LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

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MASTER IN
ACTUARIAL SCIENCE

MASTER´S FINAL WORK
THESIS

A STUDY ON THIELE'S DIFFERENTIAL EQUATION

ALICE LOUREIRO LEOCÁDIO BOTELHO DE LEMOS

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Abstract

Thiele's differential equation has a long history, dating back to an unpublished note of Thiele, 1875 (Gram, 1910). Thorvald Nicolai Thiele was a Danish researcher who worked as an actuary, astronomer, mathematician and statistician. He proved that for a whole life assurance of a single individual with benefit of amount 1, payable immediately on death, the prospective reserve satisfies a certain linear differential equation, which is extremely useful for the understanding of reality: Thiele's differential equation. In a more general framework, Thiele's differential equations for the prospective reserve are a linear system of differential equations describing the dynamics of reserves in life and pension insurance in continuous time.

This text has the main purpose of reviewing in a comprehensive way the contributions related to Thiele's equation that appeared over time, presenting the status of the art on this important topic. A revision of life insurance mathematics is first given (Dickson *et al.* 2013; Bowers *et al.*1997) and then Thiele's differential equation is derived under the classical and multiple state model of human mortality for one life and for multiple lives (Hoem 1969). After this, some illustrations are presented under different types of contracts. Following the developments in the literature, more general differential equations are obtained, including a stochastic payment process (Norberg 1992a and Møller 1993) and a diffusion process for interest rate (Norberg and Møller 1996). The technique of using Thiele's differential equation as a tool for life insurance product development (Ramlau-Hansen 1990 and Norberg 1992b) and the generalization of the equation for a closed insurance portfolio (Linnemann 1993) are also discussed. Finally, other developments are summarised (Milbrodt and Starke 1997, Steffensen 2000, Norberg 2001 and Christiansen 2008 and 2010).

Key words: life insurance, policy values, Thiele's differential equation, stochastic payment stream, diffusion process for interest rate.

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Resumo

Thorvald Nicolai Thiele foi um importante investigador dinamarquês, que trabalhou como atuário, astrónomo, matemático e estatístico (Gram, 1910). Entre os seus contributos, destaca-se em particular o facto de ter provado que para um seguro de vida inteira com benefício de valor 1, emitido sobre uma pessoa e pago imediatamente após a morte, as reservas prospetivas satisfazem uma equação diferencial linear que veio a revelar-se de grande importância para a compreensão do processo de formação das reservas: a chamada equação diferencial de Thiele. De um modo mais geral, as equações diferenciais de Thiele, para as reservas prospetivas, são um sistema diferencial linear de equações que descrevem a dinâmica das reservas nos seguros de vida e pensões em tempo contínuo.

Este texto tem como principal objetivo rever de forma tão completa quanto possível as contribuições relacionadas com a equação de Thiele que foram surgindo ao longo do tempo, dando assim '*the present state of the art*' deste relevante tópico. Começando por fazer uma revisão breve do essencial da matemática atuarial (Dickson *et al.*2013; Bowers *et al.*1997), avança depois para a derivação da equação de Thiele, considerando os dois modelos de mortalidade, o clássico e o de múltiplos estados, sobre uma pessoa e sobre várias pessoas (Hoem 1969). Algumas ilustrações, para vários tipos de contrato, são seguidamente introduzidas. Dos desenvolvimentos conhecidos, dá-se especial destaque às generalizações da equação diferencial que incluem um processo estocástico de pagamentos (Norberg 1992a e Møller 1993) e um processo de difusão para a taxa de juro (Norberg e Møller 1996). Apresenta-se também o uso da equação como ferramenta para o desenvolvimento de produtos de seguro de vida (Ramlau-Hansen 1990 e Norberg 1992b) e descreve-se uma generalização da equação diferencial para uma carteira fechada de seguros (Linnemann 1993). A última parte do trabalho faz um resumo de outros contributos relacionados com a equação (Milbrodt e Starke 1997, Steffensen 2000, Norberg 2001 e Christiansen 2008 e 2010).

Palavras-chave: seguros de vida, reserva matemática, equação diferencial de Thiele, fluxo de pagamento estocástico, processo de difusão para a taxa de juro.

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«INSURANCE – the pooling of risk helps us to lead more predictable lives. The ability to insure assumes that there is some technology capable of calculating large and complex risks, and some organizational form that can mobilize the financial resources to underwrite the calculations (…). The second half of the eighteenth century was a time of major innovations. One of the big innovations concerned life insurance. Unlike practically every other branch of insurance, it was a European invention (…). »

(Borscheid, Gugerli and Straumann 2013)

1. Introduction

Over the last two centuries life insurance theory has evolved significantly. The advance of computers has contributed to apply models and develop new products in an unprecedented way. Nowadays actuaries are able to build highly sophisticated models with powerful software to manage risks arising from insurance business.

Actuarial theory is of crucial importance for insurance business to remain solvent and to satisfy all parties of the business: shareholders, stakeholders and policyholders. The contribution of early actuaries and mathematicians to actuarial theory is of unquestionable importance. During the last century many actuaries have studied and applied early theory and contributed to new findings and new theories.

One of the earliest actuaries and a most influential scientist of his time was Thorvald Nicolai Thiele. He was born in 1838 in Denmark and was astronomer, mathematician, statistician and actuarial mathematician. Among the many contributions he made to the advance of knowledge there is a significant work in the field of actuarial theory. In particular and more important in the framework of this dissertation is the fact that his name is associated with a differential equation that will be introduced later on: Thiele's differential equation (TDE) (Lauritzen 2002).

TDE is a powerful and insightful equation that has many applications in insurance mathematics and actuarial practice. This equation was only published in Thiele obituary by Gram in 1910.

Thiele also engaged in actuarial mortality research. He introduced a mortality law capable of fitting mortality at all ages and made pioneering contributions to the theories of graduation of mortality tables. As an initiator, he was the founder of important institutions in Denmark. He founded the first Danish private insurance company in 1872, the Danish Mathematical Society in 1873 and the Danish Actuarial Society in 1901 (Norberg 2004).

The purpose of this dissertation is to present TDE and the theory behind it as well as further developments and applications of the equation. The structure of this work was carefully thought through because of the extensive literature concerning this central equation on one hand, and because of the need to fulfil the limit of pages requirement on the other hand. In order to first provide a framework to TDE, a revision of actuarial mathematics seemed to be indispensable to understand the developments and contributions that appeared over time. In that sense, this work will study in depth the fundamental developments and summarize some of the other relevant developments.

In Chapter 2 we recall life insurance mathematics as it is applied to life insurance contracts. First, the four traditional types of life insurance contracts are presented and then we introduce the actuarial assumptions that have to be considered for future cash flow projection of life insurance contracts. The valuation of life contingent benefits and premiums are then explained under the classical and multiple state models. Chapter 3 is devoted to Policy Values in discrete and in continuous time leading us to TDE. Derivation, interpretations and a numerical solution of the equation is presented as well as a few examples for some types of life insurance contracts. Chapter 4 shows a number of important developments and applications of TDE. First, the focus is on relaxing some assumptions of the original equation, e.g. considering a counting process for payments and introducing a stochastic differential equation for the discount rate. Then, some examples of the use of the equation for life insurance product development will be given. Finally, in the last part, a version of TDE considering a closed insurance portfolio is explained. The contributions related to TDE that appear in the literature are therefore reviewed on Chapter 3 and Chapter 4.

2. Life Insurance

2.1 Life insurance contracts

An insurance contract is a written agreement under which one party, the insurer, accepts a risk from another party, the policyholder, by agreeing to pay a compensation (called the benefit) if the specified uncertain future event occurs, in exchange for a premium paid by the policyholder.

Life insurance contracts cover mortality and longevity risks as well as savings. They are usually long term contracts where the benefit is commonly known at outset. Non-life insurance contracts cover a multitude of natural and man-made perils. They are usually short term coverage and the benefit is commonly unknown.

Nowadays life insurance business develops innovative and sophisticated products. For the purpose of this dissertation only the traditional life insurance contracts will be presented as they are the simplest types of contracts from which any other contract can be developed. Modern contracts, as for example contracts where the benefit depends on the performance of an investment fund, can be developed from the traditional ones. The simplest contract is the *Whole life insurance* where the benefit is paid on or after death of the policyholder. Then, under *Endowment* insurance the benefit is paid at a determined date upon survival to maturity or on death, whichever occurs first. For these two types of contracts the payment of the benefit is a certain event (ignoring surrenders). Under a *Term insurance* contract, the benefit is paid only if death occurs during the term of the contract. The last traditional type of contract is the *Pure endowment* insurance where the benefit is paid at the end of the term if the policyholder survives. For these last two types of contracts there is a positive probability that the benefit will not be paid (Dickson *et al.* 2013).

2.2 Technical bases

The projection of future cash flows under a life insurance contract for pricing and valuation purposes give rise to the need of derivation and development of actuarial assumptions, called the actuarial bases or technical bases. Actuarial assumptions have to be considered regarding future interest rates to discount cash flows to the present, future rates of mortality, future expenses and regarding any basis set on the contract (e.g. disability rates, etc.) as well as target profit (Sundt and Teugels 2004).

Traditionally, some safety margins are considered when setting the technical basis. The interest rate is fixed below the market level and a safety margin is considered to the mortality rates. However, insurance companies sell a wide variety of life insurance products and safety margins differ by type of contract. The insertion of margins implies that, on average, profit emerges over time (Ramlau-Hansen 1988).

For the purpose of this dissertation we consider both the classical approach and the multiple state approach to model mortality of a single life (Wolthuis 2003). We assume a constant force of interest (denoted δ) for the continuous time and the interest rate (denoted i) for discrete time. No expenses will be considered as they may be added by increasing premiums or decreasing benefits.

2.3 The classical approach

Actuaries model human mortality because the benefit outgo depends on the time of death of the policyholder or on survival to a predefined term. The classical approach to model the uncertainty over the duration of an individual's future lifetime is to regard the remaining life time of an individual as a random variable. Under this model the policyholder is either alive or death. The notation used is the generally accepted actuarial notation from The International Association of Actuaries (IAA).

2.3.1 The future lifetime random variable

The future lifetime of an individual aged x is represented by the continuous random variable T_x and the age at death is represented by $x + T_x$.

The cumulative distribution function of T_x to compute death probabilities at time t is $F_x(t) = \mathbb{P}\,[T_x \leq t] = \,_t\,q_x$ and the survival function to compute survival probabilities is given by $S_x(t) = 1 - F_x(t) = \mathbb{P}[T_x > t] = \,_t\,p_x$. To compute probabilities at different ages given that the individual has survived for some years connecting the random variables $\{T_x\}_{x\geq 0}$, we assume that the following relationship holds $_t\,q_x = \mathbb{P}[T_x \leq t] = \mathbb{P}[T_0 \leq x + t \mid T_0 > x\,]$ for all $x \geq 0$, where T_0 is the future life time of a baby born. Working out this relationship with probability theory, $\mathbb{P}[T_x \leq t] = \frac{\mathbb{P}[x<T_0\leq x+t]}{\mathbb{P}[T_0>x]} = \frac{\left(_{x+t}\,q_0 - \,_x\,q_0\right)}{_x\,p_0}$, and using the relationship $_t\,p_x = 1 - \,_t\,q_x$ we get an important result $_{x+t}\,p_0 = \,_x\,p_0\,\,_t\,p_x$ that can be interpreted in the following way: the survival probability of a baby born to age $x + t$ is given by the product of the survival probability from birth to age x by the survival probability from age x to age $x + t$ (Dickson *et al.* 2013).

One of the most important concepts regarding mortality is the force of mortality, denoted μ_x and defined for a life aged x as

$$\mu_x = \lim_{h\to 0} \frac{1}{h}\,\,_h\,q_x = \lim_{h\to 0^+} \frac{1}{h}\left(1 - \,_h\,p_x\right), h > 0. \qquad (2.1)$$

The force of mortality can be interpreted as the instantaneous mortality measure of a life aged x. For a short time interval h we assume $\mu_x\,h \approx \,_h\,q_x$ (Garcia and Simões 2010).

The last part of equation (2.1) shows how the force of mortality is related with the survival function. Working out equation (2.1) for any age $x + t, t \geq 0$ and knowing the force of mortality we obtain another equation to compute survival probabilities: $_t\,p_x = \exp\left\{-\int_0^t \mu_{x+s}\,ds\right\}$.

The density function of the random variable T_x can be derived using first principle $f_x(t) = \frac{d}{dt}\ {}_t q_x = -\frac{d}{dt}\ {}_t p_x$, and the relationship between the force of mortality and the survival function in equation (2.1). The density function is given by $f_t(x) = \mu_{x+t}\ {}_t p_x$. From this result we obtain an important formula that relates the future lifetime distribution function in terms of the survival function and the force of mortality ${}_t q_x = \int_0^t {}_s p_x \, \mu_{x+s} ds$ (Dickson *et al.* 2013).

The expected value and the variance of T_x can also be obtained using integration by parts. The expected value of the future lifetime random variable called the complete expectation of life and denoted $\overset{\circ}{e}_x$ is given by

$$E[T_x] = \int_0^\infty t\ {}_t p_x \, \mu_{x+t} \, dt \iff \overset{\circ}{e}_x = \int_0^\infty t f_x(t) dt. \tag{2.2}$$

The variance is given by

$$V[T_x] = 2\int_0^\infty t\ {}_t p_x \, dt - \left(\overset{\circ}{e}_x\right)^2. \tag{2.3}$$

2.3.2 Valuation of life contingent cash flows

Under a life insurance contract the payment of the benefit from the insurer and the payment of the premium from the policyholder can either be of the form of one single amount (a lump sum) or a life contingent annuity. Lump sum premiums are paid at the outset of the contract to guarantee risk coverage so they are not random. The life contingent single benefits and the life contingent annuities depend on the time of death of the policyholder. The valuation of these types of benefits and annuities is essential for the calculation of premiums and Policy Values as we shall see in Chapter 3. Some important examples are now reviewed.

The life contingent single benefit is a function of the time of death and is modelled as a random variable. Its present value depends on the actuarial basis considered. For different actuarial bases the distribution of the present value can be derived and its expected present value (EPV) and other moments can be computed. Table I presents the valuation of a benefit of amount 1 for the four traditional types of contracts, introduced in *2.1*, in continuous time and considering a n years term except for *Whole life* insurance (Bowers *et al.* 1997).

TABLE I

VALUATION OF A LIFE CONTINGENT SINGLE BENEFIT OF AMOUNT 1 IN CONTINUOUS TIME

Type of contract	Present Value	Expected Present Value
Whole life	$e^{-\delta T_x}$	$\bar{A}_x = \int_0^\infty e^{-\delta t}\ {}_t p_x\, \mu_{x+t}\, dt$
Term insurance	$\begin{cases} e^{-\delta T_x} & if\ T_x \leq n \\ 0 & if\ T_x > n \end{cases}$	$\bar{A}^{\,1}_{x:\overline{n}\rvert} = \int_0^n e^{-\delta t}\ {}_t p_x\, \mu_{x+t}\, dt$
Pure endowment	$\begin{cases} 0 & if\ T_x < n \\ e^{-n\delta} & if\ T_x \geq n \end{cases}$	$\bar{A}_{x:\overline{n}\rvert}^{\ \ \,1} = e^{-n\delta}\ {}_n p_x$
Endowment	$e^{-\delta\, min(T_x, n)}$	$\bar{A}_{x:\overline{n}\rvert} = \bar{A}^{\,1}_{x:\overline{n}\rvert} + \bar{A}_{x:\overline{n}\rvert}^{\ \ \,1}$

Source : Dickson et al. 2013

Cash flows may occur on a fraction of a year, as for example monthly or quarterly. Considering a fraction $\frac{1}{m}$, $m \geq 1$, of a year, where m can be for example 12 or 4, corresponding to months or quarters, and defining the curtate future lifetime random variable as $K_x^{(m)} = \frac{1}{m}\lfloor m\, T_x \rfloor$, where $\lfloor\ \ \rfloor$ denotes the floor function (integer part function), then life contingent single benefits can be valued in discrete time at that fraction of the year.

Table II presents the valuation of a benefit of amount 1 for the four traditional contracts in discrete time where the discount factor is $v = (1+i)^{-1}$ and ${}_{\frac{k}{m}\rvert\frac{1}{m}} q_x$ represents the probability that the life aged x survives $\frac{k}{m}$ years and then dies in the next $\frac{1}{m}$ years.

TABLE II

VALUATION OF A LIFE CONTINGENT SINGLE BENEFIT OF AMOUNT 1 IN DISCRETE TIME

Type of contract	Present Value	Expected Present Value
Whole life	$v^{K_x^{(m)}+\frac{1}{m}}$	$A_x^{(m)} = \sum_{k=0}^{\infty} v^{\frac{k+1}{m}} {}_{\frac{k}{m}\vert\frac{1}{m}}q_x$
Term insurance	$\begin{cases} v^{K_x^{(m)}+\frac{1}{m}} & if\ K_x^{(m)} \le n - \frac{1}{m} \\ 0 & if\ K_x^{(m)} > n \end{cases}$	$A^{(m)1}_{x:\overline{n}\vert} = \sum_{k=0}^{mn-1} v^{\frac{k+1}{m}} {}_{\frac{k}{m}\vert\frac{1}{m}}q_x$
Pure endowment	$\begin{cases} 0 & if\ T_x < n \\ v^n & if\ T_x \ge n \end{cases}$	$A_{x:\overline{n}\vert}^{\ \ 1} = v^n\ {}_n p_x$
Endowment	$v^{min(K_x^{(m)}+\frac{1}{m},n)}$	$A^{(m)}_{x:\overline{n}\vert} = A^{(m)1}_{x:\overline{n}\vert} + A_{x:\overline{n}\vert}^{\ \ 1}$

Source : Dickson et al. 2013

A life annuity is a life contingent series of payments. Table III gives some examples of valuation of life contingent annuities for *Life annuities* and for *Term annuities* in continuous and in discrete time period. *Life annuities* are payable as long as the annuitant survives and *Term annuities* are payable also as long as the individual survives but for a maximum of *n* years. In the discrete time period we consider an annuity of amount 1, payable in advance and a discount factor: $d = \frac{i}{1+i}$. In the continuous case we consider a rate of payment of amount 1 per year.

TABLE III

VALUATION OF LIFE CONTINGENT ANNUITIES

Type	Time period	Expected Present Value
Life	Discrete	$\ddot{a}_x^{(m)} = \sum_{k=0}^{\infty} \frac{1}{m} v^{k/m}\ {}_{\frac{k}{m}}p_x$
	Continuous	$\bar{a}_x = \int_0^{\infty} e^{-\delta t}\ {}_t p_x\, dt$
Term	Discrete	$\ddot{a}^{(m)}_{x:\overline{n}\vert} = \sum_{k=0}^{mn-1} \frac{1}{m} v^{k/m}\ {}_{\frac{k}{m}}p_x$
	Continuous	$\bar{a}_{x:\overline{n}\vert} = \int_0^{n} e^{-\delta t}\ {}_t p_x\, dt$

Source : Dickson et al. 2013

2.3.3 Premium calculation

In setting premium rates the actuary must consider a set of assumptions called the technical basis as explained in *2.2.* At the outset of the contract the basis considered at this point in time is named the *first order basis*.

In order for the company to remain solvent, premiums are expected to cover benefits paid out and expenses. For the traditional types of contracts the benchmark is to compute premiums according to the *equivalence principle*.

The random variable of interest for premium calculation is the future loss random variable, denoted $L_t,\ t \geq 0$. The future loss random variable is given by the difference between the present value of future benefit and expenses (future outgo of insurer) and the present value of future premium (future income of insurer). At the outset of the contract, $t = 0$, according to the *equivalence principle* the premium is calculated as

$$E[L_0] = E[PV\ of\ future\ outgo] - E[PV\ of\ future\ income] = 0. \tag{2.4}$$

As an example consider a *Whole life* insurance contract issued to a life aged x with an agreed benefit denoted B payable on death of the policyholder. The premium denoted P is to be paid as a *life continuous annuity*. Under the *equivalence principle* and using the results from Table I and Table III the premium is given by

$$P = B\ \bar{A}_x / \ \bar{a}_x\ . \tag{2.5}$$

At the outset of the contract both the policyholder and insurer will know the amount of P (the rate at which premium is paid) and of B (the benefit insured payable on death).

We can conclude that under the *equivalence principle* premiums and benefits will balance on average. Other premium principles may also be applied as for instance the *portfolio percentile premium principle* or the *arbitrage principle* as we shall see in Chapter 4.

2.4 The multiple state approach

The formulation of the classical approach as a stochastic continuous time model with more than two sates is important for life insurance lines of business where benefits and premiums are contingent upon the transitions of the policy between the states specified on the contract, as for example health and disability insurance.

2.4.1 The Markov chain model

To model policies with multiple states we assume that the development of an individual insurance policy is described by a continuous Markov chain model (Hoem 1969), represented by $\{\mathbb{Z}\,(t)\}_{t\geq 0}$, with finite set space $\mathbb{J}=\{0,1,\dots,J\}$ of mutually exclusive states of the policy. The transition probabilities for any state i to j, from time $s\geq 0$ to time $t\geq s$, are given by ${}_{t-s}\,p_s^{ij}=\mathbb{P}\,[Z(t)=j|\,Z(s)=i]$ where $Z(t)$ is the state of the policy at time t in the period of insurance coverage. As the process $Z(t)$ is assumed to be a Markov process, the transition probabilities satisfy the Chapman-Kolmogorov relation

$$ {}_{t-s}\,p_s^{ij}=\sum_k \left({}_{u-s}\,p_s^{ik}\,{}_{t-u}\,p_u^{kj}\right)\ for\ all\ s<u<t. \tag{2.6}$$

The transition intensities are defined for $i\neq j$ as

$$\mu_s^{ij}=\lim_{t\to s}\frac{p_{ij}(s,t)}{t-s}\ \text{ and }\ \mu_s^{i}=\sum_{j\neq i}\mu_s^{ij}=\lim_{t\to s}\frac{1-p_{ij}(s,t)}{t-s}, \tag{2.7}$$

where μ_s^i is the total intensity of transitions from state i. These limits are assumed to exist for all relevant s and the intensities are assumed to be integrable functions.

The probability that the policy will remain in state i at least until time $t\geq s$ where $Z(t)=i$ is given by

$$ {}_{t-s}\,p_s^{\overline{ii}}=exp\left\{-\int_s^t\sum_{j\neq i}\mu_u^{ij}du\right\}. \tag{2.8}$$

For simple models with few states and no re-entering possibility between states the transition probabilities can sometimes be evaluated analytically using equation (2.8), if the forces of transition are known. For models with re-entering possibilities the

Kolmogorov's forward equation has to be used to evaluate transition probabilities and the foward Euler's method (Griffiths and Higham 2010) is applied with initial condition ${}_0V = 0$ to find a numerical value. The *Kolmogorov's forward equation* result is

$$\frac{d}{dt}\ {}_t p_x^{ij} = \sum_{k\neq j}\left({}_t p_x^{ik}\ \mu_{x+t}^{kj} - {}_t p_x^{ij}\ \mu_{x+t}^{jk}\right). \tag{2.9}$$

2.4.2 Valuation of life and state contingent cash flows

In the multiple state model life contingent single benefits and life contingent annuities are valued generalizing the definitions of subsection *2.3.2.* Benefits are usually paid on making a transition between states and annuities are paid upon sojourns in certain states. An example of valuation of both types of cash flows is presented.

First, consider a life contingent benefit of amount 1 paid in each transition to state *j*, given the life is in state *i* at age *x.* The EPV of this benefit is

$$\bar{A}_x^{ij} = \int_0^\infty \sum_{k\neq j} e^{-\delta t}\ {}_t p_x^{ik}\ \mu_{x+t}^{kj}\, dt. \tag{2.10}$$

Now consider a life contingent annuity of amount 1 per year paid continuously while the life is in state *i* given that at age *x* the life was in state *j*. The EPV is

$$\bar{a}_x^{ij} = \int_0^\infty e^{-\delta t}\ {}_t p_x^{ij}\, dt. \tag{2.11}$$

For the purpose of this dissertation a generalization of valuation of premiums and benefits in the multiple state model will be considered. A general insurance policy is characterized by the following conditions:

(1) If the policyholder moves from state *i* to state *j* at time *t*, a lump-sum benefit b_t^{ij} is paid instantaneously at time *t*;

(2) While in state *i*, annuity benefits are paid continuously at a rate B_t^i ;

(3) When the policy expires at time *n*, the policyholder receives an amount B_n^i if the policy is in state *i* at maturity date;

(4) Premiums are included as negative benefit payments.

The quantities b_t^{ij}, B_t^i , B_n^i are assumed to be non-random (Hansen 1988).

3. Policy Values and Thiele's differential equation: a few insights

3.1 Policy Values

The major difference in life insurance business when compared with other businesses derived from the fact that premiums are usually received a long time before the out go of the benefit. To meet its liabilities in the long run (to pay the benefits), the insurer needs to build up its assets during the course of the policy reserving premiums to fund benefits.

On the assets side of a life insurer balance sheet, the reserved premiums appear as investments such as bonds, equity and property. Investments are held in funds from which benefits and surrender values will be paid out. On the liabilities side, as the cost of providing the benefits has to be allocated to future accounting periods, it is necessary to make provision of future benefits to the policyholder. At the end of each accounting period an estimate of the total expected future payments to the policyholders is made by an appointed actuary to ensure that the life insurance company will pay the ultimate benefits and to ensure that the business will break even over the future course of the policy. The actuarial estimate, by policy, of the amount the insurer should have in its investment is called the Policy Value. The portfolio of assets held to meet future liabilities is called the reserve (Sundt and Teugels 2004).

The Policy Value estimation at time $t > 0$ is the valuation of a policy still in force at time t updated with the information currently available at this point in time. At the outset of the contract the *first-order basis* was considered for pricing (see *2.3.3*). At time *t*, to every assumption corresponds an actual outcome. From the information available up to time *t*, the *second-order basis* is set including as well some safety margins. Relevant amounts have to be recalculated with updated information. Profit or losses may emerge from differences due to higher or lower interest rate, due to different mortality experience compared to initial basis and sometimes even from higher or lower expenses.

Policy Values can be estimated prospectively (looking into the future) or retrospectively (considering accumulated premiums received and benefits paid up to time *t*). If the actuarial bases considered are the same for pricing (at outset of the contract) and for valuation (at time *t*) the prospective and retrospective Policy Values are equal. As TDE was derived for prospective Policy Values only Policy Values calculated prospectively will be considered. From now on Policy Value will refer to prospective Policy Value.

The Policy Value at time $t > 0$, denoted ${}_tV$, is the EPV of the future loss random variable at that time (cf *2.3.3*)

$$ {}_tV = E(L_t). \tag{3.1} $$

Policy values can be estimated in discrete or in continuous time. First, the valuation of Policy Value for discrete time cash flows is presented, and then we present Policy Values for continuous time cash flows.

3.2 Policy Value in discrete time

3.2.1 Start/end of the year

Consider a policy issued to a life aged *x* and still in force at year $t \in \mathbb{N}$, where cash flows can only occur at the start or end of the year. Considering what happens between two consecutive periods from time t to time $t+1$, the future loss random variable, defined in *2.3.3,* can be written as

$$ L_t = \begin{cases} B_t(1+i_t)^{-1} - P_t & with\, probability\; q_{x+t} \\ L_{t+1}(1+i_t)^{-1} - P_t & with\, probability\; p_{x+t}. \end{cases} \tag{3.2} $$

Equation (3.2) can be interpreted as follows: if the policyholder dies from t to $t+1$ (an event with probability q_{x+t}), the premium would have been paid at start of the year (at time t) and the benefit will be paid at the end of the year (at $t+1$). If the policyholder survives during the year (event with probability p_{x+t}), the premium would have been paid as well at time t but no benefit outgo will occur at time $t+1$. The policy will remain in force and the insurer will need to have assets reserved at $t+1$ to account for future benefit outgo.

Taking the expected value of equation (3.2), it follows that

$$E[L_t] = [B_t(1+i_t)^{-1} - P_t]\, q_{x+t} + [E[L_{t+1}](1+i_t)^{-1} - P_t]\, p_{x+t}. \qquad (3.3)$$

Rearranging

$$E[L_t] = B_t(1+i_t)^{-1}q_{x+t} - P_t(q_{x+t} + p_{x+t}) + E[L_{t+1}](1+i_t)^{-1}p_{x+t}. \qquad (3.4)$$

Recognising that ${}_{t+1}V = E[L_{t+1}]$ and that $q_{x+t} + p_{x+t} = 1$ we come to

$$\left({}_{t}V + P_t\right)(1+i_t) = B_t q_{x+t} + {}_{t+1}V\ p_{x+t}. \qquad (3.5)$$

Equation (3.5) is a recursive formula to calculate Policy Value at successive points in time for policies where cash flows occur at start/end of the year. The left-hand side is the sum of the amount of assets the insurer should have at time $t+1$, the Policy Value ${}_tV$, and the premium received at start of the year, both earning interest from t to $t+1$. This in turn must be equal to the amount the insurer needs to have either to pay the benefit in case of death or to maintain the reserve in case of survival, both weighted with their respective probabilities.

This recursive relationship shows the evolution of the Policy Value. If the invested premiums (assets) earn the rate of return assumed in the *first-order basis* and the mortality experience is also the same as assumed, then at all times the assets on the balance sheet will be equal to the Policy Value.

3.2.2 The m-thly case

Contractual cash flows may occur $m > 1$ times a year as seen in *2.3.2*. Considering a version of the example presented in *2.3.3* for cash flows occurring m times a year, the Policy Value for a policy still in force at time $t + \frac{s}{m}$, $s \leq m$ where the benefit is payable at the end of the fraction of the year and the premiums are payable at start of the fraction of the year is given by

$$_{t+\frac{s}{m}}V_x = B_{t+\frac{s}{m}}\, A^{(m)}_{x+t+\frac{s}{m}} - P_{t+\frac{s}{m}}\ \ddot{a}^{(m)}_{x+t+\frac{s}{m}}. \tag{3.6}$$

A recursive formula to compute Policy Value as in (3.5) can also be easily derived.

3.3 Policy Value with continuous cash flows: Thiele's differential equation

Continuous cash flows occur when a premium rate is paid continuously and the benefit is paid immediately on death or at maturity. TDE was first derived for a *Whole life* insurance contract issued to a life aged x and still in force at time t (Lauritzen 2002). To first present Thiele's work the continuous time Policy Value is derived for a *Whole life* insurance contract.

Consider the *Whole life* insurance example in *2.3.3.* The Policy Value for this contract still in force at time t under the *equivalence principle* is

$$_{t}V_x = B\ \bar{A}_{x+t} - P\ \bar{a}_{x+t}. \tag{3.7}$$

Appling results from Table I and Table III to (3.7) and considering that the force of interest δ_t is a continuous function of time, the Policy Value at time t is

$$_{t}V_x = \int_0^\infty\ B_{t+s}\ \frac{e^{-\int_0^{t+s}\delta_z\,dz}}{e^{-\int_0^{t}\delta_w\,dw}}\ {}_{s}p_{x+t}\ \mu_{x+t+s}\ ds - \int_0^\infty\ P_{t+s}\ \frac{e^{-\int_0^{t+s}\delta_z\,dz}}{e^{-\int_0^{t}\delta_w\,dw}}\ {}_{s}p_{x+t}\ ds. \tag{3.8}$$

Equation (3.8) can be solved using numerical integration. However, Thiele turned it into a differential equation (derivation is in appendix A). The result is the so called TDE,

$$\frac{d}{dt}\ {}_{t}V_x = P_t - B_t\,\mu_{x+t} + {}_{t}V_x\ (\mu_{x+t} + \delta_t). \tag{3.9}$$

TDE is a backward differential equation. It can also be derived by direct backward construction (Wolthuis 2003). Consider a policy still in force at time t and what happens in a small time interval $[t, t+dt]$:

(1) the policyholder pays the continuous premium $P_t dt$;

(2) If he/she dies, the benefit B_t is paid with probability $\mu_{x+t}\ dt + o(dt)$;

(3) If he/she survives, with probability $\left(1 - \mu_{x+t}\ dt + o(dt)\right)$, the reserves will earn interest $e^{-\delta_t dt}\ V_{t+dt}$.

The Policy Value is then

$$ {}_tV_x = B_t\ \mu_{x+t}\ dt - P_t dt + e^{-\delta_t dt}\ {}_{t+dt}V_x\ (1 - \ \mu_{x+t}\ dt) + o(dt). \qquad (3.10)$$

Subtracting ${}_{t+dt}V_x$ on both sides and dividing by dt,

$$ \frac{{}_{t+dt}V_x - {}_tV_x}{dt} = P_t - B_t\ \mu_{x+t} - {}_{t+dt}V_x \frac{(e^{-\delta_t dt} - 1)}{dt} + e^{-\delta_t dt}\ {}_{t+dt}V_x\ \mu_{x+t} + \frac{o(dt)}{dt}. \qquad (3.11)$$

Letting $dt \rightarrow 0$ and recognising that $\lim_{dt\rightarrow 0} \frac{(e^{-\delta_t dt} - 1)}{dt} = -\delta_t$ we arrive to

$$ \frac{d}{dt}\ {}_tV_x = P_t - B_t\, \mu_{x+t} + {}_tV_x\ (\mu_{x+t} + \delta_t). \qquad (3.9)$$

TDE (3.9) shows how the rate of increase of the reserve changes per unit of time at time t and per surviving policyholder. It makes very clear that this rate is affected by the following individual factors over an infinitesimal interval:

(1) Excess of premiums over benefits: $P_t - B_t\, \mu_{x+t}$

The annual premium rate increases the reserve and the benefit will cause the reserve to decrease. The benefit B_t is paid on death of the policyholder and the expected extra amount payable in the time interval $[t, t + dt]$ is $\mu_{x+t}\, dt\, B_t$, due to the expected mortality of policyholders. So the rate of increase at time t of the benefit is $\mu_{x+t}\, B_t$, which gives the annual rate at which money is leaving the fund reserved at exact time t, due to death;

(2) The annual rate at which reserves are released by death cause: ${}_tV_x\ \mu_{x+t}$

The reserve is measured per surviving policyholder. If one policyholder dies the reserve for that policyholder is no longer needed;

(3) Interest earned on the current amount of the reserve: ${}_tV_x\ \delta_t$

The amount of interest earned in the time interval $[t, t + dt]$ is $\delta_t\ {}_tV_x\ dt$ so the rate of increase at time t is ${}_tV_x\ \delta_t$.

3.4 Thiele's differential equation: savings premium and risk premium

Another interesting insight supplied by TDE is obtained rearranging equation (3.9) for the rate of premium. It follows that

$$P_t = \frac{d}{dt}\ {}_tV_x - \delta_t\ {}_tV_x + \left(B_t - \ {}_tV_x\right)\mu_{x+t}. \tag{3.12}$$

The rate of premium from (3.12) can be decomposed into a savings premium and a risk premium (Wolthuis 2003). The savings premium is given by

$$P_t^s = \frac{d}{dt}\ {}_tV_x - \delta_t\ \ {}_tV_x \tag{3.13}$$

and shows that it is equal to the rate of change of the reserve minus the interest earned on the reserve. It is the amount that must be saved for future benefit payment if the policyholder survives. It can be interpreted as the amount needed to maintain the reserve in excess of earned interest.

The risk premium is given by

$$P_t^r = \left(B_t - \ {}_tV_x\right)\mu_{x+t}, \tag{3.14}$$

which is the amount needed to cover the benefit in excess of available reserves if the policyholder dies. If the policy becomes a claim then the extra amount $B_t - \ {}_tV_x$ is needed to increase the reserve. This amount is also called the *Death Strain at Risk.*

3.5 Thiele's differential equation: numerical solution

TDE can be used to solve numerically for premiums, given the benefits, the interest rates and boundary values for the Policy Values.

One of the approximation methods that can be applied to solve TDE numerically for Policy Value is the Euler method (Griffiths and Higham 2010). The Euler method can be applied forwards using initial condition or backwards using terminal. Applying the backward method for a small step size h, TDE (3.9) for $t = n - h$ can be written as

$$B - \ {}_{n-h}V_x \ = h(P_{n-h} - B_{n-h}\,\mu_{x+n-h} + {}_{n-h}V_x\ (\ \mu_{x+n-h} + \delta_{n-h})). \tag{3.15}$$

The equation can then be solved in order to the only unknown variable ${}_{n-h}V_x$, since all the other variables are assumed to be known. Other step sizes are then applied, $t = n - 2h$, $t = n - 3h$ and so on, to find an approximate solution (Dickson *et al.* 2013; Sundt and Teugels 2004).

3.6 Thiele's differential equation by type of contract: some illustrations

Naturally, TDE can be formulated for different types of contracts. In each case the terms of the resulting equation will depend on the cash flows that occur in the small interval of time $[t, t + dt]$. For the traditional types of contracts differences are easily detected:

(1) In *Pure endowment* contracts, as the benefit is not paid on death of the policyholder but on survival, if the policyholder dies during time dt, which happens with probability $\mu_{x+t} dt$, there is no sum assured;

(2) If premiums are paid continuously at a rate P_t per year they will be included in TDE. If a single premium is paid as a lump sum amount at the beginning of the contract no more premiums will be earned so they will not be included in TDE.

Using results from Table I and Table III some illustrations follow.

3.6.1 Term insurance

Consider a *Term insurance contract* issued to an *x* years old policyholder where the benefit *B* is payable on death if death occurs within *n* years. The premium is received continuously at rate P till death or till the end of the contract, whichever occurs first. The Policy Value for a policy still in force at time $t, 0 < t < n$, is given by

$$ {}_tV^1_{x:\overline{n}|} = B\, \bar{A}^{\,1}_{x+t:\overline{n-t}|} - P\, \bar{a}_{x+t:\overline{n-t}|}. \tag{3.16}$$

Solving and turning the equation into a differential equation, we get

$$ \frac{d}{dt}\ {}_tV^1_{x:\overline{n}|} = P - B\mu_{x+t} + \ {}_tV^1_{x:\overline{n}|}\,(\mu_{x+t} + \delta_t). \tag{3.17}$$

Considering the same *Term insurance contract* assured by a single premium at the outset of the contract, TDE for a contract still in force at time *t* is then

$$\frac{d}{dt}\ {}_tV^1_{x:\overline{n|}} = -\,B\mu_{x+t} + {}_tV^1_{x:\overline{n|}}\,(\,\mu_{x+t} + \delta_t). \tag{3.18}$$

3.6.2 Pure endowment

Under a *Pure endowment* contract, assume that a benefit *B* is payable if the policyholder aged *x* survives the *n* years contract, and a premium *P* is payable continuously until earlier death or for the term of the contract. The Policy Value of a policy still in force at time *t* is given by

$${}_tV_{x:\overset{1}{\overline{n|}}} = BA_{x+t:\overset{1}{\overline{n|}}} - P\ \bar{a}_{x+t:\overline{n|}}. \tag{3.19}$$

TDE follows,

$$\frac{d}{dt}\ {}_tV_{x:\overset{1}{\overline{n|}}} = P + {}_tV_{x:\overset{1}{\overline{n|}}}\,(\,\mu_{x+t} + \delta_t). \tag{3.20}$$

Under a *Pure endowment* contract assured by a single premium, TDE is

$$\frac{d}{dt}\ {}_tV_{x:\overset{1}{\overline{n|}}} = {}_tV_{x:\overset{1}{\overline{n|}}}\,(\,\mu_{x+t} + \delta_t). \tag{3.21}$$

3.6.3 Endowment

Consider an *Endowment contract* issued to an *x* years old life where the benefit *B* is payable on death if death occurs within the *n* years of the contract or at the end of the term if the policyholder survives. The premium is also received continuously at rate *P* till death or till the end of the contract, whichever occurs first. The Policy value at time *t* is given by

$${}_tV_{x:\overline{n|}} = B\ \bar{A}_{\ x+t:\overline{n-t|}} - P\ \bar{a}_{x+t:\overline{n-t|}}. \tag{3.22}$$

Solving the equation into a differential equation, we come to a similar result as (3.17)

$$\frac{d}{dt}\ {}_tV_{x:\overline{n|}} = P - B\mu_{x+t} + {}_tV_{x:\overline{n|}}\,(\,\mu_{x+t} + \delta_t). \tag{3.23}$$

The same reasoning used in equations (3.18) and (3.21) is applied in case of a single premium paid at outset of the contract.

3.6.4 Whole life continuous annuity

Finally consider a life insurance contract where the benefit is a deferred life continuous annuity at rate of amount 1 per year, starting in *m* years if the policyholder is then alive and continuing for life, assured by a single premium at outset of the contract. In this case we have two different TDEs: one during the deferral period (equation (3.24)) and another one during the annuity payment (equation (3.25)),

$$\frac{d}{dt}\ {}_tV_x = {}_tV_x\,(\mu_{x+t} + \delta_t), t \leq m \tag{3.24}$$

$$\frac{d}{dt}\ {}_tV_x = -1 + {}_tV_x\,(\mu_{x+t} + \delta_t), t > m. \tag{3.25}$$

3.7 Thiele's differential equation under the multiple state model

The Policy Value at time t under the multiple state model, explained in *2.4,* is the expected value at that time of future loss random variable conditional on the state of the policy. For a policy still in force at time t given that the policyholder is in state $Z(t) = i$, the Policy Value is

$${}_tV_x^i = \int_t^n e^{-\delta(u-t)} \sum_j {}_{u-t}p_t^{ij} \left[B_u^j + \sum_{k\neq j} \mu_u^{jk} b_u^{jk} \right] du + e^{-\delta(n-t)} \sum_j {}_{n-t}p_t^{ij}\ {}_nV_x^i. \tag{3.26}$$

TDE in the multiple state approach is (Hoem 1969)

$$\frac{d}{dt}\ {}_tV_x^i = \delta\ {}_tV_x^i - B_t^i - \sum_{j\neq i} \mu_t^{ij} (b_t^{ij} + {}_tV_x^j - {}_tV_x^i). \tag{3.27}$$

The last part of (3.27) is called the amount at risk, denoted R_t^{ij},

$$R_t^{ij} = b_t^{ij} + {}_tV_x^j - {}_tV_x^i \tag{3.28}$$

and it is the risk cost for transitions out of state *i* per unit of time at time t where $\left({}_tV_x^j - {}_tV_x^i \right)$ is the reserve jump associated with a transition from state *i* to state *j* at that time. TDE can then be written (Linnemann 1993 and Hoem 1988) as

$$\frac{d}{dt}\ {}_tV_x^i = -B_t^i + \delta\ {}_tV_x^i - \sum_{j\neq i} \mu_t^{ij}\ R_t^{ij}. \tag{3.29}$$

3.7.1 A classic example: The permanent disability model

Consider a *Term insurance* contract assured to a life aged *x* with benefits depending on the current state or on transition to another state: an annuity payable at rate B_t^1 per year during any period of disability, a lump sum of amount b_t^{01} payable on getting disabled and a sum assured of b_t^{i2} payable on death. The premium is payable continuously while healthy at a rate of *P* per annum. This type of multiple state model is called *The permanent disability model*. Figure 1 presents the model with state space $\mathbb{J} = \{0,1,2\}$ and transition probabilities of the form μ_x^{ij}.

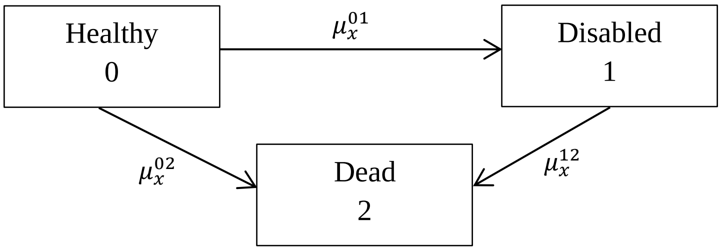

FIGURE 1 – The permanent disability model.

It is quite straight forward that TDEs under this model are,

$$\frac{d}{dt}\ {}_tV_x^{(0)} = \delta\ {}_tV_x^{(0)} + P - \mu_{x+t}^{02}\left(b_t^{02} - {}_tV_x^{(0)}\right) - \mu_{x+t}^{01}\left(b_t^{01} + {}_tV_x^{(1)} - {}_tV_x^{(0)}\right) \quad (3.30)$$

$$\frac{d}{dt}\ {}_tV_x^{(1)} = \delta\ {}_tV_x^{(1)} - B_t^1 - \mu_{x+t}^{12}\left(b_t^{12} - {}_tV_x^{(1)}\right) \quad (3.31)$$

$$\frac{d}{dt}\ {}_tV_x^{(2)} = 0. \quad (3.32)$$

While the policyholder is in state {healthy} or {disabled}, the reserve earns interest at force of interest δ. While healthy, the life insurance company receives the continuous premium *P* that increases the reserve. If the policyholder gets disabled, the company pays the contracted lump sum amount causing the reserve to decrease. Both (3.30) and (3.31) include then the outflow due to transitions just after time t, called the risk cost appearing in equation (3.28).

3.8 Thiele's differential equation under the multiple life model

So far in the text, TDE has been studied considering insurance of a single life. TDE for multiple life insurance where insurance depends on the number of survivors can also be derived under the Markov chain model (Hoem 1969). Considering *m* independent lives, the remaining life time of the x_j life is denoted $T_j, j = 1, \dots, m$. Under the multiple life model, two life statuses are particularly significant: the joint-life status and the last-survivor status. The first status is defined by having the remaining life time $T_{x_1,\dots,x_m} = min\{T_1, \dots, T_m\}$, meaning that the status (i.e. insurance coverage) terminates upon first death and the second one, the last-survivor status, is defined by having remaining life time as $T_{\overline{x_1,\dots,x_m}} = max\{T_1, \dots, T_m\}$, which means that the status (i.e. insurance coverage) terminates upon last death.

3.8.1 A classic example: The independent joint life and last survivor models

A common example is given by the independent joint life and last survivor models for two lives, (*x*) and (*y*). As in practice the two lives are partners, the lives appear in the literature as husband (*x*) and wife (*y*). Figure 2 presents the model with state space $\mathbb{J} = \{0,1,2,3\}$.

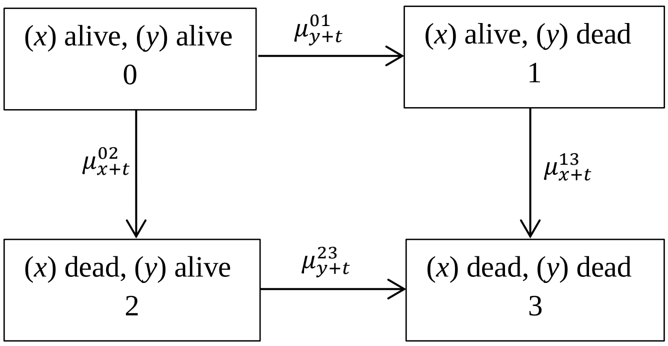

FIGURE 2 – The independent joint life and last survivor models.

Under the joint life status, TDE for an assured benefit of amount 1 payable immediately upon the first death of (*x*) and (*y*) are

$$\frac{d}{dt}\ {}_tV^{(0)} = \delta\ {}_tV^{(0)} - \left(\mu^{01}_{y+t} + \mu^{02}_{x+t}\right)(1 - {}_tV^{(0)}) \qquad (3.33)$$

$$\frac{d}{dt}\ {}_tV^{(1)} = \frac{d}{dt}\ {}_tV^{(2)} = \frac{d}{dt}\ {}_tV^{(3)} = 0. \tag{3.34}$$

Under the last survivor status, TDE for an assured annuity payable continuously at rate of amount 1 per year, while at least one of (x) and (y) is still alive are

$$\frac{d}{dt}\ {}_tV^{(0)} = \delta\ {}_tV^{(0)} - 1 - \mu_{y+t}^{01}(\ {}_tV^{(1)} -\ {}_tV^{(0)}) - \mu_{x+t}^{02}(\ {}_tV^{(2)} -\ {}_tV^{(0)}) \tag{3.35}$$

$$\frac{d}{dt}\ {}_tV^{(1)} = \delta\ {}_tV^{(1)} - 1 + \mu_{x+t}^{13}\ \ {}_tV^{(1)} \tag{3.36}$$

$$\frac{d}{dt}\ {}_tV^{(2)} = \delta\ {}_tV^{(2)} - 1 + \mu_{y+t}^{23}\ \ {}_tV^{(2)} \tag{3.37}$$

$$\frac{d}{dt}\ {}_tV^{(3)} = 0. \tag{3.38}$$

Again, differences arise first because of the number of states from which the Policy Value is different from zero. Then, also depending on the type of benefit assured, TDE has different forms.

4. Developments on Thiele's differential equation

4.1 Thiele's differential equation including payment processes

TDE (3.27) was obtained with deterministic payments. Generalizations of the equation to models with general counting processes driven payments were obtained by Norberg (Norberg 1992a) and Møller (Møller 1993).

The framework is the multiple state model (see *2.4.1)* but instead of non-random benefits, a stream of payments generated by a right continuous stochastic process is considered. The payment function is denoted B_t and represents the contractual benefits less premiums that are due immediately upon transition. The discount function is $w_t = exp(-\int_0^t \delta_s ds)$. Both functions are defined on some probability space $(\Omega, \mathcal{F}, P)$. They are adapted to a right-continuous filtration $\boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}$ where each $\mathcal{F}_t$ contains all the information available up to time *t.* Two processes of the history of the policy have to be defined: a multivariate indicator process, denoted I_t^i, that is equal to 1 or 0 according as the policy is in state *i* (or not) at time *t*, and a multivariate counting process for the number of transitions from state i to any state j, $j \neq i$, during the time interval $(0, t]$, denoted N_t^{ij}. For any small time interval $[t, t+dt], 0 < t < \infty$ the payment function generated by the life insurance policy is the stochastic differential equation

$$dB_t = \sum_i I_t^i \, dB_t^i + \sum_{i\neq j} b_t^{ij} dN_t^{ij}. \tag{4.1}$$

The future loss random variable at time t as defined in *2.3.3,* is now for the payment function (4.1)

$$L_t = \frac{1}{w_t} \int_t^\infty w_\tau \, dB_\tau. \tag{4.2}$$

The Policy Value is the expected value of (4.2) given the information available up to time *t*

$${}_tV^{\boldsymbol{F}} = \frac{1}{w_t} E\left[\int_t^\infty w_\tau \, dB_\tau \mid \mathcal{F}_t\right]. \tag{4.3}$$

Considering the Policy Value (4.3), the loss of an insurance policy in a given year can be defined as

$$L\,(s,t] = \int_s^t w_\tau\, dB_\tau + w_t\ {}_tV^{\boldsymbol{F}} - w_s\ {}_sV^{\boldsymbol{F}}, \tag{4.4}$$

where the integral accounts for the net outgoing of the period and the other terms give the difference between the reserve that has to be provided at the end of the year and the reserve released at the beginning of the year.

TDE can by derived for payment function (4.1) from Hattendorff's theorem and using loss definition (4.4) (Norberg 1992a). Hattendorff's theorem (Sundt and Teugels 2004) states that on a life insurance policy losses in different years have zero means and are uncorrelated, and that the variance is the sum of the variances of the per year losses. From the generalization of Hattendorff's theorem we come to the fact that (4.4) can be redefined as the increment over $(s,t]$ of a martingale generated by the value $L_0 = \int_0^t w_\tau\, dB_\tau$. Assuming that $E[L_t] < \infty$ for each $t \geq 0$, a martingale denoted M_t is defined

$$M_t = E(L_t|\, \mathcal{F}_t) = \int_0^t w_\tau dB_\tau + \ w_t\ {}_tV^{\boldsymbol{F}}\ . \tag{4.5}$$

Including (4.1) in (4.5) we come to

$$M_t = \ B_0^0 + \ \int_0^t w_\tau \left(\sum_i I_\tau^i\, dB_\tau^i + \ \sum_{i\neq j} b_\tau^{ij} dN_\tau^{ij}\right) + \ \sum_i I_t^i w_t\ {}_tV^i\ . \tag{4.6}$$

Assuming that $\{M_t\}_{t\geq 0}$ is square integrable, then a general representation theorem (Bremaud, 1981) says that M_t is of the form

$$M_t = M_0 + \int_0^t \sum_{i\neq j} H_\tau^{ij} (dN_\tau^{ij} - \ I_\tau^i \mu_\tau^{ij} d\tau), \tag{4.7}$$

where the H^{ij} are some predictable processes satisfying

$$E\ \left[\sum_{i\neq j} \int_0^t (H_\tau^{ij})^2 I_\tau^i \mu_\tau^{ij} d\tau\right] < \infty \tag{4.8}$$

and the variance process denoted $\langle M_t \rangle$ is given by

$$d\ \langle M_t \rangle\ (t) = \ \sum_{i\neq j} (H_t^{ij})^2 I_t^i \mu_t^{ij} dt. \tag{4.9}$$

To simplify equation (4.6) the following notation is considered

$$\tilde{b}_t^{ij} = w_t\, b_t^{ij} \tag{4.10}$$

$$\tilde{B}_t^i = \int_0^t w_\tau d\, B_\tau^i \tag{4.11}$$

$$_t\tilde{V}^i = w_t\ {}_tV^i \tag{4.12}$$

Inserting (4.10), (4.11) and (4.12) in equation (4.6) and using integration by parts to reshape the last term, then

$$M_t = B_0^0 + {}_0\tilde{V}^0 + \int_0^t \sum_i I_\tau^i\, d\big(\tilde{B}_t^i + {}_t\tilde{V}^i\big) + \int_0^t \sum_{i\neq j}(\tilde{b}_t^{ij} + {}_\tau\tilde{V}^j - {}_\tau\tilde{V}^i)dN_\tau^{ij}. \tag{4.13}$$

Upon identifying the discontinuous parts in (4.7) and (4.13) and afterwards the continuous parts of the same equations, the following theorems were obtained (Norberg 1992a):

<u>Theorem 1</u>: For any continuous discount function and any predictable contractual functions such that $E\left[(\int wdB)^2\right] < \infty$*, the variance process (4.9) is given by*

$$H_t^{ij} = \tilde{b}_t^{ij} + {}_\tau\tilde{V}^j - {}_\tau\tilde{V}^i\ . \tag{4.14}$$

The function H^{ij} *in (4.14) can be expressed as*

$$H_t^{ij} = w_t\ R_t^{ij}, \tag{4.15}$$

where $R_t^{ij} = b_t^{ij} + {}_tV^j - {}_tV^i$ *is the amount at risk.*

<u>Theorem 2</u>: For any continuous discount function and any predictable contractual functions such that $\left[E(\int wdB)^2 < \infty\right]$*, the following identity holds almost surely*

$$I_t^i d(\tilde{B}_t^i + {}_t\tilde{V}^i) + \sum_{j\neq i} H_t^{ij} I_t^i \mu_t^{ij} dt = 0. \tag{4.16}$$

The importance of this result is that (4.16) is a generalization of TDE valid for any counting process and for any predictable benefit function including a lump sum benefit upon survival.

For instance, from (4.16) we can obtain TDE (3.29). Inserting (4.11), (4.12) and (4.15) in (4.16) and using integration by parts for $d\ {}_t\tilde{V}^i = dw_t\ {}_tV^i + w_t d\ {}_tV^i = -w_t\delta_t\ {}_tV^i + w_t d\ {}_tV^i$, equation (4.16) will become

$$I_t^i d\left(\int_0^t w_\tau \, d\, B_\tau^i\right) + I_t^i(-w_t\delta_t \,{}_tV^i + w_t d\,{}_tV^i) + \sum_{j\neq i} w_t\, R_t^{ij} I_t^i \mu_t^{ij} dt = 0. \quad (4.17)$$

Dividing by w_t, rearranging and then dividing again by *dt*, then (3.29) follows

$$\frac{d}{dt}\,{}_tV^i = -B_t^i + {}_tV^i\,\delta_t - \sum_{j,j\neq i} R_t^{ij}\mu_t^{ij}.$$

The result is the same TDE obtained for deterministic payments as in *3.7.*

4.2 Thiele's differential equation including stochastic interest rates

The development of life insurance industry creates the need to adapt actuarial models to the development of financial theory. In that sense, versions of TDE can also be obtained with interest governed by stochastic processes of diffusion type, replacing the deterministic interest by a stochastic process (Norberg and Møller 1996). Introducing stochastic interest rate models on Thiele's equation opens the possibility to manage risk of long term yields on assets corresponding to the reserve.

Following the authors work, the simplest one factor diffusion model will be first studied, and then we shall include two other well-known interest models.

The model considered is the Markov chain model with a stream of payments generated by the stochastic differential equation (4.1). The deterministic discount function $w_{(t,u)} = exp(-\int_t^u \delta_s ds), t < u$ is replaced by a stochastic one, by letting the log of discount function be a continuous stochastic process adapted to some filtration $\boldsymbol{G} = \{\mathcal{G}_t\}_{t\geq 0}$, representing the economic environment. The source of randomness is modelled using a Brownian motion (Mörters and Peres 2010), W_t. The stochastic differential equation is

$$dr_t = \delta_t dt + \sigma_t\, dW_t, \quad (4.18)$$

where $r_t = -\int_0^t \delta_s ds$ and δ_t and σ_t are deterministic functions called the drift and the volatility parameters. Taking the interval $[t,u]$, we know that $w_{(t,u)} = exp(r_u - r_t)$ and that r_t has independent and normally distributed increments (from Brownian motion properties),

$$r_u - r_t \sim N(\int_t^u \delta_s ds \,, \int_t^u \sigma_s^2 ds \,). \tag{4.19}$$

The discount function is then

$$w'_{(t,u)} = E(w_{(t,u)} | \, \mathcal{G}_t) = E(e^{-(r_u - r_t)} | \, \mathcal{G}_t). \tag{4.20}$$

We can observe that the expectation is of the form $E[e^{\lambda X}]$, where λ is a constant and $X \sim N(\int_t^u \delta_s ds \,, \int_t^u \sigma_s^2 ds \,)$. Using the moment generating function of a normal variable, $M(\lambda) = exp(\mu \lambda + \frac{1}{2}\sigma^2 \lambda^2)$ with $\mu = \int_t^u \delta_s ds$ and $\sigma^2 = \int_t^u \sigma_s^2 ds$ we arrive to the discount function for the stochastic interest process (4.18),

$$w'_{(t,u)} = \; exp(\, - \int_t^u \delta_s^* ds) \tag{4.21}$$

with

$$\delta_t^* = \; \delta_t - \tfrac{1}{2} \, \sigma_t^2 \, . \tag{4.22}$$

A version of TDE is obtained by including the force of interest (4.22) in (3.29). The present model is equivalent to the one with deterministic interest.

Another version of TDE can be obtained considering the Vasicek model (Vasicek 1977) and the CIR model (Cox *et al.*1985). These are time homogeneous models, i.e., their future dynamics do not depend on what the present time t is on the calendar. The general stochastic differential equation for both models is

$$d\delta_t = k \; (t, \delta_t) dt + \; \sigma(t, \delta_t) \; dW_t. \tag{4.23}$$

The Vasicek model is an Ornstein-Uhlenberg process (Oksendal 1992), and its dynamics is given by $d\delta_t = k(\bar{\delta} - \delta_t)dt + \sigma dW_t$ where $k, \bar{\delta}$ and σ are positive constants, $\bar{\delta}$ being the long term average force of interest. The CIR model has the same form of drift parameter but includes a different volatility parameter which ensures that interest remains positive. Its stochastic differential equation is given by $d\delta_t = k(\bar{\delta} - \delta_t)dt + \; \sigma\sqrt{\delta_t} dW_t$.

Including (4.23) in Policy Value (4.3) we observe that the Policy Value is now a function depending not only on t but also on the force of interest: ${}_tV^i \; = V^i(t, \delta_t)$. To turn the Policy Value into a stochastic differential equation as it was done in (3.9), the Itô's formula (Oksendal 1992) has to be applied (Itô's formula is in appendix B),

$$dV^i(t,\delta_t) = \frac{d}{dt}V^i(t,\delta_t)dt + \frac{d}{d\delta}V^i(t,\delta_t)d\delta_t + \frac{d^2}{d\delta^2}V^i(t,\delta_t)\frac{1}{2}\,\sigma^2(t,\delta_t)dt \qquad (4.24)$$

including (4.23),

$$dV^i(t,\delta_t) = \frac{d}{dt}V^i(t,\delta_t)dt + \frac{d}{d\delta}V^i(t,\delta_t)[k\,(t,\delta_t)dt + \sigma(t,\delta_t)dW_t]$$
$$+\frac{d^2}{d\delta^2}V^i(t,\delta_t)\frac{1}{2}\,\sigma^2(t,\delta_t)dt\,. \qquad (4.25)$$

Rearranging (4.25) and knowing that $dW_t dt = 0$, then

$$\frac{d}{dt}V^i(t,\delta_t) = \frac{d}{dt}V^i(t,\delta_t) + \frac{d}{d\delta}V^i(t,\delta_t)\;k\,(t,\delta_t) + \frac{d^2}{d\delta^2}V^i(t,\delta_t)\frac{1}{2}\,\sigma^2(t,\delta_t). \quad (4.26)$$

From (4.26) we observe that the difference from the classical TDE (3.27) arises from the last two additional terms. Inserting (3.27) in (4.26), we get

$$\frac{d}{dt}V^i(t,\delta_t) = \delta\,V^i(t,\delta_t) - B_t^i - \sum_{j,j\neq i}\mu_t^{ij}(b_t^{ij} + V^j(t,\delta_t) - V^i(t,\delta_t))$$
$$+\frac{d}{d\delta}V^i(t,\delta_t)\;k\,(t,\delta_t) + \frac{d^2}{d\delta^2}V^i(t,\delta_t)\frac{1}{2}\,\sigma^2(t,\delta_t). \qquad (4.27)$$

TDE (4.27) opens the possibility to study the decomposition of the rate of change of reserves per policyholder for any state *i* where the model includes a stochastic interest rate process.

4.3 Thiele's differential equation: a tool for life insurance product development

One of the applications of TDE is the development of new products using the equation as a tool (Ramlau-Hansen 1990 and Norberg 1992b). Some illustrations follow.

First, consider a policy with non-random benefits as seen in *2.4.2.* A life insurance contract can be built setting the benefits depending on the total or on a fraction of the Policy Value and from TDE the rate of premium is obtained. To show how this technique is applied, we will make use of example *3.7.1.* Setting death benefits depending on the reserve and no disability benefit (to simplify computations), that is,

$b_t^{02} = b_t^{12} = S_1 + {}_tV_x^{(0)}$ and $B_t^1 = b_t^{01} = 0$, requiring that ${}_nV^{(0)} = {}_nV^{(1)} = S_2$ and ${}_oV^{(0)} = 0$, equations (3.30) and (3.31) become, respectively,

$$\frac{d}{dt}\ {}_tV_x^{(0)} = \delta\ {}_tV_x^{(0)} + P - \mu_{x+t}^{02}\, S_1 - \mu_{x+t}^{01}\left(\ {}_tV_x^{(1)} - \ {}_tV_x^{(0)}\right) \tag{4.28}$$

and
$$\frac{d}{dt}\ {}_tV_x^{(1)} = \delta\ {}_tV_x^{(1)} - \mu_{x+t}^{12} S_1 - \mu_{x+t}^{12}\left(\ {}_tV_x^{(0)} - \ {}_tV_x^{(1)}\right). \tag{4.29}$$

Subtracting (4.28) from (4.29) we get,

$$\frac{d}{dt}\ {}_tV_x^{(1)} - \frac{d}{dt}\ {}_tV_x^{(0)} = (\delta + \mu_{x+t}^{01} + \mu_{x+t}^{12})\left(\ {}_tV_x^{(1)} - \ {}_tV_x^{(0)}\right) - P + S_1(\mu_{x+t}^{02} - \mu_{x+t}^{12}). \tag{4.30}$$

The solution of (4.30) in order to the amount at risk as defined in (3.28) is

$$\begin{aligned} {}_tV_x^{(1)} - \ {}_tV_x^{(0)} = \ & P \int_t^n \exp(-\int_t^s \delta + \mu_{x+u}^{01} + \mu_{x+u}^{12}\, du) ds \\ & + S_1 \int_t^n \exp(-\int_t^s \delta + \mu_{x+u}^{01} + \mu_{x+u}^{12}\, du)[\mu_{x+t}^{12} - \mu_{x+t}^{02}] ds\ . \end{aligned} \tag{4.31}$$

Inserting (4.30) into (4.28) and solving for the rate of premium *P* the result is the rate of premium of the contract when death benefits depend on the Policy Value. Death benefits may also be set as a fraction $\alpha > 0$ of the Policy Value in the following way: $b_t^{02} = b_t^{12} = \alpha\ {}_tV_x$. Using the same technique, the premium is obtained for this type of contract (Ramlau-Hansen 1990).

Another approach to develop new insurance products is to set a fluctuation loading to premiums, depending on higher order conditional moments of the present value of payments. Higher order moments of present value are obtained using martingale techniques to avoid multiple integrals (Norberg 1992b).

Considering a life insurance contract with a stochastic payment function (4.1), the *qth* conditional moment of the present value of payments, denoted $V_t^{(q)i}$, given that the policyholder is in state $Z(t) = i$, is

$$V_t^{(q)i} = E\ \left[\left(\frac{1}{v_t}\int_t^n v_\tau\, dB_\tau\right)^q\ |\ I_t^i\ = 1\right]. \tag{4.32}$$

The author proves that the functions $V_t^{(q)i}$ are determined by the differential equations

$$\frac{d}{dt}V_t^{(q)i} = \left(q\delta_t^i + \mu_t^i\right) V_t^{(q)i} - q b_t^i\, V_t^{(q-1)i} - \sum_{j\neq i} \mu_t^{ij} \sum_{r=0}^{q} \binom{q}{r} \left(b_t^{ij}\right)^r V_t^{(q-r)j} \quad (4.33)$$

valid on $(0, \mathrm{n}) \setminus \mathfrak{D}$ and subject to the conditions

$$V_{t-}^{(q)i} = \sum_{r=0}^{q} \binom{q}{r} \left(B_t^i - B_{t-}^i\right)^r V_t^{(q-r)i}, t \in \mathfrak{D} \quad (4.34)$$

where $\mathfrak{D} = \{t_0, \dots, t_n\}$ is the set of times listed in chronological order, where jumps to other states can occur so that a lump sum b_t^{ij} is then payable at time t.

TDE is the particular case when *q=1* and all forces of interest δ^i are equal:

$$\frac{d}{dt}V_t^{(1)i} = \delta_t\, V_t^{(1)i} - B_t^i - \sum_{j\neq i} \mu_t^{ij} \left(b_t^{ij} + V_t^{(1)j} - V_t^{(1)i}\right). \quad (3.27)$$

Central moments, denoted $m_t^{(q)i}$, can also be obtained:

$$m_t^{(q)i} = \sum_{p=0}^{q} \binom{q}{r} (-1)^{q-p} V_t^{(p)i} \left(V_t^{(1)i}\right)^{q-p}. \quad (4.35)$$

When *q=1* the result is $m_t^{(1)i} = V_t^{(1)i}$.

As an example, premiums can include a loading proportional to the variance, as follows: $\alpha m_t^{(2)i}, \alpha > 0$.

4.4 Thiele's differential equation for a closed insurance portfolio

From TDE (3.29) derived for a single policy with non-random benefits, the Policy Value for a closed insurance portfolio can be derived (Linnemann 1993). The results may be used to make actuarial consistent projections of the development of such an insurance portfolio. It also gives the theoretical basis to perform the *Thiele control* as we shall see. First a reformulation of equation (3.29) is necessary to then come to TDE for the insurance portfolio.

Consider that the actual state of the policy at time t is $Z\,(t) = i$ and that the *second order basis* are $\hat{\delta}_t$ and $\hat{\mu}_t^{ij}$. Including the *second order basis* in equation (3.29) an additional term is added, denoted $\hat{g}_t^i$, that is the rate at which surplus accumulates per unit of time at time *t,* when $Z(t) = i$, arising from the differences between *first* and *second order* basis:

$$\frac{d}{dt}\ {}_tV_x^i + \hat{g}_t^i = \hat{\delta}_t\ {}_tV_x^i - B_t^j - \sum_{j\neq i} \hat{\mu}_t^{ij} R_t^{ij}. \tag{4.36}$$

Performing (4.36) - (3.29), we get

$$\hat{g}_t^i = (\hat{\delta}_t - \delta)\ {}_tV_x^i - \sum_{j,j\neq i}(\hat{\mu}_t^{ij} - \mu_t^{ij})R_t^{ij}. \tag{4.37}$$

The first part of the right side is the excess of interest earned and the second part is the profit or loss on transition out of state *j*.

For convenience we include in Thiele's equation a function ρ_t, representing the force of increment per unit of time corresponding to an accumulation function $\varphi_t > 0$ and differentiable on *t*, i.e.,

$$\rho_t = \frac{d}{dt}\{\ln \varphi_t\} = \frac{d}{dt}\varphi_t / \varphi_t. \tag{4.38}$$

TDE on a single policy (4.36) can be rewritten including (4.38). The result is

$$\frac{d}{dt}\left(\varphi_t\ {}_tV_x^i\right) = \varphi_t(\hat{\delta}_t + \rho_t)\ {}_tV_x^i - \varphi_t\left(B_t^i + \sum_{j,j\neq i}\hat{\mu}_t^{ij} R_t^{ij}\right) - \varphi_t\ \hat{g}_t^i. \tag{4.39}$$

Letting $\varphi_t = \widehat{w}_{(s,t)}\ {}_{t-s}\hat{p}_s^{\overline{ii}}$ where $\widehat{w}_{(s,t)} = exp\left\{-\int_s^t \hat{\delta}_u du\right\}$ and where $\widehat{p}_s^{\overline{ii}}$ refers to survival probabilities of *second order basis* and integrating, we obtain the Policy Value on a single policy in state *i* in a way that is convenient to generalize for a portfolio, say

$${}_sV_x^i = \int_s^t \widehat{w}_{(s,t)}\ {}_{u-s}\ \widehat{p}_s^{\overline{ii}}\ \widehat{g}_u^i du + \int_s^t \widehat{w}_{(s,t)}\ {}_{t-s}\ \widehat{p}_s^{\overline{ii}}\left\{B_u^i + \sum_{j\neq i}\widehat{\mu}_u^{ij}\ b_u^{ij}\right\}du + \widehat{w}_{(s,t)}\ {}_{t-s}\ \widehat{p}_s^{\overline{ii}}\ {}_tV_x^i\ . \tag{4.40}$$

The generalized TDE for closed insurance portfolio is then (cf Linnemann 1993)

$$\frac{d}{dt}\left\{\omega_t \sum_j {}_{t-s}\ \hat{p}_s^{ij}\ {}_tV_x^j\right\} = \omega_t\{\hat{\delta}_t + \alpha_t\} \sum_j {}_{t-s}\ \hat{p}_s^{ij}\ {}_tV_x^j$$

$$-\omega_t \sum_j {}_{t-s}\ \hat{p}_s^{ij}\left[B_t^j + \sum_{k\neq j}\hat{\mu}_t^{jk} b_t^{jk}\right] - \omega_t \sum_j {}_{t-s}\ \hat{p}_s^{ij}\ \hat{g}_t^j. \tag{4.41}$$

Note that, $\omega_t > 0$ is a differentiable function of *t* such that $\alpha_t = \frac{d}{dt}\{\ln \omega_t\}\backslash\omega_t$.

Letting $\omega_t = \widehat{w}_{(s,t)}$ and integrating (4.41) we arrive to the Policy Value for an insurance portfolio

$${}_sV_x^i = \int_s^t \widehat{w}_{(s,u)} \sum_j {}_{u-s}\ \widehat{p}_s^{ij}\ \widehat{g}_u^i du + \int_s^t \widehat{w}_{(s,u)} \sum_j {}_{u-s}\ \widehat{p}_s^{ij}\left\{B_u^j + \sum_{k\neq j}\widehat{\mu}_u^{jk}\ b_u^{jk}\right\}du$$

$$+ \widehat{w}_{(s,t)} \sum_j {}_{t-s}\ \hat{p}_s^{ij}\ {}_tV_x^j\ . \tag{4.42}$$

When in (4.42) $\hat{\delta}_t \equiv \delta$ or $\widehat{w}_{(s,t)} = w_{(s,t)}$, we obtain the theoretical basis for making the *Thiele control* of the increment of the Policy Value,

$$w^{t-s} \sum_j {}_{t-s}\hat{p}_s^{ij} \; {}_tV_x^j - {}_sV_x^i = \int_s^t w^{u-s} \sum_j {}_{u-s}\hat{p}_s^{ij} \left\{-b_u^j - \sum_{k \neq j} \hat{\mu}_u^{jk} \, b_u^{jk}\right\} du$$

$$+ \int_s^t w^{u-s} \sum_j {}_{u-s}\hat{p}_s^{ij} \sum_{k \neq j} \hat{\mu}_u^{jk} \, R_u^{jk} \, du - \int_s^t w^{u-s} \sum_j {}_{u-s}\hat{p}_s^{ij} \sum_{k \neq j} \mu_u^{jk} \, R_u^{jk} \, du. \quad (4.43)$$

Thiele control is required by the Danish Insurance Supervisory Authorities (Linnemann 1993). It is computed at the end of the year based on actual increments due to premiums, benefits, interest, reserve jumps and risk premiums.

4.5 Thiele's differential equation: further developments

Many other important developments of TDE appear in the literature. For completeness, this last subsection is devoted to summarize some of these developments.

Versions of TDE including a stochastic payment process and a stochastic interest rate were derived in *4.1* and *4.2*. Other assumptions of the classic TDE (3.27) may be relaxed. Milbrodt and Starke (Milbrodt and Starke 1997) proposed to jointly comprise a discrete method and a continuous method of insurance mathematics, a semi-Markov model, to account for transitions, benefits and premiums, and interest that occur in discrete time and appear in many real life insurance products. The authors modelled a policy development to account for both, discrete and continuous situations, where the dynamics of reserve for the discrete method is describe by a recursion relationship.

So far in the text, TDE has been obtained under the *equivalence principle*. Other premium principles may be applied for pricing purpose. With the development and sophistication of the financial markets, the attempt of securitization the insurance risk as an alternative to traditional exchange of risk by reinsurance contracts rises the need to consider finance principles for the calculation of life insurance premiums. Steffensen (Steffensen 2000) proposed to compute premiums under the *no arbitrage principle* redefining the Policy Value defined in *3.1* as the market price of future payments. A generalized version of TDE is obtained for insurance contracts linked to indices and marketed securities.

TDE has also been used on the study of the emergence of surplus on a life insurance contract. Norberg (Norberg 2001) defined the surplus on life insurance policy at any time *t*, for a contract still in force at that time, as the difference between the second order retrospective reserve and the first order prospective reserve. TDE is used in the process of the estimation of dividends and in bonuses prognoses.

Finally, more recent developments have been pursuit on the formulation of a sensitivity analysis of insurance reserves with respect to the technical basis, in order to improve the control of reserves (Christiansen 2008 and 2010).

5. Final thoughts

A life insurance contract is typically a long term contract where the insurer accepts risk from another party by receiving premiums and by paying a benefit if the uncertain future event occurs. As in any model to predict future events, some assumptions have to be made regarding the variables of interest, called the technical basis. Life insurance contracts depend on death or survival of the insured life or lives, on economic and financial environment, as premiums have to be invested to pay future benefits, and on any other variables considered in the contract.

The valuation of a policy still in force at any time $t, 0 < t < \infty,$ is important to assess the solvability of the business. The prospective reserve is the difference between the expected discounted future benefits outgo and expected discounted future premiums income at any point in time, named the Policy Value.

TDE appeared as the equation that decomposes the rate of increment of the Policy Value, per unit of time and per policyholder in continuous time. Since its publication, some assumptions of the original equation were relaxed and more general TDE were obtained including a stochastic payment process and a diffusion process for interest rate showing the adaptability of the equation. TDE has been also used as a tool for life insurance product development and more recently to assess the sensitivity of the technical bases considered in the contract.

This work is a survey about TDE. The equation is studied in depth and the many contributions and developments that appear in the literature are compiled. This would interest both recent graduated in actuarial science and researchers who want to broaden their knowledge about prospective reserves in continuous time. Personally, this work has contributed to deepen my knowledge in this very important topic. Further research can be pursuit generalizing the equation to the recent developments in finance.

To conclude, TDE has been generalized including new model formulations both from actuarial mathematics and finance making this equation still so important nowadays.

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APPENDIX

Appendix A

From Policy Value (3.8) TDE was derived. For completeness we present a more detailed proof.

$$ {}_tV_x = \int_0^\infty B_{t+s} \frac{e^{-\int_0^{t+s}\delta_z\,dz}}{e^{-\int_0^t \delta_w\,dw}}\ {}_s p_{x+t}\,\mu_{x+t+s}\,ds - \int_0^\infty P_{t+s}\frac{e^{-\int_0^{t+s}\delta_z\,dz}}{e^{-\int_0^t \delta_w\,dw}}\ {}_s p_{x+t}\,ds. \quad (3.8)$$

Changing the variable of integration to $r = s + t$ we get

$$ {}_tV_x = \int_t^\infty B_r \frac{e^{-\int_0^{r}\delta_s\,ds}}{e^{-\int_0^t \delta_w\,dw}}\ {}_{r-t} p_{x+t}\,\mu_{x+r}\,dr - \int_t^\infty P_r\frac{e^{-\int_0^{r}\delta_s\,ds}}{e^{-\int_0^t \delta_w\,dw}}\ {}_{r-t} p_{x+t}\,dr. \quad (A.1)$$

Rearranging and applying results from *2.3.1* of the survival function it follows that

$$ e^{-\int_0^t \delta_w\,dw}\ {}_t p_x\ {}_tV_x = \int_t^\infty B_r\, e^{-\int_0^{r}\delta_s\,ds}\ {}_r p_x\,\mu_{x+r}\,dr - \int_t^\infty P_r\, e^{-\int_0^{r}\delta_s\,ds}\ {}_r p_x dr. \quad (A.2)$$

Differentiating, then

$$ \frac{d}{dt}\left(e^{-\int_0^t \delta_w\,dw}\ {}_t p_x\ {}_tV_x\right) = -B_t\ e^{-\int_0^t \delta_w\,dw}\ {}_t p_x\,\mu_{x+t} + P_t\ e^{-\int_0^t \delta_w\,dw}\ {}_t p_x, \quad (A.3)$$

and using the rule of integration by parts it comes to

$$ e^{-\int_0^t \delta_w\,dw}\ {}_t p_x \frac{d}{dt}\ {}_tV_x + {}_tV_x \frac{d}{dt}\left(e^{-\int_0^t \delta_w\,dw}\ {}_t p_x\right) = e^{-\int_0^t \delta_w\,dw}\ {}_t p_x\,(P_t - B_t\,\mu_{x+t}). \quad (A.4)$$

Calling the right hand side of equation (A.4) $F = e^{-\int_0^t \delta_w\,dw}\ {}_t p_x\,(P_t - B_t\,\mu_{x+t})$ and using again integration by parts on the left hand side, then

$$ e^{-\int_0^t \delta_w\,dw}\ {}_t p_x \frac{d}{dt}\ {}_tV_x + {}_tV_x \left(e^{-\int_0^t \delta_w\,dw} \frac{d}{dt}\ {}_t p_x + {}_t p_x \frac{d}{dt}\, e^{-\int_0^t \delta_w\,dw}\right) = F\,. \quad (A.5)$$

Using results of life time density function from *2.3.1*, it follows

$$ e^{-\int_0^t \delta_w\,dw}\ {}_t p_x \frac{d}{dt}\ {}_tV_x + {}_tV_x \left[e^{-\int_0^t \delta_w\,dw}(-\,{}_t p_x\mu_{x+t}) + {}_t p_x\,(-\delta_t\, e^{-\int_0^t \delta_w\,dw})\right] = F. \quad (A.6)$$

Rearranging and including right hand side gives

$$ e^{-\int_0^t \delta_w\,dw}\ {}_t p_x\left[\frac{d}{dt}\ {}_tV_x - {}_tV_x\,(\mu_{x+t} + \delta_t)\right] = e^{-\int_0^t \delta_w\,dw}\ {}_t p_x\,(P_t - B_t\,\mu_{x+t}). \quad (A.7)$$

Finally rearranging again we arrive to TDE

$$ \frac{d}{dt}\ {}_tV_x = P_t - B_t\,\mu_{x+t} + {}_tV_x\,(\mu_{x+t} + \delta_t). \quad (3.9)$$

Appendix B

<u>Definition</u>: the 1-dimensional Itô processes (Oksendal 2013).

Let B_t be a 1-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. A (1-dimensional) Itô process (or stochastic integral) is a stochastic process X_t on $(\Omega, \mathcal{F}, P)$ of the form

$$X_t = X_0 + \int_0^t u(s,w)ds + \int_0^t v(s,w)dB_s, \tag{B.1}$$

where v is such that

$$P\left[\int_0^t v(s,w)^2 ds < \infty \; for\; all\; t \geq 0\right] = 1. \tag{B.2}$$

Equation (B.1) is sometimes written in the shorter differential form

$$dX_t = udt + vdB_t. \tag{B.3}$$

<u>Theorem 3</u> (The 1-dimensional Itô formula)

Let X_t be an Itô process given by

$$dX_t = udt + vdB_t.$$

Let $g(t,x) \in C^2([0,\infty] \times \mathbb{R})$ (i.e. is twice continuously differentiable on $[0,\infty] \times \mathbb{R}$). Then

$$Y_t = f(t, X_t) \tag{B.4}$$

is again an Itô process, and

$$dY_t = \frac{df}{dt}(t,X_t)dt + \frac{df}{dx}(t,X_t)dX_t + \frac{1}{2}\frac{d^2f}{dx^2}(t,X_t)(dX_t)^2 \tag{B.5}$$

where $(dX_t)^2 = (dX_t).(dX_t)$ is computed according to the rules

$$dt.dt = dt.dB_t = dB_t.dt = 0, dB_t.dB_t = dt. \tag{B.6}$$