Entangled Quantum Nonlinear Schrödinger Solitons

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Considered as a multipartite quantum system, time-multiplexed nonlinear Schrödinger solitons after collision are rigorously proved to become quantum entangled in the sense that their quadrature components of suitably selected internal modes satisfy the inseparability criterion. Clear physical insights for the origin of entanglement are given, and the required homodyne local oscillator pulse shape for optimum entanglement detection is determined.

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The quantum nonlinear Schrödinger equation (QNLSE) has been widely used as a model equation for studying the quantum effects of bosonic solitons. In particular, quantum optical solitons in photonic waveguides with Kerr nonlinearity can be accurately described by the QNLSE [1–3]. For weakly interacting ultracold atoms in a Bose-Einstein condensate, the bosonic matter wave field evolves also in the form of QNLSE, possibly with additional terms of linear or periodic potentials [4,5]. Based on the studies of the QNLSE, the possibility of squeezing generation through quantum solitons was predicted by theoretical works in 1987 [1] and subsequently confirmed by fiber soliton experiments in 1991 [6]. Some of the important theoretical approaches for solving QNLSE and other more complicated quantum nonlinear pulse propagation problems include the quantum stochastic simulation method [2], the Bethe's ansatz method [3], the quantum perturbation theory [7], the back-propagation method [8], and the cumulant expansion technique [9]. Some of the important experiments include the quantum nondemolition measurement using solitons [10], the generation of amplitude squeezed states through optical filtering [11,12] or imbalanced nonlinear interference [13], the intrasoliton photon number quantum correlation in both the spectra and time domain [14,15], and the generation of continuous variable Einstein-Podolsky-Rosen entangled states by adiabatically expanding an optical vector soliton [16]. Among them, the generation of entangled states using nonlinear Schrödinger solitons is of particular interest for possible quantum information applications.

The known optical soliton scheme of generating continuous variable Einstein-Podolsky-Rosen states is based on the generation and mixing of two independent squeezed vacuum states from a fiber squeezer [17]. In the soliton quantum nondemolition measurement schemes, soliton collision has been widely utilized to induce a quantum correlation between two solitons of different wavelengths or polarizations. The photon number noises of one soliton can be encoded to the phase noises of the other soliton through the cross-phase modulation and thus create quantum correlation between the two solitons. Recently we have also shown that the photon number correlation of two time-multiplex solitons can be directly established through nonlinear interaction [15]. The whole system is a pure state of infinite modes if no optical loss is assumed. However, if one selects one mode for each soliton and traces out all the other modes to form a two-partite system, the reduced two-partite state will be a mixed state in general. Such a reduced two-partite state is thus not guaranteed to be entangled even when the quantum correlation has been proven. The situation is also true for studying finite internal modes of a single soliton, where the quantum correlation is known to exist but the quantum entanglement is not for sure.

Even for squeezed state generation using solitons, there are still important issues left unanswered. Typically the generation of pulse squeezed vacuum states from solitons is through the use of a balanced nonlinear interferometer. The homodyne detection scheme is then used to detect the quadrature component of the output squeezed vacuum state with the mean field pulse from the other port of the interferometer as the local oscillator. Larger squeezing can be detected if one only detects the soliton parts and rejects all the continuum parts [7]. In several studies of squeezing generation using nonfundamental solitons, it has also been suggested that the continuum may help with achieving larger squeezing within some parameter space [18]. The determination of optimal local oscillators for squeezing detection has been investigated in the literature and has been related to eigenfunction problems based on the correlation properties or the transformation matrix of the multimode field operators [19–21]. Description of intrasoliton photon number correlations in terms of three simply chosen internal modes has also been shown to be effective [21]. However, since the considered soliton systems here are intrinsic, complicated multimode problems, the complete correlation properties or transformation matrices of the multimode field operators are not easy to obtain as the starting point. It is thus very desirable to develop numerically efficient algorithms that can directly determine the optimal local oscillators for squeezing or entanglement detection as well as the optimal basis functions for the mode description of quantum solitons. Recently, experimental observation of squeezed lights with 10 dB quantum noise reduction from optical parametric oscillation processes has been reported [22]. It will be interesting to see whether the soliton schemes can also generate and detect large quantum squeezing through optimization.

In the this Letter we start by developing a theory that can efficiently determine the true optimum homodyne local oscillator pulse shape for soliton squeezing detection. Our theory leads to the discovery of the ‘‘natural’’ internal squeezing modes that are minimum-uncertainty states. By thinking in terms of these internal squeezing modes, one can easily understand why there are intra- and intersoliton quantum correlations and how to optimally detect the correlation. Most importantly, based on the theory we can rigorously prove that the time-multiplexed optical solitons after nonlinear interaction are indeed quantum mechanically entangled in the sense that the ‘‘quadrature components’’ of the specially selected multipartite state can satisfy the following inseparability criterion: the uncertainty product of the inferred quadrature components is below the Heisenberg uncertainty product limit.

We start from the well known QNLSE given below: $\frac{\partial \hat{U}}{\partial z} = i\frac{1}{2}\frac{\partial^2 \hat{U}}{\partial t^2} + i\hat{U}^\dagger \hat{U}\hat{U}$. By assuming the quantum noises are much less than the mean fields, the linearization approximation can be justified. The quantum noise part of the soliton can be described by the following linearized operator equation:

$$\frac{\partial \hat{u}}{\partial z} = i\frac{1}{2}\frac{\partial^2 \hat{u}}{\partial t^2} + i2U_0{}^{*}U_0\hat{u} + iU_0^2\hat{u}^\dagger. \tag{1}$$

Here $U_0(z,t)$ is the classical solution and $\hat{u}(z,t)$, $\hat{u}^\dagger(z,t)$ are the perturbed quantum field operators. In order to calculate the quantum noises by the back-propagation method [8], the adjoint system of Eq. (1) is introduced by requiring the inner product of the solutions of the two systems to be a conserved quantity along z. Here the definition of the inner product is given by $\langle u^A(z,t)|\hat{u}(z,t)\rangle = \int \frac{1}{2}[u^{A*}(z,t)\hat{u}(z,t) + \text{H.c.}]dt$. This leads to the following classical linear adjoint evolution equation:

$$\frac{\partial u^A}{\partial z} = i\frac{1}{2}\frac{\partial^2 u^A}{\partial t^2} + i2U_0{}^{*}U_0u^A - iU_0^2u^{A*}. \tag{2}$$

Since both Eqs. (1) and (2) are linear, their solutions can be formally written as $\hat{u}(z,t) = L_{z\leftarrow 0} \otimes \hat{u}(0,t)$ and $u^A(z,t) = A_{z\leftarrow 0} \otimes u^A(0,t)$. Here $L_{z\leftarrow 0}$ and $A_{z\leftarrow 0}$ are the formal evolution operators of the two systems (linear and adjoint) from 0 to z. The symbol $\otimes$ is introduced to remind us of the fact that these formal linear differential operators operate on both $\hat{u}$ and $\hat{u}^\dagger$. With such compact notations and by assuming the initial input state is a coherent state, the detected squeezing ratio after the propagation distance z can be nicely expressed as

$$R(z) = \frac{\langle A_{0\leftarrow z} \otimes f(t)|A_{0\leftarrow z} \otimes f(t)\rangle}{\langle f(t)|f(t)\rangle}. \tag{3}$$

Here $f(t)$ is the local oscillator pulse used in the homodyne detection, and $A_{0\leftarrow z} \otimes f(t)$ is the back-propagated local oscillator pulse through the adjoint system. Mathematically Eq. (3) can be viewed as a functional of $f(t)$, and the condition for its stationary solutions can be determined by performing a variation with respect to $f(t)$. Using the fact that the inner product of two solutions is conserved along z, one has $\langle A_{0\leftarrow z} \otimes f(t)|A_{0\leftarrow z} \otimes f(t)\rangle = \langle f(t)|L_{z\leftarrow 0}A_{0\leftarrow z} \otimes f(t)\rangle$. It is then easy to show that the variational equation $\delta R(z) = 0$ leads to the following eigenvalue problem with the eigenvalue λ equal to the optimum squeezing ratio:

$$L_{z\leftarrow 0}A_{0\leftarrow z} \otimes f(t) = \lambda f(t). \tag{4}$$

Equation (4) is one of the main results in this Letter. It elegantly describes the necessary condition that the optimal local oscillator pulse shape must satisfy. Since we are mainly interested in the solution with the globally minimum eigenvalue λ, the numerical inverse power method can be applied to Eq. (4) for iteratively approaching the eigensolutions we want to find.

It is not difficult to prove that if $f(t)$ is the eigenstate of $L_{z\leftarrow 0}A_{0\leftarrow z}$ with the eigenvalue λ, then $i * f(t)$ is also the eigenstate of $L_{z\leftarrow 0}A_{0\leftarrow z}$ with the eigenvalue $1/\lambda$. This implies that if one uses $f(t)$ as the basis to project out the corresponding internal mode field operator $\hat{a}(z) = \int f^*(t)\hat{u}(z,t)dt / \int |f(t)|^2 dt$, the projected mode will be a minimum-uncertainty state with one quadrature squeezed and the other quadrature antisqueezed. So the results in Eq. (4) physically imply that the minimum-uncertainty state requirement is the necessary condition to achieve optimum squeezing detection. This is a very meaningful result that can provide us with deeper physical insights about the internal squeezing modes of the solitons. This set of internal modes is the natural basis set for describing the quantum noise properties of the soliton, in the sense that there is no quantum correlation among these internal modes. This can be easily seen from the formula for calculating the quantum correlation of two measured operators by the homodyne detection: $C_{12}(z) \propto \langle A_{0\leftarrow z} \otimes f_1(t)|A_{0\leftarrow z} \otimes f_2(t)\rangle$, where $f_1(t)$ and $f_2(t)$ are the two local oscillator functions.

With the above physical insights, we now demonstrate how to determine the optimum local oscillator pulse shapes for detecting entanglement. To illustrate, let us consider the intersoliton case and assume the classical time-multiplexed two-soliton solution is symmetric in time t. Assume $f_{\text{opt}}(t)$ is the found (symmetric) eigenfunction with the smallest eigenvalue λ_{opt} and we choose the two normalized local oscillator functions for detecting the two solitons to be

$f_1(t) \propto f_{\rm opt}(t)$ for $t > 0$ and $f_2(t) \propto f_{\rm opt}(t)$ for $t < 0$. Since $f_1(t)$ is zero for $t < 0$ and $f_2(t)$ is zero for $t > 0$, the two functions do not overlap in time and correspond to the measurements on each soliton. The four related quadrature components are $\hat{q}_1 = \langle f_1|\hat{u}\rangle$, $\hat{p}_1 = \langle if_1|\hat{u}\rangle$, $\hat{q}_2 = \langle f_2|\hat{u}\rangle$, and $\hat{p}_2 = \langle if_2|\hat{u}\rangle$. From the minimum-uncertainty state property stated above, it is not difficult to prove that the squeezing ratio for $\mathrm{Var}[\hat{q}_1 + \hat{q}_2]$ is just $\lambda_{\rm opt} \leq 1$, and the squeezing ratio of $\mathrm{Var}[\hat{p}_1 + \hat{p}_2]$ is just $1/\lambda_{\rm opt} \geq 1$. So $\hat{q}_1$ and $\hat{q}_2$ are anticorrelated, while $\hat{p}_1$ and $\hat{p}_2$ are correlated. To more accurately estimate how much $\hat{p}_1$ and $\hat{p}_2$ are correlated, we need to estimate the squeezing ratio of $\mathrm{Var}[\hat{p}_1 - \hat{p}_2]$. Note that the projection function for $\hat{p}_1 - \hat{p}_2$ is antisymmetric, and thus it will be orthogonal to $i * f_{\rm opt}(t)$. Therefore $\hat{p}_1 - \hat{p}_2$ does not contain any contribution from the optimum internal mode. The squeezing ratio of $\mathrm{Var}[\hat{p}_1 - \hat{p}_2]$ is thus upper bounded by $1/\lambda_{\rm snd}$, with $\lambda_{\rm snd}$ being the second smallest eigenvalue of the system. Based on these observations, we now have the following important result:

$$\text{Squeezing Ratio of } \mathrm{Var}[\hat{q}_1 + \hat{q}_2] \times \mathrm{Var}[\hat{p}_1 - \hat{p}_2] \leq \frac{\lambda_{\rm opt}}{\lambda_{\rm snd}} < 1.$$

Here the definition of the squeezing ratio of the uncertainty product is to compare the uncertainty product with the case of two independent coherent states.

The above result is a sufficient condition for proving that the two solitons after collision are indeed entangled in the sense that the inseparability criterion for bipartite continuous variables is satisfied [23]. The proof can be easily generalized to the more general soliton collision/interaction cases. It can also be applied to the single soliton case to determine the optimum local oscillator for detecting intrasoliton entanglement. For the case of multipartite N identical solitons, the proof still can be applied by simply noting that the squeezing ratio for $\mathrm{Var}[c_1\hat{q}_1 + \ldots + c_k\hat{q}_k + \ldots + c_N\hat{q}_N]$ is $\lambda_{\rm opt}$, if $c_k = \sqrt{\int_{t_k-\Delta/2}^{t_k+\Delta/2} |f_{\rm opt}(t)|^2 dt}$. Here the projection function for the kth soliton is chosen to be $f_{\rm opt}(t)/c_k$ within its time and to be zero elsewhere. The squeezing ratio for $\mathrm{Var}\{\hat{p}_k - [(c_1\hat{p}_1 + \ldots + c_{k-1}\hat{p}_{k-1} + c_{k+1}\hat{p}_{k+1} + \ldots + c_N\hat{p}_N)/\sqrt{|c_1|^2 + \ldots + |c_{k-1}|^2 + |c_{k+1}|^2 + \ldots + |c_N|^2}]\}$ is then upper bounded by $1/\lambda_{\rm snd}$ since its projection function is orthogonal to $f_{\rm opt}(t)$. Therefore one can use $-(c_1\hat{q}_1 + \ldots + c_{k-1}\hat{q}_{k-1} + c_{k+1}\hat{q}_{k+1} + \ldots + c_N\hat{q}_N)/c_k$ and $(c_1\hat{p}_1 + \ldots + c_{k-1}\hat{p}_{k-1} + c_{k+1}\hat{p}_{k+1} + \ldots + c_N\hat{p}_N)/\sqrt{|c_1|^2 + \ldots + |c_{k-1}|^2 + |c_{k+1}|^2 + \ldots + |c_N|^2}$ to infer $\hat{q}_k$ and $\hat{p}_k$ in order to satisfy the nonlocal criterion between the kth mode and the rest $N - 1$ modes.

As some numerical examples, let us consider the two-soliton and three-soliton collision cases illustrated in Fig. 1. The initial conditions are 2 or 3 solitons of the same phase with the soliton amplitude = 1 and separation = 5. Such a bound soliton pair/train will evolve periodically (collide, separate, and collide again as breathers). For the 2-soliton case, the optimum mode function $f_{\rm opt}(t)$ at $z = 20$ is plotted in Fig. 2. The eigenvalues $\lambda_{\rm opt} = -33.0$ dB and $\lambda_{\rm snd} = -22.4$ dB. Therefore one can expect the squeezing ratio of the inferred uncertainty product is -10.6 dB. For the 3-soliton case, the optimum mode function $f_{\rm opt}(t)$ at $z = 25$ is plotted in Fig. 3. The eigenvalues $\lambda_{\rm opt} = -35.4$ dB and $\lambda_{\rm snd} = -24.4$ dB. This time it is -11.0 dB below the Heisenberg uncertainty product limit. Actual numerical calculation of the squeezing ratio usually yields a smaller number than the predicted upper bound because the squeezing ratio of $\mathrm{Var}[\hat{p}_1 - \hat{p}_2]$ is usually less than the bound $1/\lambda_{\rm snd}$.

In practice the achievable squeezing may be limited by optical losses, nonlinear scattering noises, and detector quantum efficiency. The conventional length normalization unit (pulse-width^2/dispersion) used in the theory can be about several meters if hundreds of fs pulses are used and can be up to several hundred meters if ps pulses are used. Currently more than 6 dB squeezing has been reported with the help of gigahertz erbium-doped fiber lasers [24] and photonic crystal fibers [25]. If the soliton separation is reduced, the required propagation length as well as the achievable squeezing can also be reduced. In this way, the predicted squeezing or entanglement ratio can be adjusted to be of the order of several dB, located within the observ-

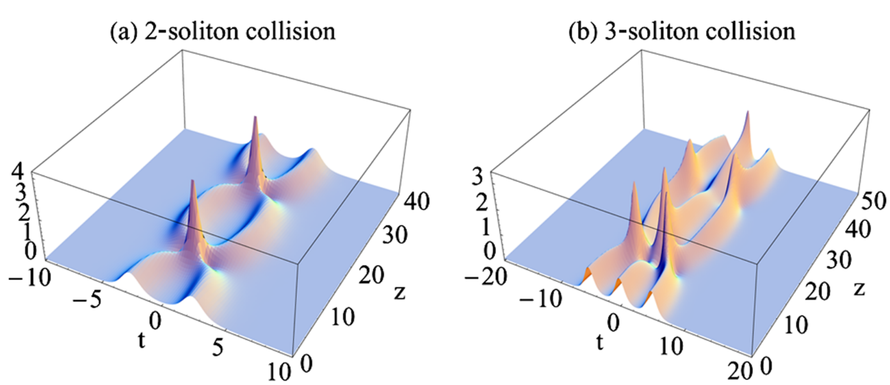

FIG. 1 (color online). Intensity evolution patterns for (a) 2 in-phase solitons and (b) 3 in-phase solitons. Soliton separation = 5.

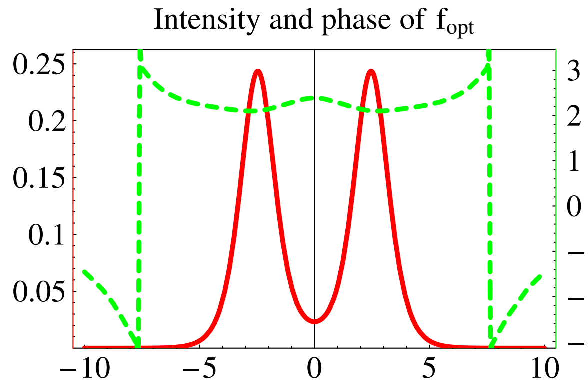

FIG. 2 (color online). $f_{\rm opt}(t)$ for the two-soliton case at $z = 20$. The solid line is for the intensity, and the dashed line for the phase.

able range of current technologies. The impacts of fiber losses on the achievable squeezing or entanglement for given local oscillators can be readily calculated by the back-propagation method [8]. Determination of the optimal local oscillator in the presence of fiber losses should also be possible with some further development of the theory. The theory developed here should also be applicable to the study of Bose-Einstein condensates. However, in contrast to the traditional soliton perturbation theory or the Bose-Einstein condensation Bogoliubov–de Gennes equation approach based on the perturbed nonlinear Schrödinger equation [4], the expansion eigenmodes employed in this Letter are not the (generalized) eigenmodes of the perturbed nonlinear equation itself. Instead, they are the eigenmodes of the cascaded (linearized + adjoint) evolution operators for a fixed propagation length.

In conclusion, we have presented an elegant theory to rigorously prove that multipartite entangled states can be directly generated by time-multiplexed solitons. The entanglement can only be detected by using specially chosen homodyne local oscillators to project out the corresponding quadrature components of the solitons. The optimum detection functions are related to the internal modes of the soliton systems, under which all the modes are uncorrelated minimum-uncertainty states. The presented theory provides the way to find the optimum local oscillator pulse shapes for detecting intra- and intersoliton entanglements and helps to clarify the physical origin of the entanglement. The theoretical concept is general and should be also applicable to other soliton squeezing and entanglement schemes. The results presented here are believed to be helpful for future quantum information experiments that require larger squeezing or entanglement factors.

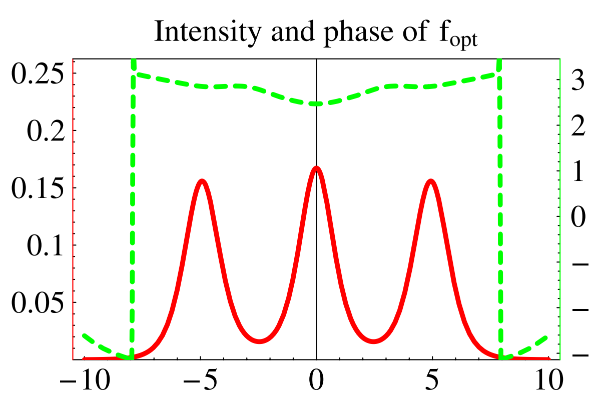

FIG. 3 (color online). $f_{\rm opt}(t)$ for the three-soliton case at $z = 25$. The solid line is for the intensity, and the dashed line for the phase.

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