Sensitivity Analysis on Generalized Fuzzy Relation Equations

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Abstract

Consider a fuzzy relation equation $\mathbf{x}\circ\mathbf{A}=\mathbf{b}$ where $\mathbf{x}$ and $\mathbf{b}$ are input and output vectors and $\mathbf{A}$ is a state matrix with max–min operator "$\circ$". This study is an extension of our previous work by relaxing the condition that the elements in vector $\mathbf{b}$ are strictly distinct and allows us to have same values of membership in $\mathbf{b}$. In this paper we propose a solution procedure from the defined quasi–characteristic matrix that not only determines the solution set but also defines the types of solution set before the solution set is determined. Therefore, it provides an insight of the structure of fuzzy relation equations. Also by means of the quasi–characteristic matrix, we do sensitivity analysis on the state matrix $\mathbf{A}$ to determine the set of state–matrices which has the same solution set with respect to the given output–vector.

Introduction

If $m,n,c \in N$, and $I=\{1,..,i,..,m\}$, $J=\{1,..,j,..,n\}$, $K=\{1,..,k,..,c\}$ and assume that an n–dimensional vector $\mathbf{b}=[b^k_j]$ called output–vector of which each k–class contains the same b values and $1\geq b^1>...>b^k>...>b^c\geq 0$, $c\leq n$, and a matrix $A=[a^k_{ij}]_{m\times n}$ called state–matrix, with $a^k_{ij}\in[0,1]$, $b^k_j\in[0,1]$ for all $i\in I$ and $j\in J$ are given, the problem of the resolution of a fuzzy relation equation is to determine an m–dimensional vector $\mathbf{x}\in\mathbf{X}$ of solution space such that

$$\mathbf{x}\circ\mathbf{A}=\mathbf{b} \qquad (1)$$

where "$\circ$" denotes max–min composition with

$$\max_{i\in I}\min(x_i\wedge a^k_{ij})=b^k_j \quad \text{for } \forall j\in J, \qquad (2)$$

and $\mathbf{X}=\{\mathbf{x}=[x_i]_{1\times m}\,|\,x_i\in[0,1]\}$. The solution set of (1) is defined by $\mathbf{X}(\mathbf{A},\mathbf{b})$.

Due to our previous studies [4,5], we know that the proposed quasi–characteristic matrix $\mathbf{C}$ of $\mathbf{A}$ represents the possible positions in determining the solutions. Thus, the solution set are characterized and the tolerance intervals of elements in $\mathbf{A}$ matrix with respect to the known solution set and output–vector are determined by this C–matrix. But in our previous studies, the elements in vector $\mathbf{b}$ are strictly distinct, it restricts the capability of applications. Therefore, in this paper, we relax this constraint. Then the extension allows us to have same values of membership in output–vector $\mathbf{b}$ so that the proposed method can be generalized.

Section 2 of this paper describes basic definitions and properties of a fuzzy relation equation. Then in Section 3 defines the quasi–characteristic matrix from a state–matrix and derives its relationship with the types of solution sets. In Section 4, the methodology of sensitivity analysis on matrix $\mathbf{A}$ is proposed. Finally, in Section 5 we conclude the study and point out the relevant issues for further studies.

Basic definitions and properties of a fuzzy relation equation

The basic definitions below will be used through out the paper until otherwise stated. Also the supporting theorems are stated without proofs.

Definition 2–1 [1]. For $\mathbf{x}^1,\mathbf{x}^2\in\mathbf{X}$, let $\mathbf{x}^1\leq\mathbf{x}^2$, then $\leq$ is partially ordering on $\mathbf{X}$, $(\mathbf{X},\leq)$ is a lattice with min and max as its meet, "$\wedge$", and join, "$\vee$", respectively. If $\mathbf{X}(\mathbf{A},\mathbf{b})=\{\mathbf{x}\,|\,\mathbf{x}\circ\mathbf{A}=\mathbf{b}\}$, i.e., $\mathbf{X}(\mathbf{A},\mathbf{b})$ denotes the set of all solutions of equation (1), when $\mathbf{A}$ and $\mathbf{b}$ are given, then $(\mathbf{X}(\mathbf{A},\mathbf{b}),\leq)$ is a subposet of $(\mathbf{X},\leq)$.

Definition 2–2 [1]. We call $\bar{\mathbf{x}}\in\mathbf{X}(\mathbf{A},\mathbf{b})$ the maximum solution of $\mathbf{X}(\mathbf{A},\mathbf{b})$, if for all $\mathbf{x}\in\mathbf{X}(\mathbf{A},\mathbf{b})$ we have $\mathbf{x}\leq\bar{\mathbf{x}}$.

Definition 2–3 [1]. We call $\underline{\mathbf{x}}\in\mathbf{X}(\mathbf{A},\mathbf{b})$ a minimal solution of $\mathbf{X}(\mathbf{A},\mathbf{b})$, if for all $\mathbf{x}\in\mathbf{X}(\mathbf{A},\mathbf{b})$, $\mathbf{x}\leq\underline{\mathbf{x}}$ implies $\mathbf{x}=\underline{\mathbf{x}}$. The set of all minimal solutions of X(A,b) is denoted by $\underline{X}$(A,b).

Definition 2–4 [2]. We define @ composition as

$$A\ @\ b^{-1}=[\bigwedge_{j=1}^{n}(a_{ij}^{k}\ \alpha\ b_j^k)] \quad (3)$$

where $a_{ij}^{k}\ \alpha\ b_j^k=\begin{cases}1 & \text{if } a_{ij}^{k}\le b_j^k,\\ b_j^k & \text{if } a_{ij}^{k}>b_j^k.\end{cases}$

and $^{-1}$ denotes transpose.

Lemma 2–1 [2]. If X(A,b)≠ϕ, then $\bar{x}=[A\ @\ b^{-1}]^{-1}$ is the maximum solution of X(A,b)5.72Xb

Theorem 2–1 [1]. If X(A,b)≠ϕ then $\underline{X}$(A,b)≠ϕ and

$$X(A,b)=\bigcup_{\underline{x}\in\underline{X}(A,b)}\{x\mid \underline{x}\le x\le\bar{x}\}.$$

Lemma 2–2 [4]. X(A,b)≠ϕ iff for each column j∈J in kth–class of **A** there exists at least an i∈I such that $\exists x_i\in[0,1]$ satisfies (i) $(x_i\wedge a_{ij}^k)=b_j^k$ and (ii) $(x_i\wedge a_{ij'}^k)\le b_{j'}^k$, ∀j'≠j.

From Lemma 2–2, we know that for each element a_{ij}^k of A two cases follow : (i) if $a_{ij}^k<b_j^k$, then no $x_i\in[0,1]$ can satisfy $(x_i\wedge a_{ij}^k)=b_j^k$;(ii) if $a_{ij}^k\ge b_j^k$ and at row i there exists an element $a_{ij'}^{k'}$, k'>k such that $a_{ij'}^{k'}>b_{j'}^{k'}$ then there is no $x_i\in[0,1]$ that can satisfy $(x_i\wedge a_{ij}^k)=b_j^k$ and $(x_i\wedge a_{ij'}^{k'})\le b_{j'}^{k'}$, ∀j'≠j, simultaneously.

Quasi–characteristic matrix for solution

In this section, we defined a quasi–characteristic matrix from a state–matrix to identify the corresponding position and properties of a solution. Then by quasi–characteristic matrix the properties of each type of solution sets are investigated.

Definition 3–1. Given **A** and **b**, a quasi–characteristic matrix **C**, $[c_{ij}^k]_{mxn}$, of a matrix A in a fuzzy relation equation (1) is called C–matrix and defined as follows:
for each row i∈I,

$$c_{ij}^k=\begin{cases}1 & \text{if } k=k(i) \text{ and } a_{ij}^k>b_j^k,\\ E & \text{if } k\ge k(i) \text{ and } a_{ij}^k=b_j^k,\\ 0 & \text{otherwise.}\end{cases}\quad \forall j\in J, \quad (4)$$

where

$$k(i)=\begin{cases}\max\{k\}, & \text{if } \{k\mid a_{ij}^k>b_j^k \text{ for a given } i\}\ne\phi,\\ 0 & \text{otherwise}.\end{cases} \quad (5)$$

From Definition 3–1, we know that each row of matrix **C** has at most one class that contains "1" elements and has zero values if k<k(i). The different of the "class" from the "column" of our previous study [4] is resultant from the relaxation of different **b** values and allow b elements having same values in the same class. This can be done easily by grouping b elements into c strictly different b values with $1\ge b^1>b^2>\cdots>b^k>\cdots>b^c\ge0$ and swapping the corresponding column of **A** (or **C**) matrix. In this way, our previous results [4,5] can be extended to specify the between–class relationship directly. Then the remaining problem is to identify the relation of elements within the same class that has the same b values. In other words, we need to prove that the results of our previous studies still hold within the class.

Lemma 3–1. At each column j in class k of C–matrix if ∃i∈I such that c_{ij}^k is E or 1 then for the column j the possible position for determining the solution x_i is at row i. That is, for column j, $x_i\in[b_j^k,b^{k(i)}]$ satisfies conditions (i) and (ii) of Lemma 2–2 when $b^0=1\ge b^1>\cdot\cdot>b^k>\cdot\cdot>b^c\ge0$.

Proof. Suppose at column j in class k, $c_{ij}^k=1$ or E then (a)if $c_{ij}^k=1$, from the definition of the C–matrix we have k=k(i) and the corresponding $a_{ij}^k>b_j^k=b^{k(i)}$, then if $x_i=b_j^k$, $(x_i\wedge a_{ij}^k)=b_j^k$ holds.
At row i, ∀j'∈class k, we have $(x_i\wedge a_{ij'}^k)\le b^k=b_{j'}^k$.
(b)if $c_{ij}^k=E$ from the definition of the C–matrix, we have k≥k(i) and the corresponding $a_{ij}^k=b_j^k$. If $x_i\in[b_j^k,b^{k(i)}]$ then $(x_i\wedge a_{ij}^k)=b_j^k$. At row i, ∀j'∈class k, if k>k(i) then $a_{ij'}^k\le b_{j'}^k$, we have $(x_i\wedge a_{ij'}^k)\le a_{ij'}^k\le b_{j'}^k$; if k=k(i) then we have $(x_i\wedge a_{ij'}^k)\le x_i=b^{k(i)}=b_{j'}^k$. □

Theorem 3–1. X(A,b)=ϕ iff ∃j∈J in class k such that $c_{ij}^k=0$ ∀i∈I.

Proof. It is trivial.

Now according to Theorem 3–1, we can detect whether there exist solutions. If there is a column with zero values in C–matrix we call the matrix "degenerate" and the corresponding solution set is empty. If C–matrix is not degenerate, from Theorem 3–1 there is at least a solution and this leads us to investigate the solution set. First, we discuss the maximum solution.

Corollary 3–1. If C is not degenerate, then the maximum solution $\bar{x}=[\bar{x}_i]$ where $\bar{x}_i=b^{k(i)}$.

Therefore, from Corollary 3–1, the value of maximum solution can be identified by checking the position of element with value 1 at each row of C–matrix and given the proper values.

Corollary 3–2. Let $x \circ A = b$ and $x' \circ A' = b$, if $\forall_{i \in I}\ k_A(i) = k_{A'}(i)$ then $\bar{x}_A = \bar{x}_{A'}$.

Now we proceed to investigate the minimal solution set $\underline{X}(A,b)$. First we decompose $c = [c_1^1, \cdots, c_j^k, \cdots, c_n^c]$ as column matrices with $c \leq n$ and define $|c_j^k|$ as index set of non–zero elements at column j in class k of C–matrix. That is, for a given column j in class k,

$$|c_j^k| = \{i \mid \text{if } c_{ij}^k = 1 \text{ or } E\ \forall i \in I\}\ \forall j \in J \text{ and } k \in K. \qquad (6)$$

If C is not degenerate, then $|c_j^k| \neq \phi\ \forall j \in J$ and $k \in K$. From our previous studies, we know that there are columns that have no influence on determining the minimal solution. So we classify the columns into critical columns and non–critical column.

Definition 3–2. Column c_j^k in C–matrix is called a non–critical column in class k, if there exists a column j_0 in class k_0 such that any of the following cases holds: (i) if $k_0 < k$, then $|c_{j_0}^{k_0}| \subseteq |c_j^k|$; (ii) if $k_0 = k$ and $j_0 < j$ then $|c_{j_0}^{k_0}| \subseteq |c_j^k|$. (iii) if $k_0 = k$ and $j_0 > j$, then $|c_{j_0}^{k_0}| \subset |c_j^k|$ and $|c_{j_0}^{k_0}| \neq |c_j^k|$.

For both simplicity of computation and consistency with our previous studies [4,5], we rearrange the columns of C–matrix by setting the critical column of each class as the leading column. Then column 1 of C–matrix is always a critical column.

Now if we define the set of path as

$$\mathbf{P} = \{p \mid p = (i_1^1, \ldots, i_j^k, \ldots, i_n^c)\ \forall i_j^k \in |c_j^k|,\ j \in J \text{ and } k \in K\}, \qquad (7)$$

then its subset $\mathbf{P}_0$ is defined as follows:

Definition 3–3. For any j, $2 \leq j \leq n$, if there is a path $p \in \mathbf{P}$ of which $\{i_1^1, .. i_{j-1}^{k'}\} \cap |c_j^k| \neq \phi$ and assume that i_d^h is the first element of $\{i_1^1, \ldots, i_{j-1}^{k'}\}$ to appear in $|c_j^k|$ then we set $i_j^k = i_d^h$; otherwise $i_j^k \in |c_j^k|$. We call the set of these path $\mathbf{P}_0$ and $\mathbf{P}_0 \subseteq \mathbf{P}$.

Now, we intend to derive a subset $\mathbf{P}^*$ of $\mathbf{P}$ and to prove that $\mathbf{P}^*$ is one to one corresponding to $\underline{X}(A,b)$. Before the definition of $\mathbf{P}^*$, we first define p^k and p_j^k as follows: for a given $k \in K$,

$$p^k = \{i^k \mid p = (i_1^1, .., i_j^k, .., i_n^c), i^k = i_j^k\ \forall\ j \in \text{class } k\} \qquad (8)$$

and for a given j in class k the subset p_j^k of p^k is

$$p_j^k = \{i_j \mid p = (i_1^1, .., i_j^k, .., i_n^c), i_j = i_{j'}^k\ \forall j' < j \text{ and } j' \in \text{class } k\}. \qquad (9)$$

Definition 3–4. A subset $\mathbf{P}^*$ of $\mathbf{P}_0$ is defined as follows: $\mathbf{P}^* = \{p \mid p = (i_1^1, .., i_j^k, .., i_n^c) \in \mathbf{P}_0$ there is no other $p' = (i'^1_1, .., i'^k_j, .., i'^c_n) \in \mathbf{P}_0$ such that there exist a j_0 in class k_0 satisfied (i) $i_j^k = i'^k_j\ \forall k < k_0$ and $j \in$ class k, and (ii) $p'^{k_0}_{j_0} \subset p^{k_0}_{j_0}$ }.

From the definition of P, $\mathbf{P}_0$ and $\mathbf{P}^*$ we know that $\mathbf{P}^* \subseteq \mathbf{P}_0 \subseteq \mathbf{P}$. Now we shall prove that $\mathbf{P}^*$ is one to one corresponding to $\underline{X}(A,b)$.

Definition 3–5. For each path $p = (i_1^1, \ldots, i_j^k, \ldots, i_n^c) \in \mathbf{P}^*$ we define a corresponding $\check{x} = [\check{x}_1, \ldots, \check{x}_i, \ldots, \check{x}_m]$ such that

$$\check{x}_i = \begin{cases} \text{Max } \{b^k\}, & \text{if } \{k:\ i_j^k = i\} \neq \phi, \\ 0, & \text{otherwise}. \end{cases} \qquad (10)$$

and $\check{X}(\mathbf{P}^*)$ is the set of $\check{x}$.

Lemma 3–2. If p, $p' \in \mathbf{P}^*$ with $p \neq p'$, p and p' correspond to $\check{x}$ and $\check{x}'$, respectively, then $\check{x} \neq \check{x}'$.

Proof. Let p and $p' \in \mathbf{P}^*$, $p = (i_1^1, \ldots, i_j^k, \ldots, i_n^c)$ and $p' = (i'^1_1, \ldots, i'^k_j, \ldots, i'^c_n)$ correspond to $\check{x} = [\check{x}_1, \ldots, \check{x}_i, \ldots, \check{x}_m]$ and $\check{x}' = [\check{x}'_1, \ldots, \check{x}'_i, \ldots, \check{x}'_m]$, respectively. If there exists a $j_0 \in J$ in class k_0 that is the first element with $i_{k_0}^{j_0} \neq i'^{j_0}_{k_0}$, then we have $\bigcup_{k=1}^{k_0-1} p^k = \bigcup_{k=1}^{k_0-1} p'^k$. By Definition 3–4 we know that $p^{k_0} \not\subset p'^{k_0}$ and $p^{k_0} \not\supset p'^{k_0}$. So there exists an $i \in I$ such that $i \in p^{k_0}$, $i \notin p'^{k_0}$ and $i \notin \bigcup_{k=1}^{k_0-1} p^k$. By Definition 3–5, we have $\check{x}_i = b^{k_0}$ and $\check{x}'_i < b^{k_0}$, then $\check{x} \neq \check{x}'$. □

Lemma 3–3. Each $\check{x} \in X(\mathbf{P}^*)$ is a minimal solution of $\underline{X}(A,b)$.

Proof. Let $\check{x} = [\check{x}_1, \ldots, \check{x}_i, \ldots, \check{x}_m]$, by Lemma 3–1 and Definitions 3–4 and 3–5, we know that $\check{x} \in \mathbf{X}(A,b)$. We shall prove that $\check{x} \in \underline{X}(A,b)$. Suppose $x = [x_1, .., x_i, \ldots, x_m] \in \mathbf{X}(A,b)$ where $x_{i'} < \check{x}_{i'}$ and $x_i = \check{x}_i\ \forall i \neq i'$. Then by Definition 3–5 we have $\{k \mid i_j^k = i'\} \neq \phi$ (otherwise, $x_i < \check{x}_{i'} = 0$.). Define $k_0 = \min\{k \mid i_j^k = i'\}$, then $\check{x}_{i'} = b^{k_0} > x_{i'}$. Because of $(x_{i'} \wedge a_{i'j}^{k_0}) \leq x_{i'} < b^{k_0}\ \forall j \in$ class k_0, if $i' \in p^{k_0}$, by Definitions 3–4 and 3–5, then $\exists j \in$class k_0 with $i_j^{k_0}=i'$ such that $(x_i \wedge a_{i\,j}^{k_0})<b^{k_0}$ $\forall i \in I$. Hence $x \notin X(A,b)$, it contradicts $x \in X(A,b)$. Therefore $\bar{x} \in \underline{X}(A,b)$. □

Lemma 3–4. For each $\underline{x} \in \underline{X}(A,b)$ there exists a $p \in P^*$ corresponding to it.

Proof. Let $\underline{x}=[\underline{x}_1,..,\underline{x}_i,...,\underline{x}_m] \in \underline{X}(A,b)$, we know that there exist some $p \in P_0$ corresponding to $\underline{x}$ [4]. From Definition 3–4, we know that each $p \in P_0$ corresponds to a path of P^*. Then by Lemma 3–3, we obtain that there is a $p \in P^*$ corresponding to $\underline{x}$. □

From Definitions 3–4,3–5 and Lemmas 3–2, 3–3, 3–4 we can conclude that each path in P^* is one to one corresponding to $\underline{X}(A,b)$, and we can obtain $\underline{X}(A,b)$ by means of P^*.

Example 3–1. Let us consider equation (1) with

$$A=\begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.3 & 0.7 & 0.5 \\ 0.8 & 0.7 & 0.9 & 0.7 & 0.1 & 0.3 \\ 0.9 & 0.9 & 0.6 & 0.7 & 0.3 & 0.1 \\ 1.0 & 1.0 & 0.3 & 0.3 & 0.9 & 0.6 \\ 1.0 & 0.9 & 0.5 & 0.8 & 0.7 & 0.7 \end{bmatrix} \text{ and } b=[0.9\ 0.9\ 0.7\ 0.7\ 0.7\ 0.6]$$

Then we have $b^0=1$; $b^1=0.9$; $b^2=0.7$; $b^3=0.6$. By the definition of C–matrix, we get

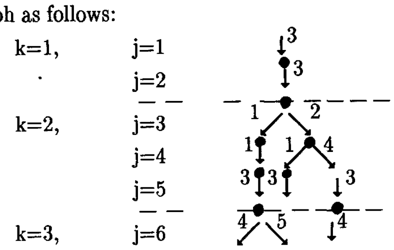
$$C=\begin{bmatrix} E & 0 & E & 0 & E & 0 \\ 0 & 0 & 1 & E & 0 & 0 \\ E & E & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & E \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

From Corollary 3–1, we get the maximum solution $\bar{x}=[\bar{x}_1,...,\bar{x}_5]$ as follows: $k(1)=0$ then $\bar{x}_1=1$; $k(2)=2$ then $\bar{x}_2=0.7$; $k(3)=0$ then $\bar{x}_3=1$; $k(4)=2$ then $x_4=0.7$; $k(5)=3$ then $\bar{x}_5=0.6$, that is, $\bar{x}=[1,0.7,1,0.7,0.6]$.

By Definition 3–2, we know that column j=2,3,5,6 are critical columns. We rearrange the columns of C–matrix by setting the critical column of each class as the leading column. Then we have

$$C=\begin{bmatrix} 0 & E & E & E & 0 & 0 \\ 0 & 0 & 1 & 0 & E & 0 \\ E & E & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 1 & 0 & E \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By Definition 3–4, we get $P^*=\{(3,3,1,1,3,4); (3,3,1,1,3,5); (3,3,2,4,3,4)\}$. The procedure to obtain P^* is shown as tree graph as follows:



Then by Lemmas 3–2, 3–3 and 3–4, we get $\underline{X}(A,b)=\{[0.7,0,0.9,0.6,0]; [0.7,0,0.9,0,0.6]; [0,0.7,0.9,0.7,0]\}$

Lemma 3–5. If C is not degenerate and $\text{card}|c_j^k|=1$ $\forall j \in C_f$ in class k then $\text{card}(\underline{X}(A,b))=1$.

Lemma 3–6. If $\text{card}(\underline{X}(A,b))=1$ and $k(i)=1$ $\forall i \in I$ then $\text{card}(X(A,b))=1$.

Sensitivity Analysis on a state–matrix A.

The state matrix in a system represented by a relation equation plays an important role, the pre–estimated values given to A matrix strongly affect the resulting solution. Therefore, for the purpose of control and management, it would be beneficial to know the tolerance intervals of the elements in A matrix with respect to the known solution set and output–vector. If the intervals are wide, then there is a great flexibility in defining the corresponding relations. If, on the contrary, the intervals are narrow, then one should pay more attention on controlling those elements, otherwise the original solution set can not be insured.

From our previous studies, we know that a solution set is characterized by the C–matrix of a state–matrix. So if we define a set of C–matrix of a state–matrix , denoted by $C(A,b)$, which has the same characteristics with respect to $X(A,b)$ then, by Corollary 3–2, each C–matrix has the same $k(i)$ $\forall i \in I$ and the same P^* for determining the same maximum and minimal solutions respectively. From the Definition 3–4, we know that P^* is completely determined by C_f and the order of column within a class has no influence on P^*. So we shall consider only the pattern of each column regardless the effect from their permutations.

From the discussions above and our previous study [5], we can derive the following algorithm for construction of

$S(A)=\{A^* \mid x \circ A^*=b, \forall x \in X(A,b)\}$.

Step 1. Determine the tolerance intervals of A for maximum solution.

For each $C^* \in C(A,b)$ the k(i) must be correspondingly equal to that of current C–matrix of A in order to have the same maximum solution.

(i) $c^k_{ij}=1$ implies $a^k_{ij} \in [b^k,1]$. But if at row i of class k $\exists j' \neq j$ with $a^k_{ij'} > b^k$, then $a^k_{ij} \in (b^k,1]$ is defined for consistency.

(ii) $c^k_{ij}=0$ and $k<k(i)$ then $a^k_{ij} \in [0,1]$.

Step 2. Determine the tolerance intervals for the minimal solutions.

In order to retain the same minimal solution set, each $C^* \in C(A,b)$ must have the same C_f that can be identified by applying Definition 3–2. Except those have been defined in Step 1, The value a^k_{ij} corresponding to the element c^k_{ij} of critical column j can be determine as follows:

(i) $c^k_{ij}=0$ implies $a^k_{ij} \in [0,b^k)$.

(ii) $c^k_{ij}=E$ and $k(i)=k$ then $a^k_{ij} \in [b^k,1]$.

(iii) $c^k_{ij}=E$ and $k(i) \neq k$ then $a^k_{ij}=b^k$.

Step 3. Determine the alternatives for non–critical columns.

From Definition 3–2, we can derive all alternatives of a non–critical column j of C–matrix by selecting a column $c^{k_0}_{j_0}$ with $j_0 \in C_f$ and $k_0 \leq k$ and setting $|c^{k_0}_{j_0}| \subseteq |c^k_j|$.

For each alternative of column j_0, the tolerance interval of each element other than those elements defined in Step 1 can be determined as follows:

(i) $c^k_{ij}=E$ and $k(i)=k$ implies $a^k_{ij} \in [b^k,1]$.

(ii) $c^k_{ij}=E$ and $k(i)<k$ implies $a^k_{ij}=b^k$.

(iii) $c^k_{ij}=0$ and $k(i) \geq k$ implies $a^k_{ij} \in [0,1]$.

(iV) $c^k_{ij}=0$ and $k(i)<k$ implies $a^k_{ij} \in [0,b^k]$.

Example 4–1. For A and b as given in Example 3–1.

By Definition 3–2, we have $C_f=\{2,3,5,6\}$ where column number is named before rearrangement in order to consistent with our original matrix A.

The step below follow the proposed algorithm to obtain S(A).

Step 1. $c^2_{23}=1$ implies $a^2_{23} \in [0.7,1]$ and a^1_{21}, $a^1_{22} \in [0,1]$.
$c^2_{44}=1$ implies $a^2_{44} \in [0.7,1]$ and a^1_{41}, $a^1_{42} \in [0,1]$.
$c^3_{56}=1$ implies $a^3_{56} \in (0.6,1]$ and a^1_{51}, a^1_{52}, a^2_{53}, a^2_{54}, $a^2_{55} \in [0,1]$.

Step 2. $C_f=\{2,3,5,6\}$

For j=2

$$[c^1_{i2}]=\begin{bmatrix} 0 \\ 0^* \\ E \\ 0^* \\ 0^* \end{bmatrix} \text{ implies } [a^1_{i2}]=\begin{bmatrix} [0,0.9) \\ [0,1]^* \\ 0.9 \\ [0,1]^* \\ [0,1]^* \end{bmatrix}.$$

*: as defined in Step 1.

For j=3

$$[c^2_{i3}]=\begin{bmatrix} E \\ 1^* \\ 0 \\ 0 \\ 0^* \end{bmatrix} \text{ implies } [a^2_{i3}]=\begin{bmatrix} 0.7 \\ [0.7,1]^* \\ [0,0.7) \\ [0,0.7) \\ [0,1]^* \end{bmatrix}.$$

For j=5

$$[c^2_{i5}]=\begin{bmatrix} E \\ 0 \\ 0 \\ 1^* \\ 0^* \end{bmatrix} \text{ implies } [a^1_{i2}]=\begin{bmatrix} 0.7 \\ [0,0.7) \\ [0,0.7) \\ [0.7,1]^* \\ [0,1]^* \end{bmatrix}.$$

For j=6

$$[c^3_{i6}]=\begin{bmatrix} 0 \\ 0 \\ 0 \\ E \\ 1^* \end{bmatrix} \text{ implies } [a^3_{i6}]=\begin{bmatrix} [0,0.6) \\ [0,0.6) \\ [0,0.6) \\ 0.6 \\ (0.6,1]^* \end{bmatrix}.$$

Step 3. $J \backslash C_f=\{1,4\}$.

For j=1 then we have $c^1_2 \in C_f$ with $j_0=2$ and $k_0 \leq 1$.

$$[c^1_{i2}]=\begin{bmatrix} 0 \\ 0 \\ E \\ 0 \\ 0 \end{bmatrix} \text{ determines } [c^1_{i1}]=\begin{bmatrix} 0 \\ 0^* \\ E \\ 0^* \\ 0^* \end{bmatrix} \text{ that implies}$$

$$[a^1_{i2}]=\begin{bmatrix} [0,0.9] \\ [0,1]^* \\ 0.9 \\ [0,1]^* \\ [0,1]^* \end{bmatrix}.$$

For j=4 then we have $j_0=2,3,5 \in C_f$ with the corresponding class $k_0 \leq 2$.

Then for $j_0=2$,

$$[c^1_{i2}]=\begin{bmatrix} 0 \\ 0 \\ E \\ 0 \\ 0 \end{bmatrix} \text{ determines } [c^2_{i4}]=\begin{bmatrix} 0 \\ 0^* \\ E \\ 0^* \\ 0^* \end{bmatrix} \text{ that implies}$$

$$[a^2_{i4}]=\begin{bmatrix} [0,0.7] \\ [0,1]^* \\ 0.7 \\ [0,1]^* \\ [0,1]^* \end{bmatrix}.$$

for $j_0=3$,

$$[c^2_{i3}]=\begin{bmatrix} E \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ determines } [a^2_{i4}]=\begin{bmatrix} E \\ E \\ 0 \\ 0 \\ 0* \end{bmatrix} \text{ that implies}$$

$$[a^2_{i4}]=\begin{bmatrix} 0.7 \\ [0.7,1] \\ [0,0.7] \\ [0,1] \\ [0,1]* \end{bmatrix}.$$

for $j_0=5$,

$$[c^2_{i5}]=\begin{bmatrix} E \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ determines } [c^2_{i4}]=\begin{bmatrix} E \\ 0 \\ 0 \\ E \\ 0* \end{bmatrix} \text{ that implies}$$

$$[a^2_{i4}]=\begin{bmatrix} 0.7 \\ [0,1] \\ [0,0.7] \\ [0.7,1] \\ [0,1]* \end{bmatrix}.$$

Note that we must keep at least one of a^2_{23} , a^2_{24} greater than $b^2=0.7$ and at least one of a^2_{44} , a^2_{45} greater than $b^2=0.7$.

Discussions and conclusions.

This study solves a generalized fuzzy relation equation and analyzes its sensitivities on its state matrix A. Since this study is an extension of our previous work [4,5] by relaxing the condition that the entries in vector b are strictly distinct, so the proposed method is generalized.

The proposed method can also be applied to interval–valued fuzzy relation equations. By means of the quasi–characteristic matrix, we can improve the method proposed by Wang & Chang [3] and we shall present it in the near future.

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