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The case of equality in the Livingstone-Wagner Theorem

David Bundy, Sarah Hart

Abstract

Let G be a permutation group acting on a set Ω of size $n \in \mathbb{N}$ and let $1 \leq k < (n - 1)/2$. Livingstone and Wagner proved that the number of orbits of G on k -subsets of Ω is less than or equal to the number of orbits on $(k + 1)$ -subsets. We investigate the cases when equality occurs.

1 Introduction

Throughout this article we let G be a permutation group acting on a set Ω of size $n \in \mathbb{N}$ and let $1 \leq k < (n - 1)/2$. In [6] Livingstone and Wagner proved the following theorem.

Theorem 1.1. *(Livingstone, Wagner) [6] The number of orbits of G on k -subsets of Ω is less than or equal to the number of orbits on $(k + 1)$ -subsets.*

Alternative proofs were subsequently given by Robinson [7] and Cameron [1] who extended the result to Ω infinite. An investigation of the cases when equality occurs for Ω infinite was then made by Cameron [1], [2] and Cameron and Thomas [5]. The case of equality also follows from a stronger “intersection property” examined by Cameron, Neumann and Saxl [4]. In this article, we will prove some similar results about the case of equality when Ω is finite.

In Section 2 we consider the case when G is intransitive. We show (see Lemma 2.1) that G must have one orbit of length at least $n - k$ and (see Proposition 2.2) that the action of G on this orbit satisfies a strong condition which in almost all cases forces G to be k -homogeneous on this orbit.

Transitive but imprimitive groups are then investigated in Section 3. In this case there are too many examples for a complete classification to be feasible, so we concentrate on finding a necessary condition for the sizes and number of blocks in a system of imprimitivity. This quickly reduces to a combinatorial problem of determining when the number of partitions of k into at most r parts of size at most s is the same as for $k + 1$. This problem is also of independent interest in invariant theory, where such partitions can be used to count the

number of linearly independent semi-invariants of degree r and weight k of a binary form of degree s . We are able to determine all the cases of equality for $r \leq 4$ (see Theorem 3.1) and conjecture that for $s \geq r \geq 5$, there are only finitely many cases of equality (see Conjecture 3.2 for details). Theorem 3.7 shows that for $s \geq r \geq 5$, equality can only occur when $2k \geq r(s-1) - 1$, that is k is close to half n . We have strong experimental evidence for believing Conjecture 3.2 to be true. We observe that for large enough fixed r and s the number of partitions of k into at most r parts of size at most s approximates to a Gaussian distribution whose peak becomes sharper for larger r and s .

In the final section we make some observations about the case when G is primitive. Aside from $(k+1)$ -homogeneous groups the only examples we know are the affine general linear groups over a field of size 2 (see Proposition 4.2) and a list of 19 further examples of degree at most 24, many of which are subgroups of M_{24} . The absence in [4] of any examples of degree greater than 24 suggests that such examples may also be rare or non-existent in our situation.

Notation and preliminary results

For each $0 \leq l \leq n$, let $\sigma_l(G)$ be the number of orbits of G on the set of l -subsets of Ω . A permutation group is said to be l -homogeneous if it is transitive in its action on l -subsets, that is $\sigma_l(G) = 1$. Let Δ be a G -invariant subset of Ω . Then G^Δ will denote the permutation group induced by G in its action on Δ .

Let H be a subgroup of a group G , χ be a character of G and ψ a character of H . Then $\chi \downarrow H$ will denote the restriction of χ to H and $\psi \uparrow G$ will denote the character induced by ψ on G . Furthermore 1_G will denote the trivial character on G .

Lemma 1.2. *Let $G \leq \text{Sym}(n)$, $0 \leq l \leq n$ and ψ_l be the character of $\text{Sym}(n)$ induced by the trivial character on $\text{Sym}(l) \times \text{Sym}(n-l)$. Then $\langle \psi_l \downarrow G, 1_G \rangle$ is the number of orbits of G on l -subsets of $\{1, \dots, n\}$ and if $0 \leq l < (n-1)/2$, then $\psi_{l+1} - \psi_l$ is an irreducible character of $\text{Sym}(n)$.*

Proof. See [7]. □

Lemma 1.3. *Let $H \leq G \leq \text{Sym}(n)$ and $1 \leq k < (n-1)/2$. Then $\sigma_{k+1}(G) - \sigma_k(G) \leq \sigma_{k+1}(H) - \sigma_k(H)$. In particular, if $\sigma_{k+1}(H) = \sigma_k(H)$, then $\sigma_{k+1}(G) = \sigma_k(G)$.*

Proof. Let $\chi := \psi_{k+1} - \psi_k$ be the irreducible character in the conclusion of Lemma 1.2. Then

$$\sigma_{k+1}(G) - \sigma_k(G) = \langle \chi \downarrow G, 1_G \rangle \leq \langle \chi \downarrow H, 1_H \rangle = \sigma_{k+1}(H) - \sigma_k(H).$$

In particular, if $\sigma_{k+1}(H) = \sigma_k(H)$, then the right-hand side is zero and by Theorem 1.1 the left-hand side is non-negative, so must also be zero. □

2 Intransitive groups with equality

In this section we investigate intransitive permutation groups which achieve equality in the Livingstone-Wagner Theorem.

Lemma 2.1. *Let $G \leq \text{Sym}(n)$ and suppose $\sigma_k(G) = \sigma_{k+1}(G)$ for some $1 \leq k < (n-1)/2$. Then G has an orbit of length at least $n-k$.*

Proof. Suppose G has no orbit of length at least $n-k$. Then $G \leq \text{Sym}(n-l) \times \text{Sym}(l) =: M$, for some $l \geq k+1$. But $\sigma_{k+1}(M) = k+2 > k+1 = \sigma_k(M)$, which contradicts Lemma 1.3. \square

Proposition 2.2. *Let $G \leq \text{Sym}(n)$ and $1 \leq k < (n-1)/2$ with $\sigma_k(G) = \sigma_{k+1}(G)$. Let Δ be an orbit of G of length at least $n-k$. Then $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$, for all $k - (n - |\Delta|) \leq l \leq \min(k, |\Delta| - k - 2)$.*

Proof. Note that an orbit of length at least $n-k$ exists by Lemma 2.1. Let $M := G^\Delta \times \text{Sym}(\Omega \setminus \Delta) \geq G$ and let $m := |\Delta|$. For $t \in \mathbb{N}$, two t -subsets of Ω are in the same M -orbit if and only if their intersections with Δ are in the same G^Δ -orbit. In particular, these intersections must be of the same size. Hence

$$\sigma_t(M) = \sum_{l=\max(0, t-(n-m))}^{\min(t, m)} \sigma_l(G^\Delta).$$

Now $m \geq n-k \geq (2k+1) - k = k+1$. Also $k - (n-m) \geq k + (n-k) - n = 0$. Therefore

$$\begin{aligned} 0 &= \sigma_{k+1}(M) - \sigma_k(M) = \sum_{l=k+1-(n-m)}^{k+1} \sigma_l(G^\Delta) - \sum_{l=k-(n-m)}^k \sigma_l(G^\Delta) \\ &= \sigma_{k+1}(G^\Delta) - \sigma_{k-(n-m)}(G^\Delta). \end{aligned}$$

That is, $\sigma_{k+1}(G^\Delta) = \sigma_{k-(n-m)}(G^\Delta)$. If $2k < m-1$ then the Livingstone-Wagner Theorem forces $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$, for each $k - (n-m) \leq l \leq k$.

On the other hand, suppose $2k \geq m-1$. Then $\sigma_{k+1}(G^\Delta) = \sigma_{m-(k+1)}(G^\Delta)$ and $m - (k+1)$ is within the range to which the Livingstone-Wagner Theorem applies. We also have that

$$(m - (k+1)) - (k - (n-m)) = (n-1) - 2k > 0.$$

Hence, by the Livingstone-Wagner Theorem, $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$, for each $k - (n-m) \leq l \leq m - k - 2$. Note that $\min(k, m - k - 2)$ is k precisely when $2k < m-1$ and $m - k - 2$ otherwise, so the proof is complete. \square

Proposition 2.2 provides the means to reduce the case of equality for an intransitive group to that of equality for a transitive group. Indeed if G is intransitive with an orbit Δ satisfying the condition of Proposition 2.2, then we nearly always have equality $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$ for several consecutive values of l . (If there is just one value of l then either G is already transitive or $n = 2k + 2$.) This almost forces G^Δ to be k -homogeneous. The only known exceptions with $k < (n - 1)/2$ are where $G^\Delta \cong M_{24}$ or M_{23} .

3 Imprimitive groups with equality

There is an abundance of imprimitive groups which achieve equality in the Livingstone-Wagner Theorem and a complete classification of them seems intractable. Nevertheless, we are able to give a condition on the block sizes which is necessary if equality in the Livingstone-Wagner Theorem holds. Observe that by Lemma 1.3, if $\sigma_k(H) = \sigma_{k+1}(H)$ holds for an imprimitive group H with r blocks of size s , then $\sigma_k(G) = \sigma_{k+1}(G)$, where $G \cong \text{Sym}(s) \wr \text{Sym}(r)$ is the full stabiliser in $\text{Sym}(rs)$ of the blocks of H . Note also that the number of orbits of G on k -subsets is equal to the number of ways, $P(r, s, k)$, to partition k into at most r parts of size at most s . We require $P(r, s, k) = P(r, s, k + 1)$. The following result is established by Lemma 3.5, Proposition 3.6 and Proposition 3.9.

Theorem 3.1. *Let $r \in \{2, 3, 4\}$ with $r \leq s$ and $1 \leq k < (rs - 1)/2$. Then $P(r, s, k) = P(r, s, k + 1)$ if and only if one of the following holds.*

(a) $r = 2$ and k is even.

(b) $r = 3$ and

$$k = \begin{cases} \frac{3s-3}{2}, & \text{if } s \text{ is odd,} \\ \frac{3s-4}{2}, & \text{if } s \equiv 0 \pmod{4}, \\ \frac{3s-2}{2} \text{ or } \frac{3s-6}{2}, & \text{if } s \equiv 2 \pmod{4}. \end{cases}$$

(c) $r = 4$ and $k = 2s - 2$ or $r = s = k = 4$.

We also make the following conjecture.

Conjecture 3.2. *Let $1 < r \leq s$, $1 \leq k < (rs - 1)/2$ and suppose $P(r, s, k) = P(r, s, k + 1)$. Then one of the following holds:*

(a) $r \in \{2, 3, 4\}$ and the possibilities for s and k are as in Theorem 3.1; or

(b) r, s and k have the values given by a column of the following table

r	5	5	5	6	6	6	6	6	7
s	6	10	14	6	7	9	11	12	10
k	14	24	34	16	20	26	32	38	34

Remark 3.3. *The quantity $P(r, s, k) - P(r, s, k - 1)$ is of interest in invariant theory. By a theorem of Cayley and Sylvester (see Satz 2.21 of [8]) it is equal to the number of linearly independent semi-invariants of degree r and weight k of a binary form of degree s . Conjecture 3.2, if proven, would then give the values of r, s and k for which no such semi-invariant exists.*

We now define some more notation which we will use in this section. Let $\mathcal{P}(r, s, k)$ be the set of partitions of k into at most r parts of size at most s , so $P(r, s, k) = |\mathcal{P}(r, s, k)|$. We will use the convention that $P(r, s, k) = 0$ if $k < 0$ or $k > rs$. By considering dual partitions we observe that $P(r, s, k) = P(s, r, k)$, so without loss we will assume that $r \leq s$. Elements of $\mathcal{P}(r, s, k)$ will be written (a_1, a_2, \dots, a_r) where $\sum_{i=1}^r a_i = k$ and $s \geq a_1 \geq \dots \geq a_r \geq 0$. Let $\mathcal{A}(r, s, k)$ be the subset of $\mathcal{P}(r, s, k)$ consisting of all partitions of the form (s, a_2, \dots, a_r) and let $\mathcal{B}(r, s, k + 1)$ be the subset of $\mathcal{P}(r, s, k + 1)$ consisting of all partitions of the form (x, x, a_3, \dots, a_r) , for some $x \leq s$. Furthermore, let $A(r, s, k) = |\mathcal{A}(r, s, k)|$ and $B(r, s, k) = |\mathcal{B}(r, s, k)|$. Note that $A(r, s, k) = P(r - 1, s, k - s)$. We will define a bijection from a subset of $\mathcal{P}(r, s, k)$ to a subset of $\mathcal{P}(r, s, k + 1)$. Let $(a_1, a_2, \dots, a_r) \in \mathcal{P}(r, s, k)$ with $s > a_1 \geq a_2 \geq \dots \geq a_r \geq 0$, and define

$$f(a_1, a_2, \dots, a_r) = (a_1 + 1, a_2, \dots, a_r).$$

Then f is a bijection from $\mathcal{P}(r, s, k) \setminus \mathcal{A}(r, s, k)$ to $\mathcal{P}(r, s, k + 1) \setminus \mathcal{B}(r, s, k + 1)$. In particular we have the following result.

Lemma 3.4. *Let $r, s, k \geq 1$. Then*

$$P(r, s, k + 1) - P(r, s, k) = B(r, s, k + 1) - A(r, s, k).$$

So the problem of determining when $P(r, s, k) = P(r, s, k + 1)$ reduces to that of determining when $B(r, s, k + 1) = A(r, s, k)$. We now consider in turn the cases when $r = 2, 3$ and 4 .

Lemma 3.5. *Let $s \geq 0$. Then*

$$P(2, s, k) = \begin{cases} 0, & \text{if } k > 2s, \text{ or } k < 0, \\ s - \lceil \frac{k}{2} \rceil + 1, & \text{if } s \leq k \leq 2s, \\ \lfloor \frac{k}{2} \rfloor + 1, & \text{if } 0 \leq k \leq s. \end{cases}$$

In particular, if $1 \leq k < s$, then $P(2, s, k) = P(2, s, k + 1)$ if and only if k is even.

Proof. Elementary. □

Proposition 3.6. *Let $s \geq 3$ and $1 \leq k < (3s - 1)/2$. Then $P(3, s, k) = P(3, s, k + 1)$ if and only if one of the following holds:*

- (a) s is odd and $k = (3s - 3)/2$,
- (b) $s \equiv 0 \pmod{4}$ and $k = (3s - 4)/2$,
- (c) $s \equiv 2 \pmod{4}$ and $k = (3s - 2)/2$ or $(3s - 6)/2$.

Proof. Let $d_k = P(3, s, k + 1) - P(3, s, k) = B(3, s, k + 1) - A(3, s, k)$. By Lemma 3.5,

$$A(3, s, k) = P(2, s, k - s) = \begin{cases} \lfloor \frac{k-s}{2} \rfloor + 1 & \text{if } s \leq k < (3s - 1)/2, \\ 0 & \text{if } k \leq s. \end{cases}$$

Moreover,

$$B(3, s, k + 1) = |\{(a, a, b) \mid s \geq a \geq b, 2a + b = k + 1\}| = \left\lfloor \frac{k+1}{2} \right\rfloor - \left\lfloor \frac{k+1}{3} \right\rfloor + 1.$$

Hence

$$B(3, s, k + 1) \geq \frac{k}{2} - \frac{k+3}{3} + 1 = \frac{k}{6} > 0.$$

So if $A(3, s, k) = 0$, then $d_k \geq k/6 > 0$. We may therefore assume that

$$s \leq k < (3s - 1)/2 \quad \text{and} \quad A(3, s, k) = \left\lfloor \frac{k-s}{2} \right\rfloor + 1.$$

Thus

$$d_k = \left\lfloor \frac{k+1}{2} \right\rfloor - \left\lfloor \frac{k+1}{3} \right\rfloor - \left\lfloor \frac{k-s}{2} \right\rfloor.$$

Suppose s is odd. Then $k + 1 \equiv k - s \pmod{2}$. Hence

$$d_k = \frac{k+1 - (k-s)}{2} - \left\lfloor \frac{k+1}{3} \right\rfloor = \frac{s+1}{2} - \left\lfloor \frac{k+1}{3} \right\rfloor.$$

Therefore

$$d_k = 0 \Leftrightarrow k \in \left\{ \frac{3s+1}{2}, \frac{3s-1}{2}, \frac{3s-3}{2} \right\}.$$

Since $k < (3s - 1)/2$, this forces $k = \frac{3s-3}{2}$.

Suppose s is even. Then

$$d_k \geq \frac{k}{2} - \frac{k+3}{3} - \frac{k-s}{2} = \frac{1}{6}(3s - 2k - 6).$$

Assume $d_k = 0$. Then $2k \geq 3s - 6$. Thus $3s/2 - 3 \leq k \leq 3s/2 - 1$ and so $\lceil \frac{k+1}{3} \rceil = \frac{s}{2}$. Therefore

$$d_k = \begin{cases} \frac{k}{2} - \frac{s}{2} - \frac{k-s}{2} = 0, & \text{if } k \text{ is even} \\ \frac{k+1}{2} - \frac{s}{2} - \frac{k-s-1}{2} = 1, & \text{if } k \text{ is odd, a contradiction.} \end{cases}$$

Thus k is even, $\frac{3s-6}{2} \leq k \leq \frac{3s-2}{2}$ and hence

$$k = \begin{cases} \frac{3s-4}{2} & \text{if } s \equiv 0 \pmod{4} \\ \frac{3s-2}{2} \text{ or } \frac{3s-6}{2} & \text{if } s \equiv 2 \pmod{4}. \end{cases}$$

□

Theorem 3.7. *Let $4 \leq r \leq s$ and $1 \leq k < (rs - 1)/2$. If $P(r, s, k) = P(r, s, k + 1)$, then $k \geq (r(s - 1) - 1)/2$ or $r = s = k = 4$.*

Proof. Suppose first that $k < s$. Then $A(r, s, k) = 0$ but $B(r, s, k) > 0$, since $r \geq 4$. Therefore by Lemma 3.4 $P(r, s, k) < P(r, s, k + 1)$. Now suppose that $k = s \geq 5$. Then

$$P(r, s, k) = P(r, k, k) = 2 + P(r, k - 2, k)$$

and

$$P(r, s, k + 1) = P(r, k, k + 1) = 3 + P(r, k - 2, k + 1).$$

Since $(r(k - 2) - 1)/2 \geq (4(k - 2) - 1)/2 = 2k - 9/2 > k$, applying Theorem 1.1 yields $P(r, k - 2, k) \leq P(r, k - 2, k + 1)$ and so $P(r, s, k) < P(r, s, k + 1)$ in this case.

It remains to show for $s < k < (r(s - 1) - 1)/2$ that $P(r, s, k) < P(r, s, k + 1)$. So we assume for a contradiction that $P(r, s, k) = P(r, s, k + 1)$ in this case. Observe that

$$P(r, s, k) = P(r, s - 1, k) + P(r - 1, s, k - s).$$

Since $k < (r(s - 1) - 1)/2$ and $k - s < (r(s - 1) - 1 - 2s)/2 < ((r - 1)s - 1)/2$, by Theorem 1.1, $P(r, s - 1, k) \leq P(r, s - 1, k + 1)$ and $P(r - 1, s, k - s) \leq P(r - 1, s, k - s + 1)$. So under our assumption we have $P(r - 1, s, k - s) = P(r - 1, s, k - s + 1)$. We now proceed by induction on r .

Suppose first that $r = 4$. Then by Proposition 3.6, $P(3, s, k - s) = P(3, s, k - s + 1)$ implies $3s/2 - 3 \leq k - s \leq 3s/2 - 1$. However $k < (4(s - 1) - 1)/2 = 2s - 5/2$, so $k - s \leq s - 3 < 3s/2 - 3$, a contradiction.

Now suppose $r > 4$ and the result holds for $r - 1$ in place of r . Since $P(r - 1, s, k - s) = P(r - 1, s, k - s + 1)$, we obtain by induction that

$$k - s \geq \frac{(r-1)(s-1)-1}{2} = \frac{rs-r-s}{2}.$$

Hence $k \geq (rs - r + s)/2 > (rs - 1)/2$, a contradiction. Therefore by induction the result holds for all $r \geq 4$. □

Proposition 3.8. *Let $s \geq 4$ and $2s - 2 \leq k \leq 2s - 1$. Then $P(4, s, k) = P(4, s, k + 1)$ if and only if $k = 2s - 2$.*

Proof. Since $r = 4$ is fixed, for this proof we will abbreviate $A(r, s, k)$ by $A(s, k)$ and $B(r, s, k)$ by $B(s, k)$. We first show that for all $s \geq 4$, $P(4, s, 2s - 2) = P(4, s, 2s - 1)$. We need to evaluate $B(s, k)$ more precisely. Now

$$\mathcal{B}(s, k) = \{(a, a, b, c) : s \geq a \geq b \geq c \geq 0, 2a + b + c = k\}.$$

Now $0 \leq b + c \leq 2a$ implies $2a \leq k \leq 4a$. Hence $\lceil \frac{k}{4} \rceil \leq a \leq \lfloor \frac{k}{2} \rfloor$. Thus

$$B(s, k) = \sum_{a=\lceil \frac{k}{4} \rceil}^{\lfloor \frac{k}{2} \rfloor} P(2, a, k - 2a).$$

By Lemma 3.5, the value of $P(2, a, k - 2a)$ depends on whether $0 \leq k - 2a \leq a$ or $a \leq k - 2a \leq 2a$. Now $2a - (k - 2a) = 4a - k \geq 0$. Also $k - 2a \geq a$ whenever $a \leq \lfloor \frac{k}{3} \rfloor$. Therefore by Lemma 3.5

$$B(s, k) = \sum_{a=\lceil \frac{k}{4} \rceil}^{\lfloor \frac{k}{3} \rfloor} (a - \lceil \frac{k-2a}{2} \rceil + 1) + \sum_{a=\lfloor \frac{k}{3} \rfloor + 1}^{\lfloor \frac{k}{2} \rfloor} (\lfloor \frac{k-2a}{2} \rfloor + 1). \quad (1)$$

It follows that

$$\begin{aligned} B(s, 2s - 1) &= \sum_{a=\lceil \frac{2s-1}{4} \rceil}^{\lfloor \frac{2s-1}{3} \rfloor} (a - \lceil \frac{2s-1-2a}{2} \rceil + 1) + \sum_{a=\lfloor \frac{2s-1}{3} \rfloor + 1}^{\lfloor \frac{2s-1}{2} \rfloor} (\lfloor \frac{2s-1-2a}{2} \rfloor + 1) \\ &= \sum_{a=\lceil \frac{s}{2} \rceil}^{\lfloor \frac{2s-1}{3} \rfloor} (2a - s + 1) + \sum_{a=\lfloor \frac{2s-1}{3} \rfloor + 1}^{s-1} (s - a) \\ &= (1 - s) (\lfloor \frac{2s-1}{3} \rfloor - \lceil \frac{s}{2} \rceil + 1) + \lfloor \frac{2s-1}{3} \rfloor (\lfloor \frac{2s-1}{3} \rfloor + 1) - \lceil \frac{s}{2} \rceil (\lceil \frac{s}{2} \rceil - 1) \\ &\quad + s (s - 1 - \lfloor \frac{2s-1}{3} \rfloor) - \frac{1}{2}(s - 1)s + \frac{1}{2} \lfloor \frac{2s-1}{3} \rfloor (\lfloor \frac{2s-1}{3} \rfloor + 1) \\ &= \lfloor \frac{2s-1}{3} \rfloor (\frac{3}{2} \lfloor \frac{2s-1}{3} \rfloor + 1 + 1 - s - s + \frac{1}{2}) + \lceil \frac{s}{2} \rceil (-\lceil \frac{s}{2} \rceil + s - 1 + 1) \\ &\quad + 1 - s + s^2 - s - \frac{1}{2}s^2 + \frac{1}{2}s \\ B(s, 2s - 1) &= \underbrace{\frac{1}{2} \lfloor \frac{2s-1}{3} \rfloor (3 \lfloor \frac{2s-1}{3} \rfloor + 5 - 4s)}_{X_3(B)} + \underbrace{\lceil \frac{s}{2} \rceil (s - \lceil \frac{s}{2} \rceil) + \frac{1}{2}(s^2 - 3s + 2)}_{X_2(B)} \end{aligned}$$

Note that $X_2(B)$ depends only on s modulo 2 and $X_3(B)$ depends only on s modulo 3.

We now work out $A(s, 2s - 2)$ in a similar fashion. Firstly note that $A(s, 2s - 2) = P(3, s, s - 2) = P(3, s - 2, s - 2)$, and

$$P(3, s - 2, s - 2) = \#\{a, b, c : a \geq b \geq c \geq 0, a + b + c = s - 2\}.$$

This implies that $\lceil \frac{s-2}{3} \rceil \leq a \leq s - 2$. Thus $A(s, 2s - 2) = \sum_{a=\lceil \frac{s-2}{3} \rceil}^{s-2} P(2, a, s - 2 - a)$. From Lemma 3.5, and noting that $s - 2 - a \geq a$ when $a \leq \lfloor \frac{s-2}{2} \rfloor$, we make the following calculation.

$$\begin{aligned} A(s, 2s - 2) &= \sum_{a=\lceil \frac{s-2}{3} \rceil}^{\lfloor \frac{s-2}{2} \rfloor} (a - \lceil \frac{s-2-a}{2} \rceil + 1) + \sum_{a=\lfloor \frac{s-2}{2} \rfloor + 1}^{s-2} (\lfloor \frac{s-2-a}{2} \rfloor + 1) \\ &= \sum_{a=\lceil \frac{s-2}{3} \rceil}^{\lfloor \frac{s}{2} \rfloor - 1} (a - (s - 2 - a)) + \sum_{a=\lceil \frac{s-2}{3} \rceil}^{s-2} (\lfloor \frac{s-a}{2} \rfloor) \\ &= \sum_{a=\lceil \frac{s-2}{3} \rceil}^{\lfloor \frac{s}{2} \rfloor - 1} (2a - s + 2) + \sum_{a=\lceil \frac{s-2}{3} \rceil}^{s-2} \binom{s-a}{2} - \frac{1}{2} \#\{i \in \{2, \dots, s - \lceil \frac{s-2}{3} \rceil\} : i \text{ odd}\} \end{aligned}$$

Now the number of odd numbers in the range $\{2, \dots, x\}$ is $\lfloor \frac{x-1}{2} \rfloor$, so the number of odd numbers in $\{2, \dots, s - \lceil \frac{s-2}{3} \rceil\}$ is $\lfloor \frac{\lfloor \frac{2s+2}{3} \rfloor - 1}{2} \rfloor = \lfloor \frac{2s-1}{6} \rfloor = \lfloor \frac{s-1}{3} \rfloor$. Therefore

$$\begin{aligned} A(s, 2s - 2) &= (2 - s) (\lfloor \frac{s}{2} \rfloor - \lceil \frac{s-2}{3} \rceil) + \lfloor \frac{s}{2} \rfloor (\lfloor \frac{s}{2} \rfloor - 1) - \lceil \frac{s-2}{3} \rceil (\lceil \frac{s-2}{3} \rceil - 1) \\ &\quad + \frac{s}{2} (s - 2 - \lceil \frac{s-2}{3} \rceil + 1) - \frac{1}{4} (s - 2)(s - 1) + \frac{1}{4} \lceil \frac{s-2}{3} \rceil (\lceil \frac{s-2}{3} \rceil - 1) - \frac{1}{2} \lfloor \frac{s-1}{3} \rfloor \\ &= \lfloor \frac{s}{2} \rfloor (2 - s + \lfloor \frac{s}{2} \rfloor - 1) + \frac{s}{2} (s - 1) - \frac{1}{4} (s - 2)(s - 1) \\ &\quad + \lceil \frac{s-2}{3} \rceil (s - 2 - \frac{3}{4} \lceil \frac{s-2}{3} \rceil + \frac{3}{4} - \frac{s}{2}) - \frac{1}{2} \lfloor \frac{s-1}{3} \rfloor \\ A(s, 2s - 2) &= \underbrace{\lfloor \frac{s}{2} \rfloor (\lfloor \frac{s}{2} \rfloor + 1 - s) + \frac{1}{4} (s - 1)(s + 2)}_{X_2(A)} + \underbrace{\frac{1}{4} \lceil \frac{s-2}{3} \rceil (2s - 5 - 3 \lceil \frac{s-2}{3} \rceil) - \frac{1}{2} \lfloor \frac{s-1}{3} \rfloor}_{X_3(A)}. \end{aligned}$$

Again note that $X_2(A)$ depends only on s modulo 2 and $X_3(A)$ depends only on s modulo 3.

Now $P(4, s, 2s - 2) = P(4, s, 2s - 1)$ if and only if $B(s, 2s - 1) = A(s, 2s - 2)$, which is if and only if $X_2(B) - X_2(A) = X_3(A) - X_3(B)$. We have

$$X_2(B) - X_2(A) = (\lceil \frac{s}{2} \rceil (s - \lceil \frac{s}{2} \rceil) + \frac{1}{2} (s^2 - 3s + 2)) - (\lfloor \frac{s}{2} \rfloor (\lfloor \frac{s}{2} \rfloor + 1 - s) + \frac{1}{4} (s - 1)(s + 2)).$$

A simple calculation shows that regardless of whether s is odd or even, $X_2(B) - X_2(A) = \frac{1}{4} (3s^2 - 9s + 6)$.

$$X_3(A) - X_3(B) = (\frac{1}{4} \lceil \frac{s-2}{3} \rceil (2s - 5 - 3 \lceil \frac{s-2}{3} \rceil) - \frac{1}{2} \lfloor \frac{s-1}{3} \rfloor) - (\frac{1}{2} \lfloor \frac{2s-1}{3} \rfloor (3 \lfloor \frac{2s-1}{3} \rfloor + 5 - 4s)).$$

Calculating for each possible value of s modulo 3 shows that in each case, $X_3(A) - X_3(B) = \frac{1}{4}(3s^2 - 9s + 6) = X_2(B) - X_2(A)$. Therefore, for all $s \geq 4$, $P(4, s, 2s-2) = P(4, s, 2s-1)$.

We now show that $P(4, s, 2s-1) < P(4, s, 2s)$ for all $s \geq 4$. Since $P(4, s, 2s-2) = P(4, s, 2s-1)$ for all $s \geq 4$, by substituting $s+1$ for s in Lemma 3.4 we have

$$A(s+1, 2s) = B(s+1, 2s+1). \quad (2)$$

Now $A(s, 2s-1) = P(3, s, s-1) = P(3, s-1, s-1)$ as no part of a partition of $s-1$ can exceed $s-1$. Similarly $A(s+1, 2s) = P(3, s+1, s-1) = P(3, s-1, s-1)$. Hence

$$A(s, 2s-1) = A(s+1, 2s). \quad (3)$$

Now we consider $B(s+1, 2s+1)$ compared to $B(s, 2s)$.

Setting $k+1 = 2s$ and $k+1 = 2s+1$, respectively, gives:

$$\begin{aligned} B(s, 2s) &= \sum_{a=\lceil \frac{s}{2} \rceil}^{\lfloor \frac{2s}{3} \rfloor} (a - (s-a) + 1) + \sum_{a=\lfloor \frac{2s}{3} \rfloor + 1}^s (s-a+1) \\ &= \sum_{a=\lceil \frac{s}{2} \rceil}^{\lfloor \frac{2s}{3} \rfloor} (2a - s + 1) + \sum_{a=\lfloor \frac{2s}{3} \rfloor + 1}^s (s-a+1); \\ B(s+1, 2s+1) &= \sum_{a=\lceil \frac{2s+1}{4} \rceil}^{\lfloor \frac{2s+1}{3} \rfloor} (a - (s-a+1) + 1) + \sum_{a=\lfloor \frac{2s+1}{3} \rfloor + 1}^s ((s-a) + 1) \\ &= \sum_{a=\lceil \frac{s+1}{2} \rceil}^{\lfloor \frac{2s+1}{3} \rfloor} (2a - s) + \sum_{a=\lfloor \frac{2s+1}{3} \rfloor + 1}^s (s-a+1). \end{aligned}$$

If $\lfloor \frac{2s}{3} \rfloor = \lfloor \frac{2s+1}{3} \rfloor$, then $B(s, 2s) - B(s+1, 2s+1) \geq \sum_{a=\lceil \frac{s}{2} \rceil}^{\lfloor \frac{2s+1}{3} \rfloor} 1 \geq \frac{2s-1}{3} - \frac{s-1}{2} > 0$.

If $\lfloor \frac{2s}{3} \rfloor < \lfloor \frac{2s+1}{3} \rfloor$ then $\lfloor \frac{2s}{3} \rfloor = \frac{2s-2}{3}$, $\lfloor \frac{2s+1}{3} \rfloor = \frac{2s+1}{3}$ and

$$\begin{aligned} B(s, 2s) - B(s+1, 2s+1) &\geq \left(\sum_{a=\lceil \frac{s}{2} \rceil}^{\lfloor \frac{2s}{3} \rfloor} 1 \right) - (2\lfloor \frac{2s+1}{3} \rfloor - s) + (s - (\lfloor \frac{2s}{3} \rfloor + 1) + 1) \\ &\geq \frac{2s-2}{3} - \frac{s-1}{2} - \frac{4s+2}{3} + 2s - \frac{2s-2}{3} \\ &= \frac{1}{6}(-3s + 3 - 8s - 4 + 12s) = \frac{1}{6}(s-1) > 0. \end{aligned}$$

Thus in any case $B(s, 2s) > B(s+1, 2s+1)$. Therefore by equations (2) and (3),

$$B(s, 2s) - A(s, 2s-1) > B(s+1, 2s+1) - A(s+1, 2s) = 0.$$

Hence by Lemma 3.4 $P(4, s, 2s-1) < P(4, s, 2s)$. \square

Proposition 3.9. *Let $s \geq 4$ and $1 \leq k \leq 2s - 1$. Then $P(4, s, k) = P(4, s, k + 1)$ if and only if $k = 2s - 2$ or $s = k = 4$.*

Proof. In the case $s = k = 4$ it be easily computed that $P(4, 4, 4) = P(4, 4, 5) = 5$. Otherwise, by Theorem 3.7, if $P(4, s, k) = P(4, s, k + 1)$, then $4(s - 1) - 1 \leq 2k$ and, since k is an integer, $2s - 2 \leq k$. We may now apply Proposition 3.8 to get the result. \square

Theorem 3.1 now follows immediately from Lemma 3.5, Proposition 3.6 and Proposition 3.9.

4 Primitive groups with equality

Primitive groups which are not $(k+1)$ -homogeneous but achieve equality in the Livingstone-Wagner Theorem for some $k < (n - 1)/2$ are fairly rare.

Remark 4.1. *The known primitive but not $(k + 1)$ -homogeneous groups G such that $\sigma_k(G) = \sigma_{k+1}(G)$, for some $k < (n - 1)/2$, are:*

- (a) $AGL(m, 2)$, for $m \geq 4, n = 2^m, k = 4$,
- (b) $ASL(2, 3)$ or $AGL(2, 3)$, for $n = 9, k = 3$,
- (c) $Sym(5), Sym(6), PGL(2, 9)$ or $P\Gamma L(2, 9)$, for $n = 10, k = 4$,
- (d) $M_{11}, PSL(2, 11), PGL(2, 11)$, for $n = 12, k = 4$,
- (e) $PSL(3, 3)$, for $n = 13, k = 4$,
- (f) $PGL(2, 13)$, for $n = 14, k = 4$,
- (g) $2^4 : Alt(6), 2^4 : Sym(6), 2^4 : Alt(7)$, for $n = 16, k = 6$,
- (h) $PGL(2, 17)$, for $n = 18, k = 6$ or 8 ,
- (i) M_{22} or $Aut(M_{22})$, for $n = 22, k = 8$,
- (j) M_{23} , for $n = 23, k = 8, 9$,
- (k) M_{24} , for $n = 24, k = 6, 8, 9$ or 10 .

Observe that many of these groups are subgroups of M_{24} .

Regarding case (a), we prove the following.

Proposition 4.2. *Let $G = AGL(m, 2)$, for $m \geq 4$, acting naturally on an m -dimensional vector space V over $GF(2)$. Then $\sigma_4(G) = \sigma_5(G) = 2$.*

Proof. Observe that the stabiliser in G of any three points of V fixes the fourth point in the unique affine plane containing these three points and is transitive on the remaining points of V . It follows that $\sigma_4(G) = 2$ and also G has a single orbit on the set of 5-subsets which contain affine planes. Let Δ be any set of five distinct points in V which does not contain any affine plane. Then Δ is not contained in an affine 3-dimensional subspace of V . Furthermore the stabiliser in G of an affine 3-dimensional subspace W is transitive on pairs (α, Λ) , where α is a point not in W and Λ is any set of four points in W which is not an affine plane. Therefore G has a single orbit on 5-subsets which do not contain any affine plane. Thus $\sigma_5(G) = 2$. \square

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