerful cross-derivation of results.

With regard to stability of equilibrium selections, we show that in parameterized GSS, monotone equilibrium selections are dynamically stable, in the following sense.

We show that in parameterized GSS, *continuous and strictly increasing equilibrium selections select strongly stable equilibria, under natural conditions.* (Intuitively, an equilibrium is strongly stable if it has a neighborhood such that every adaptive dynamic starting in this neighborhood converges to it.) In particular, small changes in the parameter are dynamically stable, because at a new parameter value, *every* adaptive dynamic starting from the old equilibrium converges to the newly selected equilibrium.

Similarly, we show that *continuous and nowhere weakly increasing equilibrium selections select equilibria that are not even weakly stable, under similar conditions.* (Intuitively, an equilibrium is weakly stable if it has a neighborhood such that some adaptive dynamic starting in this neighborhood converges to it.) In particular, changes in the parameter are dynamically unstable, because at a new parameter value, no adaptive dynamic starting from the old equilibrium converges to the newly selected equilibrium.

Thus, when considering dynamically stable equilibria (as proposed by Samuelson's Correspondence principle), we may expect monotone selections of equilibria to arise naturally in GSS. Echenique (2002) provides a similar result for parameterized GSC.

In addition to these results, we clarify two aspects of the theory of GSS.

First, we show that a GSS may not necessarily have a (pure strategy) Nash equilibrium. We present a three-player, two-action, Dove-Hawk-Chicken-type game with no pure-strategy Nash equilibrium. This shows that a GSS cannot always be viewed as an aggregative game, or as a GSC, because such games always have a PSNE. In particular, the standard technique of reversing the order on the strategy space of one player in a GSS to yield a GSC does not extend to more than two players.

Second, fixed points of the second iterate of the joint best-responses play a significant role in the analysis of GSS. These may be motivated in terms of *simply rationalizable strategies.* Intuitively, a simply rationalizable strategy profile is one that can be simultaneously rationalized by no more than two iterations of behavioral conjectures. They may be viewed

[6] In general, convergence of best response dynamics is not necessarily equivalent to convergence of adaptive dynamics. In section 2, we present a symmetric, two-player game in which their convergence behavior is different.

as outcomes of low-level rationalization, exhibiting a type of bounded rationality.[7] Simply rationalizable profiles provide a behavioral interpretation for such fixed points. In this terminology, in GSS, simple rationalization always predicts an outcome, even if a GSS has no PSNE. Moreover, if simple rationalization predicts a unique outcome, then higher-level rationalization has no additional benefit.

The paper proceeds as follows. Section 2 presents the results on global stability, and their connection to dominance solvability. Section 3 provides several applications. Section 4 presents the results on stability of monotone equilibrium selections.

2 Stability of Equilibrium

As usual, a lattice is a partially ordered set in which every two elements, x and y, have a supremum, denoted $x \vee y$, and an infimum, denoted $x \wedge y$. A complete lattice is a lattice in which every non-empty subset has a supremum and infimum in the set. A function $f : X \to \mathbb{R}$ (where X is a lattice) is *quasi-supermodular* if (1) $f(x) \geq f(x \wedge y) \implies f(x \vee y) \geq f(y)$, and (2) $f(x) > f(x \wedge y) \implies f(x \vee y) > f(y)$. A function $f : X \times T \to \mathbb{R}$ (where X is a lattice and T is a partially ordered set) satisfies *decreasing single-crossing property* in $(x; t)$ if for every $x' \succ x''$ and $t' \succ t''$, (1) $f(x', t'') \leq f(x'', t'') \implies f(x', t') \leq f(x'', t')$, and (2) $f(x', t'') < f(x'', t'') \implies f(x', t') < f(x'', t')$. The decreasing single-crossing property captures the idea of strategic substitutes, just as the single-crossing property formalizes the idea of strategic complements.[8]

Let I be a non-empty set of players, and for each player i, a strategy space that is a partially ordered set $(X^i, \preceq^i)$, and a real-valued payoff function, denoted $f^i(x_i, x_{-i})$. As usual, the domain of each f^i is the product of the strategy spaces, $(X, \preceq)$ endowed with the product order.[9] The strategic game $\Gamma = \{I, (X^i, \preceq^i, f^i)_{i \in I}\}$ is a ***game with strategic substitutes (GSS)***, if for every player i,

1. X^i is a complete lattice,
2. f^i is (jointly) upper semi-continuous, and for every x_i, f^i is order continuous in x_{-i},
3. For every fixed x_{-i}, f^i is quasi-supermodular in x_i, and
4. f^i satisfies the decreasing single-crossing property in $(x_i; x_{-i})$.[10]

[7] Nash equilibrium profiles are simply rationalizable, and simply rationalizable profiles are rationalizable; both inclusions may be strict.

[8] This property is discussed in some detail in Roy and Sabarwal (2010). Amir (1996) terms this property the dual single-crossing property.

[9] The topology on X^i is the standard order interval topology, and the topology on X is the product topology. Confer, for example, Topkis (1998). For notational convenience, we shall sometimes drop the index i from the notation for the partial order.

[10] For the general reader, it may help to keep in mind the more accessible special cases of these definitions: X^i is compact, f^i is continuous, supermodular in x_i, and satisfies decreasing differences in $(x_i; x_{-i})$.

For each player i, the ***best response of player*** i is denoted $g^i(x_{-i})$. As is well-known (see, for example, Milgrom and Shannon (1994)), for each player i, the best response of player i, $g^i(x_{-i})$, is a non-empty, complete lattice. Let $\overline{g}^i(x_{-i}) = \sup g^i(x_{-i})$ and $\underline{g}^i(x_{-i}) = \inf g^i(x_{-i})$ be the extremal best responses. As is well-known (see, for example, Topkis (1998)), for each player i, and for each profile of other player strategies x_{-i}, $g^i(x_{-i})$ is nonincreasing in x_{-i},[11] and therefore, for each player i, both $\overline{g}^i(x_{-i})$ and $\underline{g}^i(x_{-i})$ are nonincreasing functions.[12] Let $g : X \twoheadrightarrow X$, $g(x) = (g^i(x_{-i}))_{i \in I}$, denote the ***joint best-response correspondence***. Then the correspondence g is nonincreasing,[13] and the functions $\underline{g}(x) = \inf g(x)$ and $\overline{g}(x) = \sup g(x)$ are nonincreasing.

As usual, a (pure strategy) ***Nash equilibrium*** of the game is a profile of player actions x such that $x \in g(x)$. The ***equilibrium set*** of the game is given by $\mathcal{E} = \{x \in X | x \in g(x)\}$. It is possible that a GSS has no (pure strategy) Nash equilibrium. Consider the following example.

Example 1-1 (Dove-Hawk-Chicken). Consider a three-player version of a Dove-Hawk-Chicken-type game, as follows. Suppose there are three players, and each player has a choice between two actions: D (Dove) is the "lower" action (more dovish, more accommodating, less aggressive action), and H (Hawk) is the "higher" action (more hawkish, less accommodating, more aggressive action). Payoffs are given in figure 1.

Player 3: D

Player 1 \ Player 2	D	H
D	10, 10, 10	0, 20, 10
H	20, 10, 0	10, 0, 10

Player 3: H

Player 1 \ Player 2	D	H
D	10, 0, 20	10, 10, 0
H	0, 10, 10	0, 0, 0

D <------- Player 3 -------> H

Figure 1: A GSS with no (pure strategy) Nash equilibrium

Notice that each player wishes to be (weakly) less aggressive, the more aggressive are his competitors. This game is an extension of a two-player, Dove-Hawk-Chicken game. Indeed, if we fix the action of any one player, then the remaining two-player game is a version of a standard Dove-Hawk-Chicken game that has a unique Nash equilibrium − one player plays D and the other plays H. With three players, it is easy to check that this game has no (pure strategy) Nash equilibrium.

Example 1-1 clarifies another aspect of the theory of GSS. A GSS cannot always be

[11]For every x_{-i} and x'_{-i}, if $x_{-i} \preceq x'_{-i}$ then $g^i(x'_{-i}) \sqsubseteq^i g^i(x_{-i})$, where the order on nonempty subsets of X^i is the standard (induced) set order used in the literature. That is, for non-empty subsets A, B of X^i, $A \sqsubseteq^i B$ if for every $a \in A$, and for every $b \in B$, $a \wedge b \in A$, and $a \vee b \in B$, where the operations $\wedge, \vee$ are with respect to $\preceq^i$.

[12]For every x_{-i} and x'_{-i}, if $x_{-i} \preceq x'_{-i}$ then $\overline{g}^i(x'_{-i}) \preceq \overline{g}^i(x_{-i})$ and $\underline{g}^i(x'_{-i}) \preceq \underline{g}^i(x_{-i})$.

[13]In the standard induced set order, as in Topkis (1998).

viewed as an aggregative game,[14] or as a GSC, because both types of games always have a pure-strategy Nash equilibrium. In particular, the standard technique of reversing the order on the strategy space of one player in a GSS to yield a GSC does not extend to more than two players.

Let us now consider best response dynamics. When g is a best-response *function*, the notion of a best response dynamic is well-defined. Starting from a point, say, $\inf X$, the best response dynamic is generated by iterated application of g. When g is a correspondence, the question of a selection from the best responses arises. In this case, we follow Milgrom and Roberts (1990) and Echenique (2002), but adjusting the definition for strategic substitutes. In GSC, the best response dynamic starting at $\inf X$ is defined as follows: $y^0 = \inf X$, and for $k \geq 1$, $y^k = \underline{g}(y^{k-1})$. Intuitively, starting at $\inf X$, due to complementarities, the best responses to $\inf X$ are "small" or "low" or "close" to $\inf X$, so for the second term, take the smallest best response to $\inf X$. This may be viewed as a "directionally extremal" best response dynamic: if best responses remain low, take the lowest of the low best responses. Similarly, for the next term in the dynamic, and so on. For GSS, starting at $\inf X$, due to strategic substitutes, the best response to $\inf X$ is "large" or "high" or "far away" from $\inf X$, so for the second term, take the largest best response to $\inf X$. This may also be viewed as a "directionally extremal" best response dynamic: if the best responses are high, take the highest of the high best responses. For the third term in the dynamic, with substitutes, the best response to a "high" strategy profile is going to be "low," so directionally extremal would require taking the lowest of the low best responses, and so on. This is the definition we use here.

The (simultaneous) ***best response dynamic starting at*** $\inf X$ is the sequence $(y^k)_{k=0}^{\infty}$ given by $y^0 = \inf X$, and for $k \geq 1$, $y^k = \underline{g}(y^{k-1})$ if k is even, and $\overline{g}(y^{k-1})$ if k is odd. Of course, when g is a best-response *function*, (y^k) is the standard simultaneous best-response dynamic starting at $\inf X$. Similarly, the (simultaneous) ***best response dynamic starting at*** $\sup X$ is the sequence $(z^k)_{k=0}^{\infty}$ given by $z^0 = \sup X$, and for $k \geq 1$, $z^k = \overline{g}(z^{k-1})$ if k is even, and $\underline{g}(z^{k-1})$ if k is odd. Again, when g is a best-response *function*, (z^k) is the standard simultaneous best-response dynamic starting at $\sup X$.[15]

Mixtures of the sequences (y^k) and (z^k) have some nice properties, as shown in the following lemma, and they shall be useful to derive additional results. The ***lower mixture of*** $((y^k); (z^k))$ is the sequence $(\underline{x}^k)_{k=0}^{\infty}$ given by $\underline{x}^k = y^k$, if k is even, and $\underline{x}^k = z^k$, if k is odd, and the ***upper mixture of*** $((y^k); (z^k))$ is the sequence $(\overline{x}^k)_{k=0}^{\infty}$ given by $\overline{x}^k = z^k$, if k

[14] Confer Dubey, Haimanko, and Zapechelnyuk (2006), or Jensen (2010).

[15] Using the same definition as in GSC is not helpful, because it does not adjust for the opposite direction in which strategic substitutes move. Suppose $y^0 = \inf X$, and $z^0 = \sup X$, and for $k \geq 1$, $y^k = \underline{g}(y^{k-1})$, and $z^k = \overline{g}(z^{k-1})$. For GSC, this construction yields the following useful facts: (y^k) is monotone nondecreasing, (z^k) is monotone nonincreasing, and these sequences are comparable all along; that is, for every k, $y^k \preceq z^k$. For GSS, this construction does not get us very far. In this case, $y^0 \preceq z^0$, and then $y^0 \preceq y^1$, but then $y^2 \preceq y^1$, and then $y^2 \preceq y^3$, and $y^4 \preceq y^3$, and so on. Thus, a monotonic relationship in the progression of elements in either sequence does not emerge. Moreover, if we consider the second-iterate of this construction, then it is true that $y^0 \preceq y^2$, and inductively, for every k, $y^{2k} \preceq y^{2k+2}$, and therefore, the sequence (y^{2k}) is nondecreasing, and similarly, (z^{2k}) is nonincreasing. But with nonincreasing g, it does not follow that in general, $\underline{g} \circ \underline{g}(y^0) \preceq \overline{g} \circ \overline{g}(z^0)$, and therefore, a clear comparison across the sequences (y^k) and (z^k) does not emerge.

is even, and $\overline{x}^k = y^k$, if k is odd.[16]

Lemma 1. *Let* (y^k) *and* (z^k) *be the best response dynamics starting at* $\inf X$ *and* $\sup X$*, respectively, and* $(\underline{x}^k)$ *and* $(\overline{x}^k)$ *be their lower and upper mixtures, respectively.*

1. *The sequence* $(\underline{x}^k)$ *is nondecreasing and there is* $\underline{x}$ *such that* $\underline{x} = \lim_k \underline{x}^k$.

2. *The sequence* $(\overline{x}^k)$ *is nonincreasing and there is* $\overline{x}$ *such that* $\overline{x} = \lim_k \overline{x}^k$.

3. *For every* k, $\underline{x}^k \preceq \overline{x}^k$.

Proof. For statements (1) and (2), notice that if $\underline{x}^k \preceq \underline{x}^{k+1}$, then $\overline{x}^{k+1} = \overline{g}(\underline{x}^k) \succeq \overline{g}(\underline{x}^{k+1}) = \overline{x}^{k+2}$, and similarly, if $\overline{x}^k \succeq \overline{x}^{k+1}$, then $\underline{x}^{k+1} \preceq \underline{x}^{k+2}$. Thus, the sequence $(\underline{x}^k)$ is nondecreasing, and the sequence $(\overline{x}^k)$ is nonincreasing, if $\underline{x}^0 \preceq \underline{x}^1$, and $\overline{x}^0 \succeq \overline{x}^1$. But this is true, because $\underline{x}^0 = \inf X$, and $\overline{x}^0 = \sup X$. As X is complete, each of these sequences converges in X.

Statement (3) holds trivially for $k = 0$. Suppose $\underline{x}^k \preceq \overline{x}^k$. Then $\underline{x}^{k+1} = \underline{g}(\overline{x}^k) \preceq \underline{g}(\underline{x}^k) \preceq \overline{g}(\underline{x}^k) = \overline{x}^{k+1}$. ∎

Let us now focus on the structure of the set of undominated strategies in GSS. As usual,[17] a pure strategy $x_i \in X^i$ is ***strongly dominated***, if there exists $\hat{x}_i \in X^i$ such that for every x_{-i}, $f^i(x_i, x_{-i}) < f^i(\hat{x}_i, x_{-i})$. For a given set of strategy profiles $\hat{X} \subset X$, ***player i's undominated responses to*** $\hat{X}$ is the set

$$U_i(\hat{X}) = \left\{ x_i \in X^i | \forall x_i' \in X^i, \exists \hat{x} \in \hat{X}, f^i(x_i, \hat{x}_{-i}) \geq f^i(x_i', \hat{x}_{-i}) \right\}.$$

Let $U(\hat{X}) = (U_i(\hat{X})_{i \in I})$ denote the collection of undominated responses to $\hat{X}$, one for each player, and let $\overline{U}(\hat{X}) = [\inf U(\hat{X}), \sup U(\hat{X})]$ be the smallest order interval containing $U(\hat{X})$.[18] Higher-order undominated strategies are defined iteratively, as follows: $U^0(\hat{X}) = \hat{X}$, and for $k \geq 1$, $U^k(\hat{X}) = U^{k-1}(\hat{X})$. ***Serially undominated strategies*** are given by $\bigcap_{k=0}^{\infty} U^k(X)$, and a game is ***dominance solvable***, if the set of serially undominated strategies and the equilibrium set of the game are both the same singleton. The following lemma highlights the structure of the smallest order interval containing undominated strategies.

Lemma 2. *For every* $a \preceq b$ *in* X, $\overline{U}[a, b] = [\underline{g}(b), \overline{g}(a)]$.

Proof. Let us first see that $U[a, b] \subset [\underline{g}(b), \overline{g}(a)]$. Consider the contrapositive. Suppose $y \notin [\underline{g}(b), \overline{g}(a)]$. Then either, $y \not\preceq \overline{g}(a)$ or $y \not\succeq \underline{g}(b)$. Suppose $y \not\preceq \overline{g}(a)$. In particular, consider player i such that $y_i \not\preceq \overline{g}^i(a_{-i})$. Then $y_i \wedge \overline{g}^i(a_{-i})$ dominates y_i, as follows. Indeed, for every $x \in [a, b]$,

$$\begin{aligned} & f^i(y_i \vee \overline{g}^i(a_{-i}), a_{-i}) - f^i(\overline{g}^i(a_{-i}), a_{-i}) < 0 \\ \implies \quad & f^i(y_i, a_{-i}) - f^i(y_i \wedge \overline{g}^i(a_{-i}), a_{-i}) < 0 \\ \implies \quad & f^i(y_i, x_{-i}) - f^i(y_i \wedge \overline{g}^i(a_{-i}), x_{-i}) < 0, \end{aligned}$$

[16] In other words, $\underline{x}^0 = \inf X$, $\overline{x}^0 = \sup X$, and for $k \geq 1$, $\underline{x}^k = \underline{g}(\overline{x}^{k-1})$, and $\overline{x}^k = \overline{g}(\underline{x}^{k-1})$.

[17] Following Milgrom and Roberts (1990).

[18] We don't consider dominance of pure strategies by mixed strategies. Allowing for such dominance leads to a smaller set of undominated strategies, and the results here would still apply.

where the first inequality follows from the definition of $\overline{g}^i(a)$, the first implication follows from quasi-supermodularity, and the second implication follows from decreasing single-crossing property. The case $y \not\succeq \underline{g}(b)$ follows similarly. Thus, $y \notin U[a,b]$, whence $U[a,b] \subset [\underline{g}(b), \overline{g}(a)]$. Therefore, $\overline{U}[a,b] \subset [\underline{g}(b), \overline{g}(a)]$. Moreover, as g is a best response, $\underline{g}(b)$ and $\overline{g}(a)$ are in $U[a,b]$, whence $[\underline{g}(b), \overline{g}(a)] \subset \overline{U}[a,b]$. ∎

Lemma 2 shows that adjusting for strategic substitutes reverses the relationship that holds for strategic complements.[19]

To define an adaptive dynamic, we follow Milgrom and Roberts (1990). A process $(x(k))_{k\in\hat{K}}$ in Γ is an ***adaptive dynamic in*** Γ if for every K, there is K' such that for every $k \geq K'$, $x(k) \in \overline{U}[\inf P(K,k), \sup P(K,k)]$. Here, $P(K,k)$ is the set of past play from K up to (but not including) k; that is, $P(K,k) = \{x(\xi) | K \leq \xi < k\}$.

Adaptive dynamics allow for strategic behavior and learning processes based on past play, but with otherwise few restrictions.[20] All that is required is that eventually, future play should be an undominated response to the order interval determined by past play, or at least be in the order interval determined by such undominated responses. In particular, adaptive dynamics include simultaneous Cournot dynamics, sequential Cournot dynamics, and tatonnement-type price adjustment dynamics.

Moreover, adaptive dynamics include versions of fictitious play in the sense that when strategies lie in Euclidean spaces (or in many of the typical function spaces used in economics), the order interval determined by past play is a convex set, and therefore, includes the convex hull of past play, allowing for consideration of mixed strategies based on past play, and consequently, best responses to mixed strategies based on past play.

Furthermore, adaptive dynamics allow for different learning behaviors, "mistakes," and other "out-of equilibrium" dynamics, in the following sense. Suppose a player is considering past behavior of a competitor, who has played, say, 30 and 50 in the past two rounds. Then the following behaviors are admissible: the player may believe that the competitor may play something between 30 and 50, and respond appropriately (rather than responding to either 30 or 50); or the player may make a mistake in calculating his undominated response, and may play something that is only in the order interval determined by his undominated responses; or the player may play something completely irrational in the next round, and then realize his mistake and play an undominated response in the round after that. In this sense, the definition of an adaptive dynamic is fairly general.

The following result provides bounds on eventual behavior of adaptive dynamics in GSS.

Lemma 3. *Let $(\underline{x}^k)$ and $(\overline{x}^k)$ be the lower and upper mixtures of $((y^k);(z^k))$, respectively, and let $\underline{x}$ and $\overline{x}$ be their respective limits. For every adaptive dynamic $(x(k))$ in Γ,*

1. *For every N, there is K_N such that for all $k \geq K_N$, $x(k) \in [\underline{x}^N, \overline{x}^N]$.*

[19] In a GSC, the corresponding result from Milgrom and Roberts (1990) is as follows: for every $a, b \in X$ such that $a \preceq b$, $\overline{U}[a,b] = [\underline{g}(a), \overline{g}(b)]$.

[20] Notably, adaptive dynamics are not forward looking; for additional discussion, confer Milgrom and Roberts (1991).

2. $\underline{x} \preceq \liminf x(k) \preceq \limsup x(k) \preceq \overline{x}$.

Moreover, if each player has a finite strategy space, then there is K^* *such that for every* $k \geq K^*$, $\underline{x} \preceq x(k) \preceq \overline{x}$.

Proof. Statement (1) holds trivially for $N = 0$. Suppose there is K_{N-1} such that for all $k \geq K_{N-1}$, $x(k) \in [\underline{x}^{N-1}, \overline{x}^{N-1}]$. Then for all $k \geq K_{N-1}$, $[\inf P(K_{N-1}, k), \sup P(K_{N-1}, k)] \subset [\underline{x}^{N-1}, \overline{x}^{N-1}]$. Now, by definition of an adaptive dynamic, let K_N be such that for all $k \geq K_N$, $x(k) \in \overline{U}[\inf P(K_{N-1}, k), \sup P(K_{N-1}, k)]$, and consequently, for all $k \geq K_N$,

$$x(k) \in \overline{U}[\inf P(K_{N-1}, k), \sup P(K_{N-1}, k)] \subset \overline{U}[\underline{x}^{N-1}, \overline{x}^{N-1}] = [\underline{g}(\overline{x}^{N-1}), \overline{g}(\underline{x}^{N-1})] = [\underline{x}^N, \overline{x}^N],$$

where the inclusion follows from the monotonicity of $\overline{U}$, and the first equality follows from lemma 2. Statement (2) follows immediately, because $\underline{x} = \lim_N \underline{x}^N$ and $\overline{x} = \lim_N \overline{x}^N$. ∎

Lemmas 1 through 3 help formalize one of the main results in this paper.

Theorem 1. *In GSS, the following are equivalent.*

1. *Best response dynamic starting at* $\inf X$ *(or* $\sup X$*) converges*
2. *Every adaptive dynamic converges to the same Nash equilibrium*

In each case, the game has a unique Nash equilibrium.

Proof. We need only check that (1) implies (2). Let (y^k) be the best response dynamic starting at $\inf X$. Then (y^{2k}) is a subsequence of the convergent sequence $(\underline{x}^k)$ and (y^{2k+1}) is a subsequence of the convergent sequence $(\overline{x}^k)$, and therefore, if (y^k) converges, then $\underline{x} = \overline{x}$. Moreover, $\underline{x} \in g(\overline{x})$, because if $\underline{x} \notin g(\overline{x})$, then there is i, and x_i such that $f^i(x_i, \overline{x}_{-i}) - f^i(\underline{x}_i, \overline{x}_{-i}) > 0$. Let $\epsilon = f^i(x_i, \overline{x}_{-i}) - f^i(\underline{x}_i, \overline{x}_{-i}) > 0$, and let $c = f^i(\underline{x}_i, \overline{x}_{-i}) + \frac{\epsilon}{2}$. By upper semi-continuity, there is K_1 such that for all $k \geq K_1$, $f^i(\underline{x}_i^{k+1}, \overline{x}_{-i}^k) < c$. By continuity, there is K_2 such that for all $k \geq K_2$, $c < f^i(x_i, \overline{x}_{-i}^k)$. Therefore, for all $k \geq \max(K_1, K_2)$, $f^i(x_i, \overline{x}_{-i}^k) - f^i(\underline{x}_i^{k+1}, \overline{x}_{-i}^k) > 0$, contradicting the optimality of $\underline{x}_i^{k+1}$. Consequently, $\underline{x}$ is a Nash equilibrium. Lemma 3 now implies that every adaptive dynamic converges to this Nash equilibrium. The proof is similar for the best response dynamic starting at $\sup X$.

If every adaptive dynamic converges to the same Nash equilibrium, the game has a unique Nash equilibrium. For if there were two distinct Nash equilibria, then consider the following two constant sequences; each playing one of the Nash equilibria. These are two adaptive dynamics converging to distinct Nash equilibria, a contradiction. ∎

Notice that theorem 1 shows that convergence of the best response dynamic from $\inf X$ (or $\sup X$) implies a unique Nash equilibrium. It says nothing about non-convergence of this best response dynamic and existence of a Nash equilibrium. Non-convergence of this best response dynamic may be consistent with no Nash equilibrium (as in example 1-1), or it may be consistent with a unique Nash equilibrium (as in example 3-1 below).[21] The proof of

[21] Of course, a sufficient condition for non-convergence is more than one equilibrium.

theorem 1 shows that if the best response dynamic starting from $\inf X$ (or $\sup X$) does not converge, then it leads to a two-point cycle of the points $\underline{x}$, $\overline{x}$. As shown in the appendix, these points are the extremal serially undominated strategies, and also the extremal fixed points of $g \circ g$. Therefore, if the best response dynamic from $\inf X$ (or $\sup X$) does not converge, it leads to a cycle between the extremal serially undominated strategies (also the extremal simply rationalizable strategies).

Theorem 1 provides a new perspective on global stability. Say that a strategic game Γ is ***globally stable***, if it has a (pure strategy) Nash equilibrium such that every adaptive dynamic converges to this Nash equilibrium. (As shown in the proof of theorem 1, if a game is globally stable, then it has a unique Nash equilibrium.) Intuitively, in a globally stable game, all adaptive learning dynamics and strategic processes always lead to the same outcome. This is a strong definition of stability. Thus, globally stable equilibrium is robust to a large class of learning and strategic processes. Theorem 1 says that *in GSS, convergence of the best response dynamic from inf (or sup) of the strategy space is equivalent to global stability.*[22]

Theorem 1 shows that in GSS, global stability can be analyzed using a single best response dynamic, as an alternative to the traditional eigen-value approach; confer, for example, Al-Nowaihi and Levine (1985) and Okuguchi and Yamazaki (2008). Recall that eigen-value analysis requires making assumptions about each dynamic being analyzed. Theorem 1, however, yields convergence of all adaptive dynamics from knowledge of convergence of a single best response dynamic. Consequently, whether players actually play best response dynamics or not, convergence of a single best response dynamic is sufficient to conclude convergence under all adaptive behavior.

Let us now consider dominance solvability in GSS. Recall that Zimper (2007) has shown that when each player i's payoff function is supermodular in x_i and has decreasing differences in $(x_i; x_{-i})$, and when for each player i, there exists an order-continuous best response *function*, a GSS always has extremal serially undominated strategies, and a GSS is dominance solvable iff the second iterate of the (joint) best response function has a unique fixed point. The model we use here is more general. In particular, we do not make the implicit assumptions of convex strategy spaces and strictly quasi-concave payoffs to guarantee uniqueness of best responses; we work with best response correspondences. Moreover, we make assumptions on the primitive payoff functions, not on best responses. Furthermore, payoff functions here are more general: quasi-supermodular in x_i and satisfy the decreasing single crossing property in $(x_i; x_{-i})$. Finally, our proof is different, following the basic outline of Milgrom and Roberts (1990) more closely.[23] The next result shows that Zimper's results (existence of extremal serially undominated strategies and equivalence of 1 and 2 in theorem 2 below) go through for our generalizations. The proof is given in the appendix.

Theorem 2. *Let Γ be a GSS, and g be the joint best-response correspondence. The following are equivalent.*

[22] In GSC, theorem 1 is not necessarily true. Indeed, in GSC, part (1) of theorem 1 is always true.

[23] Theorem 2 is derived independently of Zimper, and includes an equivalence (item (3)) that is not in Zimper. It is our pleasure to acknowledge the earlier and independent work in Zimper (2007).

1. Γ *is dominance solvable*

2. $g \circ g$ *has a unique fixed point*

3. *Best response dynamic from* $\inf X$ *(or* $\sup X$*) converges*

In each case, the game has a unique Nash equilibrium.

Theorems 1 and 2 provide the following connections.

Corollary 1. *In GSS, the following are equivalent.*

1. *Best response dynamic starting at* $\inf X$ *(or* $\sup X$*) converges*

2. *The game is globally stable*

3. *The game is dominance solvable*

Corollary 1 brings together two different foundations for robustness of predicted outcomes in games, in terms of convergence of a single best response dynamic. Dominance solvability (and rationalizability) assumes fully informed players, using infinite iterations of rationalizing about potential (future) responses by competitors to predict a solution to a one-shot game. Global stability uses dynamic learning and strategic processes in a series of game-play over time, using past play by typically myopic players to determine present moves, and relying on limits of such learning and strategic behavior to predict an outcome robust to the dynamics. In GSS, both approaches are equivalent, and moreover, these equivalences are accessible from knowledge of convergence of a single best response dynamic.[24] Consider the following example.

Example 1-2 (Dove-Hawk-Chicken-2). Consider the Dove-Hawk-Chicken game (example 1-1), but with slightly modified payoffs, given in figure 2. The only modification is that in the top row of each matrix, player 2's payoffs are flipped. This results in D as the dominant action for player 2. Intuitively, player 2 is a type that prefers less conflict (or avoids aggression, or would prefer a more "cooperative" action).

The best response dynamic starting at (D, D, D) converges after two iterations: $(D, D, D) \mapsto (H, D, H) \mapsto (D, D, H)$. Consequently, the profile (D, D, H) is the unique Nash equilibrium, it is globally stable under all adaptive behavior, and the game is dominance solvable.[25]

[24] Using Milgrom and Roberts (1990), it is easy to see that as stated, the equivalence of parts (2) and (3) of corollary 1 is true for GSC as well. What is not true for GSC is the equivalence of (1) and (2), and the equivalence of (1) and (3)). In particular, in GSC, the best response dynamic from inf (respectively, sup) of the strategy space *always* converges to the smallest (respectively, largest) Nash equilibrium, and therefore, in GSC, convergence of either (or even both) best response dynamic does not necessarily imply global stability or dominance solvability.

[25] In contrast, the best response dynamic in example 1-1 cycles as follows: $(D, D, D) \mapsto (H, H, H) \mapsto (D, D, D)$. Example 1-1 does not have a Nash equilibrium, but even if there is a unique Nash equilibrium, best response dynamics may not necessarily converge, as shown in the simple, 3-firm Cournot oligopoly in example 3-1 below. Several additional applications are presented in the next section.

Player 3: D

Player 1 \ Player 2	D	H
D	10, 20, 10	0, 10, 10
H	20, 10, 0	10, 0, 10

Player 3: H

Player 1 \ Player 2	D	H
D	10, 10, 20	10, 0, 0
H	0, 10, 10	0, 0, 0

D <------- Player 3 -------> H

Figure 2: Dove-Hawk-Chicken-2

Recall that Moulin (1984) has shown that in all strategic games where strategy spaces are compact metric spaces and best responses are continuous *functions*, dominance solvability implies Cournot stability (convergence of all best response dynamics). Zimper (2007)'s results imply that in GSS with (order) continuous best response functions, Cournot stability is equivalent to dominance solvability. Our results extend these to show that in our more general model of GSS, global stability (convergence of every adaptive dynamic, not just best response dynamics) is equivalent to dominance solvability.[26]

Notice that the results here show that in GSS, convergence of Cournot dynamics is equivalent to convergence of adaptive dynamics. This is not necessarily true more generally. Consider the following example.

Example 1-3 (Cournot versus Adaptive Dynamics). Consider the following symmetric, two-player game.[27] Each player's strategy space is $X_i = \{0\} \cup \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$, and each player's best response function is given by $g^i(0) = 1$, and for $n \in \mathbb{N}$, $g^i(\frac{1}{n}) = \frac{1}{n+1}$. It is easy to check that every best-response dynamic converges to $(0, 0)$. It can be shown that the sequence with even terms given by $(0, 0)$ and odd terms given by $(1, 1)$ is an adaptive dynamic that does not converge.

Fixed points of $g \circ g$ play a significant role in the analysis of GSS. These may be motivated in terms of strategy profiles that are rationalizable with short cycles of justification, in the spirit of Bernheim (1984). That is, suppose $x \in g \circ g(x)$. In this case, let $y \in g(x)$ such that $x \in g(y)$. Then the profile of strategies x is rationalizable with the following cycle of conjectures. For each i, player i plays x_i because she believes her opponents shall play y_{-i}, because each of her opponents j further believes that his opponents shall play x_{-j}. Say that a profile of strategies x is ***simply rationalizable***, if there is a strategy profile y such that for every player i, x_i can be justified by such a short cycle of conjectures.

The reasoning above shows that if a profile of strategies x is a fixed point of $g \circ g$, then it is simply rationalizable. In the other direction, it is easy to check that if for each i, player

[26]It may be possible to expand the scope of these results to additional dynamics. For example, replicator dynamics, and better response dynamics, as discussed in Young (2004), and regret-matching, as in Hart and Mas-Colell (2000) are known to converge to Nash equilibrium on dominance solvable games. Expanding our results to more general dynamics is left for future work.

[27]This game is due to Hans Heller, as attributed in Zimper (2006).

i plays x_i because that is a best response to her belief that her opponents shall play y_{-i}, because each of her opponents j best responds with y_j based on his further belief that his opponents shall play x_{-j}, then the profile of strategies x is a (joint) best response to the profile of strategies y, and y is a best response to x, whence x is a fixed point of $g \circ g$.

Thus, we may view fixed points of $g \circ g$ as strategy profiles that can be (simultaneously) rationalized by no more than two iterations of behavioral conjectures.

Intuitively, simply rationalizable strategies do not rely on high orders of deduction. They may be viewed as outcomes of low-level rationalization, exhibiting a type of bounded rationality. Nash equilibria are simply rationalizable (x and y are the same), but in general, simply rationalizable strategies may include more strategies than Nash strategies.[28] Moreover, simply rationalizable strategies may form only a strict subset of all rationalizable strategies, because rationalizable strategies include conjectural cycles of all orders.[29]

In this terminology, in GSS, simple rationalization always predicts an outcome (even if a GSS has no Nash equilibrium). Notably, in example 1-1, there is no Nash equilibrium, but the profiles (D, D, D) and (H, H, H) are simply rationalizable. Zimper (2007) shows that in his model, there are always extremal serially undominated strategies, and these are simply rationalizable. In our more general model, the lemma in the appendix shows the same result. Moreover, we show that the extremal serially undominated strategies are also the extremal fixed points of $g \circ g$; that is, are the extremal simply rationalizable strategies.

Moreover, if simple rationalization predicts a unique outcome, then higher-level rationalization has no additional benefit. Furthermore, extremal simply rationalizable strategies bound limiting behavior of all adaptive dynamics.

3 Applications

Combining the equivalences in theorem 1, theorem 2, and corollary 1 allow for powerful cross-derivation of results. For example, if it is easy to know that a GSS has a unique profile of simply rationalizable strategies, then we may conclude that the game is globally stable, and every adaptive dynamic converges to the unique equilibrium. Similarly, if we can compute the convergence of the best-response dynamic from the inf (or sup) of the strategy space, we may conclude that the game is globally stable, and it is dominance solvable. A direct computation of best-response dynamics may be useful in other cases. The following applications explore these ideas.

Example 2 (Linear best responses). GSS with linear best responses arise in several

[28] Zimper (2007) provides an example of a three-player, three-action GSS with a unique Nash equilibrium, but with at least two simply rationalizable strategy profiles. A simple, textbook Cournot oligopoly example is provided in example 3 below.

[29] Indeed, in the original example in Bernheim (1984), there are rationalizable strategies that are not simply rationalizable. Our approach may be extended to define order-n rationalizable strategies as fixed points of the n-th iterate of g. Order-1 strategies are Nash profiles, order-2 strategies are simply rationalizable, and so on. We need only order-2 strategies in this paper, and mention their behavioral properties. Higher order rationalizability is not needed for this paper, and is not explored here.

contexts. They can be analyzed as follows. Consider a GSS with finitely many players, N, where each $X^i = [0, x_i^{\max}] \subset \mathbb{R}$, and $X = [0, x^{\max}] \subset \mathbb{R}^N$.[30] Suppose the joint best response function, $g : X \to X$, is such that $g(x) = a + Bx$,[31] where each component of the $N \times N$ matrix B is non-positive. Thus, g is nonincreasing. Moreover, for range of g to be in X, suppose $-Bx^{\max} \leq a \leq x^{\max}$. Notice that the game has a unique equilibrium, if, and only if, the matrix $(I - B)$ is invertible. In this case, the unique equilibrium is $x^* = (I - B)^{-1}a$. The iterated best response is given by $g \circ g(x) = a + Ba + B^2x$, and therefore, $g \circ g$ has a unique fixed point, if, and only if, the matrix $(I - B^2)$ is invertible. As determinant calculations are easy to make, this result is useful in applications. Consider the following examples.

Example 2-1 (Common-pool resources). Consider a 3-player common-pool resource game.[32] Each player has an endowment $w > 0$. There are two investment options – a common resource (such as a fishery) that exhibits diminishing marginal return, and an outside option with diminishing marginal return. If player i invests an amount $x_i \leq w$ of his endowment into the common resource, he receives a proportional share of the total output $\frac{x_i}{x_1+x_2+x_3}(a(x_1 + x_2 + x_3) - b(x_1 + x_2 + x_3)^2)$, and he receives $r(w - x_i) - s(w - x_i)^2$ on the outside investment $w - x_i$. (Here, $a, b, r, s > 0$.) Thus, payoff to player i is

$$f^i(x_1, x_2, x_3) = r(w - x_i) - s(w - x_i)^2 + \frac{x_i}{x_1 + x_2 + x_3}\left(a(x_1 + x_2 + x_3) - b(x_1 + x_2 + x_3)^2\right),$$

if $x_1 + x_2 + x_3 > 0$, and $rw - sw^2$, otherwise. Notice that best response of player i is given by $g^i(x_j, x_k) = \frac{a-r+2sw}{2b+2s} - \frac{b}{2b+2s}(x_j + x_k)$. For range of g^i to lie in $[0, w]$, we assume $\frac{a-r}{2b} \leq w \leq \frac{a-r}{2(b-s)}$.[33] Matrix B is given by $B = \begin{bmatrix} 0 & -\frac{b}{2b+2s} & -\frac{b}{2b+2s} \\ -\frac{b}{2b+2s} & 0 & -\frac{b}{2b+2s} \\ -\frac{b}{2b+2s} & -\frac{b}{2b+2s} & 0 \end{bmatrix}$. Let $\xi = (\frac{b}{2b+2s})^2$. Then $\frac{b}{2b+2s} < \frac{1}{2}$ implies $\xi < \frac{1}{4}$. Therefore, $\det(I - B^2) = (1 - 2\xi)(1 - 4\xi + \xi^2) - 2\xi^3 > (1 - 2\xi)\xi^2 - 2\xi^3 = \xi^2(1 - 4\xi) > 0$. Therefore, $I - B^2$ is invertible, the game is globally stable, the unique equilibrium is robust to all adaptive behavior, and several solution concepts all predict the same unique equilibrium outcome.

Example 2-2 (Private provision of public goods). Suppose there are finitely many consumers, indexed $i = 1, \ldots, N$, and there are two goods – good 1 is numeraire, indexed y, and good 2 is a public good, indexed x, and all variables are measured in terms of the numeraire (or in units of account). Suppose consumer i's utility is Cobb-Douglas, given by $u^i(y_i, x_1, \ldots, x_N) = y_i^\alpha(x_1 + \ldots + x_N)^\beta$, where $\alpha, \beta > 0$. Each consumer's budget constraint is $y_i + x_i = w_i$. Substituting $x_i - w_i$ for y_i, it is easy to calculate that player 1's best response function is given by $g^1(x_2, \ldots, x_N) = \frac{\beta w_1}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}(x_2 + \ldots + x_N)$, and similarly for the other players. To ensure best-responses remain non-negative, we impose the constraint $\alpha(N - 1)w^{\max} \leq \beta w_i$, for each player i. (Here, $w^{\max} = \max_i w_i$.) The matrix B is given

[30] Here, $x^{\max}$ is the vector with i-th component $x_i^{\max}$.

[31] Technically, this is an affine function, but deferring to standard terminology, we term it linear.

[32] See, for example, Ostrom, Gardner, and Walker (1994). Additional analysis of this game as a GSS is presented in Roy and Sabarwal (2010).

[33] Hence, we need $a > r$ and $b > s$.

by $B = \frac{1}{\alpha+\beta}\begin{bmatrix} 0 & -\alpha & \cdots & -\alpha \\ -\alpha & 0 & \cdots & -\alpha \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha & -\alpha & \cdots & 0 \end{bmatrix}$. It is easy to check that each entry of B^2 is non-negative, and moreover, the sum of each row of B^2 is $(\frac{(N-1)\alpha}{\alpha+\beta})^2$, which is strictly less than one. Therefore, $\sum_{n=0}^{\infty}(B^2)^n = (I - B^2)^{-1}$. Consequently, this game is globally stable, and dominance solvable. A similar example can be worked out with utility given by constant elasticity of substitution.

Example 3 (Cournot Oligopoly) The Cournot oligopoly model is used extensively in economics. The following examples analyze some commonly used variations of this model.

Example 3-1 (Cournot oligopoly; linear demand, linear cost). Consider a 3-firm Cournot oligopoly with linear inverse demand, $p = a - b(x_1 + x_2 + x_3)$, constant marginal cost, $c > 0$, and with production capacity constrained to $[0, x^{\max}]$ for each firm. For range of the best-responses to be in the strategy space, suppose $x^{\max} = \frac{a-c}{2b}$. In this case, the joint best response function is given by $g(x_1, x_2, x_3) = (\frac{a-c-b(x_2+x_3)}{2b}, \frac{a-c-b(x_1+x_3)}{2b}, \frac{a-c-b(x_1+x_2)}{2b})$, and the unique Nash equilibrium is given by $(x_1, x_2, x_3) = (\frac{a-c}{4b}, \frac{a-c}{4b}, \frac{a-c}{4b})$. Moreover, $g \circ g(x_1, x_2, x_3) = (\frac{2x_1+x_2+x_3}{4}, \frac{x_1+2x_2+x_3}{4}, \frac{x_1+x_2+2x_3}{4})$, and it is easy to see that every point on the diagonal of $[0, x^{\max}]^3$ is a fixed point of $g \circ g$. In particular, $(0, 0, 0)$ is the smallest simply rationalizable strategy and $(\frac{a-c}{2b}, \frac{a-c}{2b}, \frac{a-c}{2b})$ is the largest. Thus, this game is neither globally stable, nor Cournot stable, nor dominance solvable.[34] In fact, the best-response dynamic starting at $(0, 0, 0)$ cycles with $(\frac{a-c}{2b}, \frac{a-c}{2b}, \frac{a-c}{2b})$, and serially undominated strategies provide no help in narrowing the range of predicted outcomes.

This example may also be viewed as a game with linear best-responses, with $B = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$. It is easy to check that $I - B$ is invertible, but $I - B^2$ is not. Thus, the game has a unique Nash equilibrium, but is neither globally stable nor dominance solvable.

Example 3-2 (Cournot oligopoly; linear demand, quadratic cost). Consider the same 3-firm Cournot oligopoly, but with quadratic cost, cx_i^2, with $c > 0$. In this case, the joint best-response function is given by $g(x_1, x_2, x_3) = (\frac{a-b(x_2+x_3)}{2b+2c}, \frac{a-b(x_1+x_3)}{2b+2c}, \frac{a-b(x_1+x_2)}{2b+2c})$. Using the same technique as in example 2-1, we may conclude that this game is globally stable, the dominance solution (and several other solution concepts nested within the dominance solution) predicts a unique outcome, and the unique Nash equilibrium is robust to all adaptive behavior.

Example 3-3 (Differentiated goods Cournot oligopoly). Consider finitely many firms, indexed $i = 1, \ldots, N$, each facing demand curve $p_i = \alpha - \beta x_i - \delta(x_1 + \ldots + x_{i-1} + x_{i+1} + \ldots x_N)$, each with constant marginal cost c, and production constrained to $[0, x^{\max}]$. To introduce differentiated goods, assume that $\beta \neq \delta$; that is, the impact of self production on price is different from the impact of competitor output on price. It is easy to calculate that firm 1's

[34] Recall that Zimper (2007) provides an example of a three-player, three-action GSS with a unique Nash equilibrium, and in which every strategy profile is serially undominated.

best response function is given by $g^1(x_2, \ldots, x_N) = \frac{\alpha-c}{2\beta} - \frac{\delta}{2\beta}(x_2 + \ldots + x_N)$, and similarly for the other firms. Using the same technique as in example 2-2, we may conclude that this game is globally stable, and dominance solvable, and every adaptive dynamic converges to the unique Nash equilibrium. Notice that additional heterogeneity can be introduced in this example by varying α, β, δ, $x^{\max}$ and c by firm. Moreover, a similar example can be formulated using quadratic, or more general costs.

Example 3-4 (General symmetric Cournot oligopoly). Consider a N-firm Cournot oligopoly, with inverse demand curve given by $P(x_1 + \ldots + x_N)$, cost of firm i given by $C(x_i)$, both these functions are twice continuously differentiable, and production is constrained to $[0, x^{\max}]$. Assume that demand is downward sloping and cost is convex. The first-order condition for firm i is given by $x_i P' + P - C' = 0$, and therefore, the slope of the best response of firm i with respect to firm j's output is given by $-\frac{x_i P'' + P'}{x_i P'' + 2P' - C''}$. This is a GSS, if $x_i P'' + P' < 0$.

As in the tournaments example above, if a best response dynamic starts anywhere on the diagonal in $[0, x^{\max}]^N$, then it remains entirely on the diagonal. This reduces the problem to checking convergence only on the diagonal. Write the best-response dynamic implicitly as $yP'(y + (N-1)x) + P(y + (N-1)x) - C'(y) = 0$, where y is the best response to each competitor playing x. This process is globally asymptotically stable, if $\left|\frac{dy}{dx}\right| < 1$, which is satisfied if $|C'' - P'| > (N-2)|yP'' + P'|$. In this case, this game is dominance solvable and the unique equilibrium is globally stable under *all* adaptive behavior. This result is valid for large N, as compared to, for example, Al-Nowaihi and Levine (1985).

Let us apply this result to a specific functional form. As in Amir (1996), suppose inverse demand is given by $P(x_1 + \ldots + x_N) = \frac{1}{(x_1 + \ldots + x_N + 1)^\alpha}$, where $\alpha > 0$, and cost is given by $C(x_i) = \frac{1}{2}cx_i^2$. In this case, best responses are downward sloping, if $0 < \alpha < \frac{1}{x^{\max}}$. Moreover, the condition $|C'' - P'| > (N-2)|yP'' + P'|$ is automatically satisfied for $N = 2, 3$. More generally, it is satisfied, if $c > (N-3)\alpha$. Thus, the best response dynamic on the diagonal converges, if $\alpha < \min\left\{\frac{c}{N-3}, \frac{1}{x^{\max}}\right\}$. In particular, if we normalize production so that $x^{\max} = 1$, then the condition holds for $0 < \alpha < 1$, and $c > N - 3$. Notice that this result is valid for large N, as compared to, for example, Al-Nowaihi and Levine (1985).

Example 4 (Tournaments). Suppose a tournament[35] has 3 players, where a reward $r > 0$ is shared by the players who succeed in the tournament. If one player succeeds, he gets r for sure, if two players succeed, each gets r with probability one-half, and if all players succeed, each gets r with probability one-third. Each player chooses effort $x_i \in [0, 1]$ with probability of success x_i. Expected reward per unit for player i is

$$\pi^i(x_i, x_j, x_k) = x_i(1 - x_j)(1 - x_k) + \frac{1}{2}x_i x_j(1 - x_k) + \frac{1}{2}x_i x_k(1 - x_j) + \frac{1}{3}x_i x_j x_k.$$

The quadratic cost of effort x_i is cx_i^2. The payoff to player i is expected reward minus cost of effort. That is, $f^i(x_i, x_j, x_k) = r\pi^i(x_i, x_j, x_k) - cx_i^2$. It is easy to calculate that best response of player i is given by $g^i(x_j, x_k) = \frac{r}{2c}(1 - \frac{1}{2}(x_j + x_k) + \frac{1}{3}x_j x_k)$. Suppose, for convenience, $r = 2c$.

[35] This version is based on Dubey, Haimanko, and Zapechelnyuk (2006). Additional analysis of this game as a GSS is presented in Roy and Sabarwal (2010).

This game is globally stable, as follows. Notice that if a best response dynamic starts anywhere on the diagonal in $[0,1]^3$, then it remains entirely on the diagonal. This reduces the problem to checking convergence only on the diagonal. Using, $x_1 = x_2 = x_3 = x$, say, this reduces the problem to checking if the iterated dynamic given by $\gamma(x) = 1 - x + \frac{1}{3}x^2$ converges. (Here, γ is the best-response of an arbitrary player, reduced to one-dimension, using symmetry.) Notice that $|\gamma'(x)| = 1 - \frac{2}{3}x < 1$, if $x > 0$. Therefore, γ satisfies the contraction principle over $[\epsilon, 1+\epsilon]$ for every sufficiently small $\epsilon > 0$. In particular, the best-response dynamic starting at $(1,1,1)$ converges. Consequently, every adaptive dynamic converges, the game is globally stable, and dominance solvable. (For reference, the unique globally stable equilibrium is $(3-\sqrt{6}, 3-\sqrt{6}, 3-\sqrt{6})$.)

Example 5 (Two-player GSS and two-player GSC). For two-player GSS, global stability is equivalent to uniqueness of Nash equilibrium, as follows. It is known that in a two-player GSS, dominance solvability is equivalent to uniqueness of Nash equilibrium, because a two-player GSS can be viewed as a GSC by reversing the order on the strategy space of one of the players. Corollary 1 then yields the desired equivalence.[36]

Similarly, a two-player GSC may be viewed as a GSS by reversing the order on the strategy space of one of the players. Therefore, the results here apply to such games. In particular, (in the original order in a two-player GSC,) if the best-response dynamic starting from $(\inf X_1, \sup X_2)$ (or from $(\sup X_1, \inf X_2)$) converges, then the game is globally stable, and dominance solvable. Similarly, if a two-player GSC has a unique Nash equilibrium, it is globally stable. For example, in many commonly used specifications of Bertrand duopoly (see, Milgrom and Roberts (1990)), there is a unique equilibrium. Such duopolies are globally stable.

Example 5-1 (Two-firm R&D game). Consider the R&D game in Amir and Wooders (2000). Two *ex ante* identical firms engage in R&D investments to lower their costs. There is a non-zero probability of R&D innovation spillover from the innovator to the imitator. After innovation, the firms then compete in output markets: the model allows a unifying treatment of various versions of Cournot and Bertrand competition in the output market. Suppose each firm's (identical) unit cost is given by c, firm 1 chooses R&D activity to yield unit cost reduction x, firm two chooses unit cost reduction y, and following any such choice, there is a unique Nash equilibrium in the product market (they present sufficient conditions for this to occur). The question of interest is the equilibrium choice of R&D levels. Under their assumptions, the best response of each firm is decreasing in the other firm's choice of R&D, leading to a GSS. They show that there is a level $d \in (0,c)$, such that restricted to $[d,c] \times [0,d]$, there is a unique Nash equilibrium in R&D levels, and restricted to $[0,d] \times [d,c]$, there is a unique Nash equilibrium. In one case, firm 1 is a R&D leader and firm 2 is an imitator, the reverse is true in the other case. Therefore, starting from an *ex ante* similar situation, there is an endogenously asymmetric outcome.

[36] As shown above, this result does not extend to games with more than two players; Zimper (2007) provides a three-player, three-action counter-example, and the 3-firm Cournot oligopoly with linear demand and constant marginal cost (example 3-1) provides a counter-example. Similarly, although a Nash equilibrium is guaranteed in a 2-player GSS, existence of equilibrium does not extend to more than two players, as shown in the Dove-Hawk-Chicken game (example 1-1).

The results here provide insight into robustness of this asymmetric outcome. In particular, once choice of R&D levels reaches the box $[d, c] \times [0, d]$, the best responses stay in this box, and (because there is a unique Nash equilibrium in this box,) every adaptive dynamic converges to the unique equilibrium. (Similarly, for the box $[0, d] \times [d, c]$.) Therefore, our results show that once (sufficient) asymmetry in R&D emerges, even if play is not in equilibrium, *all* adaptive behavior reinforces the choice of Nash equilibrium related to this asymmetry. Similarly, consideration of playing (serially) undominated strategies leads to the same Nash equilibrium.[37]

A useful condition to check for dominance solvability is provided in Zimper (2006). In a GSS, if each player's strategy space is a non-empty, compact, convex interval of the reals, and each player's best response function is continuously differentiable, then if either of the following conditions is satisfied, the game is dominance solvable, and hence, globally stable: (1) for every $x \in X$, and for every player i, $\sum_{j \in I} \left| \frac{\partial g_i^2}{\partial x_j}(x) \right| < 1$, or (2) for every $x \in X$, and for every player j, $\sum_{i \in I} \left| \frac{\partial g_i^2}{\partial x_j}(x) \right| < 1$. This provides a sufficient condition useful in applications where $g \circ g$ is easy to derive.[38]

The connections shown in this paper provide several equivalent techniques to check for robustness of equilibrium predictions in GSS. Uniqueness of simply rationalizable strategies may be easy to check in games with linear best responses, convergence of best response dynamics may be easy to check in cases with non-linear best responses, especially in the presence of some symmetry, and a direct computation of best response dynamics may be useful in discrete games.

Notice that specialized results are possible in particular situations. For example, as shown in Jensen (2009), in strictly quasi-concave aggregative games with strategic substitutes, with one-dimensional strategy sets, and with a unique pure strategy Nash equilibrium, sequential best-reply dynamics converge to the unique Nash equilibrium.[39] As sequential best reply dynamics are adaptive dynamics, this result shows that in particular cases, a subset of the set of adaptive dynamics may be well-behaved. Such a result is useful, if we have in mind particular subclasses of dynamics and wish to determine if these are well-behaved.

4 Stability of Monotone Equilibrium Selections

To formalize ideas of dynamic stability of equilibria in parameterized GSS, we need some notions about best response dynamics starting at arbitrary points in the strategy space.

Let Γ be a GSS, and $y \preceq z$ be elements of X. The (simultaneous) ***best response dynamic-1*** starting at y is the sequence $(y^k)_{k=0}^{\infty}$, where $y^0 = y$, and for $k \geq 1$, $y^k = \underline{g}(y^{k-1})$ if k is even, and $y^k = \overline{g}(y^{k-1})$ if k is odd. Similarly, the (simultaneous) ***best response***

[37] Several additional applications in the same vein may be found in Amir, Garcia, and Knauff (2010).

[38] We are grateful to an anonymous referee for pointing this out.

[39] Jensen (2009)'s result does not extend to simultaneous best-reply dynamics, as shown by the Cournot oligopoly with linear demand and linear cost (example 2) above.

dynamic-2 starting at z is the sequence $(z^k)_{k=0}^{\infty}$, where $z^0 = z$, and for $k \geq 1$, $z^k = \overline{g}(z^{k-1})$ if k is even, and $z^k = \underline{g}(z^{k-1})$ if k is odd. Notice that when $y = \inf X$, best response dynamic-1 is the best response dynamic starting at $\inf X$, and when $z = \sup X$, best response dynamic-2 is the best response dynamic starting at $\sup X$. Moreover, both best response dynamics coincide when g is a function and $y = z$.

Given $y \preceq z$ and best response dynamics 1 and 2, (y^k) and (z^k), the definition of the lower and upper mixtures of $((y^k);(z^k))$ remains the same; $(\underline{x}^k)_{k=0}^{\infty}$ is given by $\underline{x}^k = y^k$, if k is even, and $\underline{x}^k = z^k$, if k is odd, and $(\overline{x}^k)_{k=0}^{\infty}$ is given by $\overline{x}^k = z^k$, if k is even, and $\overline{x}^k = y^k$, if k is odd.

The definition of an adaptive dynamic is similar. Following Echenique (2002), a process $(x(k))_{k=0}^{\infty}$ is an ***adaptive dynamic*** in the game Γ if there is $\gamma > 0$ such that for all $k \geq 0$, $x(k) \in \overline{U}[\inf P(k-\gamma, k), \sup P(k-\gamma, k)]$, where as earlier, $P(k-\gamma, k)$ is the history of past play from $k-\gamma$ to k; that is, $P(k-\gamma, k) = \{x(k-\gamma), x(k-\gamma+1), \ldots, x(k-1)\}$. By convention, when $\gamma \geq k$, we set $k - \gamma = 0$. It is easy to check that this is a special case of our earlier definition, using discrete time and a uniform bound on the length of history to affect a current decision.[40]

The following lemma presents a generalization of lemma 3. Its proof is given in the appendix. The proof builds on techniques from Echenique (2002), adjusted for the special challenges that arise when dealing with strategic substitutes.

Lemma 4. *Let $y \preceq z$, (y^k) and (z^k) be best response dynamics 1 and 2, respectively, and $(\underline{x}^k)$ and $(\overline{x}^k)$ be their lower and upper mixtures, respectively. For every $x^0 \in [y, z]$, and for every adaptive dynamic $(x(k))$ starting at x^0, the following is true.*

1. *For every N, there is K_N, such that for all $k \geq K_N$, $x(k) \in [\underline{x}^N, \overline{x}^N]$.*
2. *If $y^0 \preceq y^2$, then there exist simply rationalizable $\underline{y}$, $\overline{y}$ such that*

$$\underline{y} \preceq \liminf x(k) \preceq \limsup x(k) \preceq \overline{y}.$$

3. *If $z^2 \preceq z^0$, then there exist simply rationalizable $\underline{z}$, $\overline{z}$ such that*

$$\underline{z} \preceq \liminf x(k) \preceq \limsup x(k) \preceq \overline{z}.$$

Notice the initial monotonicity condition in parts 2 and 3. These conditions are automatically satisfied when $y^0 = \inf X$ and $z^0 = \sup X$, as was the case earlier, but may not necessarily be satisfied more generally. Below, we shall consider and motivate cases when these initial monotonicity conditions are satisfied.

Parameterized games with strategic substitutes are defined as follows. As earlier, consider a set of players I, and for each player i, a partially ordered strategy space $(X^i, \preceq^i)$, and the overall strategy space X.

[40] This specialization is helpful in generalizing lemma 3. See the double-induction argument in the proof of lemma 4.

Moreover, consider a partially ordered set of parameters, T.[41] We restrict the parameter space to satisfy a basic "density" property; that is, we assume that for every order interval $[\underline{t}, \overline{t}]$ in T and for every $\hat{t}$ such that $\underline{t} \prec \hat{t} \prec \overline{t}$, every neighborhood of $\hat{t}$ contains $t_0, t_1 \in [\underline{t}, \overline{t}]$ such that $t_0 \prec \hat{t} \prec t_1$. Notice that this property is fairly basic. In particular, a convex $T \subset \mathbb{R}^n$, as assumed in Echenique (2002), is admissible. This property rules out parameter spaces where order intervals contain isolated points.

Each player i has a payoff function, $f^i : X \times T \to \mathbb{R}$, denoted $f^i(x_i, x_{-i}, t)$. The collection $\Gamma = (I, T, (X^i, \preceq^i, f^i)_{i \in I})$ is a ***parameterized game with strategic substitutes, (parameterized GSS)***, if for every player i,

- $(X^i, \preceq^i)$ is a complete lattice,
- For every t, f^i is upper semi-continuous in (x_i, x_{-i}), and for every x_i, f^i is order continuous in (x_{-i}, t),
- For every (x_{-i}, t), f^i is quasi-supermodular in x_i,
- For every x_{-i}, f^i satisfies single-crossing property in $(x_i; t)$, and
- For every t, f^i satisfies decreasing single-crossing property in $(x_i; x_{-i})$.

As usual, single-crossing property in $(x_i; t)$ implies that each player's best response, $g^i(x_{-i}, t)$ is nondecreasing in the parameter, a standard formulation. As earlier, each player's best response is nonincreasing in other player strategies. Thus, the joint best response, $g(x, t)$ is nondecreasing in t and nonincreasing in x.

As usual, for each $t \in T$, a parameterized GSS, Γ, naturally defines a GSS, $\Gamma(t)$, with the same strategy spaces as Γ and with appropriate sections of the payoff functions. Let $\mathcal{E}(t)$ denote the set of (pure strategy) Nash equilibria in $\Gamma(t)$.

An ***equilibrium selection*** is a function $e : T \to X$ such that for every t, $e(t) \in \mathcal{E}(t)$. An equilibrium selection $e : T \to X$ is ***nowhere weakly increasing on*** $\left[\underline{t}, \overline{t}\right]$, if for every $t_0, t_1 \in \left[\underline{t}, \overline{t}\right]$, $t_0 \prec t_1$ implies $e(t_0) \npreceq e(t_1)$.[42] An equilibrium selection $e : T \to X$ is ***strictly increasing*** if it is nondecreasing and for every $t_0 \prec \hat{t} \prec t_1$, $[e(t_0), e(t_1)]$ is a neighborhood of $e(\hat{t})$ in X.[43] For notational convenience, we sometimes denote $g(\cdot, t)$ as $g_t(\cdot)$.

Consider the following conditions. An equilibrium selection $e : T \to X$ satisfies ***condition 1*** on $\left[\underline{t}, \overline{t}\right]$, if for every $t_0, \hat{t}$ in $\left[\underline{t}, \overline{t}\right]$ such that $t_0 \preceq \hat{t}$, $e(t_0) \preceq \underline{g}(\overline{g}(e(t_0), \hat{t}), \hat{t})$. An equilibrium selection $e : T \to X$ satisfies ***condition 2*** on $\left[\underline{t}, \overline{t}\right]$, if for every $\hat{t}, t_1$ in $\left[\underline{t}, \overline{t}\right]$ such that $\hat{t} \preceq t_1$, $\overline{g}(\underline{g}(e(t_1), \hat{t}), \hat{t}) \preceq e(t_1)$.

[41] For convenience, the partial order on T is denoted by the same symbol, $\preceq$, and T is assumed to have the standard order interval topology.

[42] As described in Echenique (2002), this is stronger than the negation of weakly increasing.

[43] When X is in some finite dimensional Euclidean space, as in Echenique (2002), this definition is equivalent to $t_0 \prec t_1 \Rightarrow e(t_0) \ll e(t_1)$. Another relevant case is when X is a subset of a Banach lattice that has a positive cone with a nonempty interior.

As shown in Roy and Sabarwal (2010), in GSS, conditions 1 and 2 present a natural tradeoff between a direct parameter effect and an indirect strategic substitute effect, as follows. Suppose g is a *function.* Starting from an existing equilibrium, $e(t_0)$ at $t = t_0$, an increase in t to $\hat{t}$ has two effects on, say, player i's best response function, $g^i(\cdot, \cdot)$. The direct parameter effect is an increase in g^i, because best-response is nondecreasing in t. The indirect strategic substitute effect is a decrease in g^i, because an increase in t increases the best response of the competitors of i, and their actions are strict substitutes for player i. Thus, $e(t_0) \preceq g(g(e(t_0), \hat{t}), \hat{t})$ in condition 1 says that for each player, the indirect strategic substitute effect does not dominate the direct parameter effect when the parameter goes up. Condition 2 makes the analogous statement when the parameter goes down.[44]

Notice that for parameterized GSC, both the direct and the indirect effects work in the same direction. Therefore, once the direct parameter effect is assumed to be favorable, (as formalized, for example, by a strict single crossing property in $(x_i; t)$,) the indirect strategic complement effect serves to reinforce the direct effect, and the conditions above are satisfied. In particular, Echenique (2002) does not use conditions 1 and 2, but implicitly assumes a strict single-crossing property (correspondences are assumed to be strongly increasing in t). This has the same effect.

Conditions 1 and 2 are useful to apply lemma 4, as follows. Consider $t_0 \preceq \hat{t}$. If condition 1 is satisfied, then the best response dynamic-1 given by $y^0 = e(t_0)$, and for $k \geq 1$, $y^k = \underline{g}(y^{k-1}, \hat{t})$ if n is even, and $y^k = \overline{g}(y^{k-1}, \hat{t})$ if n is odd has the feature that $y^0 \preceq y^2$. This allows us to apply item 2 of lemma 4. Similarly, consider $\hat{t} \preceq t_1$. If condition 2 is satisfied, then the best response dynamic-2 given by $z^0 = e(t_1)$, and for $k \geq 1$, $z^k = \underline{g}(z^{k-1}, \hat{t})$ if n is odd, and $z^k = \overline{g}(z^{k-1}, \hat{t})$ if n is even has the feature that $z^2 \preceq z^0$. This allows us to use item 3 of lemma 4. (For GSC, a strict single-crossing property has the same effect.)

The results here show the importance of focusing on an appropriate tradeoff between the direct parameter effect and the indirect strategic effect rather than on a strict single-crossing property. For parameterized GSS, it is precisely the reversed nature of the indirect strategic substitute effect that requires conditions 1 and 2 as the analogous conditions. Indeed, we do not require a strict single-crossing property and correspondences are not assumed to be strongly increasing in t.

To state and prove an analogue of the correspondence principle for GSS, consider the following notions of stability. Let Γ be a parameterized GSS and $t \in T$. A point $\hat{x} \in X$ is ***weakly stable at*** t, if there is a neighborhood V of $\hat{x}$ that has at least two points, such that for every $x \neq \hat{x}$ in V, there is an adaptive dynamic $(x(k))$ in $\Gamma(t)$ that starts at x and converges to $\hat{x}$.[45] A point $\hat{x} \in X$ is ***strongly stable at*** t, if there is a neighborhood V of $\hat{x}$ such that for every $x \in V$, every adaptive dynamic $(x(k))$ in $\Gamma(t)$ that starts at x converges to $\hat{x}$.[46] For notational convenience, we sometimes denote $g(\cdot, t)$ as $g_t(\cdot)$.

[44] For GSS, Roy and Sabarwal (2010) present conditions on payoff functions and on best responses under which the above conditions hold.

[45] Notice that the requirements that V have at least two points and $x \neq \hat{x}$ in V are needed, because we allow for finite lattices, and therefore, the discrete topology. In such a case, the singleton containing $\hat{x}$ is a neighborhood of $\hat{x}$, and the constant dynamic starting at $\hat{x}$ converges trivially to $\hat{x}$.

[46] Echenique (2002) terms these best-case stable and worst-case stable, respectively.

Theorem 3. ***(Correspondence Principle)*** *Let Γ be a parameterized GSS and e be a continuous equilibrium selection.*
(1) If e is nowhere weakly increasing and satisfies condition 1 on $\left[\underline{t}, \overline{t}\right]$, then for every $\hat{t}$ such that $\underline{t} \prec \hat{t} \prec \overline{t}$, $e(\hat{t})$ is not weakly stable at $\hat{t}$.
(2) If e is strictly increasing and satisfies conditions 1 and 2 on $\left[\underline{t}, \overline{t}\right]$, then for every $\hat{t}$ such that $\underline{t} \prec \hat{t} \prec \overline{t}$ and $e(\hat{t})$ is an isolated fixed point of $g_{\hat{t}} \circ g_{\hat{t}}$, $e(\hat{t})$ is strongly stable at $\hat{t}$.

Proof. Consider (1). Fix $\hat{t}$ such that $\underline{t} \prec \hat{t} \prec \overline{t}$. Consider $e(\hat{t})$, and an arbitrary neighborhood V of $e(\hat{t})$ with at least two points. By continuity of e, let t_0 be such that $\underline{t} \preceq t_0 \prec \hat{t}$ and $e(t_0) \in V$. Then, by nowhere weakly increasing, $e(t_0) \not\preceq e(\hat{t})$. Consider an arbitrary adaptive dynamic $(x(k))$ in $\Gamma(\hat{t})$ starting at $x(0) = e(t_0)$. Let $y^0 = e(t_0)$ and for $k \geq 1$, $y^k = \underline{g}_{\hat{t}}(y^{k-1})$ if n is even, and $y^k = \overline{g}_{\hat{t}}(y^{k-1})$ if n is odd. By condition 1, $y^0 \preceq y^2$. Therefore, by lemma 4, $e(t_0) \preceq \underline{y} \preceq \liminf x(k)$, whence $x(k) \not\to e(\hat{t})$.

Consider (2). Fix $\hat{t}$ such that $\underline{t} \prec \hat{t} \prec \overline{t}$ and $e(\hat{t})$ is an isolated fixed point of $g_{\hat{t}} \circ g_{\hat{t}}$. Let N be a neighborhood of $e(\hat{t})$ such that $N \cap \mathcal{E}(\hat{t}) = \left\{e(\hat{t})\right\}$. As e is continuous, let $t_0, t_1 \in \left[\underline{t}, \overline{t}\right]$ be such that $t_0 \prec \hat{t} \prec t_1$, and $e(t_0)$ and $e(t_1)$ are in N. As e is strictly increasing, $[e(t_0), e(t_1)]$ is a neighborhood of $e(\hat{t})$. Consequently, $V = [e(t_0), e(t_1)] \cap N$ is a neighborhood of $e(\hat{t})$ and $e(\hat{t})$ is the only fixed point of $g_{\hat{t}} \circ g_{\hat{t}}$ in V.

Fix $x^0 \in V$ arbitrarily, and let $(x(k))$ be an arbitrary adaptive dynamic in $\Gamma(\hat{t})$ starting at x^0. Let (y^k) and (z^k) be best response dynamics 1 and 2, respectively, with $y^0 = e(t_0)$ and $z^0 = e(t_1)$. Using conditions 1 and 2, and lemma 4, it follows that

$$e(t_0) \;=\; y^0 \;\preceq\; \underline{y} \;\preceq\; \liminf x(k) \;\preceq\; \limsup x(k) \;\preceq\; \overline{z} \;\preceq\; z^0 \;=\; e(t_1),$$

whence $\underline{y}$ and $\overline{z}$ are in $[e(t_0), e(t_1)]$. As $\underline{y}$ and $\overline{z}$ are fixed points of $g_{\hat{t}} \circ g_{\hat{t}}$, by local isolation, $\underline{y} = \overline{z} = e(\hat{t})$. Thus, $x(k) \to e(\hat{t})$, as desired. ∎

Theorem 3 provides conditions under which strict monotone comparative statics select equilibria that are dynamically stable, in the sense that for small changes in the parameter, at a new parameter value, every adaptive dynamic starting from the old equilibrium converges to the newly selected equilibrium. Moreover, nowhere increasing selections select equilibria that are dynamically unstable, in the sense that at a new parameter value, no adaptive dynamic starting from the old equilibrium converges to the newly selected equilibrium. Thus, when considering dynamically stable equilibria (as proposed by Samuelson's Correspondence principle), we may expect monotone selections of equilibria to arise naturally in GSS. The next example presents an application of this result.

Example 6 (Team projects with substitutable tasks). Suppose a project is to be accomplished by a team of 3 players,[47] each choosing task (or effort) $x_i \in [0, 1]$, with probability of success x_i and quadratic cost of effort $\frac{c}{2}x_i^2$, with $c > 0$. Tasks are substitutable in the sense that each player by herself can make the project successful. The probability of success is $1 - (1 - x_1)(1 - x_2)(1 - x_3)$. If the project is successful, player i receives a parameterized reward $r(t) > 0$ (with $t \in T$, a compact, convex order interval in $\mathbb{R}$,

[47] This version is based on Dubey, Haimanko, and Zapechelnyuk (2006).

and $r'(t) > 0$.)[48] Otherwise, the player receives zero. Therefore, the payoff to player i is $f^i(x_1, x_2, x_3, t) = r(t)(1 - (1 - x_1)(1 - x_2)(1 - x_3)) - \frac{c}{2}x_i^2$.

The best response of player i is $g^i(x_j, x_k, t) = \frac{r(t)}{c}(1 - x_j)(1 - x_k)$. For notational convenience, let $a(t) = \frac{r(t)}{c}$, and when convenient, we suppress the notation t. To ensure that best responses remain in the strategy space, we assume that $a \leq 1$ (for every t). In fact, we assume that $a < \frac{3}{4}$, the reason becoming clear below.

For each t, this game has a unique symmetric equilibrium, which is strictly increasing in t, as follows. Fix t. First, observe that in any equilibrium, no player plays 0 or 1, as follows. Suppose, $x_1 = 1$ in equilibrium. Then using player 2's and 3's best response function, $x_2 = x_3 = 0$, whence, the best response of 1 to $(0, 0)$ is $x_1 = a < 1$, a contradiction. Similarly, $x_2 \neq 1$ and $x_3 \neq 1$. Suppose $x_1 = 0$ in equilibrium. Then using player 1's best response function, either $x_2 = 1$ or $x_3 = 1$, but that contradicts $x_2 \neq 1$ and $x_3 \neq 1$. Second, observe that only symmetric equilibria (each player plays the same action) are possible, as follows. Suppose, in equilibrium, $x_1 > x_2$. Then using player 1's best response, $x_1 + ax_2 - ax_2x_3 = a - ax_3$, and using player 2's best response, $x_2 + ax_1 - ax_1x_3 = a - ax_3$, whence $(1 - a)(x_1 - x_2) = a(x_2x_3 - x_1x_3)$, a contradiction. Thus $x_1 = x_2$. Similarly, $x_2 = x_3$. Third, observe that a symmetric equilibrium exists, and is given by $\frac{2a+1-\sqrt{4a+1}}{2a}$ for each player. Fourth, as shown by Roy and Sabarwal (2008) symmetric equilibria in GSS are unique, so this game has a unique equilibrium. Finally, it is easy to check that this equilibrium selection is strictly increasing in t.

Consequently, this equilibrium selection is strictly increasing and selects locally isolated equilibria.

Let us check that conditions 1 and 2 are satisfied. Fix t_0 in the interior of T, and let x^* denote the equilibrium selection at t_0. Notice that for $t \geq t_0$, $\gamma(x^*, t) = a(t)(1 - x^*)^2$, where γ is the best-response of a player, reduced to one-dimension, using symmetry. Therefore, $\gamma(\gamma(x^*, t), t) = a(t)[1 - \frac{a(t)}{a(t_0)}a(t_0)(1 - x^*)^2]^2 = a(t)[1 - \frac{a(t)}{a(t_0)}x^*]^2$, where the last equality follows from $x^* = a(t_0)(1 - x^*)^2$. Using $a' > 0$ and $x^* < 1$, it is easy to check that $\frac{d}{dt}\gamma(\gamma(x^*, t), t)\big|_{t=t_0} > 0$, if, and only if, $x^* < \frac{1}{3}$. Using $x^* = \frac{2a+1-\sqrt{4a+1}}{2a}$, this condition is satisfied when $a < \frac{3}{4}$, as assumed above. Consequently, for every t_0 in the interior of T, there is a neighborhood $[\underline{t}, \overline{t}] \subset T$ of t_0 such that conditions 1 and 2 are satisfied on $[\underline{t}, \overline{t}]$. Applying the theorem above, for every t_0 in the interior of T, $e(t_0)$ is strongly stable.

A similar example can be constructed using tournaments as well.

[48] The parameter t can be viewed as technological improvement, or subsidy provided, or reward provided to induce an increase in effort (or probability) of task completion. As shown in the example, the best response function depends on $\frac{r(t)}{c_i}$, where c_i measures player i's costs, and therefore, $r(t)$ can be viewed as a reward enhancement parameter relative to a player's costs.

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Appendix

Lemma. *Let Γ be a GSS. Let $((y^k);(z^k))$ be the best response dynamics starting at* $\inf X$ *and* $\sup X$*, respectively, and let $\underline{x}$ and $\overline{x}$ be the limits of the lower and upper mixtures of $((y^k);(z^k))$, respectively. Then*

1. $\bigcap_{k=0}^{\infty} U^k(X) \subset [\underline{x}, \overline{x}]$*, and*

2. $\underline{x}$ *and* $\overline{x}$ *are the smallest and largest profiles of serially undominated strategies, respectively.*

Proof. Let $U^0(X) = X$, and for $k \geq 1$, let $U^k(X) = U(U^{k-1}(X))$, where $U(S)$ is the collection of undominated responses to S. It follows by induction that for $k \geq 0$, $U^k(X) \subset [\underline{x}^k, \overline{x}^k]$, as follows. This holds trivially for $k = 0$. Suppose it holds for $k-1$. Then for k,

$$U^k(X) = U(U^{k-1}(X)) \subset U[\underline{x}^{k-1}, \overline{x}^{k-1}] \subset [\underline{g}(\overline{x}^{k-1}), \overline{g}(\underline{x}^{k-1})] = [\underline{x}^k, \overline{x}^k],$$

where the first inclusion follows from the inductive hypothesis and monotonicity of U, and the second inclusion follows from lemma 1. Consequently, $\bigcap_{k=0}^{\infty} U^k(X) \subset [\underline{x}, \overline{x}]$. That is, the set of serially undominated strategies is contained in the order interval $[\underline{x}, \overline{x}]$.

Notice now that $\underline{x}$ is a best response to $\overline{x}$, and $\overline{x}$ is a best response to $\underline{x}$. That is, $\underline{x} \in g(\overline{x})$ and $\overline{x} \in g(\underline{x})$, as follows. Suppose $\underline{x} \notin g(\overline{x})$. Then there is i, and x_i such that $f^i(x_i, \overline{x}_{-i}) - f^i(\underline{x}_i, \overline{x}_{-i}) > 0$. But then, by upper semi-continuity, and continuity in the $-i$ variables, for all k sufficiently large, $f^i(x_i, \overline{x}^k_{-i}) - f^i(\underline{x}^{k+1}_i, \overline{x}^k_{-i}) > 0$, contradicting the optimality of $\underline{x}^{k+1}_i$. Similarly, $\overline{x} \in g(\underline{x})$.

Finally, note that $\underline{x}$ and $\overline{x}$ are in $\bigcap_{k=0}^{\infty} U^k(X)$, as follows. Trivially, $\underline{x}$ and $\overline{x}$ are in $U^0(X)$. Suppose $\underline{x}$ and $\overline{x}$ are in $U^k(X)$. Then $\underline{x} \in U^{k+1}(X)$, because $\overline{x} \in U^k(X)$ and $\underline{x}$ is a best response to $\overline{x}$, and $\overline{x} \in U^{k+1}(X)$, because $\underline{x} \in U^k(X)$ and $\overline{x}$ is a best response to $\underline{x}$. Thus, serially undominated strategies lie in $[\underline{x}, \overline{x}]$, and the end points are extremal serially undominated strategies. ∎

Proof of Theorem 2. Let $FP(g \circ g)$ denote the set of fixed point of $g \circ g$, and notice that $FP(g \circ g) \subset \bigcap_{k=0}^{\infty} U^k(X)$, as follows. Consider an arbitrary $x \in g \circ g(x)$. Let $y \in g(x)$ be such that $x \in g(y)$. Then, by induction, x and y are in $\bigcap_{k=0}^{\infty} U^k(X)$, as follows. Trivially, x and y are in $U^0(X)$. Suppose x and y are in $U^k(X)$. Then $x \in U^{k+1}(X)$, because $y \in U^k(X)$ and x is a best response to y, and $y \in U^{k+1}(X)$, because $x \in U^k(X)$ and y is a best response to x. Consequently, we have the following relationships: $\mathcal{E} \subset FP(g \circ g) \subset \bigcap_{k=0}^{\infty} U^k(X) \subset [\underline{x}, \overline{x}]$, the last inclusion following from the lemma above.

Notice next that both $\underline{x}$ and $\overline{x}$ are fixed points of $g \circ g$, because $\underline{x} \in g(\overline{x})$ and $\overline{x} \in g(\underline{x})$. Thus, $\underline{x}$ and $\overline{x}$ are extremal fixed points of $g \circ g$, and as shown by the previous lemma, are also extremal serially undominated strategies. With these observations, the equivalence of (1) and (2) follows immediately. For the equivalence of (2) and (3), notice that (2) is equivalent to $\underline{x} = \overline{x}$; and using lemma 3 and the proof of theorem 1 in the text, (3) is equivalent to $\underline{x} = \overline{x}$ as well. ∎

Proof of Lemma 4. To prove statement (1), consider $N = 0$. Let $K_0 = 0$. Notice that $x(0) = x^0 \in [y^0, z^0]$, by assumption, and $[y^0, z^0] = [\underline{x}^0, \overline{x}^0]$, by construction. Suppose for $0 \leq k \leq \hat{k} - 1$, $x(k) \in [\underline{x}^0, \overline{x}^0]$. Then $P(0, \hat{k}) \subset [\underline{x}^0, \overline{x}^0]$, whence

$$\begin{aligned} x(\hat{k}) &\in \overline{U}[\inf P(\hat{k} - \gamma, \hat{k}), \sup P(\hat{k} - \gamma, \hat{k})] \\ &\subset \overline{U}[\inf P(0, \hat{k}), \sup P(0, \hat{k})] \\ &\subset \overline{U}[\underline{x}^0, \overline{x}^0] \subset [\underline{x}^0, \overline{x}^0], \end{aligned}$$

where membership follows from definition of an adaptive dynamic, the first inclusion follows from $P(\hat{k} - \gamma, \hat{k}) \subset P(0, \hat{k})$ and monotonicity of $\overline{U}$, the second inclusion follows from the inductive hypothesis and monotonicity of $\overline{U}$, and the last inclusion follows trivially. Thus, for all $k \geq 0$, $x(k) \in [\underline{x}^0, \overline{x}^0]$.

Suppose the statement is true for $N - 1$. Let K_{N-1} be given by the inductive hypothesis. Let $K_N = K_{N-1} + \gamma$, where γ is from the definition of adaptive dynamic. Suppose N is even. Fix $\hat{k} \geq K_N = K_{N-1} + \gamma$. Then

$$\begin{aligned} x(\hat{k}) &\in \overline{U}[\inf P(\hat{k} - \gamma, \hat{k}), \sup P(\hat{k} - \gamma, \hat{k})] \\ &\subset \overline{U}[\inf P(K_{N-1}, \hat{k}), \sup P(K_{N-1}, \hat{k})] \\ &\subset \overline{U}[\underline{x}^{N-1}, \overline{x}^{N-1}] = [\underline{x}^N, \overline{x}^N], \end{aligned}$$

where membership follows from definition of an adaptive dynamic, the first inclusion follows from $P(\hat{k} - \gamma, \hat{k}) \subset P(K_{N-1}, \hat{k})$ and monotonicity of $\overline{U}$, the second inclusion follows from the inductive hypothesis and monotonicity of $\overline{U}$, and the equality follows from lemma 2. Thus, for all $k \geq K_N$, $x(k) \in [\underline{x}^N, \overline{x}^N]$.

To prove statement (2), notice first that $y^0 \preceq y^2$ implies that the subsequence (y^{2k}) is nondecreasing, and by completeness, there is $\underline{y}$ such that $y^{2k} \to \underline{y}$. Similarly, using $y^0 \preceq y^2 \Rightarrow y^3 = \overline{g}(y^2) \preceq \overline{g}(y^0) = y^1$, the subsequence (y^{2k-1}) is nonincreasing, and there is $\overline{y}$ such that $\lim_k y^{2k} = \overline{y}$.

Notice next that $\underline{y}, \overline{y} \in FP(g \circ g)$. This follows from the observation that $\underline{y} \in g(\overline{y})$, and $\overline{y} \in g(\underline{y})$, as follows. Suppose $\underline{y} \notin g(\overline{y})$. Then there is i, and x_i such that $f^i(x_i, \overline{y}_{-i}) - f^i(\underline{y}_i, \overline{y}_{-i}) > 0$. But then, by upper semi-continuity, and continuity in the $-i$ variables, for all k sufficiently large, $f^i(x_i, y^{2k-1}_{-i}) - f^i(y^{2k}_i, y^{2k-1}_{-i}) > 0$, contradicting the optimality of y^{2k}_i. Similarly, $\overline{y} \in g(\underline{y})$. Consequently, $\underline{y}, \overline{y}$ are simply rationalizable.

Consider an arbitrary $x^0 \in [y, z]$, and an arbitrary adaptive dynamic $(x(k))$ starting at x^0. Consider an arbitrary convergent subsequence $(x(k_l))$ of $(x(k))$. By the lemma 4, for $N = 0$, there is K_0 such that for all $k_l \geq K_0$, $y^0 \preceq x(k_l)$, whence $y^0 = \underline{x}^0 \preceq \lim_l x(k_l)$. For $N = 2$, there is K_2 such that for all $k_l \geq K_2$, $y^2 = \underline{x}^2 \preceq x(k_l)$, whence $y^2 \preceq \lim_l x(k_l)$. And by induction, for $2N$, there is K_{2N} such that for all $k_l \geq K_{2N}$, $y^{2N} = \underline{x}^{2N} \preceq x(k_l)$, whence $y^{2N} \preceq \lim_l x(k_l)$. Consequently, $\underline{y} \preceq \lim_l x(k_l)$. As $(x(k_l))$ is an arbitrary convergent subsequence, it follows that $\underline{y} \preceq \liminf x(k)$. Moreover, as $\underline{x}$ is the smallest fixed point of $g \circ g$, it follows that $\underline{x} \preceq \underline{y}$. Similarly, $\limsup x(k) \preceq \overline{y} \preceq \overline{x}$. Statement (3) follows similarly. ∎